The Dyson-Taylor commutator method applied to the CEV model: analytical and numerical results

Thesis advisor: Prof. Luca Di Persio
Assistant supervisor: Prof. Anna Mazzucato
Candidate: Federico Vesentini

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Introduction

The aim of this thesis is to illustrate how the Dyson-Taylor commutator method, developed in \cite{5}, \cite{6} and \cite{7}, works in computing the price of an option, and the corresponding Greeks $\Delta$ and $\Gamma$, assuming the CEV market model explained in section 1.3. In particular, we have focused our attention on different values of the market index $\beta$, comparing the results with those provided by other numerical methods well known in literature, described in Chapter 4.

It emerged that, considering the CEV model with leverage effect, i.e. $0 < \beta < 1$, the accuracy of Dyson-Taylor commutator method increases as $\beta$ tends to zero. Viceversa, in case of inverse leverage effect, i.e. $\beta > 1$, all the considered numerical methods failed to provide good results for values of $\beta$ relatively large. Such a loss of precision may be caused by a drastic change of the volatility coefficient, when $\beta$ overcomes 1.

Let us now briefly introduce the topics covered in the various chapters of the document. Chapter 1 presents some basic notions regarding the Black-Sholes PDE. Chapter 2 describes the problem of attributing a price to an European option and explains what is the main differences with an American one. Moreover, in this chapter we introduce the so called Method of the lines, that will be treated in details in Chapter 4.

Chapter 3 describes the theory behind the Dyson-Taylor commutator method and how to derive an explicit scheme that approximate the price of an European Call option under a general market model.

In Chapter 4, we consider the Method of the Lines, explaining how to write a Central Finite Differences approximation of a PDE, also appropriately managing boundary conditions, and how to solve it in time using both the Crank-Nicolson and the Exponential-Euler scheme. Moreover, explain how the Least Square Monte Carlo method works to price an American option.

Finally in Chapter 5 we provide all the numerical results obtained by applying all the methods shown in the Chapters 3 and 4. We studied also both accuracy and computational efforts. The contents are split into four main sections, the first one refers European options, the second one American options, the third analyses Greeks and applications in the CEV market model with $\beta > 1$, and while the last one describes a strategy that improves the accuracy of the Dyson-Taylor commutator method in pricing options with large expiry dates.
In the appendix, we provide an example of numerical implementation of Dyson-Taylor commutator method, developed as a Matlab function.
Chapter 1

Basic notions of option pricing

In what follows we introduce the elementary concepts that will allow us to introduce
and describe the main argument of this document. Basically, after having defined what
the value of an option is, we explain how to obtain the Black-Sholes PDE relatively
to an asset $S_t$ that stochastically evolves over time by following a geometrical Brown-
ian motion. Furthermore, we also recall the Costant Elasticity of Volatility (CEV) model.

1.1 Discrete time

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a stochastic filtered space, where $\Omega$ is usually called universe, $\mathbb{P}$ is the probability measure and $T \in \mathbb{N}$ indicates the maturity time of a given investment. Given $s \in [0, T]$, $\mathcal{F}_s$ can be interpreted as all the informations available until time $s$.

From now on, we assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$ which indicates the set of parts of $\Omega$ and $\mathbb{P}(\omega) > 0$ for every element $\omega \in \Omega$.

Definition 1.1.1. (Finite Market Model)

An $(\mathcal{F}_t$-adapted) stochastic process $S = (S_t)_{t \in [0, T]} = (S^0_t, S^1_t, \ldots, S^d_t)_{t \in [0, T]}$ that belongs to $\mathbb{R}^{d+1}$, $d \geq 1$ is said to be a finite market model (in discrete time).

The vector $S_t$ represents the prices of the assets $(S^0_t, S^1_t, \ldots, S^d_t)$ until time $t$, measurable with the respect to the filtration $\mathcal{F}_t$.

$S^0_t$ is the riskless asset, given by $S^0_t = (1 + r)^t$ in discrete time where $r > 0$ is the interest rate. Let $\beta_t = 1/S^0_t$, it is possible to define the discounted prices vector as follows $\tilde{S}_t = \beta_t S_t = (1, \tilde{S}^1_t, \tilde{S}^2_t, \ldots, \tilde{S}^d_t)$. 

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Definition 1.1.2. (Trading strategy)
A trading strategy is a predictable stochastic process \( \phi = (\phi_t^0, \phi_t^1, \ldots, \phi_t^d)_{t=0,1,\ldots,T} \in \mathbb{R}^{d+1} \), where each element denotes the number of shares of the \( i \)-th asset held in our portfolio.

Recall that to be predictable means that \( \phi_t^i \) is, for every \( i = 0, \ldots, d \) measurable with respect to the filtration \( \mathcal{F}_0 \) and \( \phi_t^i \) is measurable with respect to \( \mathcal{F}_{t-1} \) for every \( t \geq 1 \).

Definition 1.1.3. (Value of the portfolio)
The value of the portfolio is given by the following scalar product

\[
V_t(\phi) = \phi \cdot S_t = \sum_{i=0}^{d} \phi_t^i S_t^i
\]

If \( S_t \) is replaced with \( \tilde{S}_t \), from the previous definition it is possible to obtain the discounted value of the portfolio, denoted by \( \tilde{V}_t(\phi) \). Basically, \( \tilde{V}_t(\phi) = \beta_t V_t(\phi) \). Moreover, if for any \( t \in \{0, 1, \ldots, T-1\} \) the following equality holds true

\[
\phi_t \cdot S_t = \phi_{t+1} \cdot S_t
\]

the strategy is said to be self-financing.

Proposition 1.1.1. The following statements are equivalent:

1. the strategy is self-financing
2. the value of the portfolio is

\[
V_t(\phi) = V_0(\phi) + \sum_{j=1}^{t} \phi_j \Delta S_j \quad t \in \{0, 1, \ldots, T-1\}
\]

where \( \Delta S_j = S_j - S_{j-1} \).
3. the value of the portfolio is

\[
\tilde{V}_t(\phi) = V_0(\phi) + \sum_{j=1}^{t} \phi_j \Delta \tilde{S}_j \quad t \in \{0, 1, \ldots, T-1\}
\]

where \( \Delta \tilde{S}_j = \beta_j S_j - \beta_{j-1} S_{j-1} \).

Proof. See proposition 1.1.2, page 2 of [P].
For any predictable process \( \{ (\phi_1^t, \ldots, \phi_d^t) \}_{t=0,1,\ldots,T} \) and any variable \( V_0 \) that is \( \mathcal{F}_0 \)-measurable, it is possible to find a unique predictable process \( \{ \phi_0^t \}_{t=0,1,\ldots,T} \) such that the strategy given by \( \{ (\phi_0^t, \phi_1^t, \ldots, \phi_d^t) \}_{t=0,1,\ldots,T} \) is self-financing with initial value \( V_0 \). Strictly related to the value of the portfolio \( V_t(\phi) \), there are a few very important definitions concerning the trading strategy that are the following.

**Definition 1.1.4. (Admissible strategy)** If a self-financing strategy \( \phi \) is such that \( V_t(\phi) \geq 0 \), for every \( t \), namely the value of the option is always non-negative, then the strategy is said to be admissible.

![Figure 1.1: Admissible strategy with maturity T=10 (units of time)](image)

**Definition 1.1.5. (Arbitrage strategy)** If an admissible strategy \( \phi \) is such that \( V_0(\phi) = 0 \) and \( V_T(\phi) > 0 \), with \( T \in \mathbb{N} \) being the maturity, or expiry, time of our investment, then the strategy is said to be an arbitrage or, equivalently, that the market allows for arbitrage opportunity.

A market that does not present any possibility of arbitrage strategies is called a **free of arbitrage market**, shortly FOA. A market can be FOA if and only if there exists a probability measure \( \mathbb{P}^* \) equivalent to \( \mathbb{P} \) such that the discounted asset prices \( \tilde{S}_t \) are \( \mathbb{P}^* \)-martingales. This is possible only in the case of a complete market. But when a market is complete? Let us introduce some arguments to introduce before answering the question.
Definition 1.1.6. \textit{(Contingent claim)}

A contingent claim, with maturity time $T$, is characterized by a payoff function $\varphi$, that is a $\mathcal{F}_T$-measurable non-negative random variable whose form depends on the type of the considered option.

Indeed, for example, for an European Call option

$$\varphi = \max(0, S_T^i - K) = (S_T^i - K)^+$$

where $K$ is called the strike price and $T$ is the expiry time. Analogously, we can define an European Put option as follows

$$\varphi = \max(0, K - S_T^i) = (K - S_T^i)^+$$

A contingent claim is said to be \textbf{attainable} if there exists and admissible strategy that worths $\varphi$ at maturity $T$. If \textbf{every contingent claim is attainable}, then the market is said to be a \textbf{complete market}. Observe that, in a F.O.A market, the only thing needed to have an attainable contingent claim is a self-financing strategy worthing $\varphi$ at maturity: if $\phi$ is a self-financing strategy and $\mathbb{P}^*$ is a probability measure, equivalent to $\mathbb{P}$, that makes the discounted asset prices to be $\mathbb{P}^*$-martingales, then also $\tilde{V}_t(\phi)$ is a $\mathbb{P}^*$-martingale too.

Indeed, for any (discrete) time instant, $\tilde{V}_t(\phi) = \mathbb{E}^*(\tilde{V}_T|\mathcal{F}_T)$ where $\mathbb{E}^*$ is the expectation taken under the probability measure $\mathbb{P}^*$. Then, for the attainability, $\tilde{V}_T(\phi) = \varphi \geq 0$ (by definition) and the strategy is admissible.

Remark 1.1.1. \textit{The existence of $\mathbb{P}^*$ is granted by the Girsanov’s Theorem, that will be stated, without proof, later on in the continuous time case, see theorem 8.6.4 of \cite{2}.}

Thus, from now on, we consider the market to be FOA and complete, $\mathbb{P}^*$ being equivalent to the physical measure $\mathbb{P}$ that makes the discounted prices to be martingales. Let
φ be an $\mathcal{F}_T$-measurable non-negative random variable and $\phi$ an admissible strategy such that $V_T(\phi) = \varphi$ (hence it replicates the contingent claim), then

$$V_0(\phi) = \mathbb{E}^*(\tilde{V}_T(\phi)) = \mathbb{E}^*(\frac{\varphi}{S^0_T} | \mathcal{F}_T).$$

(1.2)

More generally, for any $t \in \{0, 1, \ldots, T\}$, it holds

$$V_t(\phi) = S^0_t \mathbb{E}^*(\frac{\varphi}{S^0_T} | \mathcal{F}_T).$$

(1.3)

Equation (1.3) represents the value of the option at time $t$, hence the wealth at time $t$ that the strategy $\phi$ needs in order to replicate $\varphi$ at maturity.

Equation (1.2) defines the so called fair price, that makes the investor to be indifferent between choosing the contingent claim or the replicating strategy: perhaps, he can sell the option at the initial time $t = 0$, with value $V_0$, then following a replicating strategy $\phi$ that grants an amount $\varphi$ at expiry, in order to be hedged perfectly.

Remark 1.1.2. Let us underline that:

1. the probability measure $\mathbb{P}^*$ is called risk-neutral probability.

2. in eq. (1.2) there would be an $S^0_0$ term multiplying $\mathbb{E}^*(\ldots)$, but since in general $S^0_0 = (1 + r)^t$, if $t = 0$ then $S^0_0 = 1$.

1.2 Continuous time models: The Black-Sholes model

This section explains the Black-Sholes model and how it can be derived. Before going into details, let us recall some fundamental results that will be later used.

1.2.1 Preliminary notions

Let us start recalling the definition of a Stochastic Differential Equation (SDE)

**Definition 1.2.1. (Stochastic differential equation)**

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space, $b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $x_0$ $\mathcal{F}_0$-measurable random variable, $W_t$ a standard $\mathcal{F}_t$-Wiener Process. Then a SDE can be defined as follows: for $t \in [0, T]$, where $T > 0$, represents the expiry time, there exists either its integral form

$$X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad s \in [0, T].$$

(1.4)
or its “differential form”

\[
\begin{cases}
    dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t & t \in [0, T] \\
    X_0 = x_0
\end{cases}
\tag{1.5}
\]

where \(b(\cdot, \cdot)\) and \(\sigma(\cdot, \cdot)\) are continuous functions, called drift and diffusion coefficients, respectively, while \(T\) represents, again from a financial point of view, the maturity/expiry time of our investment.

Roughly speaking, an SDE can be seen as an Ordinary Differential Equation (ODE) affected by the differential of a random noise. In particular, “\(dW_t\)” usually refers to the (stochastic) of a Brownian motion (or Wiener process), here expressed by the term “\(dW_t\)”. See, e.g. […] for a formal introduction to the subject.

Since to go deep into the characterization of \(W_t\) is not the purpose of this document, we shall only point out that it is a stochastic process with the following important properties:

- for every \(t \in [0, T]\), \(W_t\) is normally distributed, with mean equal to zero, and variance equal to \(t\). In other words, for any \(t \in [0, T]\), \(W_t \approx N(0, t)\), \(W_0 = 0\) and the variance grows linearly in time.

- it is a diffusive Markov process, with independent increments, drift coefficient equal to zero and diffusion coefficient equal to one.

- it is a martingale, namely for every \(0 \leq s \leq t \leq T\), we have \(\mathbb{E}(W_t|\mathcal{F}_s) = W_s\).

**Theorem 1.2.1. (Existence and uniqueness of solution of an SDE)**

Consider the coefficients \(b(\cdot, \cdot)\) and \(\sigma(\cdot, \cdot)\) as defined previously, if there exist two finite constant \(K\) and \(C\) such that the following condition hold

1. \(|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K|x - y| \quad x, y \in \mathbb{R}^n, t \in [0, T]\)

2. \(|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|) \quad x \in \mathbb{R}^n, t \in [0, T]\)

3. \(\mathbb{E}(x_0^2) < +\infty\)

then, there exists a unique stochastic process \((X_s)_{0 \leq s \leq T}\) solution to eq. (1.5), in \([0, T]\). Moreover, \(X\) has the following properties

- \((X_s)_{0 \leq s \leq T}\) is a \(t\)-continuous stochastic process adapted to the filtration \(\mathcal{F}_t^{x_0}\), generated by \(x_0\) and \(W_t\).

- \(\mathbb{E}\left(\int_0^T |X_s|^2 ds\right) < +\infty\)
Proof. See Theorem 5.2.1 page 66 of [2].

Condition 2 of Theorem 1.2.1 is called **Linear Growth condition**, and it basically requires that the sum of the modulus of coefficients is, for every \( x \in \mathbb{R}^n \), bounded from above by a certain finite constant, growing at most linearly.

Now, it is necessary to introduce a very important result, essential for most of the incoming computations.

**Theorem 1.2.2. (The one-dimensional Itô-Döblin Formula)**

Let \( X_t \) be an Itô process given by \( dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \) and consider a function \( g(t, x) \in C^2([0, \infty) \times \mathbb{R}) \).

Then, \( Y_t = g(t, X_t) \) is again an Itô process, and

\[
dY_t = \partial_t g(t, X_t)dt + \partial_x g(t, X_t)dX_t + \frac{1}{2}\partial_{xx} g(t, X_t)(dX_t)^2
\]

(1.6)

where \((dX_t)^2 = (dX_t) \cdot (dX_t)\) is computed according to the following rules

\[
dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt \simeq 0, \quad dW_t \cdot dW_t \simeq dt
\]

(1.7)

**Proof.** See Theorem 4.1.2, pages from 44 to 48, of [2].

An Itô process is often called Itô diffusion. Roughly speaking, the Itô-Döblin Formula (1.6) can be seen as a “stochastic derivation rule”, that allows to write the differential of a function \( g(t, x) \) whose “space variable” depends on time \( t \) in a non-deterministic way. Rules in (1.7) are directly justified in the proof of the theorem, plus \( dW_t \cdot dW_t = dt \) could be numerically shown very fast, by using computation enviroments like Matlab or GNU Octave.

**Theorem 1.2.3. (The Girsanov’s Theorem)**

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}) \) be a filtered probability space, \( b : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \sigma : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times m} \).

Let \( Y_t \in \mathbb{R}^n \) be an Itô process of the form

\[
dY_t = b(t, \omega)dt + \sigma(t, \omega)dW_t, \quad t \in [0, T],
\]

\( W_t \in \mathbb{R}^n \) is a standard Wiener Process. Suppose that there exist two processes \( \{\theta_t(\omega)\}_{0 \leq t \leq T} \) and \( \{a_t(\omega)\}_{0 \leq t \leq T} \) such that \( \sigma(t, \omega)\theta_t(\omega) = b(t, \omega) - a_t(\omega) \) and, in particular, assume that \( \theta_t \) satisfies the Novikov’s condition

\[
\mathbb{E}\left[ \exp\left( \frac{1}{2} \int_0^T \theta_s^2(\omega)ds \right) \right] < +\infty
\]

(1.8)
Defining
\[ L_t = \exp\left( -\int_0^t \theta_s(\omega)dW_s - \frac{1}{2} \int_0^t \theta_s^2(\omega)ds \right) \quad t \in [0, T] \] (1.9)
and \( dQ(\omega) = L_T(\omega)dP(\omega) \) on \( \mathcal{F}_T \).

Then
\[ \tilde{W}_t := \int_0^t \theta_s(\omega)ds + W_t, \quad t \in [0, T], \] (1.10)
is a Wiener process (or Brownian Motion) w.r.t. \( Q \). Moreover in terms of \( \tilde{W}_t \), the process \( Y_t \) has the following stochastic differential representation
\[ dY_t = a_t(\omega)dt + \theta_t(\omega)d\tilde{W}_t \] (1.11)

Proof. See Theorem 8.6.4, pages from 157 to 159, of [2]. \( \square \)

We underline that the Girsanov’s Theorem says that, given a filtered probability space and a standard Brownian motion \( B_t \), with the respect to probability measure \( P \), it is always possible to find another probability measure \( Q \) equivalent to \( P \) under which the stochastic process defined in eq. (1.10) is a standard Brownian motion.

1.2.2 The Black-Sholes model

The so called Black-Sholes model has been introduced by Fisher Black and Myron Sholes in [3] to capture . In what follows, and without loss of generality, we will consider a realisation of the Black-Sholes model, consisting of one risky asset \( S_t \) and one risk-less asset \( S^0_t \). In particular, \( S^0_t \) evolves accordingly to
\[ \begin{cases} dS_t^0 = rS_t^0dt & t \in [0, T], \quad r > 0 \\ S_0^0 = 1 \end{cases}, \] (1.12)
here following a first order ODE with costant coefficients. Equation (1.12) represents the time evolution of, e.g., a bank account whose revenue is characterized by a costant interest rate \( r \). By separation of variables, we have
\[ \int_{S_0}^{S_t^0} \frac{1}{S_s^0} dS_s^0 = r \int_0^t ds, \quad s \in [0, T], \]
hence
\[ S_t^0 = e^{rt}, \quad t \in [0, T], \quad r > 0. \] (1.13)
Concerning the risky asset, it evolves according to the following SDE
\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0, T], \] (1.14)
where $\mu \in \mathbb{R}$, respectively $\sigma > 0$, represents the drift, respectively the diffusion, coefficients being assumed to be constant. Let us recall that $\sigma$ is financially referred as the market volatility parameter. Note that if $\sigma = 0$, then also $S_t$ becomes a non-risky type investment.

