Means and Divided Differences

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1. Introduction

Let $f \in C^{2n}(0, \infty)$, with $f^{(2n)}(x) \neq 0$ on $(0, \infty)$. For $a$ and $b$ positive real numbers with $a \neq b$, let $f[b^{[n]}, a^{[n]}]$ denote the $(2n - 1)$st order divided difference of $f$ with $n$ occurrences of $b$ and $n$ occurrences of $a$. For example, $f[b, a]$ is the usual difference quotient $(f(b) - f(a))/(b - a)$, while

$$f[b^{[2]}, a^{[2]}] = f[b, b, a, a] = \frac{f''(b) - 2f[b, a] + f'(a)}{(b - a)^2}.$$ 

In general, divided differences at distinct points are defined inductively by

$$f[x_j, \ldots, x_0] = \frac{f[x_j, \ldots, x_i] - f[x_{j-1}, \ldots, x_0]}{x_j - x_0} \quad \text{with } f[x_0] = f(x_0).$$

For suitably differentiable $f$ we can allow some of the nodes to coalesce, in which case certain derivatives are involved. In particular, for $f \in C^{2n-1}[a, b], f[x_{2n-2}, \ldots, x_0]$ is a continuous function on $[a, b]^n$ (see [IK]). For the precise expansion of $f[b^{[n]}, a^{[n]}]$ in terms of function and derivative values of $f$ at $a$ and $b$, see Lemma 1 below.

Now by the mean-value theorem for divided differences (see [IK, p. 252, Corollary 2]), there is at least one point $c$ in $(a, b)$ such that $f[b^{[n]}, a^{[n]}] = f^{(2n-1)}(c)/(2n - 1)!$. (Technically we only know from the cited reference that $c \in [a, b]$. However, for $f^{(2n-1)}$ strictly monotone, it follows
that \( f[x_0, \ldots, x_{2n-1}] \) is a strictly monotone function of its arguments. If \( c = a \), say, then \( f^{(2n-1)}(a)/(2n - 1)! = f[b^{[n]}, a^{[n]}] > f[a^{[n]}, a^{[n]}] = f^{(2n-1)}(a)/(2n - 1)! \), a contradiction. Similarly we cannot have \( c = b \).

Since \( f^{(2n-1)} \) is strictly monotone, such a \( c \) is unique, and this defines a

**mean** \( M_f(a, b) \) in \( a \) and \( b \). Hence

\[
M_f(a, b) = (f^{(2n-1)})^{-1}((2n - 1)!f[b^{[n]}, a^{[n]}]).
\]  

(1)

Of course \( M_f \) depends on \( n \) as well, but we suppress this dependence in our notation. The case \( n = 1 \) was considered by Stolarsky [S] and Mays [M], in which case \( M_f(a, b) = (f')^{-1}((f(b) - f(a))/(b - a)) \).

The means we consider in this paper are all symmetric, i.e., \( M_f(a, b) = M_f(b, a) \). One could also produce non-symmetric means by considering \((f^m)^{-1}(r!f[b^{[m]}, a^{[m]}])\), where \( m \neq n \) and \( r = m + n - 1 \). For general \( n \) it also follows that if \( f(x) = x^p \), then \( M_f \) is a homogeneous mean—i.e., \( M_f(ka, kb) = kM_f(a, b) \), and we denote \( M_f \) by \( M_p \). For \( p \in \{0, 1, \ldots, 2n - 1\} \), \( M_p \) must be defined by taking a limit since in that case \( f^{(2n-1)} \) is not strictly monotonic. Equivalently, one could define \( M_p = M_f \), where \( f(x) = x^p \log x \), when \( p \in \{0, 1, \ldots, 2n - 1\} \).

In Section 2 we give a useful integral representation for \( M_f \) derived using the Peano Kernel Theorem (Theorem 1). In Section 3 we examine the special cases \( p = 2n, -1, \) and \( (2n - 1)/2 \). We prove that \( M_{2n}(a, b) = (a + b)/2 \), the arithmetic mean (Theorem 2), \( M_{-1}(a, b) = \sqrt{ab} \), the geometric mean (Theorem 3), and \( M_{(2n-1)/2}(a, b) = ((\sqrt{a} + \sqrt{b})/2)^2 \) (Theorem 4). For \( n = 1 \) it was noted by Mays that \((f')^{-1}((f(b) - f(a))/(b - a)) \)

never gives the harmonic mean. We prove that this is the case for all \( n \) (Theorem 7), using a series representation for \( M_p \) (Corollary 1 to Theorem 5), in Section 4.

Finally, it is interesting to examine the behavior of \( M_f \) as \( n \) approaches \( \infty \). In Section 5 we prove that \( M_p(a, b) \to \sqrt{ab} \) for any fixed \( p \) (Theorem 12). For a function such as \( e^x \), however, \( M_f(a, b) \to (a + b)/2 \) (Theorem 13). We do not know the limiting behavior in general for \( f(x) \neq x^p \).

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**2. Integral Representation**

First we prove a formula which expresses \( f[b^{[n]}, a^{[n]}] \) as a linear combination of the values of \( f \) and its first \( n - 1 \) derivatives at \( a \) and \( b \).

