Efficient approximation schemes to the solution of pricing problems for American type options

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# Contents

| Acknowledgments | iii |
| Introduction | v |
| 1 Introduction to option pricing | 1 |
| 1.1 Discrete-time models | 1 |
| 1.2 Continuous-time models: The Black-Scholes model | 4 |
| 2 Approximation of the Green's function for second order parabolic equation by the Dyson-Taylor Commutator Method | 9 |
| 2.1 Kernel expansion at z=x | 20 |
| 3 American Option Pricing | 23 |
| 3.1 American Option as a Free Boundary Problem | 23 |
| 3.2 American Put Option | 24 |
| 3.3 The Dyson-Taylor Commutator method applied to American Option pricing | 26 |
| 3.4 The Implicit Euler method applied to American Option pricing | 31 |
| 3.4.1 Finite-difference Approximations | 31 |
| 3.5 The Monte Carlo method applied to American Option pricing | 34 |
| 4 Numerical Implementations | 37 |
| 4.1 Pricing an American put option under the Black and Scholes model | 38 |
| 4.2 Pricing an American put option under the CEV model | 44 |
| A Preliminary notions | 53 |
| A.1 Introduction to stochastic calculus | 53 |
| A.1.1 Quadratic Variation | 55 |
| A.1.2 Construction of the stochastic integral and Itô calculus | 56 |
| A.1.3 Stochastic differential equations | 58 |
| A.1.4 Markov property | 60 |
| A.1.5 Change of probability and Girsanov theorem | 60 |
CONTENTS

Bibliography 63
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Introduction

The aim of this thesis is to find the price of an American Put option using the Dyson-Taylor commutator method introduced in [4] and further developed in [9]. This method is used to approximate the Green’s function of second order parabolic equations. In [9] we can see how this method works and how it is applied to European option pricing. In Chapter 6 of [9] is described an algorithm in order to extend the use of the Dyson-Taylor commutator methods also for American option pricing. Our aim is to implement this method to price American put options.

The thesis is structured as follow: in the first chapter, we give some basic theoretical notions useful to understand how to obtain a partial differential equation (PDE) associated to the type of option pricing problem we are dealing with. In the second chapter, we give a summary of the Dyson-Taylor commutator method and we explain how to find the approximations at the first and the second order of the Green’s function of a second order parabolic equation exploiting this method. In the first part of Chapter 3 we introduce how to find the free boundary problem associated to the pricing of an American option, in particular for an American Put option. Then, we explain how to rewrite the related free boundary problem in order to solve it using the algorithm explained in Chapter 6 of [9] which lends itself to application of the Dyson-Taylor commutator method. In the last part of Chapter 3 we introduce other two methods, namely the Implicit Euler and the Monte Carlo methods, which have been used for various type of performance comparisons versus the Dyson Taylor approach. The latter numerics has been collected in Chapter 4. In particular we have performed aforementioned numerical tests considering the pricing problem of an American put option under the Black and Scholes model, and the so-called CEV model (Constant Elasticity of Variance), a local volatility model [16].
INTRODUCTION
Chapter 1

Introduction to option pricing

In what follows we recall some basic notions about option pricing, mainly following [1] and [3]. We refer to Appendix A for basic notions in the theory of stochastic processes.

1.1 Discrete-time models

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space and let $(\mathcal{F}_t)_{0 \leq t \leq T}$ be a filtration, where $T \in \mathbb{N}$ is defined to be the time horizon that correspond to the maturity of a given investment. $\mathcal{F}_s$ can be seen as the information available at time $s$. From now on, we will assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$ which is the set of parts of $\Omega$, and we also assume $\mathbb{P}(\{\omega\}) > 0$, for $\omega \in \Omega$.

**Definition 1.1.1.** (Discrete-time finite market model). An adapted stochastic process $S = \{S_t\}_{t=0,1,...,T} = \{(S^0_t, S^1_t, \ldots, S^d_t)\}_{t=0,1,...,T}$ taking values in $\mathbb{R}^{d+1}$ is called a discrete-time finite market model.

The vector $S_t = (S^0_t, S^1_t, \ldots, S^d_t)$ is defined to be the vector of prices of a portfolio of assets at time $t$, which are measurable with respect to $\mathcal{F}_t$.

The asset $S^0$ is the risk-less asset and, if we set $S^0_0 = 1$, we will obtain $S^0_t = (1 + r)^t$, where $r > 0$ is the interest rate, while the other assets $S^i$, $i = 1, \ldots, d$, are the risky assets.

**Definition 1.1.2.** (Trading strategy) A trading strategy is defined as a stochastic process

$$\phi = (\phi^0_t, \phi^1_t, \ldots, \phi^d_t)_{0 \leq t \leq T}$$

in $\mathbb{R}^{d+1}$, where $\phi^i_t$ denotes the numbers of shares of the $i$-th asset held in our portfolio at time $t$. 


The sequence $\phi$ is assumed to be predictable, i.e.

$$
\begin{cases}
\phi_0 \text{ is } \mathcal{F}_0 - \text{measurable} \\
\text{and, for } t \geq 1, \; \phi_t \text{ is } \mathcal{F}_{t-1} - \text{measurable}
\end{cases}
\forall i \in \{0, 1, \ldots, d\}
$$

Let us outline that the latter assumption means that the position in the portfolio at time $t$, namely $\phi_0^t, \phi_1^t, \ldots, \phi_d^t$, are decided with respect to the information available at time $(t-1)$ and kept until time $t$.

**Definition 1.1.3.** (Value of portfolio). The value of the portfolio at time $t$ is given by the following scalar product

$$V_t(\phi) = \phi_t \cdot S_t = \sum_{i=0}^{d} \phi_i^t S_i^t.$$ 

Let us thus define the discounted price processes by rescaling the vector of prices w.r.t. the risk-less asset,

$$\tilde{S}_t = (1, \beta_t S_1^t, \ldots, \beta_t S_d^t)$$

where $\beta_t = \frac{1}{S_t}$. Then, the discounted value of the portfolio is defined as

$$\tilde{V}_t(\phi) = \beta_t (\phi_t \cdot S_t) = \phi_t \cdot \tilde{S}_t$$

**Definition 1.1.4.** (Self-financing strategy). A strategy is said to be self-financing if the following equation is satisfied for all $t \in \{0, 1, \ldots, T-1\}$:

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t.$$ 

**Proposition 1.1.1.** The following are equivalent:

1. The strategy $\phi$ is self-financing;
2. For any $t \in \{1, \ldots, T\}$,
   $$V_t(\phi) = V_0(\phi) + \sum_{j=1}^{t} \phi_j \cdot \Delta S_j,$$
   where $\Delta S_j$ is the vector $S_j - S_{j-1}$.
3. For any $t \in \{1, \ldots, T\}$,
   $$\tilde{V}_t(\phi) = V_0(\phi) + \sum_{j=1}^{t} \phi_j \cdot \Delta \tilde{S}_j,$$
   where $\Delta \tilde{S}_j$ is the vector $\tilde{S}_j - \tilde{S}_{j-1} = \beta_j S_j - \beta_{j-1} S_{j-1}.$
1.1. DISCRETE-TIME MODELS

Proposition 1.1.2. For any predictable process \((\phi^0_t, \ldots, \phi^d_t)_{0 \leq t \leq T}\) and for any \(\mathcal{F}_0\)-measurable variable \(V_0\), there exists a unique predictable process \((\phi^0_t)_{0 \leq t \leq T}\) such that the strategy \(\phi = (\phi^0, \phi^1, \ldots, \phi^d)\) is self-financing and its initial value is \(V_0\).

Definition 1.1.5. (Admissible strategy) A strategy \(\phi\) is admissible if it is self-financing and if \(V_t(\phi) \geq 0\) for any \(t \in \{0, 1, \ldots, T\}\).

Definition 1.1.6. (Arbitrage). An arbitrage strategy is an admissible strategy with zero initial value and non-zero final value.

Conversely, we say that a market is free of arbitrage (FOA), if the market does not admit an arbitrage possibility.

Theorem 1.1.1. (Fundamental Theorem of Asset Pricing) The market is FOA if and only if there exists a probability measure \(\mathbb{P}^*\) equivalent to \(\mathbb{P}\) such that the discounted prices of assets are \(\mathbb{P}^*\)-martingales.