Developing equation (1.14), the derivation rules for multiplications applies, yielding
\[
d\tilde{S}_t = -e^{-rt}S_t dt + e^{-rt}dS_t = -e^{-rt}S_t dt + e^{-rt}(\mu S_t dt + \sigma S_t dW_t)
\]
now, using the definition of discounted stock price, we write
\[
d\tilde{S}_t = \tilde{S}_t[(\mu - r)dt + \sigma dW_t] = \sigma \tilde{S}_t d\tilde{W}_t. \tag{1.14^*}
\]

We need to find a risk-neutral probability $\mathbb{P}^*$, that makes the discounted stock price $\tilde{S}_t = e^{-rt}S_t$ to be a martingale, because it is not granted that $\tilde{W}_t$ in eq. (1.14*) is still a standard Brownian motion for the probability measure $\mathbb{P}$.

Applying the Girsanov Theorem, with $\theta_t = \frac{(\mu - r)}{\sigma} t$, is possible to find a probability measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, under which $\tilde{W}_t = \mu - r \frac{t}{\sigma} + W_t$, is a standard Brownian motion, thus a martingale.

The associated fair price is given by the so called Expected Discounted Payoff (EDP), namely
\[
V_0 = \mathbb{E}^*(e^{-rT}\varphi(S_T)|S_0 = s_0). \tag{1.15}
\]
and one aims to find the related self-financing portfolio, which means to find $V \in C^{1,2}([0, +\infty) \times \mathbb{R})$ such that
\[
V = V(t, S_t) = V(t, S). \tag{1.16}
\]
Then by definition (1.17), the Itô-Döblin Formula applied with $f(t, x) = V(t, S)$, where $dS = \mu S dt + \sigma S dW$, we have
\[
V = V(0, S_0) + \int_0^t \partial_{\xi} V(\xi, S) d\xi + \int_0^t \partial_{S} V(\xi, S) dS + \frac{1}{2} \int_0^t \partial_{SS}^2 V(\xi, S) (dS)^2
\]
whose differential form reads
\[
dV = \partial_t V(t, S) dt + \partial_S V(t, S) dS + \frac{1}{2} \partial_{SS}^2 V(t, S) dS^2
\]
where $(dS)^2 \simeq \sigma^2 S^2 dt$ thanks to (1.7), seen in Theorem 1.2.2.

Therefore,
\[
V = \sigma S \partial_S V dW_t + \left( \mu S \partial_S V + \partial_t V + \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 V \right) dt, \tag{1.17}
\]
which gives the value $V$ in terms of a deterministic part and a one. In order to hedge the risk, it is necessary to find out a way to eliminate the first addend in eq. (1.18), hence to construct a portfolio whose value is

$$\Pi = V - \Delta S,$$

(1.18)

with $\Delta$ chosen in such a way to eliminate the risk. Differentiating eq.(1.19), and exploiting the expression already obtained from $S_t$, we have

$$d\Pi = dV - \Delta dS = \sigma S(\partial_S V - \Delta)dW + \left(\mu S\partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_{SS}^2 V + \partial_t V - \mu \Delta S\right)dt,$$

hence, choosing $\Delta = \partial_S V = \frac{\partial V}{\partial S}$, leads to

$$d\Pi = r\Pi dt = \left(\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_{SS}^2 V\right)dt,$$

(1.19*)

that presents a fully deterministic PDE between brackets.

By the previous assumption of being in a FOA market, the portfolio has to evolve like a riskless asset with deterministic interest rate $r > 0$, i.e. $d\Pi = r\Pi dt$, thus, by (1.19) and (1.19*) we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

(1.19)

representing the Black-Sholes partial differential equation.

**Remark 1.2.1.** Note that the Black-Sholes PDE (1.19) does not depend on the drift parameter $\mu$, depending only on the interest rate $r$ and the volatility coefficients $\sigma$. It means that even if two users may differ in the estimation of $\mu$ or, eventually, its scaling parameter (a very useful application of Girsanov’s Theorem) they should anyway be in agreement about the option value.

Let us also underline that $\Delta = \partial V / \partial S$ can be seen as the rate of change of the option value with the respect to the asset price. It represents the correlation between $V$ and $S$.

Before to conclude this subsection, a step back: remember that an European option is a contract between two parties, the **holder** and the **writer**. The first owns the right without obligation to buy (call) or sell (put) a certain underlying asset $S$ at a strike price $K$, according with an expiry time $T$, while the second is obliged to sell or buy that asset if the holder decides to exercise his right.

In the case of an European Call, the holder exercises his right to buy the stock at a strike price $K$ if the value of the stock is strictly greater than $K$ at maturity $T$, earning an amount of $S_T - K$. Otherwise, $S_T \leq K$, he will not exercise his right.

Conversely, in the case of an European Put the holder exercises his right to sell the stock
at a strike price $K$ if the value of the stock is strictly lower than $K$, at time $T$, earning an amount of $K - S_T$. Otherwise, $S_T \geq K$ he will not exercise his right.
To obtain the exact solution to the problem of pricing an European Call, respectively Put, option it is necessary to solve the Black-Sholes PDE (1.19) adding the terminal condition $V(T, S_T) = (S_T - K)^+$, respectively $V(T, S_T) = (K - S_T)^+$.

Exploiting these conditions to solve equation (1.19), gives the Call price:

$$C(t, S) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

respectively, the Put price

$$P(t, S) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where $N$ is the cumulative distribution function of a standard Gaussian random variable, hence

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy,$$

and $d_1$ and $d_2$ defined as follows

$$d_1 := \frac{ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}},$$

$$d_2 := \frac{ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}.$$

We refer to chapter 3, pages from 46 to 48 of [3], and references therein for further details.

**Remark 1.2.2. (about American options)**

Let us underline that while European options require to wait until expiry time $T$ before being exercised, American ones can be exercised at any time $t \in [0, T]$. Briefly, this implies that the value of an American option is generally greater or equal than a corresponding European one, hence one searches for the time $\tilde{t} \in [0, T]$ which maximizes the option’s revenue. We will see later, in section 2.2, how this latter problem can be correctly solved.

### 1.3 Final considerations: introducing the CEV model

The Black-Sholes model is very popular among economists and market analysts due to its simplicity, but it is limited: from series analysis it was discovered that the volatility generally does not remain constant, specially in the long run. Therefore, it has become
necessary to study more realistic market models, in which volatility is generally not constant. A possible generalization of the Black-Sholes model is obtained allowing the volatility to be stochastic, in particular, depending on another stochastic process

\[
\begin{align*}
    dX_t &= \mu(t, X_t) dt + \sigma(t, Y_t) dW_1(t) \\
    dY_t &= \bar{\mu}(t, Y_t) dt + \sigma(t, Y_t) dW_2(t)
\end{align*}
\quad t \in [0, T],
\]

where \( \mu, \bar{\mu} \) and \( \sigma \) are real-valued continuous functions, \( T > 0 \), and \( W_1(t) \) and \( W_2(t) \) are independent Brownian motions.

**Remark 1.3.1.** The Black-Sholes PDE allows to define the following linear differential operator

\[
L = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS \frac{\partial}{\partial S} - r
\]

(1.22)

that can be seen like a measure of the difference between two terms: the first one represents the return on a hedged portfolio, it is given by \( \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} \). The second one represents the return on a bank account and it is given by \( rS \frac{\partial}{\partial S} - r \). In order to avoid arbitrage, for an European option the difference described just above has to be zero.

Therefore, to better capture real market behaviour, it is natural to consider a model in which \( \sigma \) is a certain function showing how the volatility changes in the time and/or according to the asset price; namely a model of the form

\[
dS_t = \mu S_t dt + \sigma(t, S_t) dW_t,
\]

(1.23)

often referred as *local volatility model*, because the volatility coefficient depends on \( S_t \) and not on another stochastic process. Considering the Black-Sholes PDE, we associate to eq. (1.23) the following

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(t, S_t) S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.
\]

(1.24)

In particular, if the volatility coefficient \( \sigma \) is given by

\[
\sigma(t, S_t) = \sigma S_t^{\beta - 1}, \quad 0 < \beta < 1, \quad \sigma > 0,
\]

(1.25)

hence a function that depends only on the underlying asset \( S_t \). Equation (1.22) becomes

\[
dS_t = \mu S_t dt + \sigma S_t^{\beta - 1} S_t dW_t
\]

(1.26)

that defines the so called *Constant Elasticity of Volatility* model (CEV), see [13] for further details.
Chapter 2

Option pricing for European and American options

2.1 European options

As already seen in section 1.2.2, valuing an european option means that we have to solve the Black-Sholes PDE together with some conditions on \( V(t, S) \). More precisely, equation (1.20) requires two boundary conditions

\[
\begin{align*}
&V(t, 0) = V_0(t), \\
&V(t, \infty) = V_\infty(t),
\end{align*}
\]

where \( S \in [0, \infty) \) and \( t \in [0, T] \), \( T \) being the option maturity time, thus two conditions that specify the solution behaviour with respect to \( S \). Moreover, (1.20) needs also an initial condition \( V(0, S) = V_0(S) \) or a terminal condition \( V(T, S) = V_T(S) \), fixed according to the particular type of PDE to be solved.

Since the Black-Sholes PDE (1.20) is a backward parabolic equation for a function \( V(t, S) \), then it establish a specific relationship between \( V \) and its partial derivatives with respect to the independent variables \( S \) and \( t \). In the simplest case, the highest derivative with respect to \( S \) is a second order derivative and the highest derivative with respect to \( t \) is only a first derivative. If the equation is linear, and the sign of these derivatives agree when they appear on the same side of the equation, then the equation is said to be backward parabolic.

Backward equations need to be solved starting from \( T \) and proceeding back in time until \( t = 0 \), so they require a terminal condition to be specified.

From a financial point of view it is more realistic to start with a specified initial condition, then we consider to change the time variable (reversing it) to turn the Black-Sholes PDE (1.20) into a forward equation.
Therefore, taking $\tau = T - t$ we have

$$V(\tau, S) = V(t - T, S)$$

$$\frac{\partial V}{\partial t} = \frac{\partial V(S, T - t)}{\partial t} = -1 \cdot \frac{\partial V}{\partial \tau}$$

that allows to re-write (1.20) as follows

$$\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV$$

namely as a forward equation that can be equipped with the initial condition $V(0, S)$ given by the payoff function, that corresponds to the $t = 0$ evolution of the solution.

Hence, for an European Call option with strike price $K$, we now have

$$\begin{cases}
\frac{\partial V}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV, & t \in [0, T] \ S \in [0, \infty] \\
V(t, 0) = 0 \\
\lim_{S \to \infty} V(t, S) \sim 0 \\
V(0, S) = \max(S - K, 0)
\end{cases}$$

(2.2)

analogously for an European Put option, it holds

$$\begin{cases}
\frac{\partial V}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV, & t \in [0, T] \ S \in [0, \infty] \\
V(t, 0) = Ke^{-r0} \\
\lim_{S \to \infty} V(t, S) = 0 \\
V(0, S) = \max(K - S, 0)
\end{cases}$$

(2.3)

where, both in (2.2) and (2.3), the initial condition datum fits the boundary conditions.

As to make an example, for a Put option at $t = 0$, if $S = 0$ then $V(0, 0) = \max(K, 0) = K$ that corresponds to $K \cdot 1 = Ke^{-r0}$, viceversa if $S \to \infty$ then $V(0, S) = \max(-\infty, 0) = 0$, since the strike price $K$ is a fixed, positive and finite number.

Initial and boundary conditions are essential in solving the problem of finding a solution $V$ that makes sense, tipically indeed a PDE itself may have a lot of solutions.

In our case, for example, also the trivial solution $V(t, S) \equiv 0$ solves the Black-Scholes PDE but clearly it is completely useless.

For more details about PDEs theory we refer, e.g., to [4] and [16].

### 2.2 American options

American style options give to the holder the possibility of early exercise at any time $t \in [0, T]$, $T < 0$, hence the problem of pricing this type of options is slightly different
compared to the European case. In particular, the Black-Sholes PDE no longer holds, nevertheless considering the risk-free portfolio

$$\Pi = V - \Delta S$$

with $\Delta = \partial V/\partial S$ choosen to eliminate the risk in $d\Pi$, we have that the inequality $d\Pi \leq r\Pi dt$ has still to hold, otherwise there would be a risk free asset with a return strictly greater then the riskless rate $r$, which implies the existence of an arbitrage opportunity. Recall that this latter situation has to be avoided because of the early assumption of being in a FOA market. Before going into details about such an issue, recall that

$$d\Pi = \left\{ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right\} dt,$$

with $\Delta = \partial V/\partial S$, so $d\Pi \leq r\Pi dt$ implies

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \leq r \left( V - \frac{\partial V}{\partial S} S \right)$$

namely

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \leq 0 \quad (2.4)$$

hence there is an inequality that needs to be fulfilled. Moreover since at any time there is the possibility of early exercise, the following holds true

$$V(t, S) \geq \varphi = \text{payoff}(S) \quad (2.5)$$

indeed, assume by contraddiction that $V(t, S) < \varphi$, where $\varphi$ is the payoff of an american put option, then one can buy the stock at price $S$ (that is strictly smaller than $K$ since the option is “in-the-money”) and at the same time he can buy the option at a price $P = V(t, S)$ and exercise it immediately. The put indeed gives the right to sell the stock at price $K$, allowing the optionholder to make an immediate net profit of

$$\xi = K - S - P > 0$$

without any risk, hence realizing an arbitrage opportunity, contraddicting the hypothesis of a being in a FOA market.

Note that since (2.4) is an inequality because of the opportunity either

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad (2.6)$$
or else

\[ V(S, t) = \text{payoff}(S). \]  

(2.7)

In fact the early exercise opportunity implies that if

- \( V(t, S) = \varphi \), then it is optimal to exercise the option, and we say that the option is in the *exercise region*,

else

- \( V(t, S) > \varphi \), then \( V \) satisfies the Black-Sholes PDE, so it is optimal to held the option, and we can say that the option is in the *held region*.

Equations (2.6) and (2.7) can be expressed in a compact form as

\[
\left( \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \right) \cdot \left( V(t, S) - \text{payoff}(S) \right) = 0 \quad (2.8)
\]

The conditions given by equations (2.4), (2.5) and (2.8), together with the usual initial payoff condition, define the so called *linear complementary problem*, that can be solved using the numerical schemes explained in chapter 4.

The opportunity of early exercise cancels any possibility to find an explicit solution to American option pricing problems, thus it is really important to have good numerical schemes to count on. From a physical point of view, pricing an American style option corresponds to the so called “Obstacle Problem”, for more details about this kind of differential problem, we refer to, e.g., chapter 7 of [3] pages from 108 to 114.

### 2.3 Remarks

Let us underline that from the financial point of view the asset price \( S \) can grow indefinitely, so theoretically \( S \in [0, \infty] \). Though, computationally speaking this is not possible. Hence previous interval has to be replaced with an appropriate finite one. If \( K \) is the strike price, then the asset price \( S \) should grow in units of \( K \), thus \( S \in [0, S_{\text{max}}] \), where \( S_{\text{max}} = nK \) \( n \in \mathbb{N}^+ \). Tipical choices are \([0, 2K]\) or \([0, 3K]\).

Consequently, problem (2.2), respectively (2.3), becomes

\[
\begin{align*}
\frac{\partial V}{\partial t} &= \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV, \quad t \in [0, T], \ S \in [0, S_{\text{max}}] \\
V(t, 0) &= 0 \\
V(t, S_{\text{max}}) &= S_{\text{max}} - Ke^{-rt} \\
V(0, S) &= \max(S - K, 0)
\end{align*}
\]

(2.9)
respectively

\[
\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV, \quad t \in [0, T], \quad S \in [0, S_{\text{max}}] \\
V(t, 0) &= Ke^{-rt} \\
V(t, S_{\text{max}}) &= 0 \\
V(0, S) &= \max(K - S, 0)
\end{aligned}
\]

leading to consider two Advection-Diffusion-Reaction (ADR) problems. They can be solved using the so called “Method of the Lines”, which consists of the following two steps:

- discretizing the Black-Sholes PDE (in its forward formulation) in space, using the central finite differences method discussed in section (4.1), then
- computing the evolution in time by using one of the methods described in sections (4.2) and (4.3)

Considering the American option pricing problem, latter method can be still used, provided some changes due to the early exercise opportunity. In particular, discretizing both space and time into \(m\) and \(n\) discrete nodes, we obtain a grid of points \((x_i, t_j) = (idx, jdt)\) at which the option value has to be checked. If the option value \(v_j\) is lower than the payoff from early exercise, then the value \(v_j\) is replaced by \((K - x)^+\).
Chapter 3

The Dyson-Taylor commutator method

3.1 Introduction

In what follows we introduce the so called Dyson-Taylor commutator method, providing a rapid overview about the theoretical concept on which it is based on, see, e.g., [5], [6] and [7].

We start considering the following problem

\[
\begin{aligned}
&\frac{\partial}{\partial t}U(t, x) - LU(t, x) = 0, \quad t \in [0, T] \\
&U(0, x) = h(x), \quad x > 0
\end{aligned}
\]

(3.1)

where \(U(0, x) = h(x)\) represents an initial datum and \(L(t, x)\) is a second order linear differential operator that may assume various forms, but that can be generally expressed as

\[
L(t, x) := \frac{1}{2} a(t, x)^2 \partial_x^2 + b(t, x) \partial_x + c(t, x)
\]

(3.2)

where coefficients \(a, b\) and \(c\) are smooth and bounded functions with their derivatives. In particular, we assume that \(a(\cdot, x) > 0\), for every \(x > 0\), and that there exists a constant \(\gamma > 0\) such that \(a(x) \geq \gamma > 0\), i.e. \(L(t, x)\) is a strongly elliptic operator.