**Lemma 1.**

\[
f[b^{[n]}, a^{[n]}] = \frac{1}{(n - 1)!^2} \frac{1}{(b - a)^{2n-1}} \sum_{k=0}^{n-1} \binom{n - 1}{k} \times (2n - 2 - k)!(-1)^{n-1-k} (f^{(k)}(b) - (-1)^k f^{(k)}(a))(b - a)^k.
\]
Proof.

\[ f[b^{n+1}, a^{n+1}] = \frac{1}{(n!)^2} \frac{\partial^n}{\partial a^n} \left( \frac{\partial^n}{\partial b^n} f[b, a] \right) \]

(see [IK, p. 254, Corollary 7]). Now

\[
\frac{\partial^n}{\partial b^n} f[b, a] = \frac{\partial^n}{\partial b^n} \left( \frac{f(b) - f(a)}{b - a} \right)
\]

\[ = \sum_{k=0}^{n} \binom{n}{k} (f(b) - f(a))^k \left( \frac{1}{b - a} \right)^{(n-k)} \quad \text{(by Leibniz' rule)} \]

\[ = \sum_{k=1}^{n} \binom{n}{k} f^{(k)}(b)(-1)^{n-k}(n-k)!(b - a)^{-n+k-1} + \frac{f(b) - f(a)}{(b - a)^{n+1}} (-1)^n n! . \]

Hence

\[ \frac{\partial^n}{\partial a^n} \left( \frac{\partial^n}{\partial b^n} f[b, a] \right) = \sum_{k=1}^{n} \binom{n}{k} f^{(k)}(b)(-1)^{n-k}(n-k)!(b - a)^{-2n+k-1} \]

\[ \times (2n-k) \cdots (n-k+1) - (-1)^n n! \sum_{k=1}^{n} \binom{n}{k} f^{(k)}(a) \]

\[ \times (b - a)^{-2n+k-1}(2n-k) \cdots (n+1) \]

\[ + (-1)^n n! (f(b) - f(a))(b - a)^{-2n-1}(2n) \cdots (n+1) \]

\[ = \frac{1}{(b - a)^{2n+1}} \sum_{k=0}^{n} \binom{n}{k} (2n-k)! (-1)^{a-k} \]

\[ \times (f^{(k)}(b) - (-1)^kf^{(k)}(a))(b - a)^{k} . \]

Replacing \( n \) by \( n - 1 \) finishes the proof of Lemma 1.

Theorem 1. For any \( f \in C^{2n-1}[a, b] \),

\[ f[b^n, a^{n}] = \frac{1}{(n-1)!^2 (b - a)^{2n-1}} \int_a^b f^{(2n-1)}(t)((b - t)(t - a))^{n-1} dt . \]

Proof. Let \( L \) be the linear functional defined by \( L(f) = f[b^n, a^{n}] \) for fixed \( a < b \). Then \( L \) annihilates \( \pi_{2n-2} \), the polynomials of degree \( \leq 2n - 2 \). Hence by the Peano Kernel Theorem (see [D])

\[ L(f) = \int_a^b f^{(2n-1)}(t) K_n(t) \, dt \quad \text{for } f \in C^{2n-1}[a, b], \]  

(2)

where \( K_n(t) = (1/(2n-2)!) L_n[(x-t)^{2n-2}] \), with \((x-t)^{2n-2} = (x-t)^{2n-2}\) for \( t \leq x \) and \( = 0 \) for \( t > x \).

Applying \( L \) to the function \((x-t)^{2n-2}\) gives, by Lemma 1,

\[
\frac{1}{(n-1)!^2 (b-a)^{2n-1}} \times \sum_{k=0}^{n-1} \binom{n-1}{k} (2n-2-k)! (-1)^{n-1-k} \\
\times (b-t)^{2n-2-k}(2n-2) \cdots (2n-k-1)(b-a)^k
= \frac{(2n-2)!}{(n-1)!^2 (b-a)^{2n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} \\
\times (b-t)^{2n-2-k}(b-a)^k
= \frac{(-1)^{n-1}(2n-2)!}{(n-1)!^2 (b-a)^{2n-1}} \\
\times (b-t)^{2n-2} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left( \frac{b-a}{b-t} \right)^k
\]

(by the Binomial Theorem).

The theorem then follows from (2).

**Remark.** Lemma 1 is of interest in its own right since it gives an explicit formula for computing the means \( M_f \), assuming one can compute \( (f^{(2n-1)})^{-1} \). Theorem 1 can be proven more easily, however, without Lemma 1. By [OS, p. 11, formula (9)], \( f[b^{[n]}, a^{[n]}] = (1/(n-1)!^2) \int_0^1 f^{(2n-1)} \left((1-t)a + tb\right)^{(n-1)} dt \). Letting \( u = (1-t)a + tb \) gives Theorem 1.

It is important to note that by (1) and Theorem 1, the means \( M_f \) can now be written as

\[ M_f(a, b) = g^{-1} \left( \int_a^b g(t) E_n(t) \, dt \right); \]

(3)

where \( g(t) = f^{(2n-1)}(t) \) and \( E_n(t) = ((2n-1)!/(n-1)!^2 (b-a)^{2n-1}) \\
\times ((b-t)(t-a))^{n-1} \).