A contingent claim with maturity \(T\) can be characterized by its payoff \(\varphi\), which is a non-negative, \(\mathcal{F}_T\)-measurable random variable. For example, an European call option on the underlying \(S_i\) with strike price \(K\), is defined by the following payoff function
\[
\varphi = (S^i_T - K)^+,
\]
while a European Put on the same underlying and with the same strike price, is defined by
\[
\varphi = (K - S^i_T)^+.
\]

Definition 1.1.7. (Attainable contingent claim). The contingent claim defined by \(\varphi\) is said to be attainable if there exists an admissible strategy worth \(\varphi\) at time \(T\).

Remark 1.1.1. In a FOA market, we only need a self-financing strategy worth \(\varphi\) at maturity to say that the contingent claim is attainable. Indeed, if \(\phi\) is a self-financing strategy and if \(\mathbb{P}^*\) is an equivalent probability measure to \(\mathbb{P}\) under which the discounted prices are martingales, then \((\tilde{V}_t(\phi))\) is also a \(\mathbb{P}^*\)-martingale (see Theorem A.1.8). Hence, for \(t \in \{0, 1, \ldots, T\}\), \(\tilde{V}_t(\phi) = \mathbb{E}^*(\tilde{V}_T(\phi) | \mathcal{F}_t)\), where \(\mathbb{E}^*\) is the expectation under the probability measure \(\mathbb{P}^*\). Then, if \(\tilde{V}_T(\phi) = \varphi \geq 0\), the strategy \(\phi\) is admissible.

Definition 1.1.8. (Complete market). The market is complete if every contingent claim is attainable.

Theorem 1.1.2. A FOA market is complete if and only if there exists a unique probability measure \(\mathbb{P}^*\) equivalent to \(\mathbb{P}\), under which the discounted prices are martingale.
CHAPTER 1. INTRODUCTION TO OPTION PRICING

So, from now on, we consider a FOA and complete market, and we denote by \( \mathbb{P}^* \) the unique probability measure under which the discounted prices are martingales. Let \( \varphi \) be an \( \mathcal{F}_T \)-measurable, non-negative random variable and \( \phi \) be an admissible strategy replicating the contingent claim, i.e.,

\[
V_T(\phi) = \varphi.
\]

Since the sequence \((\tilde{V}_t)_{0 \leq t \leq T}\) is a \( \mathbb{P}^* \)-martingale, then

\[
V_0(\phi) = \mathbb{E}^* \left( \tilde{V}_T(\phi) \right),
\]

i.e. \( V_0(\phi) = \mathbb{E}^* \left( \frac{\varphi}{S_T} \right) \) and, more generally, we have

\[
V_t(\phi) = S^0_t \mathbb{E}^* \left( \frac{\varphi}{S_T} \mid \mathcal{F}_t \right), \quad t = 0, 1, \ldots, T.
\]

\( V_t(\phi) \) is called the value of option at time \( t \): that is the wealth needed at time \( t \) to replicate \( \varphi \) at time \( T \) by the strategy \( \phi \).

The number \( V_0(\phi) = \mathbb{E}^* \left( \tilde{V}_T(\phi) \right) \) is called the \textit{fair price}, i.e., is the price that makes an investor indifferent from choosing the contingent claim or the replicating strategy (if the investor, at time 0, sells the option at price \( V_0 \), he can follow a replicating strategy \( \phi \) to generate an amount \( \varphi \) at time \( T \), i.e., the investor is perfectly hedged).

The probability measure \( \mathbb{P}^* \) equivalent to \( \mathbb{P} \), under which the discounted prices are martingales, is called the \textit{risk-neutral probability measure}.

1.2 Continuous-time models: The Black-Scholes model

Now we’ll introduce the Black-Scholes model, described by Black and Scholes in [15]. The Black-Scholes model is a continuous-time model with one risky asset \( S_t \) and a risk-less asset \( S^0_t \). We suppose that \( S^0_t \) evolves according to the following differential equation:

\[
dS^0_t = rS^0_t dt,
\]

where \( r \geq 0 \) is the constant interest rate. We set \( S^0_0 = 1 \), so that \( S^0_t = e^{rt} \) for \( t \geq 0 \).

We assume that \( S_t \) evolves according to the following stochastic differential equation:

\[
dS_t = S_t(\mu dt + \sigma dB_t),
\]

where \( \mu \) and \( \sigma \) are two constants and \( B_t \) is the standard Brownian motion. The constant \( \sigma \) is called the volatility of the asset. We set \( S_0 = s_0 \), i.e., \( s_0 \) is the spot price observed at time 0. The model is considered in the interval \([0, T]\), where \( T \) is the maturity of the option.
1.2. CONTINUOUS-TIME MODELS: THE BLACK-SCHOLES MODEL

We want to find a probability measure under which the discounted stock price \( \tilde{S}_t = e^{rt} S_t \) is a martingale, i.e., we want to find the risk-neutral probability characterizing the model. Exploiting equation (1.2), we have

\[
d\tilde{S}_t = -re^{-rt} S_t dt + e^{-rt} dS_t = \tilde{S}_t((\mu - r) dt + \sigma dB_t).
\]

(1.3)

So, if we set \( W_t = B_t + (\mu - r)t/\sigma \),

\[
d\tilde{S}_t = \tilde{S}_t \sigma dW_t,
\]

(1.4)

then, by Girsanov Theorem (A.1.8), applied with \( \theta_t = (\mu - r)/\sigma \), there exists a probability \( \mathbb{P}^* \) equivalent to \( \mathbb{P} \), under which \( (W_t)_{0 \leq t \leq T} \) is a standard Brownian motion. Then, under probability \( \mathbb{P}^* \), we deduce from (1.4) that \( (\tilde{S}_t) \) is a martingale and then the unique fair value of the option is given by the expected discounted payoff:

\[
V_0 = \mathbb{E}^{\mathbb{P}^*}(e^{-rT} \varphi(S_T)|S_0 = s_0).
\]

To find the value of the option replicating it by a self-financing portfolio, suppose to have an option, the value \( V(S_t, t) \) of which depends on \( S_t \) and \( t \), \( V \in C^{1,2}([0, \infty \times \mathbb{R}) \). From now on, we write \( S \) instead of \( S_t \), for the sake of clarity. Using the Itô-Doeblin Lemma (A.1.4) and equation (1.2), we can write

\[
dV = \sigma S \frac{\partial V}{\partial S} dW + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt,
\]

(1.5)

and we can construct the portfolio, consisting of one option and a number \( \Delta \) of the underlying asset to hedge the risk, with value

\[
\Pi = V - \Delta S.
\]

(1.6)

Since it is self-financing, we have

\[
d\Pi = dV - \Delta dS,
\]

so that, putting together (1.2), (1.5), (1.6), we obtain

\[
d\Pi = \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) dW + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt.
\]

(1.7)

Since we want to construct a risk-less portfolio, we can eliminate the random component by choosing

\[
\Delta = \frac{\partial V}{\partial S},
\]

(1.8)
and by (1.8) in (1.7), we have

\[ d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \] (1.9)

Now, since the portfolio is deterministic and using the non-arbitrage condition, we must have that the evolution of the portfolio equals the risk-less asset with deterministic interest rate \( r > 0 \). The return of an amount \( \Pi \) invested in the risk-less asset would grow by \( r\Pi dt \) in a time \( dt \). Then, if the right-hand side of (1.9) were greater than this quantity, an arbitrager could make a secure risk-less profit by borrowing an amount \( \Pi \) and investing it in the portfolio. The return for this risk-free strategy would be greater than the cost of borrowing. Instead, if the right-hand side of (1.9) is less than this quantity, then the arbitrager would short the portfolio and invest \( \Pi \) in the bank. In either way the arbitrager could make a profit without risk. Then, we must have

\[ r\Pi dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \] (1.10)

Substituting (1.6), (1.8) into (1.10), we obtain the Black-Scholes partial differential equation, i.e,

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \] (1.11)

Remark 1.2.1. The delta

\[ \Delta = \frac{\partial V}{\partial S}, \]

is the rate of change of the value of the option \( V \) with respect to \( S \). It measures the correlation between the movements of the option and the movements of the underlying asset.