Denoting with \(G_t(x, y)\) the Green function or fundamental solution of the operator \(\partial_t - L\), the solution \(U(t, x)\) to (3.1), reads as follows

\[
U(t, x) = \int G_t(x, y)f(y)dy.
\]

(3.3)

Therefore by approximating the Green function \(G_t(x, y)\), it is possible to directly approximate the solution \(U(t, x)\).
Remark 3.1.1. (about existence of exact solutions)

It is worth to mention that problem (3.1) does not always have an explicit solution, which, indeed, exists only under specific conditions. This is due to the fact that the Green Kernel $G_t(x,y)$ has an explicit expression only under suitable assumptions for $L$ and $h$. The latter is the case if one assumes $L(t,x)$ originated by the Black-Sholes PDE, to be considering a Geometrical Brownian Motion market model. In that case, the Green’s function is given by

$$G_{BSM}^t = \frac{e^{-rt}}{y\sqrt{2\pi\sigma^2 t}} \left[ -\frac{|\ln(x/y) + (r - \sigma^2/2)t|^2}{2\sigma^2t} \right],$$

and, e.g., the solution (3.3) for the European Call option case with payoff function $h(x) = \max(x - K, 0)$, equals

$$U_{BSM}^t(x) = \int_0^\infty G_t(x,y)(y-K)^+dy = N(d_+)x - Ke^{-rt}N(d_-),$$

where $N(x) = 1/\sqrt{2\pi} \int_{-\infty}^{0} e^{-z^2/2}dz$ is the cumulative distribution function of a standard Gaussian normal random variable and

$$d_\pm = \frac{\ln(x/K) + (r \pm \sigma^2/2)t}{\sigma\sqrt{t}}$$

and $t \in [0,T]$, as we have already derived in section 1.2.2.

3.2 Approximation of Green’s function for second order parabolic equations

If problem (3.1) admits an unique solution, then the linear operator, called Solution Operator, that maps the initial datum $h$ into the solution $U(t,x)$ is well defined. Considering a solution operator with costant coefficients $L_0$, it defines a semigroup denoted by $e^{tL_0}$, for $t > 0$.

Let us state some definitions, that will turn out to be useful later, see section 3.3.

Definition 3.2.1. (Semigroup properties)

Let $X$ be a Banach space. Consider a family of bounded linear operators $\{T(s)\}_{s \geq 0}$ that maps $X$ into $X$. This family is said to be a semigroup if the following properties are satisfied.
• $T(0)x = x$, for every $x \in X$;

• $T(s + t)x = T(s)T(t)x = T(t)T(s)x$, for every $x \in X$ and $t, x \geq 0$.

In our case, see section 2 of [5], these properties apply as follows

1. $e^{tL_0}|_0 = I$ where $I$ is the identity operator;

2. $e^{t_1L_0}e^{t_2L_0} = e^{(t_1+t_2)L_0}$, $t_1, t_2 > 0$.

This result holds true for any time-independent type of operator $L$ with variable coefficients, but in case of a time-dependent operator $L = L(t)$ though, the solution operator defines an evolution system $S(t_1, t_2)$ for which the property 2 is replaced with

$$S(t_1, t_2)S(t_2, t_3) = S(t_1, t_3), \text{ for every } 0 \leq t_3 \leq t_2 \leq t_1.$$

**Definition 3.2.2. (Parabolic rescaling)**

Let $z \in \mathbb{R}$ be a fixed and arbitrary point, $s > 0$ be a parameter and $f(t, x)$ a function. Then the so called parabolic rescaling by $s$ of function $f$ about $z$ is given by

$$f^{s,z}(t, x) := f(s^2t, z + s(x - z))$$

and $z$ is said to be the “basepoint” of the rescaling.

If $f$ does not depend on $t$, then

$$h^{s,z}(x) := h(z + s(x - z))$$

Definition 3.2.2 allows to rescale $L$, finding

$$L^{s,z}(t, x) := \frac{1}{2}a^{s,z}(t, x)^2\partial_x^2 + sb^{s,z}(t, x)\partial_x + s^2c^{s,z}(t, x). \quad (3.4)$$

Therefore, if $U(t, x)$ actually solves problem (3.1), then $U^{s,z}(t, x)$, parabolic rescaling of $U$, solves the “rescaled problem”

$$\begin{cases}
\partial_t U^{s,z}(t, x) - L^{s,z}(t, x)U^{s,z}(t, x) = 0, \\
U^{s,z}(0, x) = h^{s,z}.
\end{cases} \quad (3.5)$$

The same happens for the Green function. In fact, naming $G^L_t(x, y)$ as the Green’s function of operator $\partial_t - L$, then the (rescaled) Green’s function of $\partial_t - L^{s,z}$ is given by

$$G^L_t(x, y) = s^{-1}G^{L^{s,z}}_{s^{-2}t}(z + s^{-1}(x - z), z + s^{-1}(y - z)) \quad (3.6)$$

and if $s = t^z$, then

$$G^L_t(x, y) = t^{-1/2}G^{L^{\sqrt{t}z}}_1(z + t^{-1/2}(x - z), z + t^{-1/2}(y - z)). \quad (3.7)$$
Computing the Green function $G^{s,z}_{t}$ of (3.5), when $t = 1$, by deriving the $n^{th}$-order Taylor’s expansion of (3.5), where $n \in \mathbb{N}$, we have

$$L^{s,z} = \sum_{k=0}^{n} s^k L^z_k + s^{n+1} L^s_{n+1}(t, x)$$

(3.8)

where the secondo addend is called the remainder term, of the Taylor’s expansion, and the $L^z_k$ are given by

$$L^z_k = \frac{1}{z!} \left. \left( \frac{d^z}{ds^z} \right) \right|_{s=0},$$

being all independent on $s$.

In this setting, we aim to obtain an explicit 2nd-order Taylor’s expansion of the Green function of $L$. From now on, we simplify notations denoting, given a generic function $g(t, x)$,

$$\dot{g}(t, x) = \frac{\partial g(t, x)}{\partial t},$$

$$g'(t, x) = \frac{\partial g(t, x)}{\partial x}$$

to indicate first derivative of $g$ w.r.t time and space, respectively.

Thus, the parabolic rescaling of a certain function $f$, $f^{s,z}$, at $s = 0$ is

$$f^{s,z}(t, x) = f(0, z) + s(x - z)f'(0, z) + s^2 t f''(0, z) + \frac{1}{2} s^2(x - z)^2 f'''(0, z) + s^3 r$$

(3.9)

and $s^3 r = s^3 r(s; t; x; z)$ is a remainder term.

In what follows we require that all the coefficients are evaluated at $(0, z)$, then whenever a term like $b$ shows up, it means $b = b(0, z)$ and the same holds true for all the other coefficients.

The second order Taylor’s expansion of $L^{s,z}$ w.r.t $s$ at $s = 0$ is given by

$$L^{s,z}(t, x) = L^{s,z}_0 + s L^{s,z}_1(x) + s^2 (L^{s,z}_{2,x}(x) + t L^{s,z}_{2,t}) + s^3 L^{s,z}_{3}(t, x)$$

(3.10)

where

$$L^{s,z}_0 := \frac{1}{2} a^2 \partial_x^2,$$

(3.11)

$$L^{s,z}_1 := a a'(x - z) \partial_x^2 + b \partial_x,$$

(3.12)

$$L^{s,z}_2 = L^{s,z}_{2,x}(x) + t L^{s,z}_{2,t},$$

(3.13)

$L^{s,z}_{2,x} := \frac{1}{2} (a^2 + a'')(x - z)^2 \partial_x^2 + b'(x - z) \partial_x + c$, $L^{s,z}_{2,t} := a a \partial_t^2$ and $L^{s,z}_{3}(t, x)$ representing the remainder term of the expansion. To approximate the Green function $G^L_t$ for the
problem described at the very beginning of the chapter, aiming to obtain the solution \( U(t, x) \), we start, decomposing \( L(t) \) into

\[
L(t) = L_0 + V(t),
\]

where \( L_0 \) is the second-order constant coefficients operator previously mentioned, for which there exist an explicitly form for the solution operator, and \( V(t) \) is a time-dependent and variable-coefficients second order operator.

Since \( L \) is uniformly strongly elliptic and all its coefficients are smooth and bounded functions together with their derivatives, by *Duhamel’s principle*, see e.g. section 4.1 of [7], the Green function can be written \( G^L \) can be written as

\[
G^L_t = e^{tL_0} + \int_0^t e^{(t-\tau)L_0} V(\tau) G^L_{\tau} d\tau. \tag{3.14}
\]

Then, repeating the application of eq. (3.14), we obtain

\[
G^L_t = e^{tL_0} + \int_0^t e^{(t-\tau)L_0} V(\tau_1) e^{\tau_1 L_0} d\tau_1 + \int_0^t \int_0^{\tau_1} e^{(t-\tau_1)L_0} V(\tau_1) e^{(\tau_1-\tau_2)L_0} V(\tau_2) e^{\tau_2 L_0} d\tau_2 d\tau_1 + \ldots
\]

\[
\ldots + \int_0^t \int_0^{\tau_1} \ldots \int_0^{\tau_{d-1}} e^{(t-\tau_1)L_0} V(\tau_1) e^{(\tau_1-\tau_2)L_0} V(\tau_2) \ldots V(\tau_d) e^{\tau_d L_0} d\tau_d + \ldots
\]

\[
\ldots + \int_0^t \int_0^{\tau_1} \ldots \int_0^{\tau_{d+1}} e^{(t-\tau_1)L_0} V(\tau_1) e^{(\tau_1-\tau_2)L_0} V(\tau_2) \ldots V(\tau_{d+1}) G^L_{\tau_{d+1}} d\tau_d,
\tag{3.15}
\]

where the term \( d\bar{\tau} \) incorporates all the \( d\tau_i \) for \( i = 1, 2, ..., k \).

Letting \( d \rightarrow \infty \), we are left with an asymptotic, time-ordered, expansion of the Green’s function. Such an expansion is called *Dyson series*. The integer \( d \) is said to be the iteration level in the time-ordered expansion and, in general, it differs from the degree \( n \) of the Taylor’s expansion. It is possible to prove that, to obtain consistency, \( n \) must be greater or equal than \( d \). For further details, refer to [5]. From now on we take \( n = d = 2 \) for consistency.

Analogously proceeding as before, but for the Green function \( G^L_{t}^{s,z} \), we start writing

\[
L^{s,z}(t) = L_0^z + V^{s,z}(t),
\]

and then, using (3.10), we have

\[
V^{s,z}(t) := L^{s,z}(t) - L_0^z = sL_1^z(x) + s^2 L_2^z(t, x) + s^3 L_3^{s,z}(t, x), \tag{3.16}
\]

where \( L_3^{s,z}(t, x) \) is the remainder term, and the other ones are given respectively by (3.11), (3.12) and (3.13).

Since we have assumed \( d = n = 2 \), taking \( t = 1 \), we have

\[
G^L_{1}^{s,z} = e^{L_0} + sI_1^z + s^2 (I_{1,1}^z + I_{x,z}^z + I_{t,z}^z) + R^{s,z}
\tag{3.17}
\]

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where

\[ I_{z1}^z = \int_0^1 e^{(1-\tau_1)L_0} L_1^z e^{\tau_1 L_0} d\tau_1, \]

\[ I_{z1,1}^z = \int_0^1 \int_0^{\tau_1} e^{(1-\tau_1)L_0} L_1^z e^{(\tau_1-\tau_2)L_0} L_1^z e^{\tau_2 L_0} d\tau_2 d\tau_1, \]

\[ I_{z2,x}^z = \int_0^1 e^{(1-\tau_1)L_0} L_{2,x}^z e^{\tau_1 L_0} d\tau_1, \]

\[ I_{z2,t}^z = \int_0^1 e^{(1-\tau_1)L_0} L_{2,t}^z e^{\tau_1 L_0} d\tau_1, \]

(3.18)

terms of higher order being all included into the remainder one.

The main objective is to obtain an approximation to the Green’s function \( G^L_t \) for (3.1). Before introducing the so called Distribution Kernel associated to the operator \( e^{L_0} + sI_{z1}^z + s^2(I_{z1,1}^z + I_{z2,x}^z + I_{z2,t}^z) \), given by

\[ T^{s,z}(x,y) := G_0(x,y,z) + sG_1(x,y,z) + s^2G_2(x,y,z) \]

we need the following definition

**Definition 3.2.3. (Admissible function)**

In our theoretical framework, a function \( z : \mathbb{R}^{2N} \rightarrow \mathbb{R}^N \) defined by \( z = z(x,y) \) is said to be admissible if

- \( z(x,x) = x; \)
- all its derivatives are bounded.

So, the desired second order approximation reads as follows

\[ G^L_t^{[2]}(x,y) = t^{-1/2}T^{s,z}(z + (x - z)/\sqrt{t}, z + (y - z)/\sqrt{t}), \]

(3.19)

for an ammissible function \( z = z(x,y) \).

There is an important lemma that needs to be stated, that will allow to carry out the calculations for integrals given by (3.18), but before doing that some theoretical notions are required

**Definition 3.2.4. (Space of Differentials)**

For any \( a, b \) non-negative integers, denote by \( D(a,b) \) the vector space of all the differentiations of degree at most \( a \) and order at most \( b \). If either \( a \) or \( b \) is negative, set \( D(a,b) = \{0\} \).

For \( A \in D(a,b) \), “degree of \( A \)” means the highest power of the polynomials appearing as coefficients in \( A \).
Definition 3.2.5. (Adjoint representation)

Given two differentiations \( A_1 \in D(a_1, b_1) \) and \( A_2 \in D(a_2, b_2) \), the commutator between these two is defined as \([A_1, A_2] := A_1 A_2 - A_2 A_1\). The so called “adjoint representation” is

\[
ad_{A_1}(A_2) := [A_1, A_2]
\]

and for any \( j \geq 1 \), \( ad^j_{A_1}(A_2) \) is recursively defined by

\[
ad^j_{A_1}(A_2) := ad_{A_1}(ad^{j-1}_{A_1}(A_2))
\]

Lemma 3.2.1. (Backer-Campbell-Hausdorff formula)

Let \( A \) and \( B \) be two operators, then

\[
[e^A, B] = \left( [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots \right)e^A,
\]

which indicates how to commute \( e^A \) and \( B \):

\[
e^A B = \left( B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots \right)e^A.
\]

Proposition 3.2.1. Suppose \( A_1 \in D(a_1, b_1) \) and \( A_2 \in D(a_2, b_2) \). Then, for any integer \( k \geq 1 \),

\[
ad^k_{A_1}(A_2) \in D(k(a_1 - 1) + a_2, k(b_1 - 1) + b_2)
\]

Proof. See Proposition 3.3.6 page 55 of [7]

Lemma 3.2.2. Let \( m, k \) be fixed integers greater or equal than 1. Let \( L^z_0 \in D(0, 2) \) the constant coefficients second order operator and \( L^z_m \in D(m, 2) \) be the operator above, then

\[
ad^k_{L^z_0}(L^z_m) \in D(m - k, k + 2)
\]

In particular, whenever \( k > m \) then

\[
ad^k_{L^z_0}(L^z_m) = 0
\]

holds true.

Proof. See Lemma 3.3.7 page 56 of [7]

Thus, the latter case \([L^z_0, L^z_2] = 0\), \( ad^2(L^z_1) = 0 \) and \( ad^3_{L^z_0}(L^z_2) = 0 \).
Lemma 3.2.3. Let $L_0$ and $L_m$ defined as above, then for any $\theta \in (0,1)$,
$$e^{(1-\theta)L_0} L_m(\theta) = P_m(\theta, x - z, \partial)e^{(1-\theta)L_0}$$
where
$$P_m(\theta, x - z, \partial) := L_m(\theta) + \sum_{i=1}^{m} \frac{(1-\theta)^i}{i!} ad_{L_0}^i(L_m(\theta)) \in \mathcal{D}(m, m + 2)$$
is a finite sum of terms with the form $a(z)(1-\theta)^i(x-z)^k \partial_x^\alpha$, in which $a(z)$ and all its derivatives are bounded and $\alpha$ is a multi-index.

Proof. See, e.g., Lemma 3.3.8, pages 56 and 57, of [7]. □

It follows that integrals appearing in (3.18) can be computed exploiting Lemma 3.2.2 and 3.2.3, to obtain
$$I_1 = \int_0^1 e^{(1-\tau_1)L_0} L_1 e^{\tau_1 L_0} d\tau_1 = \int_0^1 (L_1 + (1 - \tau_1)[L_0, L_1]) e^{L_0} d\tau_1 = (L_1 + \frac{1}{2}[L_0, L_1]) e^{L_0},$$

$$I_{1,1} = \int_0^1 \int_0^{\tau_1} e^{(1-\tau_1)L_0} L_1 e^{(\tau_1-\tau_2)L_0} L_1 e^{\tau_2 L_0} d\tau_2 d\tau_1 =$$
$$= \int_0^1 \int_0^{\tau_1} (L_1 + (1 - \tau_1)[L_0, L_1])(L_1 + (1 - \tau_2)[L_0, L_1]) e^{L_0} d\tau_2 d\tau_1 =$$
$$= \left(\frac{1}{2}(L_1)^2 + \frac{1}{3}L_1[L_0, L_1] + \frac{1}{6}[L_0, L_1]^2 \right) e^{L_0},$$

$$I_{2,x} = \int_0^1 e^{(1-\tau_1)L_0} L_2 e^{\tau_1 L_0} d\tau_1 =$$
$$= \int_0^1 (L_2 + (1 - \tau_1)[L_0, L_2]) e^{L_0} d\tau_1 =$$
$$= \left( L_2 + \frac{1}{2}[L_0, L_2] + \frac{1}{6}[L_0, L_0, L_2] \right) e^{L_0},$$

$$I_{2,x} = \int_0^1 e^{(1-\tau_1)L_0} L_2 e^{\tau_1 L_0} d\tau_1 = \int_0^1 \tau_1 L_2 e^{L_0} d\tau_1 = \frac{1}{2} L_2 e^{L_0}. (3.23)$$

See, e.g., page 913 of [5] and pages from 95 to 97 of [7].

Then, the expansion of the parabolic rescaling of the Green function given by (3.17) becomes
$$e^{L_{s,z}} = (1 + sQ_1 + s^2 Q_2)e^{L_0} + \mathcal{R}_{s,z}$$

(3.24)
where we have defined
\[ Q_1 = L_1^z + \frac{1}{2}[L_0^z, L_1^z] \]
\[ Q_2 = \frac{1}{2}(t^2)^z + \frac{1}{3}L_1^z[L_0^z, L_1^z] + \frac{1}{6}[L_0^z, L_1^z]L_1^z + \frac{1}{8}[L_0^z, L_1^z]^2 + L_2_{,x} + \]
\[ + \frac{1}{2}[L_0^z, L_2_{,x}] + \frac{1}{6}[L_0^z, L_0^z, L_2_{,x}] + \frac{1}{2}L_2^z \]
while \( R_{s,z} \) is the remainder term and we are left with the computation of the commutators appearing into the definitions of \( Q_1 \) and \( Q_2 \), to get the second order approximation of \( G_t^{L^{zz}} \).

**Remark 3.2.1.** In analytical mechanics, commutators are also known as Lie Brackets, that in a certain sense show “how two vector fields commute”. See, e.g., Chapter 1, page 40, of [17].