Since \( \int_a^b E_n(t) \, dt = 1 \), this takes the form \( g^{-1} (\int_a^b g(t) \, d\mu(t)) \), where \( d\mu \) is a probability measure. When \( \mu \) is concentrated at a finite set of points, one obtains a class of means discussed in [HLP] (See also the discussion in [M].)
It follows easily from (3) that the means $M_p$ are homogeneous. In the next section we examine $M_p$ for certain values of $p$.

3. Special Cases

**Theorem 2.** $M_{2n}(a, b) = (a + b)/2$.

*Proof.* If $f(x) = x^m$, $m$ a positive integer $\geq n$, then $f[x_0, \ldots, x_{m-1}] = x_0 + \cdots + x_{m-1} = (n-1)^{m-1} f[x_0, \ldots, x_{n-1}]$ (see [OS, p. 3]). For $m = 2n$, $x_0 = \cdots = x_{n-1} = a$, and $x_n = \cdots = x_{2n-1} = b$, $f[b^{n-1}, a^{n-1}] = n(a + b)$. Thus, by (1), $M_f(a, b) = (f^{(2n-1)})^{-1}[(2n - 1)! f[b^{n-1}, a^{n-1}] = (1/(2n)) (2n - 1)! n(a + b) = (a + b)/2.$

**Theorem 3.** $M_{-1}(a, b) = \sqrt{ab}$.

*Proof.* If $f(x) = 1/x$, then it is easy to prove, using induction, that $f[x_0, \ldots, x_n] = (-1)^n/(x_0 \cdots x_n)$ (or see [OS, p. 11, formula (4)]), and we omit the proof. It follows, then, that $f[b^{n-1}, a^{n-1}] = (-1)^n/b^n a^n$. Since $f^{(2n-1)}(x) = (-1)^{2n-1}(2n - 1)!/x^{2n}$, $M_f(a, b) = (f^{(2n-1)})^{-1}[(2n - 1)! f[b^{n-1}, a^{n-1}] = (b^n, a^n)^{1/2n} = \sqrt{ab}$.

**Remark.** It is interesting to note that the function $f(x) = 1/x$ gives the same mean, the geometric mean, for all $n$. A similar phenomenon occurs for $1/x$ in [H1], [H2], where the mean in that case is the harmonic mean.

One could also prove Theorems 2 and 3 using (3), but the proofs given above are easier. For the case $p = (2n - 1)/n$, however, there is no simple formula for the divided differences of $x^p$. Hence we use the integral representation in (3). First we need the following lemma.

**Lemma 2.** $\int_a^b [(b - t)(t - a)]^{n-1} t^{-n+1/2} \, dt = (2^{2n-1}/(2n - 1)! (n - 1)! (\sqrt{b} - \sqrt{a})^{2n-1}$.

*Proof.* We claim

$$p(x) \equiv \int_1^x [(x^2 - t)(t - 1)]^{n-1} t^{-n+1/2} \, dt$$

is a polynomial in $x$ of degree $\leq 2n - 1$.

To prove (4), first

$$(x^2 - t)^{n-1}(t - 1)^{n-1} = \left( \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} t^{n-1-k} \right) \left( \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{n-1-j} t^j \right).$$
Then the highest power of \( x \) occurring in \( p \) is obtained by taking \( j = n - 1 \) in the second summation. This yields

\[
\left( \sum_{k=0}^{n-1} \binom{n-1}{k} x^{2k} (-1)^{n-1-k} \right) \left( \int_1^x t^{n-1-k} t^{n-1} t^{-n+1/2} dt \right) = \sum_{k=0}^{n-1} \binom{n-1}{k} x^{2k} (-1)^{n-1-k} x^{2n-2k-1} = \left( \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} \right) x^{2n-1}.
\]

Other values of \( j \) in the second summation yield lower powers of \( x \) in \( p(x) \). Since \( \int_1^x t^{n-1-k} t^{-n+1/2} dt = (x^{j-2k+1} - 1)/(j - k + \frac{1}{2}) \), \( p \) is a polynomial in \( x \). This proves (4). Now \( p(x) = q(x^2) \), where

\[
q(u) = \int_1^u [(u - t)(t - 1)]^{n-1} t^{-n+1/2} dt.
\]

We claim

\[
q^{(j)}(1) = 0 \quad \text{for } j = 0, 1, \ldots, 2n - 2.
\]

To prove (6), by (5), \( q^{(j)}(u) = (n - 1) \cdots (n - j) \int_1^u (u - t)^{n-1-j} (t - 1)^{n-1} t^{-n+1/2} dt \) for \( j = 1, \ldots, n - 1 \). This follows from the formula \((\partial/\partial u) \int_a^u f(t) K(u, t) dt = \int_a^u f(t) (\partial/\partial u) (K(u, t)) dt + K(u, u) \). Hence \( q^{(j)}(1) = 0 \) for \( j = 0, 1, \ldots, n - 1 \). Now \( q^{(n-1)}(u) = (n - 1)! \int_1^u (t - 1)^{n-1} t^{-n+1/2} dt \), and hence \( q^{(n)}(u) = (n - 1)! (u - 1)^{n-1} u^{-n+1/2} \Rightarrow q^{(n)}(1) = 0 \), and \( q^{(j)}(1) = 0 \) for \( j = n + 1, \ldots, 2n - 2 \) by Leibniz' rule. This proves (6).