Remark 1.2.2. The action of the linear differential operator \( L \) given by

\[ L = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS \frac{\partial}{\partial S} - r, \] (1.12)

can be interpreted as measuring of the difference between the return on a hedged option portfolio (given by the first two terms) and the return on a bank account (given by last two terms). In order to avoid arbitrage, this difference has to be zero for a European option. We will see later on that the latter is not a necessary condition when one has to deal with options of American type, see Chapter 3.

Remark 1.2.3. From equation (1.11), we can see that the growth parameter \( \mu \) is not present. This means that the value of an option is independent of the speed of growth of an asset. The only parameter of the stochastic differential equation (1.2) for the asset that appears in (1.11) is the value \( \sigma \). This means that, even if two people may differ in the estimation for \( \mu \), they are nevertheless agree on the value of the option.
We give now the exact solution of the European call/put option problems, solving (1.11) with terminal condition respectively \( V(S_T, T) = (S_T - K)^+ \) for the call case and \( V(S_T, T) = (K - S_T)^+ \) for the put one (see [3]).

We recall that a European Option is a contract between two parties, the holder and the writer. The holder has the right, but no obligation, to buy or sell a certain underlying asset \( S \) for a determined price \( K \) (it is called strike price), at a fixed future time \( T \) (it is called the expiration time). Conversely, the writer has the obligation to sell, buy (respectively) the asset if the holder decides to exercise his right to buy, sell, respectively, it.

We have already seen that the payoff of a European Call option is \( \varphi(S_T) = (S_T - K)^+ \), i.e., the holder will exercise his right (buy the stock at \( K \)) if the value of the stock is greater than \( K \) at \( T \). In this case, the holder earn \( S_T - K \). Conversely, if the value of the stock is lower than \( K \) at \( T \), the holder will not exercise his right.

The payoff of a European Put option is \( \varphi(S_T) = (K - S_T)^+ \), i.e., the holder will exercise his right (sell the stock at \( K \)) if the value of the stock is lower than \( K \) at \( T \). In this case, the holder earn \( K - S_T \). Conversely, if the value of the stock is greater than \( K \) at \( T \), the holder will not exercise his right.

Then, in the case of a European Call, solving 1.10 with final condition \( C(S_T, T) = (S_T - K)^+ \), we obtain, see Section 3.8 in [3],

\[
C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2),
\]

(1.13)

where \( N \) is the cumulative distribution function for a standardized normal random variable, given by

\[
N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy,
\]

and where

\[
d_1 := \frac{\ln(S_T) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},
\]

while

\[
d_2 = \frac{\ln(S_T) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.
\]

In the case of a European Put, solving 1.10 with final condition \( P(S_T, T) = (K - S_T)^+ \), we obtain, see Section 3.8 in [3],

\[
P(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1).
\]

Remark 1.2.4. Until now we have assumed to have a constant volatility value \( \sigma \). However, the volatility is not a constant in real financial market. In order to derive a model that could be closer to real markets data, a
natural extension is to consider $\sigma$ as a function depending on time and on
the underlying. This choice leads to the so-called local volatility models,
where we have:

$$\sigma = \sigma(t, S_t).$$

Therefore, the equation (1.2) for the underlying becomes:

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t dW_t$$  \hspace{1cm} (1.14)

Extending to the present case the approach developed through the previous
paragraph, in the hypothesis of the Feynman-Kac Theorem [A.1.9] we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(t, S_t) S_t^2 \frac{\partial^2 V}{\partial S^2} + r S_t \frac{\partial V}{\partial S} - r V = 0.$$  \hspace{1cm} (1.15)

An example is the model proposed by Cox in (see [16]), where it was called
the Constant Elasticity of Variance (CEV) model (see also [25]). In this
model we have that the volatility depends on the underlying asset according
to

$$\sigma(t, S_t) = \sigma S_t^{\beta - 1},$$

where $\beta > 0$, and then (1.14) becomes:

$$dS_t = \mu S_t dt + \sigma S_t^{\beta - 1} S_t dW_t.$$  \hspace{1cm} (1.16)

We will restrict to the case $0 < \beta < 1$, since it is the most interesting. In
this case, typical of equity markets, the volatility increases when the price
decreases, i.e., we have a leverage effect. We will return on this CEV model
in Chapter 4.
Chapter 2

Approximation of the Green’s function for second order parabolic equation by the Dyson-Taylor Commutator Method

In this chapter we want to briefly introduce the Dyson-Taylor Commutator method developed in [4], i.e., a method used to obtain a small-asymptotic expansion for the Green’s function for a second order parabolic equation. In what follows we refer also to [5] and [7], to better explain the aforementioned approach that we then exploit throughout the next chapter. We restrict to discuss the method in one space dimension.

Consider the Initial-Valued Problem (IVP):

\[
\begin{aligned}
&\partial_t U(t,x) - L(t,x)U(t,x) = 0 \quad t > 0, x \in (0, +\infty), \\
&U(0,x) = f(x) \quad x \in (0, +\infty).
\end{aligned}
\] (2.1)

where \(L\) is the operator:

\[
L(t,x) := \frac{1}{2} a(t,x)^2 \partial_x^2 + b(t,x) \partial_x + c(t,x).
\] (2.2)

From now on, we assume that \(a(x) > 0\) for \(x > 0\), the coefficients \(a, b, c\) are smooth and bounded functions and their derivatives are bounded too. We also assume that \(L\) is strong elliptic, i.e., there exists a constant \(\gamma > 0\) such that \(a(x) \geq \gamma > 0\), see Section 1 and Definition 2.1 in [4], [5] and [7] for time-dependent coefficients.

We denote by \(G_t(x,y)\) the Green’s function or the fundamental solution of
the operator $\partial_t - L$. Since the solution to the IVP $\text{[2.1]}$ is given by, see Chapter 2 in $[21]$, 
\[ U(t, x) = \int G_t(x, y)f(y)dy, \]
then, through the approximation of the Green’s function, we can obtain also an approximation of the solution $U(t, x)$.

If the IVP $\text{[2.1]}$ admits a unique solution, then the linear operator that maps the initial data $f$ to the solution $U$ at time $t$ is well defined and we refer to it as the solution operator. If $L$ has coefficients independent of time, then it is possible to show it generates a semigroup of operators, denoted $\{e^{tL_0}, t > 0\}$.

We recall that:

**Definition 2.0.1.** (Semigroup) A family of bounded linear operators $\{T(t)\}_{t \geq 0}$ mapping $X$ to $X$ is called a semigroup if the following are satisfied:

- $T(0)x = x, x \in X$;
- $T(s + t)x = T(s)T(t)x = T(t)T(s)x, x \in X, \ t, x \geq 0$.

In our case, we have:

1. $e^{tL_0}|_0 = I$;
2. $e^{t_1L_0}e^{t_2L_0} = e^{(t_1 + t_2)L_0}, \ t_1, t_2 > 0$.

If we consider a time-dependent operator $L = L(t)$, then the solution operator does not define a semigroup, but an evolution system $S(t_1, t_2)$. It means that the property $\text{[2]}$ is replaced by $S(t_1, t_2)S(t_2, t_3) = S(t_1, t_3)$, if $0 \leq t_3 \leq t_2 \leq t_1$ (see $[7]$).

A basic tool used in the Dyson-Taylor commutator method is a certain parabolic rescaling (see $[5]$).

**Definition 2.0.2.** (Parabolic rescaling) Let $z \in \mathbb{R}$ be a fixed but arbitrary point, $s > 0$ be a parameter and $f(t, x)$ a function. Then

\[ f^{s,z}(t, x) := f(s^2t, z + s(x - z)), \]

is the parabolic rescaling by $s$ of the function $f$ about $z$. If we have a function $h$ that does not depend on $t$, we have

\[ h^{s,z}(x) := h(z + s(x - z)). \]

We will refer to $z$ as the basepoint for the rescaling; $t = 0$ is always assumed as the time basepoint.