Since all the functions, as well as their derivatives, appearing above are evaluated in \((0, z)\), we have
\[ [L_0^z, L_1^z] = \frac{1}{2}a^2 \partial_x^2 (a a'(x - z) \partial_x^2 + b \partial_x) \]
\[ = \frac{1}{2}a^2 \partial_x (a a'(x - z)^2 \partial_x^2 + b \partial_x) \]
\[ = \frac{1}{2}a^2 \partial_x (a a' \partial_x^2 + aa'(x - z) \partial_x) + b \partial_x^2 \]
\[ = \frac{1}{2}a^3 a' \partial_x^3 + \frac{1}{2}a^3 a'(x - z) \partial_x^4 + \frac{1}{2}a^3 a' + \]
\[ + \frac{1}{2}a^2 b \partial_x^2 - \frac{1}{2}a^3 a'(x - z) \partial_x^4 - \frac{1}{2}b a^2 \partial_x^2 = a^3 a' \partial_x^2 \]

Analogously, we derive
\[ [L_0^z, L_1^z]^2 = a^6 a^2 \partial_x^6 \]
\[ L_1^z[L_0^z, L_1^z] = a^4 a^2 (x - z) \partial_x^5 + b a^3 a' \partial_x^4 \]
\[ [L_0^z, L_1^z]L_1^z = a^4 a^2 (x - z) \partial_x^5 + (b + 3aa') a^3 a' \partial_x^4 \]
\[ [L_0^z, L_2_{,x}] = a^2 (a^2 + aa'')(x - z) \partial_x^3 + a^2 \left( b' + \frac{1}{2}a^2 + \frac{1}{2}aa'' \right) \partial_x^2 \]
\[ [L_0^z, [L_0^z, L_2_{,x}]] = a^4 (a^2 + aa'') \partial_x^2 \]
and finally
\[ (L_1^z)^2 = (a a'(x - z))^2 \partial_x^4 + 2(a^2 a'^2 + a a'b)(x - z) \partial_x^3 + (aa'b + b^2) \partial_x^2 \]

Let us underline that above computations show that the key to obtain \( G_t^{L^{zz}} \) is to apply a differential operator with polynomial coefficients to the Green’s function of \( e^{tL_0} \). Before going on, let us introduce the following definition
Definition 3.2.6. (Convolution operator)
Given two smooth functions \( \phi \) and \( f \), the Convolution operator with \( \phi \), denoted by \( C_\phi \) is defined by

\[
C_\phi f(x) := \phi * f(x) = \int \phi(x - y) f(y) dy
\]

The convolution operator shows that the distribution kernel of \( C_\phi \) is \( C_\phi(x, y) = \phi(x - y) \).
Then \( \partial_x C_\phi = C_\phi \partial_x \) while \( C_\phi \partial_x = -C_\phi \partial_x \phi \).
Since \( L_0 \) is a costant coefficients operator, its related Green’s function can be explicitly written as follows

\[
e^{L_0}(x, y) = \frac{1}{\sqrt{2\pi a^2}} \exp\left(-\frac{|x - y|^2}{2a^2}\right), \quad a = a(0, z),
\]

which shows that, \( e^{L_0} \) is a convolution operator. Moreover, exploiting previous definition, we obtain

\[
\partial_x^k e^{L_0}(x, y) = H_k(\theta) e^{L_0}(x, y), \quad \theta = \frac{x - y}{a^2},
\]

where \( H_k \) indicates the \( k \)th-order Hermite polynomial w.r.t. \( \theta \).

Remark 3.2.2. (Hermite polynomials)
Let us underline that, in the present case, Hermite polynomials can be inductively generated by the following expression

\[
H_{k+1}(\theta) = -\theta H_k(\theta) + H_k'(\theta)/a^2,
\]

taking into account that \( H_0 = 1 \). For example, fixing \( k = 3 \), we have

\[
H_3(\theta) = -\theta^3 + \frac{3\theta}{a^2}
\]

In our framework, until the 6th degree Hermite polynomials are required, to find \( G_i \) for \( i = 0, 1, 2 \), terms part of the Distribution Kernel definition \( T^{s,z}(x, y) \).
In particular, we have

\[
G_0(x, y, z) = e^{L_0} = \frac{1}{\sqrt{2\pi a^2}} \exp\left(-\frac{|x - y|^2}{2a^2}\right),
\]

\[
G_1(x, y, z) = \left(aa'(x - z)H_2(\theta) + bH_1(\theta) + \frac{1}{2}a^3a'H_3(\theta)\right)e^{L_0}(x, y) = \frac{1}{\sqrt{2\pi a^2}} e^{-\frac{|x - y|^2}{2a^2}} \left[(x - y) \left(\frac{3aa' - 2b}{2a^2}\right) - \frac{a'}{2a^3} (x - y)^3 + a'(x - z)\left(\frac{(x - y)^3 - a^2}{a^3}\right)\right],
\]

29
and since $G_2(x, y, z) = (Q_2 e^{L_0^2})(x, y)$, then

$$G_2(x, y, z) = \left( \frac{1}{2} L_{2,x}^2 + L_{2,x}^2 + \frac{1}{2} [L_0^2, L_{2,x}^2] + \frac{1}{6} [L_0^2, [L_0^2, L_{2,x}^2]] + \frac{1}{2} (L_1^2)^2 + \frac{1}{3} L_1^2 [L_0^2, L_1^2] + \frac{1}{6} [L_0^2, L_1^2] L_1^2 + \frac{1}{8} [L_0^2, L_1^2]^2 \right) e^{L_0^2} =$$

$$= \left( P_0 + \sum_{i=1}^{6} P_i H_i(\theta) \right) e^{L_0^2} (x, y)$$

(3.29)

where all the $P_i$ terms are polynomials in the variable $x - z$ whose coefficients are given by the functions $a, b, c$ and their derivatives, all evaluated in $(0, z)$. Namely

$$P_0 = c = c(0, z), \quad P_1 = b'(x - z),$$

$$P_2 = \frac{1}{2} \left[ \frac{1}{2} a^3 a'' + a^2 b' + \frac{a^2 a'^2}{2} + b^2 + a^2 (x - z)^2 + a(ba' + a + a''(x - z)^2) \right],$$

$$P_3 = a(x - z) \left( a'b + \frac{1}{2} a^3 a'' + \frac{3}{2} aa'^2 \right),$$

$$P_4 = a^2 \left[ \frac{1}{2} a^3 a'' + 2a^2 a'^2 + \frac{3}{2} aa'b + \frac{3}{2} a^2 (x - z)^2 \right],$$

$$P_5 = a^4 a(x - z),$$

$$P_6 = \frac{1}{8} a^6 a'^2.$$

(3.30)

To complete the technicalities behind the Dyson-Taylor commutator method, it is important to introduce a generalization of the Sobolev’s spaces.

**Definition 3.2.7. (Exponentially weighted Sobolev spaces)**

Define $\langle x \rangle = (1 + x^2)^{1/2}$. Then, for $m \in \mathbb{Z}^+$, $1 < p < \infty$, and $a \in \mathbb{R}$, the space

$$W_{a,m}^{m,p}(\mathbb{R}) := \{ u : \mathbb{R} \rightarrow \mathbb{C}, \partial^j_x(e^{a(x)}u(.)) \in L^p(\mathbb{R}), j \leq m \},$$

is called “Exponentially Weighted Sobolev Space” (EWSS).

**Remark 3.2.3.** Taking $a = 0$ in Definition 3.2.7, we recover the usual definition of Sobolev Space, see, e.g., Definition 2.5 pag 916 of [5] for further details.

EWSS are particularly useful to work with when dealing with typical option pricing settings. As to make example, considering the problem of pricing an European Call option under the Black-Sholes-Merton model, after the change of variables $x = e^y$, the payoff function becomes exponentially increasing and belongs to $W_{a,m}^{m,p}$. We refer to, e.g.,
Introducing now the rescalings:
\[
\begin{align*}
  x_t &= z + (x - z)/\sqrt{t}, \\
y_t &= z + (y - z)/\sqrt{t},
\end{align*}
\]

let us state the following result:

**Theorem 3.2.1.** Let \( L \) be given by equation (3.2) and \( z = z(x, y) \) be an admissible (in the sense of definition 3.2.3) function. Assume that function \( 1/a \), where \( a \) is the first coefficient of \( L \), and all its \( k \)-th, for \( k \geq 0 \), derivatives are bounded on \( \mathbb{R} \). Define
\[
G_t^{[2]}(x, y) = t^{-1/2}G_0(x_t, y_t, z) + t^{1/2}G_1(x_t, y_t, z) + tG_2(x_t, y_t, z)
\]
where \( z = z(x, y) \) and the functions \( G_j \) are those defined by (3.27)-(3.29). Moreover, define the error term \( \mathcal{E}_t^{[2]} \) in the Green function approximation by
\[
e^{tL}f(x) = \int_{\mathbb{R}^N} G_t^{[2]}(x, y)f(y)dy + t^{3/2}\mathcal{E}_t^{[2]}f(x),
\]
then, for any \( f \in W_{a}^{m,p}(\mathbb{R}) \), \( s \in \mathbb{R} \), \( m \geq 0 \), \( 1 < p < \infty \), the norm of the error is given by
\[
\|\mathcal{E}_t^{[2]}f\|_{W_{a}^{m+k,p}} \leq Ct^{-k/2}\|f\|_{W_{a}^{m,p}}
\]
for any \( t \in [0, T] \), \( 0 < T < \infty \), \( k \in \mathbb{Z}^+ \) with the constant \( C \) completely independent on \( t \in [0, T] \).

**Proof.** See Theorem 4.1.6, page 100, of [7] and section V of [6].

The general result about the \( n \)-th-order can be found by referring to Theorem 1.2.1 of [7] and Theorem 0.1 in [8].

**Remark 3.2.4.** Let us highlight that in CEV case, with \( 0 < \beta < 1 \), see section 1.3, the hypothesis on coefficients stated in Theorem 3.2.1, are not satisfied. Indeed, the CEV model is characterized by a volatility coefficient which is clearly not a bounded function on \( \mathbb{R} \).

### 3.3 Kernel expansion at z=x

In what follows, we will show how to explicitly compute \( G_t^{[2]} \) at \( z = x \), and how to find an expression for the approximated price of an European Call option. This choice leads
to a simplified expression for the approximation, since the terms \((x - z)\) disappear. Accordingly, equation (3.28) simplifies to
\[
G_1(x, y, z = x) = \frac{x - y}{\sqrt{2\pi a^2}} e^{-\frac{(x-y)^2}{2a^2}} \left( \frac{3aa' - 2b}{2a^2} - \frac{a'}{2a^3}(x-y)^2 \right),
\] (3.31)
and, consequently eq. (3.29) becomes
\[
G_2(x, y, z = x) = \frac{x - y}{\sqrt{2\pi a^2}} e^{-\frac{(x-y)^2}{2a^2}} \left\{ \frac{1}{8}a^6a'^2H_6(\theta) + \frac{a^3}{6}(a^2a'' + 4aa'^2 + 3ba')H_4(\theta) + \right. \\
+ \left. \frac{1}{4}(a^3a'' + 2a^2b' + 2aa + 2aa'b + a^2a^2 + 2b^2)H_2(\theta) + c \right\}
\] (3.32)
where \(\theta = \frac{x - y}{a(0, z)^2}\), and \(H_i, i = 2, 4, 6\) are the Hermite polynomials.

Therefore, the first order is given by
\[
G^{[1]}_t(x, y) = t^{-1/2}(G_0(x_t, y_t, z) + t^{1/2}G_1(x_t, y_t, z)),
\] (3.33)
while the second order approximation reads
\[
G^{[2]}_t(x, y) = t^{-1/2}(G_0(x_t, y_t, z) + t^{1/2}G_1(x_t, y_t, z) + tG_2(x_t, y_t, z)),
\] (3.34)

Taking into account the above introduced change of variables, i.e.,
\[
\begin{align*}
  x_t &= z + (x - z)/\sqrt{t}, \\
y_t &= z + (y - z)/\sqrt{t},
\end{align*}
\]
hence,
\[
G^{[1]}_t(x, y) = \frac{1}{\sqrt{2\pi ta^2}} e^{-\frac{(x-y)^2}{2ta^2}} \left\{ 1 + \frac{3aa' - 2b}{2a^2}(x-y) - \frac{a'}{2ta^3}(x-y)^3 \right\}
\] (3.35)
and
\[
G^{[2]}_t(x, y) = G^{[1]}_t(x, y) + \frac{1}{\sqrt{2\pi ta^2}} e^{-\frac{(x-y)^2}{2ta^2}} \left\{ tc + \left[ \frac{(x-y)^2}{2a^4} - \frac{t}{2a^2} \right] \\
+ \left[ \frac{1}{2}a^3a'' + a^2b' + \frac{a^2a'^2}{2} + b^2 + aba' + aa \right] + \\
+ \left[ \frac{(x-y)^4}{3ta^6} - \frac{2(x-y)^2}{a^4} + \frac{t}{a^2} \right] \left[ \frac{1}{2}a^3a'' + 2aa'^2 + \frac{3}{2}a'a'b \right] + \\
+ \frac{1}{8}a^2 \left[ \frac{(x-y)^6}{a^6t^2} - \frac{15(x-y)^4}{t a^4} + \frac{45}{4}a(x-y)^2 - 15t \right] \right\}
\] (3.36)

So that, by equations (3.35) and (3.36), it is possible to find the first and second order forms approximating the price \(U(t, x)\) for an European Call option under the Black-Sholes-Merton model, the CEV model and the corresponding time-dependent generalizations just computing the derivatives of the coefficients \(a, b\) and \(c\) at \((0, x)\). To see the general forms for \(U^{[n=1,2]}(t, x)\), refer to formula (3.5) at page 921 of [5].
Chapter 4

The Method of the Lines and Least Square Monte Carlo

In what follows we provide a quick description of some of the most relevant numerical methods that are often used in solving problems usually encountered in finance. We will omit proofs about existence, uniqueness and regularity of solutions, since they are out of our scopes, nevertheless, we refer the interested reader to [9] for details.

4.1 Finite differences

4.1.1 BVPs

The “Finite Differences” approximation method is commonly used in the so called boundary value problems, shortly BVPs, i.e. problem that can be formulated as follows

\[
\begin{align*}
    u''(x) &= f(x, u(x), u'(x)) & x \in (a, b) \\
    &+ \text{boundary condition at } a \\
    &+ \text{boundary condition at } b
\end{align*}
\]

where variable \( x \) represents the space, \( u(x) \) a sufficiently regular real valued function, while \( f \) is typically assumed to be at least continuous.

Concerning boundary conditions, we can have

- **Dirichlet’s conditions**, prescribing the value of the solution \( u(x) \) at the end points \( a \) and \( b \), i.e. \( u(a) = u_a \) or \( u(b) = u_b \);
- **Neumann’s conditions**, prescribing the value of the derivative of solution \( u'(x) \) at the end points \( a \) and \( b \), i.e. \( u'(a) = u_a \) or \( u'(b) = u_b \);
Robin’s Conditions, prescribing a linear combination of Dirichlet and Neumann conditions, e.g., \( \alpha u(a) + \beta u'(a) = \gamma_a \) or \( \alpha u(b) + \beta u'(b) = \gamma_b \).

**Remark 4.1.1.** Sometimes may happen that prescribed boundary values are zero, in that case the conditions, despite their main type, are said to be homogeneous. Moreover, there exist problems in which the first two kinds of conditions appear at the same time, for example there might be a Dirichlet’s condition at “a” and a Neumann’s condition at “b”.

### 4.1.2 Second order Central Finite Differences

In what follows, we consider \( u \in C^3([a, b]; \mathbb{R}) \), and we discretize the space interval \([a, b]\) into a certain number \( m \in \mathbb{N} \) of equidistant nodes, \( x_i \), so that we have

\[
x_i = a + (i - 1)h, \quad i \in \{1, \ldots, m\},
\]

where \( h = \frac{(b - a)}{(m - 1)} \) is called discretization step.

Considering problem (4.1), we aim at trying a discretization of \( u'(x_i) \). Therefore, we consider the Taylor’s series expansion of \( u \) evaluated at the nodes namely \( x_{i+1} \) and \( x_{i-1} \),

\[
u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2!}u''(x_i) + \frac{h^3}{3!}u^{(3)}(\hat{x}_i),
\]

\[
u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2!}u''(x_i) - \frac{h^3}{3!}u^{(3)}(\tilde{x}_i).
\]

Then, performing a simple term-by-term subtraction between \( (4.2) \) and \( (4.3) \), we have

\[
u(x_{i+1}) - \nu(x_{i-1}) = 0 + 2hu'(x_i) + 0 + \frac{2h^3}{3!}u^{(3)}(\bar{x}_i),
\]

from which

\[
u'(x_i) = \frac{\nu(x_{i+1}) - \nu(x_{i-1})}{2h} - \xi^{(1)}(\bar{x}_i) = \Delta u(x_i)
\]

where the term \( \xi^{(1)}_i = \frac{h^2}{6}u^{(3)}(\bar{x}_i) \), for a certain \( \bar{x}_i \) given by the Lagrange’s Mean Value Theorem, is called local error, and we write

\[
u'(x_i) \approx \frac{\nu(x_{i+1}) - \nu(x_{i-1})}{2h} =: \Delta u(x_i) \tag{4.4*}
\]

Assuming \( u \in C^4([a, b]; \mathbb{R}) \), we can analogously discretize \( u''(x) \), taking care about the fact that now the Taylor’s expansion requires a further step, namely

\[
u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2!}u''(x_i) + \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\bar{x}_i),
\]
\[
\begin{align*}
    u(x_{i-1}) &= u(x_i) - h u'(x_i) + \frac{h^2}{2!} u''(x_i) - \frac{h^3}{3!} u'''(x_i) + \frac{h^4}{4!} u^{(4)}(\bar{x}_i), \\
    \end{align*}
\] 

(4.6)

from which, summing term-by-term between (4.5) and (4.6), we find

\[
\begin{align*}
    u''(x_i) &= \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} - \xi^{(2)}_i(\bar{x}_i), \\
    \end{align*}
\] 

(4.7)

where \(\xi^{(2)}_i = \frac{h^2}{24} u^{(4)}(\bar{x}_i)\), \(\bar{x}_i\) having the same mean value property as before.

Again, neglecting the local error term, one can briefly write

\[
\begin{align*}
    u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} =: \Delta^2 u(x_i). \\
    \end{align*}
\] 

(4.7*)

As we can easily deduce, we have obtained a “second order” scheme since the local error is proportional to the second power of \(h\). Moreover, this method is “central” since nodes are equidistant and moreover the discretizations are centered in \(x_i\). Namely both in \(u'(x_i)\) and in \(u''(x_i)\) there are the same number of discretization points on both the right and the left hand side of \(x_i\).