By the Chain Rule, \( p^{(j)}(x) = \sum_{k=0}^j \alpha_k(x) q^{(k)}(x^2) \), where the \( \alpha_k \) are monomials in \( x \). Equation (6) then implies that \( p^{(j)}(1) = 0 \) for \( j = 0, 1, \ldots, 2n - 2 \). Since \( p \in \pi_{2n-1} \) (by (4)), \( p(x) = C_n(x - 1)^{2n-1} \) for some constant \( C_n \). Now let \( h(b, a) = \int_a^b [(b - t)(t - a)]^{n-1} t^{-n+1/2} dt \), the integral in the lemma. Then \( h(ka, kb) = \int_{ka}^{kb} [(kb - t)(t - ka)]^{n-1} t^{-n+1/2} dt = k^{n-1/2} \int_a^b [(b - u)(u - a)]^{n-1} u^{-n+1/2} du \) (letting \( u = t/k \) \( k^{n-1/2} h(b, a) \). Now \( h(b, 1) = p(\sqrt{b}) \) (see (4)) \( = C_n(\sqrt{b} - 1)^{2n-1} \), and thus \( h(b, a) = h(a \cdot (b/a), a \cdot 1) = a^{n-1/2} C_n(\sqrt{b}/a - 1)^{2n-1} = C_n(\sqrt{b} - \sqrt{a})^{2n-1} \). It remains to determine the constant \( C_n \). So let \( b = 1 \) and \( a = 0 \) to get \( C_n = \int_0^1 (1 - t)^{-n-1/2} dt = B(\frac{1}{2}, n) = B(n, \frac{1}{2}) = 2^{2n-1} B(n, n) \) (see [3W]) \( = 2^{2n-1}(n - 1)!/(2n - 1)! \), where \( B \) is the beta function. This completes the proof of Lemma 2.

**Theorem 4.** \( M_{(2n-1)/2}(a, b) = ((\sqrt{b} + \sqrt{a})/2)^2 \).

**Proof.** Let \( f(t) = t^{(2n-1)/2} \Rightarrow f^{(2n-1)}(t) = (\text{constant}) t^{-n+1/2} \). By (3),
\[ M_{12n-1}^{1/2}(a, b) = \left( \frac{(2n - 1)!}{(n - 1)!^2 (b - a)^{2n-1}} \int_a^b ((b - t)(t - a))^{n-1} t^{n+1/2} dt \right)^{1/(n+1/2)}. \]

By Lemma 2 we get

\[ \left( 2^{2n-1} \left( \frac{\sqrt{b} - \sqrt{a}}{b - a} \right)^{2n-1} \right)^{1/(n+1/2)} = \left( \frac{\sqrt{b} + \sqrt{a}}{2} \right)^2. \]

4. Series Expansion and Some Comparisons

**Theorem 5.** Let \( f \) be analytic with \( f^{(2n)}(x) \neq 0 \) in \((0, \infty)\) for some positive integer \( n \). For fixed \( a < x \), let \( M(x) = M_f(a, x) \). Also, let \( g(x) = f^{(2n-1)}(x) \) with \( a_j = g^{(j)}(a)/j! \). Then

(i) \( M'(a) = \frac{1}{2} \).

(ii) \( M''(a) = \frac{1}{2(2n + 1) a_1} \).

(iii) \( M'''(a) = \frac{-6a_3^3 + 9a_2 a_4}{4(2n + 1) a_1^3} \).

(iv) \( M^{(iv)}(a) = \frac{3a_3^2 a_2 (2n + 1)(4n + 7) - 9a_3 a_2 a_4 (2n + 1)(2n + 3) + a_3^2 (2n + 3)(8n + 3)}{2a_1^3 (2n + 3)(2n + 1)^2} \).

**Proof.** By (3),

\[ M(x) = g^{-1} \left( \int_a^x g(t) E_n(t) \, dt \right), \]

\[ E_n(t) = \frac{(2n - 1)!}{(n - 1)!^2 (x - a)^{2n-1}} ((x - t)(t - a))^{n-1}. \]

Hence

\[ g(M(x)) = \frac{(2n - 1)!}{(n - 1)!^2 (x - a)^{2n-1}} \int_a^x g(t) ((x - t)(t - a))^{n-1} dt. \quad (7) \]

Note that if \( f \) is analytic, then \( f(x^n, a^{[n]}) \) is an analytic function of \( x \). Thus \( M(x) = (f^{(2n-1)})^{-1}((2n - 1)! f(x^n, a^{[n]}) \) is as well, and we can differentiate \( M(x) \) as many times as we wish.
Now
\[(x - t)^{n-1}(t - a)^{n-1} = ((x - a) + (a - t))^{n-1}(t - a)^{n-1}\]
\[= \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (a - t)^k (x - a)^{n-1-k}\right)(t - a)^{n-1}\]
\[= (x - a)^{n-1}(t - a)^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{(t - a)^k}{(x - a)^k}.\]

Then
\[g(t)(x - t)^{n-1}(t - a)^{n-1} = \left(\sum_{m=0}^{\infty} a_m(t - a)^m\right)\]
\[\times \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{(t - a)^k}{(x - a)^k}\right)(x - a)^{n-1}(t - a)^{n-1}.\]