\footnote{We call it Green’s function since we solve it in $(0, +\infty)$ and not in $(-\infty, +\infty)$, even if the boundary conditions are automatically satisfied.}
Using this definition, we can define the rescaled operator $L^{s,z}$ by:

$$L^{s,z}(t,x) := \frac{1}{2}a^{s,z}(t,x)\partial_t^2 + sb^{s,z}(t,x)\partial_x + s^2c^{s,z}(t,x).$$  \(2.3\)

If $U$ solves the IVP \(2.1\), then $U^{s,z}$ solves the rescaled problem, see Section 2 in \[5\],

$$\begin{aligned}
\partial_t U^{s,z}(t,x) - L^{s,z}(t,x)U^{s,z}(t,x) &= 0, \\
U^{s,z}(0,x) &= f^{s,z}(x).
\end{aligned}  \(2.4\)

Then we have that the Green’s functions of the operator $\partial_t - L$ and the Green’s function of the rescaled operator $\partial_t - L^{s,z}$ are related as follows:

$$G^L_t(x,y) = s^{-1}G^{L^{s,z}}_s-z\left(z + s^{-1}(x-z), z + s^{-1}(y-z)\right).$$  \(2.5\)

If $s = t^{\frac{1}{2}}$, then

$$G^L_t(x,y) = t^{-\frac{1}{2}}G^{\sqrt{t\cdot z}}_t\left(z + t^{-\frac{1}{2}}(x-z), z + t^{-\frac{1}{2}}(y-z)\right),$$  \(2.6\)

see formula 2.6 in \[5\]. Then, by \(2.6\), is sufficient to calculate the Green’s function $G^{s,x}_t$ of the rescaled problem \(2.3\) when $t = 1$. In order to do so, we shall consider the Taylor expansion with refer to the parameter $s$ at $s = 0$ up to order $n \in \mathbb{N}$ of the rescaled operator $L^{s,x}$ in \(2.3\). Consequently, the operator $L^{s,x}$ can be written in the following way, see formulas (33) and (34) in \[4\]:

$$L^{s,x} = \sum_{k=0}^{n} s^k L^k_k + s^{n+1} L^{s,z}_{n+1}(t,x),$$  \(2.7\)

where

$$L^k_k = \frac{1}{k!} \left( \frac{d^k}{ds^k} L^{s,z} \right) \bigg|_{s=0},$$

and $L^{s,z}_{n+1}(t,x)$ contains all the remainder terms given by the Taylor expansion of the coefficients. We can note that the terms $L^k_k$ are independent of $s$. Our aim is to calculate explicitly the second order approximation of the Green’s function of $L$, then we fix from now on $n = 2$. For a simpler notation, if $g$ is our function, we denote by

$$\dot{g}(t,x) = \frac{\partial}{\partial t} g(t,x),$$

the first derivative of $g$ with respect to $t$ and by

$$g’(t,x) = \frac{\partial}{\partial x} g(t,x),$$

the first derivative of $g$ with respect to $x$.

So, the second-order Taylor expansion with refer to $s$ of the rescaling parabolic function $f^{s,x}$ at $s = 0$ is given by

$$f^{s,z}(t,x) = f(0,z) + s(x-z)f’(0,z) + s^2 t f''(0,z)$$

$$+ \frac{1}{2} s^2(x-z)^2 f''(0,z) + s^3 r,$$  \(2.8\)
where \( s^3 r = s^3 r(s; t; x; z) \) is the remainder term.

In the following calculations, all functions are calculated at \((0, z)\), so we will write, for notation convenience, \( a \) instead of \( a(0, z) \) (the same for all other functions \( b, c \) etc. . . ).

Hence the second order Taylor expansion of \( L^{s,z} \) in \( s \) at \( s = 0 \) is:

\[
L^{s,z}(t, x) = L^s_0 + sL^s_1(x) + s^2(L^s_{2, x}(x) + tL^s_{2, t}) + s^3L^s_{3, x}(t, x),
\]

(2.9)

where:

\[
L^s_0 := \frac{1}{2}a^2 \partial^2_x,
\]

(2.10)

\[
L^s_1 := aa'(x - z)\partial^2_x + b\partial_x,
\]

(2.11)

\[
L^s_{2, x} := \frac{1}{2}(a^2 + aa'')(x - z)^2\partial^2_x + b'(x - z)\partial_x + c, \quad L^s_{2, t} := aa\partial^2_x,
\]

(2.12)

and \( L^{3, z}(t, x) \) is the remainder term.

We want to approximate the Green’s function \( G^L_t \) of the IVP (2.1) in order to obtain the solution \( U(t, x) = \int G^L_t(x, y)f(y)dy \). In order to do this, we decompose \( L \) in this way, see (2.11) in [3]:

\[
L(t) = L_0 + V(t),
\]

(2.13)

where \( L_0 \) is a constant-coefficients second-order operator, for which we can compute explicitly the solution operator, and \( V(t) \) is a time-dependent, variable-coefficient, second-order operator.

By Lemma 2.3 in [7], we have:

\[
G^L_t = e^{tL_0} + \int_0^t e^{(t-\tau)L_0} V(\tau)G^L_\tau d\tau.
\]

(2.14)

We repeat the application of Duhamel’s formula, i.e., the previous equation (2.14), to obtain a recursive representation of \( G^L_t \), see equation (2.13) in [3]:

\[
G^L_t = e^{tL_0} + \int_0^t e^{(t-\tau)L_0} V(\tau_1)e^{\tau_1 L_0} d\tau_1
\]

\[
+ \int_0^t \int_0^{\tau_1} e^{(t-\tau)L_0} V(\tau_1)e^{(\tau_1 - \tau_2)L_0} V(\tau_2)e^{\tau_2 L_0} d\tau_2 d\tau_1 + \cdots
\]

\[
+ \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{d-1}} e^{(t-\tau)L_0} V(\tau_1)e^{(\tau_1 - \tau_2)L_0} V(\tau_2) \cdots V(\tau_d)e^{\tau_d L_0} d\tau d\tau_1 \cdots d\tau_{d-1}
\]

\[
+ \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{d+1}} e^{(t-\tau)L_0} V(\tau_1)e^{(\tau_1 - \tau_2)L_0} V(\tau_2) \cdots V(\tau_{d+1})G^L_{\tau_{d+1}} d\tau,  
\]

(2.15)
where we have set \( d\tau_1 \cdots d\tau_d \tilde{\tau}_1 = d\tilde{\tau} \).

Since, as we have assumed before, \( L \) is uniformly strongly elliptic and all the coefficients of \( L \) and their derivatives are bounded, then this expansion can be justified rigorously with a bound on the remainder term. Furthermore, when \( d \to \infty \), we obtain an asymptotic time-ordered series for the Green’s function (also called a Dyson series), see pag. 911 in [5]. The integer \( d \) is the iteration level in the time-ordered expansion, which can be different from the order \( n \) of the Taylor expansion of the operator \( L \). For consistency we need \( n \geq d \) (see pag. 911 in [5], for more details [7], [4]). We set, from now on, \( d = n = 2 \).

We can apply the same procedure to the Green’s function \( G^{L_2,z}_{t} \) of the solution operator for the rescaled problem \((2.4)\). We have already seen in \((2.6)\) that it’s enough to compute the approximate Green’s function at \( t = 1 \). As in \((2.13)\), we can write

\[
L^{s,z}_1(t) = L^z_0 + V^{s,z}(t),
\]

and then, using \((2.9)\),

\[
V^{s,z}(t) := L^{s,z}_1(t) - L^z_0 = sL^1_1(x) + s^2L^2_2(t, x) + s^3L^{s,z}_3(t, x),
\]

where \( L^z_0, L^z_1 \) and \( L^z_2 \) are given respectively by equations \((2.10), (2.11), (2.12)\) and \( L^{s,z}_3(t, x) \) is the remainder term.