**Remark 4.1.2. (other types of finite differences)** As already mentioned above, there are several types of finite differences methods that can be used. For example, if one guesses in advance that the solution of a certain problem would be mostly flat on the interior of \([a,b]\) and slightly less on the boundary, it would be better to use the so called Chebychev’s nodes, see e.g., [18], instead of equidistributed ones or, to extend the computations stated before, to consider not only the nearest neighbors \(x_{i-1}, x_{i+1}\), but also more “distant” nodes. Of course, in these ways the finite differences are no more centered and so in Taylor’s expansions one has to consider a space dependent discretization step \(h_i\) for \(i \in \{1,...,m-1\}\) instead of a fixed \(h\). We do not consider latter scenarios, referring the interested reader to, e.g., Chapter 7 of [17].

Exploiting the discretization scheme that lead us to (4.4*) and (4.7*), we have that the discretized problem shown below

\[
\begin{align*}
    \Delta^2 u_i &= f(x_i, u_i, \Delta u_i) & i \in \{2,...,m-1\} \\
    u_1 &= u_a \\
    u_m &= u_b \\
    \end{align*}
\] 

(4.8)

is the one that will be solved, approximating the original one. Notice that the index \(i\) goes from 2 to \((m - 1)\) because the first and the last rows of the discretized system need to be treated separately in order to adapt them to boundary conditions. Indeed, to find a solution to (4.8), means to solve a system of \(m\) linear equations w.r.t to the \(u_i\).
unknown variables, for \( i \in \{1, \ldots, m\} \). Then, we have

\[
\Delta u = \frac{1}{2h} \begin{bmatrix}
* & * & * & * & * & * \\
-1 & 0 & 1 & 0 & \cdots & 0 \\
0 & -1 & 0 & 1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \cdots & 0 \\
0 & \cdots & 0 & -1 & 0 & 1 \\
* & * & * & * & * & *
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{m-1} \\
u_m
\end{bmatrix} = \frac{1}{2h} Au,
\]

for the first derivative, respectively. Looking at the matrices in equations (4.8*) and (4.8**), notice that they are both sparse and tridiagonal, also assuming a precise pattern except for the first and the last row, which as already said, have to be treated separately. That pattern, called “stencil”, equals \((-1, 0, 1)\) for \(u'\) and \((1, -2, 1)\) for \(u^{''}\).

**Remark 4.1.3. (existence and uniqueness of solutions)** As already mentioned, the main of the present thesis is not to go into details covering latter topic, nevertheless we underline that if the coefficients (that can be either constants or functions of \(x\)) characterizing the original problem are sufficiently regular, the existence and uniqueness of the analytical solution is granted. On the other hand, if the discretized problem can be expressed in the form \(Au = g\), where \(g\) is at least continuous and \(A\) is non-singular, the uniqueness of \(u\) is granted, too.

### 4.1.3 Boundary conditions

**Dirichlet's conditions**

Whenever the boundary conditions are of the type \(u(a) = u_a\) or \(u(b) = u_b\), it is convenient to start the discretization of the original problem without carrying about these conditions and then, directly modify the first or last row to fit the prescribed conditions.

For example, consider the following system

\[
\begin{align*}
u''(x) = 2, & \quad x \in (a, b), \\
u(a) = u_a, \\
u(b) = u_b,
\end{align*}
\]

(4.9)
then, discretizing it without taking into account the boundary conditions, we obtain

\[
\begin{bmatrix}
-2 & 1 & 0 & 0 & \cdots & 0 \\
1 & -2 & 1 & 0 & \ddots & \vdots \\
0 & 1 & -2 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & -2 & 1 \\
0 & \cdots & 0 & 0 & 1 & -2
\end{bmatrix}
\frac{1}{h^2}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{m-1} \\
u_m
\end{bmatrix}
=
\begin{bmatrix}
2 \\
2 \\
\vdots \\
2 \\
2
\end{bmatrix}
\] (4.10)

Then, adjusting the boundary conditions values, we can easily derive the correct discretized problem

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
1 & -2 & 1 & 0 & \ddots & \vdots \\
0 & 1 & -2 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & -2 & 1 \\
0 & \cdots & 0 & 0 & 0 & 1
\end{bmatrix}
\frac{1}{h^2}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{m-1} \\
u_m
\end{bmatrix}
=
\begin{bmatrix}
u_a/h^2 \\
2 \\
\vdots \\
u_{m-1} \\
u_b/h^2
\end{bmatrix}
\] (4.11)

**Remark 4.1.4. (penalty method)** There is disadvantage in what we have provided so far, namely: the matrix loses its symmetry. Thus, it is no more possible to apply all those decomposition methods for symmetric matrices that allow the problem to be less heavy from a computational point of view. To recover symmetry, one can use the so-called “penalty method”, numerically equivalent to scheme (4.11), that consist in solving

\[
\begin{bmatrix}
M & 1 & 0 & 0 & \cdots & 0 \\
1 & -2 & 1 & 0 & \ddots & \vdots \\
0 & 1 & -2 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & -2 & 1 \\
0 & \cdots & 0 & 0 & 0 & 1
\end{bmatrix}
\frac{1}{h^2}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{m-1} \\
u_m
\end{bmatrix}
=
\begin{bmatrix}
u_a \\
2 \\
\vdots \\
u_{m-1} \\
u_b
\end{bmatrix}
\]}

for a very large \(M\), plus if the matrix is positive definite, it’s possible to use the Cholesky factorization, see, e.g., Section 9.2 of [9] for further details.

**Neumann’s conditions**
Treating this type of boundary conditions implies the need to discretize \(u'(x_1)\) and/or
eventually \( u'(x_m) \). Recall that \( u'(x_i) \) depends on the previous point \( x_{i-1} \) and on the next point \( x_{i+1} \), but in this case if \( i = 1 \) then \( x_{i-1} = x_0 \) and if \( i = m \), \( x_{i+1} \) would be equal to a certain \( x_{m+1} \). The problem is that those points are out of the bounds of space discretization. This technical issue can be overcome defining a “ghost node”, i.e. considering \( x_1 = a \), then defining \( u_0 \approx u(a-h) \) in order to find

\[
u'(a) \approx \frac{u_2 - u_0}{2h} = u'_a \tag{4.12}\]

that immediately provides \( u_0 = u_2 - 2hu'_a \). This definition has then to be used in every discretization stencil.

For example, considering the problem \( u''(x) - u'(x) = 2 \) (which is a slight modification of the example (4.9)) and willing to put the Neumann’s condition \( x = a \), we have

\[
\frac{u_0 - 2u_1 + u_2}{h^2} - \frac{u_2 - u_0}{2h} = \frac{u_2 - 2hu'_a - 2u_1 + u_2}{h^2} - \frac{u_2 - u_2 + 2hu'_a}{2h} = \\
= \frac{2u_2 - 2u_1 - 2hu'_a}{h^2} - u'_a = 2,
\]

from which

\[
\frac{2u_2 - 2u_1}{h^2} = 2 + \left( \frac{2}{h} + 1 \right) u'_a \tag{4.13}\]

4.1.4 Resolution methods

Linear systems
Tipically, linear tridiagonal systems of the form \( Ax = b \), with \( x \) to be found and \( A \) non-singular matrix are easy to solve. In fact, it is sufficient to compute \( A^{-1} \) and then find \( x \) directly, by \( x = A^{-1}b \). Usually, computation enviroments like Matlab or GNU Octave, automatically perform an \( LU \) decomposition of \( A \) in these cases, moreover if the matrix is symmetric it would be useful to opt for a Cholesky decomposition that allows to write \( A = LL^T \) where \( L \) is a lower triangular matrix, see, e.g., Chapter 9 of [9] for further details about such type of operatorial decompositions.

Non-linear systems
To solve systems of non-linear equations, one of the most used methods is the so called Newton’s method. Consider the following system of \( m \in \mathbb{N} \) non-linear equations

\[
\begin{align*}
f_1(x_1, \cdots, x_m) &= 0 \\
& \vdots \\
f_N(x_1, \cdots, x_m) &= 0
\end{align*}
\]
that can be expressed in the following short form \( f(x) = 0 \).

Given an initial datum, \( x^{(1)} \), the Newton’s Method for \( x^{(l+1)} \) gives

\[
J^{(l)} \delta x^{(l)} = -f(x^{(l)}),
\]

\[
x^{(l+1)} = x^{(l)} + \delta x^{(l)},
\]

where \( J^{(l)} \) is the Jacobian Matrix of the system, obtained directly by deriving \( f \) w.r.t. the unknown variable. Once a fixed approximation level has been decided, then the methods end according to the stoppin criterion

\[
||\delta x^{(l)}|| \leq \text{tolerance},
\]

which allows, at least in principle, to decide how our approximation will be precise.

### 4.2 Single-step methods

The next pages deal with the so called single-step methods, tipically used in solving systems of ODEs with an initial datum \( y_0 \). As in the previous chapter, these pages are intended to provide an introduction able, without entering into details, to give a clear idea of how these approximation schemes work.

#### 4.2.1 IVPs

Single-step methods are tipically applied to solve the so called initial value problems, shortly IVPs, i.e. problems that are defined by

\[
\begin{align*}
  y'_1(t) &= f_1(t, y_1(t), y_2(t), \ldots, x, y_d(t)) \\
  & \vdots \\
  y'_d(t) &= f_1(t, y_1(t), y_2(t), \ldots, x, y_d(t))
\end{align*}
\]

with initial datum,

\[
\begin{align*}
  y_1(t_0) &= y_{10} \\
  & \vdots \\
  y_d(t_0) &= y_{d0}
\end{align*}
\]

that can be expressed in compact form

\[
\begin{align*}
  y'(t) &= f(t, y(t)), \quad t > 0, \\
  y(t_0) &= y_0
\end{align*}
\]

assuming that the initial value is \( y_0 \in \mathbb{R}^d \) and \( f : [t_0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d \) is a globally Lipschitz function in the second variable, namely \( ||f(t, x) - f(t, y)|| \leq \lambda ||x - y|| \) for every
Then, the system (4.18) has a unique solution starting from $y_0$. Generally, systems like these appear in non-autonomous form, but however they can be turned in autonomous form introducing a new variable $y_{d+1} = t$ and re-written as

$$
\begin{align*}
  \begin{cases}
    y'(t) = f(y_{d+1}(t), y(t)), & t > t_0, \\
    y'_{d+1}(t) = 1, \\
    y(t_0) = y_0, \\
    y_{d+1}(t_0) = t_0
  \end{cases}
\end{align*}
$$

(4.19)

System of order $d > 1$, that prescribe also the initial value of derivatives of $y$, like

$$
\begin{align*}
  \begin{cases}
    y^{(d)}(t) = f(t, y(t), y'(t), \ldots, y^{(d-1)}(t)) & t > t_0, \\
    y(t_0) = y_{0,0}, \\
    y'(t_0) = y_{0,1}, \\
    \vdots \\
    y^{(d-1)}(t_0) = y_{0,d-1}
  \end{cases}
\end{align*}
$$

(4.20)

can be reconducted to a first order ODEs system by first considering

$$
\begin{align*}
  \begin{cases}
    y_1(t) = y(t) \\
    y_2(t) = y'(t) \\
    \vdots \\
    y_d(t) = y^{(d-1)}(t)
  \end{cases}
\end{align*}
$$

which leads to

$$
\begin{align*}
  \begin{cases}
    y'(t) = f(t, y(t)) \\
    y(t_0) = [y_{0,0}, y_{0,1}, \ldots, y_{0,d-1}]^T
  \end{cases}
\end{align*}
$$

(4.21)

where $f(t, y(t)) = [y_2(t), y_3(t), \ldots, y_d(t), f(t, y_1(t), y_2(t), \ldots, y_{d-1}(t))]^T$.

Within the family of single-step methods, one can consider the theta-methods ones. In particular, consider the time interval $[0, T]$ where $T > 0$ and divide it into a certain number of discrete points $t_0 = 0, t_1 = t_0 + k, t_2 = t_1 + k = t_0 + 2k, \ldots, t_n = t_0 + nk$ where $n \in \mathbb{N}$ and $k$ is usually called time step, then for the $i^{th}$ time instant, the numerical scheme is given by

$$
\begin{align*}
  \begin{cases}
    y_{i+1} = y_i + k(1 - \theta)f(t_i, y_i) + k\theta f(t_{i+1}, y_{i+1}) \\
    y_0 = y(t_0)
  \end{cases}
\end{align*}
$$

(4.22)

where $\theta \in [0, 1]$, and $y_i = y(t_i)$. As the theta parameter varies, different numerical schemes can be defined, some of them
are of explicit type, others are implicit. The first ones are relatively easy to solve, since, in this case \( y_{i+1} \) only depends on the terms related to the previous time step. While, in the second case to find \( y_{i+1} \) is necessary to solve a system of non-linear equations that depends on \( y_{i+1} \) itself. To do so, a good choice could be Newton’s Method described in the previous section.

Theta-methods also differ in the convergence order, again related to the local error that is proportional to a certain power \( p \) of the time step: a scheme is of order \( p \) if its local error is proportional to \( k^p \). It means that, as the number of time steps grows up, the distance between the exact and the approximated solution goes to zero like a straight line of slope \(-p\).

- \( \theta = 0 \) yields \( y_{i+1} = y_i + kf(t_i, y_i) \) which constitutes an explicit scheme, called forward Euler, of order 1,
- \( \theta = 1/2 \) gives \( y_{i+1} = y_i + \frac{1}{2}[f(t_i, y_i) + f(t_{i+1}, y_{i+1})] \), hence an implicit method called Crank-Nicolson scheme, of order 2,
- \( \theta = 1 \) allows \( y_{i+1} = y_i + kf(t_{i+1}, y_{i+1}) \), and it provides an implicit method called backward Euler, of order 1.

4.3 Exponential Euler

The problems of absolute stability for linear IVPs leads to quest for new methods. A very useful one is the so called Exponential-Euler, an explicit scheme that belong to a particular class of methods called Exponential Integrators.

Consider the following differential problem

\[
\begin{cases}
y'(t) = Ay(t) + b, & t > 0, \\
y(t_0) = y_0
\end{cases}
\] (4.23)

then, the associated analytical solution is given by

\[
y(t) = \exp((t-t_0)A)y_0 + (t-t_0)\varphi_1((t-t_0)A)b = y_0 + (t-t_0)\varphi((t-t_0)A)(Ay_0 + b),
\] (4.23*)

Since \( y(t_0) = y_0 \) and, observing that

\[
\frac{d}{dt}[(t-t_0)\varphi((t-t_0)A)b] = \exp((t-t_0)A)b = (t-t_0)A\varphi((t-t_0)A)b + b
\]

then, both the exponential and \( \varphi \) function may be approximated exploiting what we shall state at the end of this section. In particular, for a problem of the form

\[
\begin{cases}
y'(t) = Ay(t) + b(t, y(t)), & t > 0, \\
y(t_0) = y_0
\end{cases}
\] (4.24)
the “Exponential-Euler” is given by the following equation

\[ y_{n+1} = y_n + k \varphi(kA)(Ay_n + b(t_n, y_n)), \]  

(4.25)

where \( k \) is the time-step and \( \varphi \) is a function that approximates the exponential of the matrix \( A \).

Exponential-Euler is generally an explicit method of order 1, furthermore, it is possible to prove that it is a convergent (thus stable) method. Moreover, since it solves exactly linear problems, then Exponential-Euler is also absolutely stable and its absolute-stability region is \( |r(z)| = |e^z| < 1 \) and thus \( \mathbb{C}^- \).

### 4.3.1 Exponential of a matrix and possible approximations

In what follows, as already anticipated in Section 4.3, we show how to approximate the function \( \varphi \) that calculates the exponential of a matrix.

Given a square matrix \( A \in \mathbb{R}^{N \times N} \), define

\[ \exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!} \]

which is a convergent series for every matrix \( A \), being \( A \) a linear operator between Banach spaces and having a convergence radius \( \infty \). If \( A \) and \( B \) commutes, i.e. \( AB = BA \), then \( \exp(A + B) = \exp(A)\exp(B) \).

Like in solving linear systems, the “definitive method” to compute \( \exp(A) \) doesn not exist, instead several one can be developed optimizing the results to particular situations. The following listed below apply to those matrices for which direct methods for solving linear systems are used.

**Spectral decomposition**

If the matrix is diagonalizable, thus \( A = VDV^{-1} \), then \( \exp(A) = V\exp(D)V^{-1} \), where \( \exp(D) \) is the diagonal matrix with elements \( e^{d_1}, e^{d_2}, ..., e^{d_N} \). It is indeed sufficient to observe that

\[ A^2 = (VDV^{-1})^2 = (VDV^{-1})(VDV^{-1}) = VD^2V^{-1}, \]

and write \( \exp(A) \), as in the Taylor series case. In general, the computational cost of these decomposition is \( O(N^3) \).

**Padé’s approximation**

Consider a rational approximation of exponential function

\[ e^z \approx \frac{a_1 z^{p-1} + a_2 z^{p-2} + ... + a_p}{b_1 z^{-1} + b_2 z^{-2} + ... + b_q}, \]
where $b_q = 1$ for convention. It is called *diagonal* when $p = q$. Once the approximation grade is fixed, one can develop the Taylor series of the exponential function and let as many coefficients as possible to coincide. For example, let us fix $p = q = 2$, then

\[
\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots\right) (b_1 z + 1) \approx a_1 z + a_2,
\]

\[
b_1 z + 1 + b_1 z^2 + z + \frac{z^2}{2} + \ldots \approx a_1 z + a_2,
\]

from which

\[
\begin{cases}
1 = a_2 \\
b_1 + 1 = a_1 \\
b_1 + \frac{1}{2} = 0
\end{cases}
\]

Padé’s approximation can be easily extended to the matricial case. As to make an example, if $p = q = 2$ then

\[\exp(A) \approx B = (b_1 A + I)^{-1} (a_1 A + a_2 I),\]

thus $B$ is the solution of the linear system $(b_1 A + I) B = a_1 A + a_2 I$. In this case, the approximated solution of a differential problem of the type $y'(t) = Ay(t)$, becomes

\[y(t) = \exp(tA) y_0 \approx (I - tA/2)^{-1} (I + tA/2) y_0,\]

which coincides with the resulition obtained by using trapezoidal method. Padé’s approximation is accurate only when $|z| < 1/2$, equivalently, in matricial case, when $|A| < 1/2$.

For the exponential function, there exists a technique called *scaling and squaring* that allows to twist around the problem. It uses the property

\[e^z = (e^{z/2})^2 = (e^{z/2^j})^{2^j},\]

hence if $|z| > 1/2$, then $|z|/2^j < 1/2$ for $j > \log_2(|z|) + 1$. Next, compute Padé’s approximation of $e^{z/2^j}$ after that rise to the square $j$ times. For $\varphi$ function holds

\[\varphi(z) = \frac{1}{2} (e^{z/2} + 1) \varphi(z/2)\]

that is a duplication formula. Matricial Padé’s approximation has a computational cost that equals $O(N^3)$. See, e.g., Chapter 4, page 48, of [19] for further details.