A little work then gives
\[\int_a^x g(t)(x - t)^{n-1}(t - a)^{n-1} \, dt\]
\[= \sum_{m=0}^{\infty} a_m(x - a)^{n+m} \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{1}{n+k+m}\right)(x - a)^{n-1}\]
\[= \left(\sum_{m=0}^{\infty} a_m(x - a)^m \frac{(n-1)!(n+m-1)!}{(2n+m-1)!}\right)(x - a)^{2n-1}.\]

Then by (7) we have
\[g(M(x)) = \sum_{m=0}^{\infty} a_m e_m(x - a)^m, \quad \text{where} \quad e_m = \frac{(2n-1)!(n+m-1)!}{(n-1)!(2n+m-1)!}.\]  
(8)

Then
\[g'(M(x))g'(x) = \sum_{m=1}^{\infty} ma_m e_m(x - a)^{m-1}\]  
(9)

\[\Rightarrow g'(a)M'(a) \quad \text{(Note that} \quad M(a) = a = a_1 e_1 \Rightarrow M'(a) = e_1 = \frac{1}{2}. \text{One then proceeds to differentiate both sides of (9) with respect to} \quad x, \quad \text{plug in} \quad x = a, \quad \text{and then solve for} \quad M''(a). \text{Continuing in this fashion one obtains as}\]
many derivatives of $M$ at $a$ as one likes, though the process becomes increasingly cumbersome. The author will supply the rest of the details of the proof of Theorem 5 upon request.

**Remark.** Theorem 5 holds if $f$ has enough continuous derivatives—$f$ need not be analytic, though that assumption makes the proof easier to write out. It is sufficient for our purposes to note that if $f \in C^{2n+1}(0, \infty)$, then (i) and (ii) still hold. This follows immediately from the fact that $g \in C^2 \Rightarrow g'(t) - g'(a) = g'(\xi)(x - a)$ and $g''(t) - g''(a) = g''(\xi)(x - a)$.

**Corollary 1.** For $M(x) = M_p(x, 1)$, we have

(i) $M''(1) = (p - 2n)/2(2n + 1)$. 

(ii) $M^{\text{iv}}(1) = (3(p - 2n)/16(2n + 3)(2n + 1)^2)((p - 2n - 1)(p - 2n - 2)(2n + 1)(4n + 7) - 6(p - 2n)(p - 2n - 1)(2n + 1)(2n + 3) + (p - 2n)^3(2n + 1)(8n + 3)).$

**Theorem 6.** Let $m$ and $n$ be distinct positive integers, and $p$ and $q$ real numbers. Let $M(a, b)$ be the mean $M_p(a, b)$ corresponding to $m$, and let $N(a, b)$ be the mean $M_q(a, b)$ corresponding to $n$. Suppose that $M = N$. Then one of the following holds:

(A) $p = q = -1 \Rightarrow M = N = \sqrt{ab}$.

(B) $p = 2m$ and $q = 2n \Rightarrow M = N = (a + b)/2$.

(C) $p = (2m - 1)/2$ and $q = (2n - 1)/2 \Rightarrow M = N = ((\sqrt{b} + \sqrt{a})/2)^2$.

**Proof.** Since $M''(1) = N''(1)$, by Corollary 1 we have

$$p = (2m + 1) \left( \frac{1}{2n + 1} q + \frac{2m}{2m + 1} - \frac{2n}{2n + 1} \right).$$  \hspace{1cm} (10)

Setting $M^{\text{iv}}(1) = N^{\text{iv}}(1)$ gives nothing new, so we proceed to $M^{\text{iv}}(1) = N^{\text{iv}}(1)$, which implies

$$R(q) = -3(q + 1)(2q^2 - (6n - 1)q + 4n^2 - 2n) = 0.$$ \hspace{1cm} (11)

The solutions of (11) are easily seen to be $q = -1, 2n,$ and $(2n - 1)/2$. Substituting those values for $q$ into (10) gives $p = -1, 2m,$ and $(2m - 1)/2$, respectively. Theorems 2, 3, and 4 then give the possibilities for $M$ and $N$ above.

Note that for $n = 1$ and $p = 0$, which really corresponds to $f(x) = \log x$, we have $M_p(a, b) = L(a, b) = (b - a)/(\log b - \log a)$, the logarithmic mean. An immediate consequence, then, of Theorem 6 is that $M_p \neq L$ for
any \( n > 1 \). Conspicuous by its absence in all our previous theorems is the harmonic mean. The following theorem shows why.