Now, using \((2.15)\) with \( d = n = 2 \) and \( t = 1 \), we have

\[
\mathcal{G}_1^{L_{s,z}} = e^{L^z_0} + s\mathcal{I}_1 + s^2(\mathcal{I}_{1,1} + \mathcal{I}_{2,x} + \mathcal{I}_{2,t}) + \mathcal{R}^{s,z},
\]

where

\[
\begin{align*}
\mathcal{I}_1 &= \int_0^1 e^{(1-r_1)L^z_0} L^z_1 e^{r_1 L^z_0} d\tau_1, \\
\mathcal{I}_{1,1} &= \int_0^1 \int_0^{\tau_1} e^{(1-r_1)L^z_0} L^z_1 e^{(r_1 - r_2)L^z_0} L^z_1 e^{r_2 L^z_0} d\tau_2 d\tau_1, \\
\mathcal{I}_{2,x} &= \int_0^1 \int_0^{\tau_1} e^{(1-r_1)L^z_0} L^z_2 e^{r_1 L^z_0} d\tau_1, \\
\mathcal{I}_{2,t} &= \int_0^1 \int_0^{\tau_1} e^{(1-r_1)L^z_0} L^z_2 e^{r_1 L^z_0} d\tau_1.
\end{align*}
\]

All the higher-order terms are contain in \( \mathcal{R}^{s,z} \) in \((2.17)\) and we include this term in the remainder, see pag. 912 in [5].

Since we want to obtain the approximation of the Green’s function \( \mathcal{G}_t^L \) of the IVP \((2.1)\), we let, following Chapter 2 in [5],

\[
\mathcal{T}^{s,z}(x, y) := G_0(x, y, z) + sG_1(x, y, z) + s^2G_2(x, y, z),
\]

\((2.19)\)
be the distribution kernel of the operator $e^{L_{0}t} + sI_{1}z + I_{2}\delta + I_{3}\delta_{t}$. We have, see equation (2.16) in [5]:

$$G_{t}^{[n]}(x, y) = G_{t}^{[0]}(x, y, z) + t^{\frac{1}{2}} G_{t}^{[1]}(x, y, z) + \cdots + t^{\frac{n}{2}} G_{t}^{[n]}(x, y, z), \quad (2.20)$$

where $G_{t}^{[n]}$ is the sum of the first $n$ terms of the expansion and represents the $n$th-order approximate kernel. Since this formula is obtained with an error bound, the fixed basepoint $z$ can be replaced by $z = z(x, y)$ satisfying some conditions, i.e., $z(x, y) = x$ and all its derivatives are bounded (we refer to [5] for more details). The kernels $G_{t}^{[n]}$ in (2.20) are given by

$$G_{t}^{[n]}(x, y) := t^{-\frac{1}{2}} G_{n}(z + (x - z)/\sqrt{t}, z + (y - z)/\sqrt{t}, z(x, y)),$$

and then, our second-order approximation is given by:

$$G_{t}^{[2]}(x, y) = t^{-\frac{1}{2}} T^{\sqrt{t}, z}(z + (x - z)/\sqrt{t}, z + (y - z)/\sqrt{t}). \quad (2.21)$$

The aim of the Dyson-Taylor commutator method is that, commutating $e^{L_{0}t}$ with $L_{i,j}$, we can calculate the terms $I_{i,j}$ in equations (2.18) in a closed-form.

Now, in order to state Lemma (2.0.3), we need some theoretical notions (we refer to [7]):

**Definition 2.0.3.** (Spaces of Differentials). For any nonnegative integers $a, b$ we denote by $D(a, b)$ the vector space of all differentiations of degree at most $a$ and order at most $b$. We extend this definition to negative indices by defining $D(a, b) = \{0\}$ if either $a$ or $b$ is negative. By degree of $A$ we mean the highest power of the polynomials appearing as coefficients in $A$.

**Definition 2.0.4.** (Adjoint Representation). For any two differentiations $A_{1}, A_{2} \in D(a_{1}, b_{1})$ and $A_{2} \in D(a_{2}, b_{2})$ we define $ad_{A_{1}}(A_{2})$ by

$$ad_{A_{1}}(A_{2}) := [A_{1}, A_{2}] = A_{1}A_{2} - A_{2}A_{1}, \quad (2.22)$$

and for any integer $j \geq 1$ we define $ad_{A_{1}}^{j}(A_{2})$ recursively by

$$ad_{A_{1}}^{j}(A_{2}) := ad_{A_{1}}(ad_{A_{1}}^{j-1}(A_{2})). \quad (2.23)$$

**Lemma 2.0.1.** (Backer-Campbell-Hausdorff formula)

Let $A$ and $B$ be two operators, then

$$[e^{A}, B] = \left( [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \cdots \right) e^{A}. \quad (2.24)$$

This formula tells us how to commute $e^{A}$ and $B$:

$$e^{A}B = \left( B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \cdots \right) e^{A}. \quad (2.25)$$
Proposition 2.0.1. Suppose $A_1 \in \mathcal{D}(a_1, b_1)$ and $A_2 \in \mathcal{D}(a_2, b_2)$. Then, for any integer $k \geq 1$,
\[ ad^k_{A_1}(A_2) \in \mathcal{D}(k(a_1 - 1) + a_2, k(b_1 - 1) + b_2). \]

Lemma 2.0.2. Let $m, k$ be fixed integers $\geq 1$. Let $L_0 \in \mathcal{D}(0, 2)$ be the constant coefficient operator and $L_m \in \mathcal{D}(m, 2)$ be the operator given above, then
\[ ad^k_{L_0}(L_m) \in \mathcal{D}(m - k, k + 2). \]
In particular,
\[ ad^k_{L_0}(L_m) = 0 \quad \forall k > m. \]

In our case, we have:
\[ [L_0, L_2] = 0, \quad \text{and} \quad ad^2_{L_0}(L_1) = 0, \]
and
\[ ad^3_{L_0}(L_2) = 0. \]

Lemma 2.0.3. Let $L_0$ and $L_m$ defined as above, then for any $\theta \in (0, 1)$,
\[ e^{(1-\theta)L_0}L_m(\theta) = P_m(\theta, x - z, \partial)e^{(1-\theta)L_0}, \]
where
\[ P_m(\theta, x - z, \partial) := L_m(\theta) + \sum_{i=1}^{m} \frac{(1-\theta)^i}{i!} ad^i_{L_0}(L_m(\theta)) \in \mathcal{D}(m, m + 2), \]
is a finite sum of terms with the form $a(z)(1-\theta)^i(1-x-z)^k\partial^\alpha_x$, in which $a(z)$ and all its derivatives are bounded, $\alpha$ is a multi-index.

Proof. See Lemma 3.8 in [7].

Now, we can compute the integrals in (2.18), using lemma (2.0.2) and Lemma (2.0.3). We thus have:
\[ I_1 = \int_0^1 e^{(1-\tau_1)L_0}L_1e^{\tau_1L_0}d\tau_1 = \int_0^1 (L_1^2 + (1-\tau_1)[L_0, L_1])e^{L_0}d\tau_1 = (L_1^2 + \frac{1}{2}[L_0, L_1])e^{L_0}, \]
(2.25)
\[ I_{1,1} = \int_0^1 \int_0^{\tau_1} e^{(1-\tau_1)L_0}L_1e^{(\tau_1-\tau_2)L_0}L_1e^{\tau_2L_0}d\tau_2d\tau_1 \]
\[ = \int_0^1 \int_0^{\tau_1} (L_1^2 + (1-\tau_1)[L_0, L_1])(L_1^2 + (1-\tau_2)[L_0, L_1])e^{L_0}d\tau_2d\tau_1 \]
\[ = \left( \frac{1}{2}L_1^2 + \frac{1}{3}[L_0, L_1]L_1^2 + \frac{1}{6}[L_0, L_1]^2L_1 + \frac{1}{8}[L_0, L_1]^3 \right) e^{L_0}, \]
(2.26)
\[ I_{2,x}^z = \int_0^1 e^{(1-\tau_1)\mathcal{L}_0^z L_{2,x}^z} e^{\tau_1 \mathcal{L}_0^z} d\tau_1 \]

\[ = \int_0^1 (L_{2,x}^z + (1 - \tau_1)[\mathcal{L}_0^z, L_{2,x}^z] + \frac{(1 - \tau_1)^2}{2} [\mathcal{L}_0^z, [\mathcal{L}_0^z, L_{2,x}^z]]) e^{\mathcal{L}_0^z} d\tau_1 \]

\[ = \left( L_{2,x}^z + \frac{1}{2}[\mathcal{L}_0^z, L_{2,x}^z] + \frac{1}{6} [\mathcal{L}_0^z, [\mathcal{L}_0^z, L_{2,x}^z]] \right) e^{\mathcal{L}_0^z}, \]