### 4.4 Least Square Monte Carlo methods

The standard Monte Carlo method is a very useful tool for option pricing, due to its simplicity of implementation, but it has a limitation: it cannot be used for pricing
American style options, because of the early exercise is possibility. To make it works, it has to be modified and there are several ways to do it, the simplest possible is to perform a “Monte Carlo on Monte Carlo” simulation, in which for every realisation, it is necessary to predict its possible evolution step-by-step in time. This generates a huge amount of realisations, hence it is not a feasible alternative because of its large computational cost.

A class of Monte Carlo methods that is often used in finding the fair price of American options, is the one based on folding back the price paths history, aiming to find the optimal stopping time i.e. the one that maximizes the option value \( V(S,t) \).

In particular, we have focused our attention on the model proposed by Longstaff and Schwartz in 2001, see [10], which estimated the so called “conditional expected cash-flow” at every possible exercise date using a polynomial regression technique.

4.4.1 Least Square Monte Carlo mathematical framework

Before entering into details about the concrete realisation of a Least Square Monte Carlo (LSMC) approach to the American option price problem, some assumptions are needed:

1. \((\Omega, \mathcal{F}, \mathcal{P})\) is a complete probability space, where every element \( \omega \) of \( \Omega \) represents a possible market realisation within a finite time horizon \( T \), and \( \mathcal{F} = \{\mathcal{F}_t|0 \leq t \leq T\} \) is the associated filtration.

2. There are no arbitrage opportunities and the market is complete, so there exist a unique probability measure \( \mathcal{P}^* \) equivalent to \( \mathcal{P} \) that make the discounted stock prices to be martingales.

3. The option is exercisable at any \( 0 < t_0 < t_1 < \ldots < t_i < \ldots < t_{n-1} < t_n = T \) of the finite time horizon.

4. The underlying model \((S_i)_{0 \leq i \leq n}\) enjoys the Markov property.

5. The payoff \( P_i(\omega) \) at date \( t_i \) for \( i = 0,...,n \) is a square-integrable random variable, namely \( \mathbb{E}[(P_i(\omega))^2] < \infty \) for every \( i \), and \( P_i(\omega) \in L^2(\Omega, \mathcal{F}_i, Q_{S_i}) \) where \( Q_{S_i} \) is the probability measure of \( S_i(\omega) \) on \( \mathcal{F}_i \) and \( L^2 \) Hilbert space.

The investor, once gotten to maturity date, exercises the option if it is “in the money” or it allows it to expire otherwise. However, at a possible exercise time \( t_i \) prior to the final maturity date, the optionholder must choose whether to exercise immediately or to continue the life of the option and revisit the exercise decision at \( t_{i+1} \).

The point is: the cash flows from immediate exercise at time \( t_i \) are known, they are given directly by payoff function at \( S(t_i) \), but the cash-flows from continuation are not. They
have to be estimated. Specifically, at time $t_k$, the value of continuation $CV(\omega, t_i)$ can be expressed in the following form

$$CV(\omega, t_i) = \mathbb{E}_{P^*} \left[ \sum_{j=k+1}^{n} e^{\int_{t_k}^{t_j} r(\omega, s) ds} C(\omega, t_j, t_k, T) \mathcal{F}_{t_k} \right], \quad (4.26)$$

where $P^*$ is the probability measure mentioned in point 2), $C(\omega, t_j, t_k, T)$ are the discounted cash-flows and $r(\omega, t)$ is the, possibly stochastic, interest rate.

Concluding, from the maximization of the revenue point of view, the optimal stopping time is given by

$$\tau_0^* = \inf \{ i \geq 0 | CV_i(\omega) = P_i(\omega) \} \quad (4.27)$$

### 4.4.2 Regression framework

In the following section, we explain what means to fold back the prices history in order to calculate the continuation value (4.26). The starting point of the algorithm, described at page 5 of [11], is the maturity date $t_n = T$, for which there are no continuation values to estimate. The next problem to solve, then, is to find the best estimation for the continuation value $CV(\omega, t_{n-1})$. The LSMC approach uses least squares to approximate the conditional expectation function for continuation values at $t_{n-1}, t_{n-2}, ..., t_2, t_1$. More precisely, we assume that at time $t_{n-1}$ the unknown functional form of $CV(\omega, t_{n-1})$ in equation (4.25) is a linear combination of a countable set of $\mathcal{F}_{t_{n-1}}$-measurable basis functions. This assumption is formally justified from point 5) given in section 4.4.1. Infact, an Hilbert space possesses a countable orthonormal basis and the conditional expectation function can be represented as a linear combination of its elements.

There are several possible choices of basis functions, one is the Laguerre polynomials

$$L_n(X) = e^{-X/2} \frac{d^n}{n!} (X^n e^{-X}),$$

where $X$ is the value of the asset underlying the option and that $X$ follows a Markov Process. With this specification, $CV(\omega, t_{n-1})$, can be represented by

$$CV(\omega, t_{i-1}) = \sum_{j=0}^{\infty} a_j L_j(X), \quad i = n, ..., 2, \quad (4.27^*)$$

for some coefficients $a_j$ to be estimated. Other choices are possible, as for the Hermite, Legendre, Chebyshev, Gegenbauer and Jacobi polynomial, see e.g., page 122 of [10].

In the real world however it is not possible to use a countable number of basis functions, so it is necessary to truncate the series of the type in (4.27*) at an appropriate level $M < \infty$, such that $CV_M(\omega, t_{n-1})$ could be considered a good estimator of $CV(\omega, t_{n-1})$, provided a given accuracy level.

45
In our analysis, see section 5.3, we have derived a regression using simple powers of the state variables. The code related to this document perform a regression using simple powers of the state variables. In particular, the basis we have used is given by \( \{1, X, X^2\} \), hence obtaining

\[
CV(\omega, t_{i-1}) = \sum_{j=0}^{2} a_j X^j, \quad i = n, \ldots, 2 \tag{4.28}
\]

Numerical tests indicate that also this basis choice ensure convergence and provides accurate results, see [11]. To see all the details about convergence and accuracy, we refer to the bibliography of [10], pages from 124 to 126.

4.4.3 Code discussion

In what follows we give a description of the computational steps we performed to treat American Put Option price problem. As already mentioned, the regression polynomial used reads as follows \( E[Y|X] = a_0 + a_1 X + a_2 X^2 \), where \( X \) is the value of the asset and \( Y \) is the expected cash-flow.

1. Generate the Stock Price Matrix: remember that if \( r \) is the riskless interest rate and \( \sigma \) the asset volatility, in the case of a GBM model \( dS_t = rS_t dt + \sigma S_t dW_t \), then \( S_t = S_0 \exp\{(r - 1/2\sigma^2)t + \sigma W_t\} \) holds true. For other market models an explicit form may not exist. In that case, realisations need to be computed differently, e.g. using the Euler-Maruyama Method that will be discussed as last argument of this section.

2. Create a vector for the payoff at maturity: in the case of a Put option, we have that its payoff equals \( (K - S(t_n = T))^+ \). If one discounts the mean value of the elements of this vector, then would find the equivalent european value for the option.

3. Perform the regression folding back the Stock Price Matrix: for every \( t_1 < t_2 < \ldots < t_i < \ldots < t_{n-2} < t_{n-1} \), execute the following instructions

   (i) discount the actual payoff of just a period;
   (ii) compute the current exercise values for the option;
   (iii) calculate the regression coefficients, using only the paths that actually are in-the-money (Matlab/GNU Octave instructions like find and polyfit may turn out to be useful);
   (iv) use the coefficients just computed to find the continuation values of the in-the-money paths (Matlab/GNU Octave instruction polyval is helpful);
   (v) compare the actual exercise values with continuation values and, if the second ones are smaller, update the payoff vector with the first ones.
4. **Find the option fair price**: discount the payoff average to time $t = 0$.

We refer to [11] for more details about the just described procedure.

**Euler-Maruyama scheme**

The Euler-Maruyama method is a numerical explicit scheme allowing to derive approximate solutions to a stochastic differential equation. It generalizes the Euler method for ordinary differential equations, to the SDEs case. In particular, consider the following SDE

$$\begin{cases}
    dX_t = \mu(t, X_t)X_t dt + \sigma(t, X_t)X_t dW_t, & t \in [0, T] \\
    X_0 = x_0
\end{cases} \tag{4.29}$$

where $\mu$ and $\sigma$ are respectively the drift and the diffusion coefficient, $T$ is the finite time horizon and $W_t$ is a standard Wiener Process (or Brownian Motion). Then the Euler-Maruyama approximation to the true solution $X$, is the Markov chain $Y$ defined as follows:

- partition the interval $[0, T]$ into $N$ equal subintervals of width $\Delta t > 0$ so that $t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ and $\Delta t = T/n$;
- set $Y_0 = x_0$;
- recursively define $Y_i$ for $1 \leq i \leq n$ by
  $$Y_{i+1} = Y_i + \mu(t_i, Y_i)Y_i \Delta t + \sigma(t_i, Y_i)Y_i \Delta W_i$$

where $\Delta W_i = W_{i+1} - W_i$, the random variables $\Delta W_i$ are independent and identically distributed Gaussian random variables, with expected value zero and variance $\Delta t$.

It can be proved that the local error $\tau$ between the exact (or reference) solution and the approximate one is proportional to $O(\Delta t^{1/2})$, see [12] for further details.
Chapter 5

Numerical tests results

This chapter shows results of various numerical implementations of Dyson-Taylor commutator method compared with the other schemes we have introduced in Chapter 4. Let us underline, for the sake of completeness, that all the reported computations have been performed on a LG1151 socket equipped with a Skylake Intel Core i5 6400 running at 2.70 Ghz, 8GB of 2133Mhz DDR4 RAM and a Gigabyte GTX 1060 3GB GDDR5 as GPU accelerator.

5.1 European style - accuracy test

The section shows results about the convergence of Dyson-Taylor commutator method in pricing an European Call option under two different types of market models. Moreover, it holds true also for an equivalent European Put option directly, by mean of the Put/Call parity formula

\[ P_t = C_t - S_t + Ke^{-r(T-t)}, \]  

(5.1)

where \( P_t \) and \( C_t \) are the values of Put and Call at time \( t \), respectively, \( S_t \) is the current underlying price and \( K \) is the strike.

5.1.1 European Call option under Black-Sholes-Merton model

In this case, the problem has an exact solution that can be used for a direct comparison. Infact, we have

\[ C(S, t) = SN(d_+) - Ke^{-rt}N(d_-), \]  

(5.2)

where

\[ d_\pm = \frac{\log(S/K) \pm (r + \sigma^2/2)t}{\sigma \sqrt{t}}, \]
and $N(x)$ represents the cumulative distribution function of a standard Gaussian random variable, see the derivation recalled in Chapter 3.

The tests have been performed assuming $r = 0.05$ as interest rate, a volatility coefficient of $\sigma = 0.40$, a strike price $K = 10$ and various different expiry dates $T \in \{1, 0.5, 0.4, 0.3, 0.2, 0.1\}$. The space discretization counts $m = 50$ equidistant nodes in a range $[0, S_{\text{max}}]$ where $S_{\text{max}} = 2K$.

The first order Dyson-Taylor approximation gave the following results

| Expiry | $||DT_{1}\text{st} - \text{exact}||_{\infty}$ |
|--------|---------------------------------------------|
| 1.00   | $3.791 \cdot 10^{-1}$                      |
| 0.50   | $1.962 \cdot 10^{-1}$                      |
| 0.40   | $1.640 \cdot 10^{-1}$                      |
| 0.30   | $1.318 \cdot 10^{-1}$                      |
| 0.20   | $9.550 \cdot 10^{-2}$                      |
| 0.10   | $5.000 \cdot 10^{-2}$                      |

while the second order Dyson-Taylor approximation, provided the result listed just below

| Expiry | $||DT_{2}\text{nd} - \text{exact}||_{\infty}$ |
|--------|---------------------------------------------|
| 1.00   | $8.200 \cdot 10^{-1}$                      |
| 0.50   | $3.946 \cdot 10^{-1}$                      |
| 0.40   | $4.500 \cdot 10^{-2}$                      |
| 0.30   | $2.182 \cdot 10^{-2}$                      |
| 0.20   | $8.260 \cdot 10^{-3}$                      |
| 0.10   | $1.764 \cdot 10^{-3}$                      |

We can easily notice that the infinity norm of the error decreases in the limit of $T \to 0$ in either the two schemes, but in general the second order approximation is really more accurate. Images below graphically show the results displayed above

### 5.1.2 European Call option under CEV model

When the CEV model is assumed to be the reference market model, there are no more exact solutions to count on, thus it is necessary to construct a so called reference solution with another approximation scheme known in litterature. In these settings, the reference solution has been constructed via the Method of the Lines, see sections (4.1) and (4.2), discretizing the space by using second order central finite difference and finally solving the discretized problem with respect to time, by using the Crank-Nicolson scheme.

In the previous section, the tests have been performed assuming $r = 0.05$ as interest rate, a volatility coefficient of $\sigma = 0.40$, a strike price $K = 10$ and various different expiry dates $T \in \{1, 0.5, 0.4, 0.3, 0.2, 0.1\}$. The space discretization counts $m = 50$ equidistant
nodes in a range \([0, S_{\text{max}}]\) where \(S_{\text{max}} = 2K\), while the time discretization \(n = 100\) equidistant points in \([0, T]\).

This time though, several values of the market parameter \(\beta \in (0, 1)\) have been tested out, in particular \(\beta \in \{\frac{3}{4}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}\).

Results are the following

| \(\beta = \frac{3}{4}\) | Expiry | \(|DT1_{\text{nd}} - \text{ref}|_\infty\) | \(|DT2_{\text{nd}} - \text{ref}|_\infty\) |
|-------------------------|--------|-----------------------------|-----------------------------|
| 1.00                    | 5.044 \cdot 10^{-1} | 3.487 \cdot 10^{-1} |
| 0.50                    | 2.565 \cdot 10^{-1} | 2.549 \cdot 10^{-1} |
| 0.40                    | 2.305 \cdot 10^{-1} | 2.292 \cdot 10^{-1} |
| 0.30                    | 2.003 \cdot 10^{-1} | 1.993 \cdot 10^{-1} |
| 0.20                    | 1.637 \cdot 10^{-1} | 1.630 \cdot 10^{-1} |
| 0.10                    | 1.146 \cdot 10^{-1} | 1.143 \cdot 10^{-1} |

| \(\beta = \frac{2}{3}\) | Expiry | \(|DT1_{\text{nd}} - \text{ref}|_\infty\) | \(|DT2_{\text{nd}} - \text{ref}|_\infty\) |
|-------------------------|--------|-----------------------------|-----------------------------|
| 1.00                    | 5.116 \cdot 10^{-1} | 2.871 \cdot 10^{-1} |
| 0.50                    | 2.531 \cdot 10^{-1} | 2.107 \cdot 10^{-1} |
| 0.40                    | 2.020 \cdot 10^{-1} | 1.895 \cdot 10^{-1} |
| 0.30                    | 1.672 \cdot 10^{-1} | 1.647 \cdot 10^{-1} |
| 0.20                    | 1.359 \cdot 10^{-1} | 1.345 \cdot 10^{-1} |
| 0.10                    | 9.441 \cdot 10^{-2} | 9.380 \cdot 10^{-2} |

Figure 5.1: (Dyson-Taylor 2\textsuperscript{nd} order approximation when \(T = 1.00\) and \(T = 0.40\) respectively.)
\[ \beta = \frac{1}{2} \]

| Expiry | \(| DT_1^{nd} - \text{ref} |_\infty \) | \(| DT_2^{nd} - \text{ref} |_\infty \) |
|--------|----------------|----------------|
| 1.00   | 5.123 \cdot 10^{-1} | 1.980 \cdot 10^{-1} |
| 0.50   | 2.531 \cdot 10^{-1} | 1.437 \cdot 10^{-1} |
| 0.40   | 2.020 \cdot 10^{-1} | 1.293 \cdot 10^{-1} |
| 0.30   | 1.211 \cdot 10^{-1} | 1.214 \cdot 10^{-1} |
| 0.20   | 1.005 \cdot 10^{-1} | 9.143 \cdot 10^{-2} |
| 0.10   | 6.408 \cdot 10^{-2} | 6.302 \cdot 10^{-2} |

\[ \beta = \frac{1}{3} \]

| Expiry | \(| DT_1^{nd} - \text{ref} |_\infty \) | \(| DT_2^{nd} - \text{ref} |_\infty \) |
|--------|----------------|----------------|
| 1.00   | 5.123 \cdot 10^{-1} | 1.370 \cdot 10^{-1} |
| 0.50   | 2.531 \cdot 10^{-1} | 9.811 \cdot 10^{-2} |
| 0.40   | 2.020 \cdot 10^{-1} | 8.860 \cdot 10^{-2} |
| 0.30   | 1.511 \cdot 10^{-1} | 7.710 \cdot 10^{-2} |
| 0.20   | 1.005 \cdot 10^{-1} | 6.267 \cdot 10^{-2} |
| 0.10   | 5.012 \cdot 10^{-2} | 4.348 \cdot 10^{-2} |

\[ \beta = \frac{1}{4} \]

| Expiry | \(| DT_1^{nd} - \text{ref} |_\infty \) | \(| DT_2^{nd} - \text{ref} |_\infty \) |
|--------|----------------|----------------|
| 1.00   | 5.123 \cdot 10^{-1} | 1.200 \cdot 10^{-1} |
| 0.50   | 2.531 \cdot 10^{-1} | 8.136 \cdot 10^{-2} |
| 0.40   | 2.020 \cdot 10^{-1} | 7.376 \cdot 10^{-2} |
| 0.30   | 1.511 \cdot 10^{-1} | 6.445 \cdot 10^{-2} |
| 0.20   | 1.005 \cdot 10^{-1} | 5.270 \cdot 10^{-2} |
| 0.10   | 5.012 \cdot 10^{-2} | 3.720 \cdot 10^{-2} |

From the data shown above it is possible to deduce that also in the case of the CEV market model both approximation schemes converge for \( T \to 0 \).

Moreover, note that the accuracy of either the two schemes increases as \( \beta \to 0 \).

Looking at these tables below, that show the average distance between the reference solution and the Dyson-Taylor \( n^{th} \)-order scheme (with \( n = 1, 2 \)) through various decreasing beta values

<table>
<thead>
<tr>
<th>Scheme</th>
<th>( \beta = 3/4 )</th>
<th>( \beta = 2/3 )</th>
<th>( \beta = 1/2 )</th>
<th>( \beta = 1/3 )</th>
<th>( \beta = 1/4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>D-T1</td>
<td>2.450 \cdot 10^{-4}</td>
<td>2.273 \cdot 10^{-1}</td>
<td>2.138 \cdot 10^{-1}</td>
<td>2.115 \cdot 10^{-1}</td>
<td>2.065 \cdot 10^{-1}</td>
</tr>
<tr>
<td>D-T2</td>
<td>2.182 \cdot 10^{-4}</td>
<td>1.800 \cdot 10^{-1}</td>
<td>1.230 \cdot 10^{-1}</td>
<td>8.499 \cdot 10^{-2}</td>
<td>7.160 \cdot 10^{-2}</td>
</tr>
</tbody>
</table>

it is possibile to deduce that \(| DT_{n^{th}} - \text{exact} |_\infty \) indeed decreases where the second order approximation however shows to be more accurate than the first one.
5.2 European style - efficiency test

5.2.1 European Call option under BSM model

In what follows, several different schemes have been compared to the second order Dyson-Taylor approximation in accuracy, with respect to the exact solution to the problem, and in computational time.