**Theorem 7.** For any positive integer \( n \), \( M_p(a, b) \) is never the harmonic mean \( H(a, b) = 2ab/(a + b) \).

**Proof.** Let \( M(x) = H(a, x) \) for fixed \( a > 0 \). A simple computation gives \( M''(a) = -1/2a \). By Theorem 5, \( M''(a) = (p - 2n)/(4(2n + 1)a) \), and hence \( (p - 2n)/(4(2n + 1)a) = -1/2a \), which implies that \( p = -2n - 2 \). Then \( g(x) = f^{2n-1}(x) = Cx^{-4n-1} \) for some constant \( C \). There is no loss of generality in taking \( C = 1 \). Then \( a_j = g^{(j)}(1)/j! \), and we have \( a_j = ((-1)^j/j! \prod_{i=1}^{j} (4n + i) \). Now \( M^{(iv)}(1) = -\frac{3}{2} \) (again, considering \( M^{(iv)} \) gives no new information). By Corollary 1, after substituting for \( a_1, \ldots, a_4 \), we obtain

\[
M^{(iv)}(1) = -\frac{3}{2} \left( \frac{n + 3}{2n + 3} \right) = \left( \frac{n + 3}{2n + 3} \right) = 1 \Rightarrow n = 0,
\]

which contradicts the fact that \( n \) is a positive integer.

One might wonder if Theorem 7 holds if \( f(x) \) is not of the form \( cx^p \) (which gives the same mean as \( x^p \) for any \( c \neq 0 \)). If \( f \in C^{2n+1}(0, \infty) \), then by Theorem 5(i) and (ii) (see also the remark following Theorem 5), \( M''(a) = 1/4(2n + 1)(g''(a)/g'(a)) \). Now if \( M_f(a, b) = 2ab/(a + b) \), then \( M''(a) = -1/2a \) implies that \( g''(a)/g'(a) = 4(2n + 1)(-1/2a) \) for all \( a > 0 \). Solving that differential equation gives \( g(x) = cx^{-4n-1} \). One can then use Theorem 7 as above.

**Remark.** It should be noted that the harmonic mean does arise among the more general class of means defined by \( (f_1(2n)/f_2(2n))^{-1}((f_1(b^{2n}), a^{2n}))/((f_2(b^{2n}), a^{2n})) \). For example, if \( n = 1, f_1(x) = x^a, \) and \( f_2(x) = x^b \), one obtains Stolarsky’s means \( (\beta(b^\alpha - a^\alpha)/\alpha(b^\beta - a^\beta))^{1/(\alpha - \beta)} \) (see [S]), which give the harmonic mean for \( \alpha = -2 \) and \( \beta = -1 \).

We now prove a result concerning the comparability of the means \( M_{f_1} \) and \( M_{f_2} \).

**Theorem 8.** Let \( n \) be a positive integer, and suppose \( f_1 \) and \( f_2 \) are in \( C^{2n+1}(0, \infty) \) with \( f_1^{(2n)} > 0 \) and \( f_2^{(2n)} > 0 \) on \( (0, \infty) \). Then \( M_{f_1}(a, b) \leq M_{f_2}(a, b) \) for all \( 0 < a < b \) if and only if \( f_1^{(2n+1)}(x)/f_2^{(2n)}(x) \leq f_2^{(2n+1)}(x)/f_1^{(2n)}(x) \) for all \( x > 0 \).

**Proof.** Suppose \( M_{f_1}(a, b) \leq M_{f_2}(a, b) \) for all \( 0 < a < b \). For fixed \( a \), let \( M_i(b) = M_i(a, b), i = 1, 2 \). Since \( M_1(a) = M_2(a) = a \) and \( M_2(a) = M_2(a) = \frac{1}{2} \) (by Theorem 5(i) and the remark following), \( (M_1(b) - M_1(a)/(b - a)^2 \leq (M_2(b) - M_2(a) - M_2(a))/(b - a)) \).
\[(b - a)^2, M_1[b, a, a] \leq M_2[b, a, a] \Rightarrow M_1'(a) \leq M_2'(a),\] which by Theorem 5 implies that \[f_i^{2n+1}(a)/f_i^{2n}(a) \leq f_2^{2n+1}(a)/f_2^{2n}(a).\] Since \(a\) can be any positive number, this completes the proof of \((\Rightarrow)\). To prove \((\Leftarrow)\), by (3), \[M_i(a, b) = g_i^{-1}(\int_a^b g_i(t) \, E_n(t) \, dt),\] where \(g_i(t) = \int_t^{2n+1}(t), \quad i = 1, 2,\) and \(\int_a^b E_n(t) \, dt = 1.\) Approximate \(f_i^{2n} g_i(t) E_n(t) \, dt\) by a Riemann sum of the form \(S_i = \sum_{j=1}^m \lambda_j g_i(a_j)\) with \(a \leq a_1 \leq \cdots \leq a_m \leq b\) and \(\sum_{j=1}^m \lambda_j = 1\) (we are looking at the integral as a Stieltjes integral with respect to a probability measure).

Now
\[
\frac{f_i^{2n+1}(x)}{f_i^{2n}(x)} \leq \frac{f_2^{2n+1}(x)}{f_2^{2n}(x)} \iff \frac{g_i'(x)}{g_i''(x)} \leq \frac{g_2'(x)}{g_2''(x)},
\]

which is easily seen to be equivalent to \(g_2 \circ g_1^{-1}\) being convex. By [HLP, Theorem 92], \(g_1^{-1}(S_1) \leq g_2^{-1}(S_2)\) for any choice of the \(a_i\)'s and \(\lambda_j\)'s. Taking the limit as the Riemann sum \(S_i\) approaches \(\int_a^b g_i(t) E_n(t) \, dt\) gives \(M_i(a, b) \leq M_i'(a, b).\)

**Theorem 9.** \(M_i(a, b) = M_i'(a, b)\) for all \(0 < a < b\) if and only if \(f_2 = cf_1 + p\) for some nonzero constant \(c\) and some \(p \in \pi_{2n-2}.\)

**Proof.** \((\Rightarrow)\) If \(M_i = M_i',\) then by Theorem 8, \(f_i^{2n+1}(x)/f_i^{2n}(x) = f_2^{2n+1}(x)/f_2^{2n}(x)\) for all \(x > 0,\) which implies that \(f_2^{2n}(a) = cf_1^{2n}, c \neq 0.\) Integrating \(2n\) times gives \(f_2 = cf_1 + p\) for some nonzero constant \(c\) and some \(p \in \pi_{2n-2}.\) \((\Leftarrow)\) This is easy and we omit the proof.