(2.27)

\[ I_{3,x}^z = \int_0^1 e^{(1-\tau_1)\mathcal{L}_0^z L_{3,x}^z} e^{\tau_1 \mathcal{L}_0^z} d\tau_1 = \int_0^1 \tau_1 L_{2,x}^z e^{\mathcal{L}_0^z} d\tau_1 = \frac{1}{2} L_{2,x}^z e^{\mathcal{L}_0^z}. \]

(2.28)

Hence, (2.17) becomes:

\[ e^{\mathcal{L}_0^z} = (1 + s\mathcal{Q}_1 + s^2\mathcal{Q}_2)e^{\mathcal{L}_0^z} + \mathcal{R}^{s,z}, \]

(2.29)

where

\[ Q_1 = L_1^z + \frac{1}{2}[\mathcal{L}_0^z, L_1^z], \]

\[ Q_2 = \frac{1}{2}(L_1^z)^2 + \frac{1}{3}L_1^z[L_0^z, L_1^z] + \frac{1}{6}[\mathcal{L}_0^z, L_1^z]L_1^z + \frac{1}{8}[\mathcal{L}_0^z, L_1^z]^2 + L_{2,x}^z, \]

\[ + \frac{1}{2}[L_0^z, L_{2,x}^z] + \frac{1}{6}[\mathcal{L}_0^z, [\mathcal{L}_0^z, L_{2,x}^z]] + \frac{1}{2}L_{2,t}^z, \]

(2.30)

and \( \mathcal{R}^{s,z} \) is the error term as in (2.17).

Therefore, we have to compute the commutators in (2.30) to get the second-order approximation of \( G_{s,z}^z \). We recall that all functions in the commutator formulas below are evaluated at \( (0, z) \) (as before, \( a = a(0, z), a' = a'(0, z), \) etc.

We will show the calculations only for the first formula:

\[ [\mathcal{L}_0^z, L_1^z] = \frac{1}{2} a^2 \partial_x^2 (aa'(x - z) \partial_x^2 + b \partial_x^2) \]

\[ = \frac{1}{2} a^2 \partial_x^2 (aa'(x - z) \partial_x^2 + b \partial_x^2) \]

\[ = \frac{1}{2} a^2 \partial_x^2 (aa' \partial_x^2 + aa'(x - z) \partial_x^2 + b \partial_x^2) \]

\[ = \frac{1}{2} a^2 \partial_x^2 (aa' \partial_x^2 + \frac{1}{2} a^3 \partial_x^2 + \frac{1}{2} a^3 a' \partial_x^2) \]

\[ + \frac{1}{2} a^2 b \partial_x^3 - \frac{1}{2} a^2 a' (x - z) \partial_x^4 - \frac{1}{2} b a^2 \partial_x^3 \]

\[ = a^3 a' \partial_x^3, \]

(2.31)

and, in the same way, we obtain, see also equations (2.25)-(2.29) in [5],

\[ [\mathcal{L}_0^z, L_3^z]^2 = a^6 a'^2 \partial_x^6, \]

(2.32)
\[ L_1^2[L_0, L_1^2] = a^4 a''^2 (x - z) \partial_x^2 + b a^3 a' \partial_x^4, \quad (2.33) \]

\[ [L_0, L_1^2] = a^4 a'' (x - z) \partial_x^2 + (b + 3aa') a^3 a' \partial_x^4, \quad (2.34) \]

\[ [L_0^2, L_2^2] = a^2 (a^2 + aa'')(x - z) \partial_x^3 + a^2 \left( b + \frac{1}{2} a'^2 + \frac{1}{2} aa'' \right) \partial_x^2, \quad (2.35) \]

\[ [L_0^2, [L_0^2, L_2^2]] = a^4 (a^2 + aa'') \partial_x^4, \quad (2.36) \]

and finally

\[ (L_1^2) = (aa' (x - z))^2 \partial_x^4 + 2(a^2 a'^2 + aa'b)(x - z) \partial_x^3 + (aa'b + b^3) \partial_x^2. \quad (2.37) \]

Then, the applications of a differential operator with polynomial coefficients to the Green’s function of \( e^{L_0} \), which is known explicitly, since \( L_0 \) has constant coefficients, gives the approximation kernel of \( G_{1,x}^{+,-} \).

We next recall that: if \( \phi \) is a smooth function, we denote by \( C_\phi \) the convolution operator with \( \phi \). Then

\[ C_\phi f(x) := \phi * f(x) = \int \phi(x - y)f(y)dy, \nonumber \]

which shows that the distribution kernel of \( C_\phi \) is

\[ C_\phi(x, y) = \phi(x - y). \nonumber \]

Then

\[ \partial_x C_\phi = C_{\partial_x \phi}, \quad (2.38) \]

while \( C_\phi \partial_x = -C_{\partial_x \phi} \). Since, as we have just said, \( L_0^2 \) is a constant coefficient operator, its Green’s function can be computed explicitly:

\[ e^{L_0^2}(x, y) = \frac{1}{\sqrt{2\pi a^2}} \exp \left(-\frac{|x - y|^2}{2a^2}\right), \quad a = a(0, z), \quad (2.39) \]

and hence \( e^{L_0^2} \) is a convolution operator (see [5]). Then, by equation (2.38),

\[ \partial_x^k e^{L_0^2}(x, y) = H_k(\Theta) e^{L_0^2}(x, y), \quad \Theta = \frac{x - y}{a^2}, \]

where \( H_k \) are the rescaled Hermite polynomials satisfying \( H_0 = 1 \) and \( H_{k+1}(\Theta) = -\Theta H_k(\Theta) + H'_k(\Theta)/a^2 \) (see [5]). The polynomials \( H_k \) can be computed by induction, so we have, see equations (2.32) in [5]:

\[ H_1(\Theta) = -\Theta, \quad H_2(\Theta) = \Theta^2 - \frac{1}{a^2}, \quad H_3(\Theta) = -\Theta^3 + \frac{3\Theta}{a^2}, \]

\[ H_4(\Theta) = \Theta^4 - \frac{6\Theta^2}{a^2} + \frac{3}{a^4}, \quad H_5(\Theta) = -\Theta^5 + \frac{10\Theta}{a^2} - \frac{15\Theta}{a^4}, \quad (2.40) \]

\[ H_6(\Theta) = \Theta^6 - \frac{15}{a^2} \Theta^4 + \frac{45}{a^4} \Theta^2 - \frac{15}{a^6}. \]
Then we have

\[ G_0(x, y, z) = e^{L_0} = \frac{1}{\sqrt{2\pi a^2}} e^{\exp \left(-\frac{|x - y|^2}{2a^2}\right)}, \quad (2.41) \]

and using (2.40), we have:

\[ G_1(x, y, z) = \left( L_1^z + \frac{1}{2}[L_0^z, L_1^z] \right) e^{L_0^z}(x, y) \]

\[ = \left( aa'(x - z)\partial_x^2 + b\partial_x + \frac{1}{2}a^3 a'\partial_x^3 \right) e^{L_0^z}(x, y) \]

\[ = \left( aa'(x - z)H_2(\Theta) + bH_1(\Theta) + \frac{1}{2}a^3 a' H_3(\Theta) \right) e^{L_0^z}(x, y) \]

\[ = \frac{1}{\sqrt{2\pi a^2}} e^{-\frac{|x - y|^2}{2a^2}} \left[ (x - y) \left( \frac{3aa' - 2b}{2a^2} \right) - \frac{a'}{2a^3} (x - y)^3 \right. \]

\[ + \left. a'(x - z) \left( \frac{(x - y)^2 - a^2}{a^3} \right) \right], \quad (2.42) \]

and, since

\[ G_2(x, y, z) = (Q_2 e^{L_0^z})(x, y), \]

with \( Q_2 \) given by (2.30), then

\[ G_2(x, y, z) = \left( \frac{1}{2} L_{2,x}^z + L_{2,x}^z + \frac{1}{2}[L_0^z, L_{2,x}^z] + \frac{1}{6}[L_0^z, [L_0^z, L_{2,x}^z]] + \frac{1}{2}(L_1^z)^2 \right. \]

\[ + \frac{1}{3} L_1^z[L_0^z, L_1^z] + \frac{1}{6}[L_0^z, L_1^z]L_1^z + \frac{1}{8}[L_0^z, L_1^z]^2 \right) e^{L_0^z} \]

\[ = \left( P_0 + \sum_{i=1}^{6} P_i H_i(\Theta) \right) e^{L_0^z}(x, y), \quad (2.43) \]

where \( P_i \) are polynomials in \( x - z \) the coefficients of which are given by the
functions $a, b,$ and $c$ and their derivatives, all evaluated in $(0,z)$ as follows:

- $P_0 = c = c(0,z)$,
- $P_1 = b'(x-z)$,
- $P_2 = \frac{1}{2} \left[ \frac{1}{2}a^3 a'' + a^2 b' + \frac{a^2 a'^2}{2} + b^2 + a^2(x-z)^2 + a(ba' + a' + a''(x-z)^2) \right]$,
- $P_3 = a(x-z) \left( a'b + \frac{1}{2} a^3 a'' + \frac{3}{2} a a'^2 \right)$,
- $P_4 = \frac{a^2}{3} \left[ \frac{1}{2} a^3 a'' + 2 a^2 a'^2 + \frac{3}{2} a a'b + \frac{3}{2} a^2(x-z)^2 \right]$,
- $P_5 = \frac{1}{2} a^4 a'^2(x-z)$,
- $P_6 = \frac{1}{8} a^6 a'^2$.

(2.44)

In order to continue, we need to recall the definition of admissible basepoint function $z$ and we need to give the definition of exponentially weighted Sobolev spaces $W_{a}^{m,p}(\mathbb{R})$. We need to introduce Sobolev spaces $W_{a}^{m,p}(\mathbb{R})$ since we want to work with the typical initial conditions that appear in option pricing, for example the payoff functions of European/American options. For example, if we consider the problem of pricing a European Call option under the Black and Scholes model, after making the change of variable $x = e^y$, the payoff function becomes exponentially increasing and it belong to $W_{a}^{m,p}(\mathbb{R})$. The reason of this change of variable is that it reduces the Black and Scholes PDE to a constant coefficient one. Instead, in a more general case, when the change of the coordinates is not helpful in order to reduce the degeneracy of the coefficients of the operator $L$, we need an exponential weight, and then we tend to use the exponentially weighted Sobolev space $W_{a}^{m,p}(\mathbb{R})$, for more details see also Section 2 in [22]. In the following definitions we refer to Section 2 in [5].

**Definition 2.0.5.** (Admissible function). We call a function $z: \mathbb{R}^{2N} \to \mathbb{R}^N$ admissible if $z(x,x) = x$ and all derivatives of $z$ are bounded.

Now, let

$$\langle x \rangle := (1 + x^2)^{\frac{1}{2}}.$$

Then, we define

$$W_{a}^{m,p}(\mathbb{R}) := \{ u : \mathbb{R} \to \mathbb{C}, \partial_x^j (e^a(x)u(\cdot)) \in L^p(\mathbb{R}), j \leq m \}, \quad (2.45)$$

for $1 < p < \infty$, $m \in \mathbb{Z}_+$ and $a \in \mathbb{R}$. When $a = 0$, we recover the usually Sobolev spaces.

We next introduce the notation

$$\begin{cases} x_t = x + (x-z)/\sqrt{t} \\ y_t = y + (y-z)/\sqrt{t} \end{cases}.$$  

(2.46)
And, from Theorem 2.5 in [5], we have

**Theorem 2.0.1.** Let $L$ be given by (2.2) and $z = z(x, y)$ be an admissible function. Let us assume that the function $1/a$ and all the derivatives $a^{(k)}, b^{(k)}, c^{(k)}$ $k \geq 0$ are bounded on $\mathbb{R}$.

Define

$$G^2_t(x, y) = t^{-\frac{1}{2}}(G_0(x_t, y_t, z) + t^\frac{1}{2}G_1(x_t, y_t, z) + tG_2(x_t, y_t, z)),$$

where $z = z(x, y)$ and the functions $G_j$ are given by (2.41)-(2.44). Also, let us define the error term $E^{[2]}_t$ in the approximation of the Green’s function by

$$e^{tL}f(x) = \int_{\mathbb{R}^N} G^{[2]}_i(x, y)f(y)dy + t^\frac{3}{2}E^{[2]}_t f(x).$$

Then, for any $f \in W^{m,p}_a(\mathbb{R})$, $s \in \mathbb{R}$, $m \geq 0$, $1 < p < \infty$, we have

$$\|E^{[2]}_tf\|_{W^{m+k,p}_a} \leq Ct^{-\frac{k}{2}}\|f\|_{W^{m,p}_a},$$

for any $t \in [0, T]$, $0 < T < \infty$, $k \in \mathbb{Z}_+$, with $C$ independent of $t \in [0, T]$.

The general results for $n$th-order can be found in Theorem 1.2.1 in [9], Theorem 0.1 in [7].

**Remark 2.0.1.** In the case of the CEV model with $0 < \beta < 1$, introduced in Remark 1.2.4, we have that the hypothesis on the coefficients of the previous Theorem are not satisfied. Numerical tests in Chapter 4 in [9] shows that the result should extend to this case. Then, in Section 4.2, we will give only a numerical study of the American put price under this CEV model, even if it is not supported by theoretical results. For more details and for other methods used to solve this kind of problem see, for example, [23], [24].

### 2.1 Kernel expansion at $z=x$.

Now, we will se the approximation of the Green’s function $G^L_t$, computing $G^{[2]}_t$ at $z = x$. This choice leads us to have a simplified expression for the approximation, since the terms in $(x - z)$ disappears. In the next chapter we will see that this kind of approximation allows us not to need numerical quadrature in evaluating the integrals to find the price of an American put option, giving faster calculations and more precision.

The equation (2.42) becomes:

$$G_1(x, y, z = x) = \frac{x - y}{\sqrt{2\pi a^2}}e^{-\frac{(x-y)^2}{2a^2}} \left( \frac{3aa' - 2b}{2a^2} - \frac{a'}{2a^3}(x - y)^2 \right), \quad (2.47)$$
2.1. KERNEL EXPANSION AT Z=X.

and (2.43) becomes:

\[ G_2(x, y, z = x) = \frac{x - y}{\sqrt{2\pi a^2}} e^{-\frac{(x-y)^2}{2a^2}} \left[ \frac{1}{8} a^6 a^2 H_6(\Theta) + \frac{a^3}{6} (a^2 a'' + 4aa'2 + 3ba') H_4(\Theta) + \frac{1}{4} (a^3 a'' + 2a^2 b' + 2aa' + a^2 a^2 + 2b^2) H_2(\Theta) + c \right], \]

(2.48)

where \( \Theta = \frac{x-y}{a(0, z)^2} \), and \( H_2, H_4, H_6 \) are given by equations in (2.40).

Now, recalling that:

\[ G_1^{[1]}(x, y) = t^{-\frac{1}{2}} (G_0(x_t, y_t, z) + t \frac{1}{2} G_1(x_t, y_t, z)), \]

and

\[ G_1^{[2]}(x, y) = t^{-\frac{1}{2}} (G_0(x_t, y_t, z) + t \frac{1}{2} G_1(x_t, y_t, z) + t G_2(x_t, y_t, z)), \]

where

\[ \begin{cases} x_t = z + (x - z)/\sqrt{t} \\ y_t = z + (y - z)/\sqrt{t} \end{cases} \]

always in \( z = x \), we obtain:

\[ G_1^{[1]}(x, y) = \frac{1}{\sqrt{2\pi t a^2}} e^{-\frac{(x-y)^2}{2ta^2}} \left[ 1 + \frac{3aa' - 2b}{2a^2} (x - y) - \frac{a'}{2ta^3} (x - y)^3 \right], \]

(2.49)

and

\[ G_1^{[2]}(x, y) = \frac{1}{\sqrt{2\pi t a^2}} e^{-\frac{(x-y)^2}{2ta^2}} \left\{ \begin{array}{l} 1 + \frac{3aa' - 2b}{2a^2} (x - y) - \frac{a'}{2ta^3} (x - y)^3 + \frac{(x - y)^2}{2a^4} \left[ \frac{1}{2} a^3 a'' + a^2 b' + \frac{a^2 a'^2}{2} + b^2 + aba' + a\right] + \frac{(x - y)^4}{3ta^6} - \frac{2(x - y)^2}{a^4} + \frac{t}{a^2} \left[ \frac{1}{2} a^3 a'' + 2a^2 a'^2 + 3\right] \right. \\
\left. + \frac{1}{8} \frac{(x - y)^6}{a^8 t^2} - \frac{15 (x - y)^4}{a^4} + \frac{45}{a^2 (x - y)^2} - 15t \right\}. \]

(2.50)

We will use these approximations in Chapter 4, in order to use the Dyson-Taylor commutator method to price American Put options.
Chapter 3

American Option Pricing

In the previous chapter we have briefly introduced the Dyson-Taylor Commu-
tator method, and in Chapter 4 of \[9\] we can find its application to European
options pricing. Now, we want to extend this approach to the case of Amer-
ican put options. In all this chapter we refer to Chapter 7 of \[3\].