More precisely, \( N = 50000 \) Monte Carlo simulations and two versions of the Method of the lines have been implemented: both use the central finite differences of the second order as space discretization, while one uses Crank-Nicolson and the other uses Exponential Euler in solving the (discretized) problem through the time.

Once again, the tests have been performed assuming \( r = 0.05 \) as interest rate, a volatility coefficient of \( \sigma = 0.40 \), a strike price \( K = 10 \) and various different expiry dates \( T \in \{0.1, 0.2, 0.3, 0.4, 0.5\} \). The space discretization counts \( m = 100 \) equidistant nodes and \( m = 200 \) then, in a range \([0, S_{\text{max}}]\) where \( S_{\text{max}} = 2K \).

The time discretization counts \( n = 1000 \) equidistant points in a range of \([0, T]\) where \( T \) is the expiry date.

For \( m = 100 \) the results are the following:

<table>
<thead>
<tr>
<th>( T = 0.1 )</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>1.757 ( \cdot 10^{-3} )</td>
<td>4.203 ( \cdot 10^{-4} )</td>
<td>4.203 ( \cdot 10^{-4} )</td>
<td>2.283 ( \cdot 10^{-2} )</td>
</tr>
<tr>
<td>Time</td>
<td>0.13s</td>
<td>0.47s</td>
<td>0.19s</td>
<td>1.32s</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( T = 0.2 )</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>8.270 ( \cdot 10^{-3} )</td>
<td>3.047 ( \cdot 10^{-4} )</td>
<td>3.052 ( \cdot 10^{-4} )</td>
<td>3.842 ( \cdot 10^{-2} )</td>
</tr>
<tr>
<td>Time</td>
<td>0.14s</td>
<td>0.48s</td>
<td>0.19s</td>
<td>1.34s</td>
</tr>
</tbody>
</table>
#### $T = 0.3$

<table>
<thead>
<tr>
<th>Distance</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>2.184 · $10^{-2}$</td>
<td>5.026 · $10^{-4}$</td>
<td>5.026 · $10^{-4}$</td>
<td>4.198 · $10^{-2}$</td>
</tr>
</tbody>
</table>

#### $T = 0.4$

<table>
<thead>
<tr>
<th>Distance</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>4.501 · $10^{-2}$</td>
<td>2.512 · $10^{-3}$</td>
<td>2.512 · $10^{-3}$</td>
<td>4.895 · $10^{-2}$</td>
</tr>
</tbody>
</table>

#### $T = 0.5$

<table>
<thead>
<tr>
<th>Distance</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>8.203 · $10^{-2}$</td>
<td>6.952 · $10^{-3}$</td>
<td>6.952 · $10^{-3}$</td>
<td>5.003 · $10^{-2}$</td>
</tr>
</tbody>
</table>

on the other hand, for $m = 200$ one finds

#### $T = 0.1$

<table>
<thead>
<tr>
<th>Distance</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>1.763 · $10^{-3}$</td>
<td>1.040 · $10^{-4}$</td>
<td>1.041 · $10^{-4}$</td>
<td>2.621 · $10^{-2}$</td>
</tr>
</tbody>
</table>

#### $T = 0.2$

<table>
<thead>
<tr>
<th>Distance</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>8.282 · $10^{-3}$</td>
<td>7.534 · $10^{-5}$</td>
<td>7.576 · $10^{-5}$</td>
<td>4.129 · $10^{-2}$</td>
</tr>
</tbody>
</table>

#### $T = 0.3$

<table>
<thead>
<tr>
<th>Distance</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>2.184 · $10^{-2}$</td>
<td>5.026 · $10^{-4}$</td>
<td>5.026 · $10^{-4}$</td>
<td>4.496 · $10^{-2}$</td>
</tr>
</tbody>
</table>

#### $T = 0.4$

<table>
<thead>
<tr>
<th>Distance</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>4.501 · $10^{-2}$</td>
<td>2.512 · $10^{-3}$</td>
<td>2.512 · $10^{-3}$</td>
<td>5.164 · $10^{-2}$</td>
</tr>
</tbody>
</table>

#### $T = 0.5$

<table>
<thead>
<tr>
<th>Distance</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>8.203 · $10^{-2}$</td>
<td>6.952 · $10^{-3}$</td>
<td>6.952 · $10^{-3}$</td>
<td>5.830 · $10^{-2}$</td>
</tr>
</tbody>
</table>

From these results it is easy to realize that, although it is not the most accurate one, Dyson-Taylor approximation scheme is surely the fastest even if one doubles the number of points of the space discretization of the same $[0, S_{max}]$ interval.

Concerning the other methods, it can be said that the most precise one is Crank-Nicolson, but notice that even though it is actually of order 2 yields results very close to those
provided by Exponential Euler, that is generally of order 1 though. It happens because for problems of the form

\[
\begin{cases}
  y'(t) = Ay(t) + b(t, y(t)), & t > 0 \\
  y(t_0) = y_0
\end{cases}
\]

where \( b(t, y(t)) = b \) and \( b \) is a constant term, equal to zero in our case, Exponential Euler is an exact method. Concluding, note that Crank-Nicolson is not considerable to be slow, but Exponential-Euler is faster because, being an explicit scheme, it does not require to invert any matrix in order to solve a linear system.

5.2.2 European Call option under CEV model

In what follows, we consider different steps, w.r.t. ones used in previous section, to perform the tests of interest. In particular, implementations are divided into two parts. In the first one, the time required to give the price of the option by Dyson-Taylor, Crank-Nicolson and Exponential-Euler methods is tested, through space discretizations that count \( m = 100, m = 200 \) and \( m = 400 \) equidistant nodes, hence becoming increasingly dense. The second part is about to discover how distant these methods are with respect to each other, due to the fact that no exact solution exist. For the rest, we kept other parameter unchanged, except for the \( \beta \) parameter of the CEV model, which has been fixed to \( \beta = 1/2 \), finite time horizons \( T \in \{0.1, 0.25, 0.50\} \).

The results of the first part are listed below

<table>
<thead>
<tr>
<th>( m ) = 100</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0.58s</td>
<td>0.89s</td>
<td>0.61s</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( m ) = 200</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0.57s</td>
<td>1.32s</td>
<td>0.63s</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( m ) = 400</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0.58s</td>
<td>4.38s</td>
<td>1.09s</td>
</tr>
</tbody>
</table>

It is possible to notice that, even if the number of discretization points quadruples, the Dyson-Taylor commutator method turns out not to be just the fastest one, but the unique one with a time performance that remains practically unchanged.

The results of the second part are listed below

| \( m \) = 100 | \( |DT - CN|_\infty \) | \( |DT - EE|_\infty \) |
|--------------|-----------------|-----------------|
| \( T = 0.10 \) | \( 1.046 \cdot 10^{-3} \) | \( 1.064 \cdot 10^{-3} \) |
| \( T = 0.25 \) | \( 1.451 \cdot 10^{-3} \) | \( 1.451 \cdot 10^{-3} \) |
| \( T = 0.50 \) | \( 4.300 \cdot 10^{-3} \) | \( 4.300 \cdot 10^{-3} \) |
hence confirming what shown in section 5.1.2, namely: if the expiration date moves away from zero, the output provided by the Dyson-Taylor method tends to move further and further away from those provided by Crank-Nicolson and Exponential-Exponential, whose results are in any case very close one to the other, as seen in the previous sections. As the number of discretization points increases, however, the distance between Dyson-Taylor method and the other two methods reduces significantly, while maintaining the same divergent behaviours.

Monte Carlo is treated separately, being based on simulations of market realizations carried out by Euler-Maruyama method. In particular, tests have been realized assuming $m = 50$ discretization point $[0, S_{\text{max}}]$, and with $M = 10000$, $M = 20000$, and finally with $M = 50000$ simulations, as in the latter case,

| $M = 10000$ | $|DT - MC|_\infty$ | Time  |
|--------------|-----------------|-------|
| $T = 0.10$   | $8.848 \cdot 10^{-3}$ | 53.67s |
| $T = 0.25$   | $1.153 \cdot 10^{-2}$ | 53.65s |
| $T = 0.50$   | $2.852 \cdot 10^{-2}$ | 53.29s |

| $M = 20000$ | $|DT - MC|_\infty$ | Time  |
|--------------|-----------------|-------|
| $T = 0.10$   | $8.129 \cdot 10^{-3}$ | 106.70s |
| $T = 0.25$   | $1.463 \cdot 10^{-2}$ | 106.03s |
| $T = 0.50$   | $1.423 \cdot 10^{-2}$ | 107.99s |

| $M = 50000$ | $|DT - MC|_\infty$ | Time  |
|--------------|-----------------|-------|
| $T = 0.10$   | $5.003 \cdot 10^{-3}$ | 270.19s |
| $T = 0.25$   | $9.223 \cdot 10^{-3}$ | 278.02s |
| $T = 0.50$   | $1.136 \cdot 10^{-2}$ | 275.29s |

From these data it can be seen how the distance between the result provided by Monte Carlo and the one related to Dyson-Taylor is much smaller as the number $M$ of simulated trajectories increases.
Figure 5.4: Dyson-Taylor and Monte Carlo values for an European Call options under CEV model ($\beta = 1/2$) when $M = 10K$ and $M = 20K$, respectively.

However, as one may expect in view of previous tests, the aforementioned distance increases as $T$ moves far from zero.

5.3 American style - accuracy test

Within this section we present results about both accuracy and efficiency of the numerical methods discussed previously, but pricing an american option. Recall that, for such a problem, not only there is no exact solutions, but Dyson-Taylor commutator method does not provide any closed form for the price.

The tests focus in pricing a Put option, because the value of an American Call option that does not pay any dividend would be identical to the corresponding European Call. The reason, however, lies in the fact that due to the nature of the payoff function, early exercising these type of option is never an optimal choice. This last fact can be proved in a more rigorous manner, see, e.g., Chapter 3 of [3] for details.
5.3.1 American Put option under BSM model

As already specified, since in this case we cannot rely on exact solutions, the only thing that can be done is a direct comparison between the solutions provided by the various types of numerical schemes and see what is their mutual distance.

The first set of tests provides a direct comparison between Dyson-Taylor commutator method and Crank-Nicolson scheme, since Exponential-Euler returns almost identical values to those provided by the latter which, in section (5.1), turned out to be the most accurate one. The assumed market data are \( r = 0.05 \) as interest rate, \( \sigma = 0.40 \) as volatility coefficient, \( K = 10 \) as strike price. The space discretization counts \( n = 1000 \) timesteps into \([0, T]\) where \( T \in \{0.10, 0.25, 0.50\} \) while the space discretization counts \( m = 100, m = 200 \) and \( m = 400 \), hence becoming increasingly dense.

| Expiry Time | \(|DT - CN|_\infty (m=100)\) | \(|DT - CN|_\infty (m=200)\) | \(|DT - CN|_\infty (m=400)\) |
|-------------|-----------------|-----------------|-----------------|
| \( T = 0.10 \) | \(1.869 \cdot 10^{-2}\) | \(1.976 \cdot 10^{-2}\) | \(2.028 \cdot 10^{-2}\) |
| \( T = 0.25 \) | \(4.100 \cdot 10^{-2}\) | \(4.060 \cdot 10^{-2}\) | \(4.144 \cdot 10^{-2}\) |
| \( T = 0.50 \) | \(8.898 \cdot 10^{-2}\) | \(8.898 \cdot 10^{-2}\) | \(8.898 \cdot 10^{-2}\) |

Again, the Least Square Monte Carlo method has been treated separately due to its nature based on simulations, the space discretization counts \( m = 50 \) points in \([0, S_{\text{max}}]\), the time one \( n = 1000 \) and a number \( M \in \{10000, 20000, 50000\} \) of simulations have been performed.

| \( m = 50 \) | \(|DT - MC|_\infty (T=0.10)\) | \(|DT - MC|_\infty (T=0.25)\) | \(|DT - MC|_\infty (T=0.50)\) |
|-------------|-----------------|-----------------|-----------------|
| \( M = 10000 \) | \(1.931 \cdot 10^{-2}\) | \(4.242 \cdot 10^{-2}\) | \(8.243 \cdot 10^{-2}\) |
| \( M = 20000 \) | \(1.640 \cdot 10^{-2}\) | \(3.143 \cdot 10^{-2}\) | \(8.207 \cdot 10^{-2}\) |
| \( M = 50000 \) | \(1.395 \cdot 10^{-2}\) | \(2.670 \cdot 10^{-2}\) | \(8.177 \cdot 10^{-2}\) |
From these results it is clear that even in the case of an American-type option, Dyson-Taylor retains its divergent behavior for maturity dates increasingly distant from $T = 0$. In particular, for the LSMC, it is possible to notice that, by fixing the value of the expiry date $T$, the distance decreases as $M$ increases, but in any case the divergent behavior detected by the results just above remains the same as simulated trajectories.

![American Put Option under BSM model](image)

Figure 5.6: Comparison between Dyson-Taylor and Crank-Nicolson results for $T = 0.10$ (left) and $T = 0.50$ (right)

### 5.3.2 American Put option under CEV model

When the assumed market model is CEV, the results obtained by assuming $\beta = 1/2$ substantially lead to the same conclusions just discussed for the Black-Sholes-Merton model.

| Expiry Time | $|DT - CN|_{\infty}$ (m=100) | $|DT - CN|_{\infty}$ (m=200) | $|DT - CN|_{\infty}$ (m=400) |
|-------------|-------------------------------|-------------------------------|-------------------------------|
| $T = 0.10$  | $1.869 \cdot 10^{-2}$         | $1.976 \cdot 10^{-2}$         | $2.028 \cdot 10^{-2}$         |
| $T = 0.25$  | $4.100 \cdot 10^{-2}$         | $4.060 \cdot 10^{-2}$         | $4.144 \cdot 10^{-2}$         |
| $T = 0.50$  | $8.898 \cdot 10^{-2}$         | $8.898 \cdot 10^{-2}$         | $8.898 \cdot 10^{-2}$         |

The only difference is that the two methods seem to be closer in distance than before, and the LSMC results confirm such conclusion

| $m = 50$ | $|DT - MC|_{\infty}$ (T=0.10) | $|DT - MC|_{\infty}$ (T=0.25) | $|DT - MC|_{\infty}$ (T=0.50) |
|----------|-------------------------------|-------------------------------|-------------------------------|
| $M = 10000$ | $6.261 \cdot 10^{-3}$         | $1.993 \cdot 10^{-2}$         | $3.688 \cdot 10^{-2}$         |
| $M = 20000$ | $1.276 \cdot 10^{-2}$         | $1.674 \cdot 10^{-2}$         | $3.587 \cdot 10^{-2}$         |
| $M = 50000$ | $8.975 \cdot 10^{-3}$         | $1.959 \cdot 10^{-2}$         | $3.331 \cdot 10^{-2}$         |
due to the fact that Dyson-Taylor commutator method has shown to be more accurate when $\beta \rightarrow 0$, and of course $\beta = 1/2$ is closer to zero than $\beta = 1$, that characterizes the BSM model.

Figure 5.7: Comparison between Dyson-Taylor and LSMC results for $T = 0.10$ (left) and $T = 0.50$ (right), with $M = 10000$ simulated trajectories

5.4 American style - efficiency test

In what follows, we report the results of the tests related to the efficiency of the methods implemented so far. Dyson-Taylor, Crank-Nicolson and Exponential-Euler are stressed by enlarging the number of both the space and time nodes, starting from a base of $m = 100$ space points and $n = 500$ time points and then doubling their quantity twice. As usual, LSMC is treated aside fixing $m = 50$ space nodes and $M \in \{10000, 20000, 50000\}$. In every test, expiry date is assumed to be $T = 0.25$.

5.4.1 American Put option under BSM model

For Black-Scholes-Merton market model, the results are the following

<table>
<thead>
<tr>
<th>n = 500</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential-Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>m = 100</td>
<td>0.69s</td>
<td>0.71s</td>
<td>0.57s</td>
</tr>
<tr>
<td>m = 200</td>
<td>0.71s</td>
<td>0.93s</td>
<td>0.58s</td>
</tr>
<tr>
<td>m = 400</td>
<td>0.74s</td>
<td>2.32s</td>
<td>0.93s</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 1000</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential-Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>m = 100</td>
<td>0.84s</td>
<td>0.94s</td>
<td>0.58s</td>
</tr>
<tr>
<td>m = 200</td>
<td>0.88s</td>
<td>1.28s</td>
<td>0.61s</td>
</tr>
<tr>
<td>m = 400</td>
<td>0.93s</td>
<td>4.17s</td>
<td>1.17s</td>
</tr>
</tbody>
</table>
Even if in this case Dyson-Taylor method does not turn out to be the fastest, its performances are still very solid because fast even when the density of the discretization becomes very dense, something that can not be said speaking about Crank-Nicolson: despite is the most accurate method, it is proved to be quite sensitive to increasing the size of the instances. Exponential-Euler seems to be a good compromise, and the LSMC yields

<table>
<thead>
<tr>
<th>m = 50</th>
<th>Least Square Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>M = 10000</td>
<td>63.66s</td>
</tr>
<tr>
<td>M = 20000</td>
<td>89.82s</td>
</tr>
<tr>
<td>M = 50000</td>
<td>189.82s</td>
</tr>
</tbody>
</table>

confirming to be the slowest one.