**Theorem 10.** The means \(M_p\) are increasing in \(p,\) for each \(n.\)

**Proof.** For \(p_1 < p_2,\) let \(f_i(x) = x^p, i = 1, 2.\) Then
\[
\frac{f_i^{2n+1}(x)}{f_i^{2n}(x)} = \frac{p_1 - 2n}{x} \leq \frac{p_2 - 2n}{x} = \frac{f_2^{2n+1}(x)}{f_2^{2n}(x)}.
\]

By Theorem 8, \(M_{p_i}(a, b) \leq M_{p_2}(a, b)\) if neither \(p_1\) nor \(p_2\) is an integer between \(0\) and \(2n - 1.\) For \(p \in \{0, 1, \ldots, 2n - 1\}\) Theorem 8 follows by taking a limit, or one may apply Theorem 8 with \(f_i(x) = x^p, \log x.\)

**Theorem 11.** For each \(n,\) \(\lim_{p \to -\infty} M_p(a, b) = \max\{a, b\}\) and \(\lim_{p \to -\infty} M_p(a, b) = \min\{a, b\}.

**Proof.** Let \(f(x) = x^p, g = f^{(2n-1)}.\) As noted earlier we can approximate \(\int_a^b g(t) E_n(t) \, dt\) by a Riemann sum of the form \(S = \sum_{j=1}^m \lambda_j g(a_j)\) with \(a \leq a_1 \leq \cdots \leq a_m \leq b\) and \(\sum_{j=1}^m \lambda_j = 1.\) By [HLP, Theorem 4], the means \(g^{-1}(S)\) approach \(\max\{a_1, \ldots, a_m\}\) as \(p \to -\infty\) and approach \(\min\{a_1, \ldots, a_m\}\) as \(p \to -\infty.\) Assume without loss of generality that \(a < b.\) Then letting
$a = a_1$ and $b = a_n$ and letting the Riemann sums $S$ approach the Stieltjes integral $\int_a^b g(t)E_n(t)\,dt$ give Theorem 11.

5. ASYMPTOTICS

In this section we examine the behavior of the means $M_f$ as $n \to \infty$.

**Theorem 12.** $\lim_{n \to \infty} M_p(a, b) = \sqrt{ab}$ for each $p$.

**Proof.** By (3), for $p$ not an element of $\{0, 1, \ldots, 2n - 1\}$, we have

$$M_p(a, b) = \left( \int_a^b t^{p-2n+1} E_n(t)\,dt \right)^{1/(p-2n+1)}, \quad \text{where } E_n(t)$$

$$= \frac{(2n-1)!}{(n-1)!^2 (b-a)^{2n-1}((b-t)(t-a))^{n-1}}. \quad (12)$$

Now make the change of variable $t = (by + a)/(y + 1) \Rightarrow dt = ((b - a)/(y + 1)^2)\,dy$. Then (12) becomes

$$M_p(a, b) = \frac{(2n-1)!}{(n-1)!^2} \left( \int_0^\infty \frac{(by + a)^{p-2n+1}(y + 1)^{-(p+1)y^{n-1}}\,dy}{y} \right)^{1/(p-2n+1)}. \quad (13)$$

Let $I_1 = \int_{a/b}^\infty (by + a)^{p-2n+1}(y + 1)^{-(p+1)y^{n-1}}\,dy$ and $I_2 = \int_0^{a/b} (by + a)^{p-2n+1}(y + 1)^{-(p+1)y^{n-1}}\,dy$. Rewrite $I_1$ as

$$\int_{a/b}^\infty \left( \frac{y}{(by + a)^2} \right)^n \left( \frac{by + a}{y + 1} \right)^{p+1} \frac{1}{y}\,dy.$$

We now use Laplace’s method to estimate $I_1$. We use the notation in [OL, Sect. 7.2] with $n$ replacing $x$ and $y$ replacing $t$. Let $p(y) = \log((by + a)^2/y)$, $q(y) = (1/y)((by + a)/(y + 1))^{p+1}$, so that $I_1 = \int_{a/b}^\infty e^{-nq(y)} q(y)\,dy$. Then $p'(a/b) = 0$ and $p''(a/b) = b^2/2a^2 > 0$. It follows that $p$ is increasing on $(a/b, \infty)$ and attains its unique minimum at $a/b$. Also, $p(y) - p(a/b) \sim (b^2/4a^2)(y - a/b)^2$ as $y \to a/b +$ and $q(y) \sim q(a/b)(y - a/b)^0$ since $q(a/b) \neq 0$. Then by [OL, Theorem 7.1],

$$I_1 \sim \frac{1/2(ab/(a + b)\Gamma(1/2)e^{-n \log(4a)}}{(b^2/4a^2)^{1/2}} \quad \text{as } n \to \infty. \quad (14)$$
Now write
\[ I_2 = \int_{0}^{\frac{a}{b}} \left( \frac{y}{(by + a)^2} \right)^n \left( \frac{by + a}{y + 1} \right)^{p+1} \frac{1}{y} dy. \]

If we let \( u = -y \), \( I_2 \) becomes
\[ \int_{-\frac{a}{b}}^{0} \left( \frac{-u}{(-bu + a)^2} \right)^n \left( \frac{-bu + a}{-u + 1} \right)^{p+1} \left( \frac{-1}{u} \right) du. \]

Letting \( p(u) = \log((-bu + a)^2/-u) \), \( q(u) = (-1/u)((-bu + a)/(-u + 1))^{p+1} \), as earlier by Laplace's method we have
\[ I_2 \sim \frac{1}{2} (2ab/(a + b))^{p+1}(b/a)\Gamma(\frac{1}{2})e^{-n \log(4ab)} \frac{(b/4a^2)^n}{((b^2/4a^2)^n)^{1/2}} \quad \text{as } n \to \infty. \quad (15) \]

By (14) and (15),
\[ I_1 + I_2 \sim \frac{(2ab/(a + b))^{p+1}(b/a)\Gamma(\frac{1}{2})e^{-n \log(4ab)}}{((b^2/4a^2)^n)^{1/2}} \quad \text{as } n \to \infty \]
\[ \Rightarrow \]
\[ (I_1 + I_2)^{1/(p-2n+1)} \sim (4ab)^{-n/(p-2n+1)} \to 2\sqrt{ab} \quad \text{as } n \to \infty \]

(it follows easily that the other terms in the asymptotic expression for \( I_1 + I_2 \) tend to 1 when raised to the 1/(p - 2n + 1) power). Thus we have shown
\[ \left( \int_{0}^{\infty} (by + a)^{p-2n+1}(y + 1)^{-p+1}y^{n-1} \frac{1}{y} dy \right)^{1/(p-2n+1)} \sim 2\sqrt{ab} \quad (16) \]

(Note that the integral in (16) equals \( I_1 + I_2 \).) A simple application of Stirling's formula shows that \( ((2n-1)!/(n-1)!)^{1/(p-2n+1)} \sim \frac{1}{2} \) as \( n \to \infty \), by (13) and (16), this finishes the proof of Theorem 12 if \( p \not\in \{0, 1, \ldots, 2n - 1\} \). If \( p \in \{0, 1, \ldots, 2n - 1\} \), Theorem 12 follows immediately from Theorem 10 and the case just proved.

It is interesting to ask what means arise from \( \lim_{n \to \infty} M_f \) when \( f \) is not of the form \( x^p \). We do not know the answer to this in general, but we now show that the geometric mean is not the only such mean.
THEOREM 13. If $f(x) = e^x$, then $\lim_{n \to \infty} M_f(a, b) = (a + b)/2$.

Proof. By (3), $M_f(a, b) = \log(\int_a^b e^t E_n(t) \, dt)$. Again letting $t = (by + a)/(y + 1)$, we have

$$M_f(a, b) = \log \left( \frac{(2n - 1)!}{(n - 1)!^2} \int_0^\infty \left( \frac{y}{(y + 1)^2} \right)^n y^{-1} e^{(by + a)/(y + 1)} \, dy \right).$$

Let

$$I_1 = \int_1^\infty \left( \frac{y}{(y + 1)^2} \right)^n y^{-1} e^{(by + a)/(y + 1)} \, dy = \int_1^\infty e^{-n \log(y/(y + 1)^2)} y^{-1} e^{(by + a)/(y + 1)} \, dy.$$

Again we use Laplace's method. Letting $p(y) = \log(y/(y + 1)^2)$ and $q(y) = y^{-1} e^{(by + a)/(y + 1)}$, by [OL, Theorem 7.1], we have

$$I_1 \sim \frac{1}{2} e^{(a+b)/2} \frac{1}{\sqrt{n}} \frac{2}{(2n - 1)!} \Gamma \left( \frac{1}{2} \right) \frac{\sqrt{n}}{(4n)!^{1/2}} \quad \text{as } n \to \infty$$

by a simple application of Stirling's formula. Hence,

$$\frac{(2n - 1)!}{(n - 1)!^2} \int_1^\infty \left( \frac{y}{(y + 1)^2} \right)^n y^{-1} e^{(by + a)/(y + 1)} \, dy \sim \frac{1}{2} e^{(a+b)/2} \quad \text{as } n \to \infty.$$

Similarly,

$$\frac{(2n - 1)!}{(n - 1)!^2} \int_0^1 \left( \frac{y}{(y + 1)^2} \right)^n y^{-1} e^{(by + a)/(y + 1)} \, dy \sim \frac{1}{2} e^{(a+b)/2} \quad \text{as } n \to \infty.$$

Adding the two asymptotic limits and taking the natural logarithm finish the proof of Theorem 13.

REFERENCES