### 5.4.2 American Put option under CEV model

For the CEV model, Dyson-Taylor, Crank-Nicolson and Exponential-Euler methods yield

<table>
<thead>
<tr>
<th>n = 500</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential-Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>m = 100</td>
<td>0.83s</td>
<td>0.75s</td>
<td>0.56s</td>
</tr>
<tr>
<td>m = 200</td>
<td>0.97s</td>
<td>0.95s</td>
<td>0.59s</td>
</tr>
<tr>
<td>m = 400</td>
<td>1.06s</td>
<td>2.51s</td>
<td>0.88s</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 1000</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential-Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>m = 100</td>
<td>1.07s</td>
<td>0.92s</td>
<td>0.58s</td>
</tr>
<tr>
<td>m = 200</td>
<td>1.24s</td>
<td>1.36s</td>
<td>0.61s</td>
</tr>
<tr>
<td>m = 400</td>
<td>1.54s</td>
<td>4.41s</td>
<td>1.10s</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 2000</th>
<th>Dyson-Taylor</th>
<th>Crank-Nicolson</th>
<th>Exponential-Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>m = 100</td>
<td>1.65s</td>
<td>1.21s</td>
<td>0.60s</td>
</tr>
<tr>
<td>m = 200</td>
<td>1.92s</td>
<td>2.11s</td>
<td>0.68s</td>
</tr>
<tr>
<td>m = 400</td>
<td>2.56s</td>
<td>8.37s</td>
<td>1.56s</td>
</tr>
</tbody>
</table>

The conclusions are substantially the same, the only difference being that the Dyson-Taylor method turns out to be slightly slower than the previous case due to nature of the PDE coefficients, derived from the market model itself. However, its solidity remains intact and the LSMC this yielding
<table>
<thead>
<tr>
<th>m = 50</th>
<th>Least Square Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>M = 10000</td>
<td>62.69s</td>
</tr>
<tr>
<td>M = 20000</td>
<td>121.37s</td>
</tr>
<tr>
<td>M = 50000</td>
<td>301.23s</td>
</tr>
</tbody>
</table>

### 5.5 Greeks and CEV model with $\beta > 1$

#### 5.5.1 Delta and Gamma derivatives of $u(t, x)$

This section presents the accuracy results in computing Delta and Gamma functions of the option value $u(t, x)$, defined respectively as follows

$$
\Delta := \frac{\partial u(t, x)}{\partial x}, \quad \Gamma := \frac{\partial^2 u(t, x)}{\partial x^2}, \quad x \in [0, S_{max}]
$$

It is important to precise that Monte Carlo method has been excluded from these tests, being notoriously bad in computing Greeks, see, e.g., [7].

The true value for Greeks in European options is based on an explicit formula developed by Shroder for $0 < \beta < 1$ and by Emanuel and MacBeth for $\beta > 1$ around 1989, see [13]. This formula is taken as exact solution and it is developed by means of non-central Chi-square distributions.

For call options, the formulas are

$$
C = Se^{-\alpha t}Q(2y, 2 + 2/(2 - \beta), 2x) - Ee^{-rt}(1 - Q(2x, 2/(2 - \beta), 2y)), \tag{5.3}
$$

for $0 < \beta < 1$, where $E$ is the strike price, $\alpha$ is the dividends ratio, degrees of freedom and non-centrality parameters are $x = kS^{2-\beta}e^{(r-\alpha)(2-\beta)t}$ and $y = kE^{2-\beta}$, respectively, where $k$ is

$$
k = \frac{2(r - \alpha)}{\sigma^2(2 - \beta)[e^{(r-\alpha)(2-\beta)t} - 1]}.
$$

Similarly, for $\beta > 1$ one gets

$$
C = Se^{-\alpha t}Q(2x, 2/(2 - \beta), 2y) - Ee^{-rt}(1 - Q(2y, 2 + 2/(\beta - 2), 2x)), \tag{5.4}
$$

for further details, we refer to [13].

The test are performed with the usual values for interest rate and volatility, hence we are taking a space discretization that counts $m = 100$ equidistant nodes, a CEV index $\beta = 2/3$ and $T \in \{0.10, 0.25, 0.50, 1.00\}$. Greeks are computed by using Central Finite Differences of $4^{th}$ order, with the appropriate corrections to the stencil when it comes to manage the boundaries. The theory behind is the same viewed in Chapter 4 for CFDs of order 2, the only difference is about the stencils coefficients that can be easily obtained by refering to [14].

Results are listed below.
These results underlines that Crank-Nicolson is, on average, the most accurate method and its precision seems to increase as \( T \) moves away from zero. Exponential-Euler, in particular, gives results really similar to those provided by Crank-Nicolson, but just when maturity is close to zero: as \( T \) grows up, there is a significant loss of precision in either the two Greeks, especially \( \Gamma \). Dyson-Taylor commutator is the most precise method when expiry date is close to zero, because of its nature of being a short-time asymptotic expansion, indeed it is the only that returns an error whose infinity norm is proportional to \( 10^{-6} \). As \( T \) increases, however, precision decreases as one may expect refering to previous sections. The good thing is that, even if the value of maturity date is really far from those that guarantee a good precision. Therefore it is no more possible to speak about ”short-time”, Greeks are anyway really accurate.

### 5.5.2 CEV model with inverse leverage effect

The section shows how Dyson-Taylor method, together with the other methods for a comparison, behaves in computing the price of an European Call option under a CEV market model whose index \( \beta \) exceeds the “critical value” \( \beta = 1 \), that corresponds to the standard Black-Sholes-Merton model, with respect to the explicit formula given by (5.4) taken as exact solution. The space discretization counts \( m = 100 \) points in \([0, S_{max}]\) while the time \([0, T]\) one counts \( n = 1000 \) nodes.

The values of interest rate, volatility and strike price are unchanged wit respect to previous sections, \( T \in \{0.10, 0.25, 0.50\} \) and \( \beta \in \{\frac{11}{10}, \frac{4}{3}, \frac{3}{2}\} \).

<table>
<thead>
<tr>
<th>( T )</th>
<th>( 10^{5} )</th>
<th>( 10^{-5} )</th>
<th>( 10^{-3} )</th>
<th>( 10^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>3.650</td>
<td>3.650</td>
<td>3.650</td>
<td>3.650</td>
</tr>
<tr>
<td>0.25</td>
<td>3.760</td>
<td>3.760</td>
<td>3.760</td>
<td>3.760</td>
</tr>
<tr>
<td>0.50</td>
<td>4.947</td>
<td>4.947</td>
<td>4.947</td>
<td>4.947</td>
</tr>
</tbody>
</table>

| \( T \) | \( 10^{-5} \) | \( 10^{-3} \) | \( 10^{-2} \) |
|---|---|---|
| 0.10 | 3.650 | 3.650 | 3.650 |
| 0.25 | 3.760 | 3.760 | 3.760 |
| 0.50 | 4.947 | 4.947 | 4.947 |

<table>
<thead>
<tr>
<th>( T )</th>
<th>( 10^{-3} )</th>
<th>( 10^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>3.650</td>
<td>3.650</td>
</tr>
<tr>
<td>0.25</td>
<td>3.760</td>
<td>3.760</td>
</tr>
<tr>
<td>0.50</td>
<td>4.947</td>
<td>4.947</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( T )</th>
<th>( 10^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>3.650</td>
</tr>
<tr>
<td>0.25</td>
<td>3.760</td>
</tr>
<tr>
<td>0.50</td>
<td>4.947</td>
</tr>
</tbody>
</table>
The results clearly show that for a value of $\beta$ that is indeed strictly greater but anyway pretty close to 1, all the methods behave similarly to those obtained in the case of the standard BSM market model seen in section (5.2.1). Unfortunately, as $\beta$ move relatively away from 1, every method shows that its distance w.r.t the norm $|\cdot|_\infty$ badly increases towards not acceptable values. Nevertheless, it is interesting to notice that Dyson-Taylor commutator method seems to be the most stable in a certain sense. Monte Carlo method has been launched $M \in \{10000, 20000, 50000\}$, fixing a space discretization of $m = 50$ points and a maturity date $T = 0.5$.

| $\beta = 4/3$ | $|CN - ref|_\infty$ | $|EE - ref|_\infty$ | $|DT - ref|_\infty$ |
|--------------|------------------|------------------|------------------|
| $T = 0.10$   | $1.625 \cdot 10^{-2}$ | $1.625 \cdot 10^{-2}$ | $1.933 \cdot 10^{-2}$ |
| $T = 0.25$   | $2.129 \cdot 10^{-1}$ | $2.129 \cdot 10^{-1}$ | $7.689 \cdot 10^{-2}$ |
| $T = 0.50$   | $6.954 \cdot 10^{-1}$ | $6.954 \cdot 10^{-1}$ | $1.351 \cdot 10^{-1}$ |

| $\beta = 3/2$ | $|CN - ref|_\infty$ | $|EE - ref|_\infty$ | $|DT - ref|_\infty$ |
|--------------|------------------|------------------|------------------|
| $T = 0.10$   | $2.048 \cdot 10^{-1}$ | $2.048 \cdot 10^{-1}$ | $9.826 \cdot 10^{-2}$ |
| $T = 0.25$   | $9.469 \cdot 10^{-1}$ | $9.469 \cdot 10^{-1}$ | $1.812 \cdot 10^{-1}$ |
| $T = 0.50$   | $2.059$           | $2.059$           | $3.695 \cdot 10^{-1}$ |

As it is possible to see, when $\beta$ is relatively close to 1, Monte Carlo method seems to show some kind of convergence as the number $M$ of simulated trajectories increases, since error consequently decreases. Once again, though, when $\beta$ moves away from 1 numerical instability occurs, making literally impossible to compute the requested call option value.

The reason seems to lie in the fact that Euler-Maruyama method is not able to properly compute every trajectory when $\beta$ too large and underlying price is close to $S_{\text{max}}$. A reasonable conjecture behind such bad behaviours, of all the numerical schemes, in computing $u(t, x)$ once the CEV market model is assumed, might lie in the nature of the diffusion coefficient itself, that drastically changes as $\beta$ overcomes 1. Indeed, if $0 < \beta < 1$ the volatility assumes the form

$$\sigma(t, X_t) = \sigma X_t^{\beta - 1} = \sigma X_t^\xi, \quad \sigma > 0,$$

where $\xi < 0$, meaning that it decreases as the asset price grows up. Vice versa, $\beta > 1$ implies $\xi > 0$ so its behaviour totally changes and the volatility increases within the growth of $X_t$. 

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This means that also the properties of the operator \( L(t, x) = \frac{1}{2} \sigma x^2 \partial_x^2 + r x \partial_x - r \) change, namely the diffusion coefficient increases much faster than it does by assuming the Black-Scholes-Merton model. Moreover, if the following system

\[
\begin{bmatrix}
  y_{n,1} \\
y_{n,2} \\
  \vdots \\
y_{n,m-1} \\
y_{n,m}
\end{bmatrix}
= \begin{bmatrix}
  C_{1,1} & C_{1,2} & \cdots & \cdots & C_{1,m} \\
  C_{2,1} & C_{2,2} & \ddots & \cdots & C_{2,m} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  C_{m,1} & C_{m,2} & \cdots & \cdots & C_{m,m}
\end{bmatrix}
\begin{bmatrix}
  u_{n,1} \\
u_{n,2} \\
  \vdots \\
u_{n,m-1} \\
u_{n,m}
\end{bmatrix}
\]

where \( C \in M_{m \times m}(\mathbb{R}) \approx L(t, x) \), represents the discretized version of the first equation in (2.2), it is possible to show that the stiffness ratio sensitively increases within the growth of \( \beta \) itself. In particular, let \( \lambda_1, \ldots, \lambda_m \) be the eigenvalues of matrix \( C \) such that \( |\lambda_1| < \ldots < |\lambda_m| \), then the stiffness coefficient, given by \( S = \frac{R(\lambda_m)}{R(\lambda_1)} \), increases as \( \beta \) does too.

Summing up, since solutions \( u(t, x) \) are bad for \( \beta > 1 \), the methods fail in computing the Greeks too.

Figure 5.8: (Delta and Gamma functions at \( T = 0.50 \), computed by Dyson-Taylor commutator method)
5.6 Future research: the Bootstrap scheme

In section 5.5.1 we have pointed out the principal disadvantage of the Dyson-Taylor commutator method, namely that the related asymptotic expansion of the Green function is accurate for small times, i.e. when $T \to 0$, which is not a realistic point of view, at least for standard financial instruments.

In what follows we present a strategy, called Bootstrap scheme, that allows to overcome such issue, producing accurate solutions for arbitrary large expiry dates $T$.

Let us illustrate this strategy assuming a time-independent market model. As already mentioned in section 3.2, the Solution Operator forms a semigroup. Thus, in this case, Definition 3.2.1 yields

$$e^{tL} = \left(e^{\frac{t}{m}L} \right)^m. \quad (5.5)$$

If $m$ is sufficiently large, then $e^{\frac{t}{m}L}$ will be accurately approximated at least assuming a strongly elliptic operator $L(t, x)$, i.e. for $a(t, x) \geq \gamma > 0$ for every $x > 0$.

The Bootstrap scheme uses $(G^{[2]}_{t/m})^m$ to approximate $e^{tL}$ where $G^{[2]}_t$ is the second order approximation of the Green function $G_t(x, y)$, see section 3.3, eq. (3.36).

5.6.1 Error estimate

Once the number $m \in \mathbb{N}$ of points of the time discretization is fixed, the error of $(G^{[2]}_{t/m})^m$ is of the order $O\left(\left(\frac{t}{m}\right)^{3/2}\right)$, thus the total error is indeed a compound error given by

$$O\left(\left(\frac{t}{m}\right)^{3/2}\right) \times m = O\left(\frac{t^{3/2}}{\sqrt{m}}\right). \quad (5.6)$$

Therefore, for fixed $T > 0$, it becomes smaller and smaller as the number of points $m$ increases. This approach requires a quadrature formula, hence higher computational efforts than the closed-form Dyson-Taylor commutator method described in Chapter 3, adding an extra error. We refer to section 5.6.2.

The Bootstrap strategy does not work using $G^{[1]}_t$, whose error is $O\left(\frac{t^3}{m}\right)$, because the compound error would be

$$O\left(\frac{t}{m}\right) \times m = O(t), \quad (5.7)$$

implying that it does not tend to zero, as $m \to \infty$.

We made precise the above heuristic argument in the following theorem

**Theorem 5.6.1.** Let $G^{[2]}_t$ be the second order approximation to the solution operator $e^{tL}$, and fix $m \in \mathbb{Z}^+$. Then there exists costants $\omega$, $\bar{\omega}$ and $C > 0$ independent on $m$ and $t$ such that

$$||e^{tL} - (G^{[2]}_{t/m})^m||_{W^{m,p}} \leq Ce^{(\omega + \bar{\omega})t} \frac{t^{3/2}}{\sqrt{m}}.$$
Proof. See Lemma 4.4.2 and Lemma 4.4.3, pages from 115 to 118, of [7].

5.6.2 The strategy

The Bootstrap scheme works as follows

- discretize the time interval \([0, T]\), for \(T > 0\), into \(m \in \mathbb{N}\) equidistant points, i.e. \(t_0 = 0, t_1, ..., t_{m-1}, t_m = T\);

- compute the evolution of the solution at time \(t_1\) by using the Dyson-Taylor commutator method, given by the closed form (3.5), page 921 of [5];

- for every \(t_i, i = 2, ..., m\), calculate the following integral

\[
U^{[2]}(t_i, x) = \int_{0}^{\infty} \left( C_{t/m}^{[2]}(x, y) \right)^{m} \cdot \text{evolution}_{t_{i-1}}(y) dy,
\]

where with “\(\text{evolution}_{t_i}\)” we intend the solution computed at time \(t_{i-1}\), by adopting an appropriate quadrature formula.

As the expiry date \(T\) increases, the Bootstrap scheme requires a greater number of discretization points \(m \in \mathbb{N}\) to preserve a given accuracy level. Such approach is affected by the following drawbacks:

1. it is an iterative procedure, so there are no more closed-forms after \(t_1\),

2. if \(m \in \mathbb{N}\) is the number of points of the discretization, it requires \(m - 1\) numerical integrations, to compute the integrals given by eq. (5.8), by using a quadrature formula.

Concluding, we aim to obtain an increment in accuracy allowing the distance, between the exact solution and our approximation to be, at least, proportional to \(10^{-3}\), accordingly to results shown in section 5.1.1.
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This chapter contains an example of numerical implementation of Dyson-Taylor commu-
tator method, applied in pricing an European option assuming the Black-Sholes-Merton
model.
The code has been developed as Matlab function, input parameters are the space dis-
cretization $S$, the expiry date $T$, the strike price $K$, the interest rate $r$, the volatility
$sigma$ and finally the order of the Dyson-Taylor approximation.
Output results are

- Eurocall - price of European Call option
- Europut - price of European Put option

function [Eurocall,Europut] = DT_BSM(S,T,K,r,sigma,order)
if order == 1
    % FIRST ORDER APPROXIMATION
    % Coefficients
    S = S(2:end);  \% to avoid NaN results
    a = @(x) sigma*x;
    b = @(x) r*x;
    % FIRST ORDER APPROXIMATION
    U1 = @(t,x) (sqrt(t)/(2*sqrt(2*pi)))*exp(-(x-K).^2./(2*a(x).^2*t));
    U2 = @(t,x) (2*a(x)-sigma*(x-K));
    U3 = @(t,x) 0.5*(erf((x-K)./(sqrt(2*t)*a(x)))+1).*((b(x)*t+x-K);
    % Dyson-Taylor Price (1st order)
    U1 = U1(T,S); U2 = U2(T,S); U3 = U3(T,S);
    U1_DTBSM = U1.*U2+U3;
    S = [0;S]; \% restore the original space vector
    % Prices
    Eurocall = [0;U1_DTBSM];
    Europut = Eurocall - S + K*exp(-r*T); \% Put-Call parity
% else if order == 2
% SECOND ORDER APPROXIMATION
% Coefficients
S = S(2:end); % to avoid divisions by zero, that produces NaN results
a = @(x) sigma*x;
b = @(x) r*x;
% FIRST ORDER APPROXIMATION
U1 = @(t,x) (sqrt(t)/(2*sqrt(2*pi)))*exp(-(x-K).^2./(2*a(x).^2*t));
U2 = @(t,x) (2*a(x)-sigma*(x-K));
U3 = @(t,x) 0.5*(erf((x-K)./(sqrt(2*t)*a(x)))+1).*b(x)*t+x-K);
% Dyson-Taylor Price (1st order)
U1 = U1(T,S); U2 = U2(T,S); U3 = U3(T,S);
U1_DTBSM = U1.*U2+U3;
% We’ll still need U1-DT, but we’ll add some other extra terms
% we divide the second order approximation form into the sum of some
% terms in order to make everything less tedious
U4 = @(t,x) (0.5*r*t)*((erf((x-K)./(sqrt(2*t)*a(x)))+1).*K-x);
U5 = @(t,x) (sqrt(t)/(2*sqrt(2*pi*t)))*exp(-(x-K).^2./(2*a(x).^2*t));
u51 = @(t,x) -a(x).*(r+sigma^2/12)*t.^2;
u52 = @(t,x) (t./a(x)).*(t*b(x).^2-(x-K).^2*sigma^2/6);
u53 = @(t,x) ((sigma*(x-K).^2)*t.*a(x))./a(x).^2;
u54 = @(t,x) (sigma^2*(x-K).^4)./(4*a(x).^3);
% Dyson-Taylor price (2nd order)
% Terms evaluation
U4 = U4(T,S); U5 = U5(T,S); u51 = u51(T,S); u52 = u52(T,S);
u53 = u53(T,S); u54 = u54(T,S);
U2_DTBSM = U1_DTBSM + U4 + U5.*(u51+u52+u53+u54);
S = [0;S]; % restore the original space vector
% Prices
Eurocall = [0;U2_DTBSM];
Europut = Eurocall - S + K*exp(-r*T);

else
    disp('<< ATTENTION: not possible choice >>')
end
end
end