First of all, we have to explain what is an American Option.
An American call/put option has the same features of a call/put European
option, except we can exercise it during all its life and not only at maturity
as a European one. Since the American option gives the right to early exer-
cise, and so it gives more rights respect to an European one, it has, at least,
an equal or higher value.

The value of an American option is more complicate to find, since, differently
from the European case, to price an American option we have also to taken
into account possible early exercise. Indeed, at each time \(t\), there is a specific
value of the stock price \(S\) that determines two regions: one where the option
should be held, and one where it should be exercised. We denote this value
of \(S\) by \(S_f(t)\), and we call it the optimal exercise boundary. Since we do
not know \(S_f(t)\) a priori, we lose an information with refer to the problem
to evaluate an European option: in the last case we know both which kind
of boundary conditions we would have to apply and when we would have to
apply them. Then, for this reason, the unknown exercise boundary \(S_f(t)\) is
a free boundary. So, we can rewrite the problem of evaluating an American
option as a free boundary problem.

3.1 American Option as a Free Boundary Problem

We have just seen that we can write the American option valuation problem
as a free boundary problem. It can be described by a set of constraints, as
we can seen in Section 7.3 of \[3\]. In particular:

- The option value must be greater or equal than the payoff function;
The option pricing equation is replaced by an inequality;

- the option value must be a continuous function of the stock price $S$;
- the $\Delta$ defined in equation (1.8) must be continuous.

We briefly explain what each constraint means.

The first constraint allows us to avoid any arbitrage opportunity: indeed, if we have a put option and its value is $V_P(S,t) < (K - S)^+$, we can buy the asset in the market at price $S$ and, at the same time, we can buy the option at price $P$. Then we immediately exercise the option, and so we can sell the asset at price $K$, making a risk-free profit equal to $K - S - V_P$, contrary to the arbitrage-free assumption on the market.

So, in the case of an American Option in which early exercise is permitted, to avoid arbitrage, we have to impose:

$$V_P(S,t) \geq (K - S)^+. \tag{3.1}$$

Then we have that, either the option value is equal to payoff, and then the option should be exercised, or, if it is greater than payoff, it has to satisfy the Black and Scholes equation (see also [8]). Then, putting together this two fact, we obtain the inequality presented in the second constraint. The third constraint derives from arbitrage, indeed, if we have a discontinuity that in $S$ that persists more than an infinitesimal time, we can have profit without risk. Since we want to determine an unique $S_f$, we need an other constraint about the option function, i.e., that the option $\Delta$ has to be continuous too (the fourth constraint). For this constraint we give a briefly explanation for the case of an American put, that is the case in which we are interested in.

### 3.2 American Put Option

We consider now an American put option with value $P(S,t)$ and exercise boundary $S_f$. Suppose that the stock price $S_t = S$ evolves according to the following stochastic differential equation, as in (1.2),

$$dS_t = S_t(\mu dt + \sigma dB_t),$$

where $\mu$ and $\sigma$ are two constants and $B_t$ is the standard Brownian motion. We have already seen that the option should be exercised if $S < S_f$ and it should be held otherwise. We focus now on the second constraint, i.e., the Black and Scholes inequality. We know that the Black and Scholes equation derives from an arbitrage argument, see Section 1.2, and it yields an equality in the case of an European
3.2. **AMERICAN PUT OPTION**

Therefore, in the case of an American option, it yields only an inequality, see Section 7.4 of [3]. So we have:

\[
\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP \leq 0. \tag{3.2}
\]

The equality, and then the Black and Scholes equation, holds when it is optimal to hold the option and then the constraint (3.1) has to be satisfied. Conversely, when it is optimal to exercise the option, then the inequality in (3.2) holds, and the equality in the constraint (3.1) is satisfied (i.e., the option value equals the payoff). We can see an example, choosing as parameters \( K = 15 \), \( \sigma = 0.3 \), \( r = 0.1 \) as we will see next in Chapter 4. When \( P = K - S \), and then we are in the second case discussed above, for \( S < K \), substituting into (3.1), we have

\[
\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = -rK = -1.5 < 0.
\]

Now, we return on the fourth constraint, i.e., that the \( \Delta \) has to be continuous. Considering that \( S_f < K \), we have that the derivative with refer to \( S \) of the payoff function is \( -1 \). Then there are three possibilities for the delta of the option, \( \frac{\partial P}{\partial S} \), at \( S = S_f(t) \):

- \( \frac{\partial P}{\partial S} < -1 \);
- \( \frac{\partial P}{\partial S} > -1 \);
- \( \frac{\partial P}{\partial S} = -1 \);

We want to prove that the correct possibility is the third, see Section 7.4 in [3].

We begin to discuss the first: if it holds, then, as \( S \) increases, \( P(S,t) \) goes under the payoff function since its slope is more negative. This contradicts our earlier first constraint \( P(S,t) > \max(K - S, 0) \), and then the first possibility is wrong. We can pass to the second one: if it were true, then we would have a sub-optimal option value. In this case, this slope does not give the maximum value for the option, as according to Black and Scholes risk-free hedging strategy and the first constraint (3.1), see Section 7.4 in [3]. Now we can write the America put problem as a free boundary problem. For each \( t \) we can consider two regions:

- \( 0 \leq S < S_f(t) \);
- \( S_f(t) \leq S < \infty \).

The first region is the so called *exercise region* and here early exercise is optimal. Then in this region the following hold:

\[
P = K - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP < 0. \tag{3.3}
\]
The second region is the so called *continuous region* and here it is optimal to hold the option. Then in this region the following holds:

\[ P > K - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0. \]  

(3.4)

At \( S = S_f(t) \) we have the boundary conditions given by the third and fourth constraint, i.e., that \( P \) and \( \frac{\partial P}{\partial S} \) are continuous in \( S \):

\[ P(S_f(t), t) = \max(K - S_f(t), 0), \quad \frac{\partial P}{\partial S}(S_f(t), t) = -1. \]  

(3.5)

### 3.3 The Dyson-Taylor Commutator method applied to American Option pricing

We want to apply now the previous argue to *local volatility models*. Then we suppose that the stock price \( S_t \) evolves according to the following stochastic differential equation, as in (1.14),

\[ dS_t = \mu S_t dt + \sigma(S_t) S_t dW_t. \]

Then, as we have just seen in Remark (1.2.4), if we denote with \( u(\tau, x) \) the option price where \( x = S_\tau \) is the stock price, the associated backward Kolmogorov equation is given by

\[ \frac{\partial u(\tau, x)}{\partial \tau} + \frac{1}{2} \sigma^2(\tau, x) x^2 \frac{\partial^2 u(\tau, x)}{\partial x^2} + rx \frac{\partial u(\tau, x)}{\partial x} - ru(\tau, x) = 0. \]  

(3.6)

Therefore, putting \( t = T - \tau \) where \( T \) is the maturity, we find the following parabolic equation, a forward Kolmogorov equation:

\[ \frac{\partial u(t, x)}{\partial t} - \frac{1}{2} \sigma^2(t, x) x^2 \frac{\partial^2 u(t, x)}{\partial x^2} - r x \frac{\partial u(t, x)}{\partial x} + ru(t, x) = 0. \]  

(3.7)

Now, we can rewrite the free-boundary problem introduced in the previous section for the present case. Let us denote by:

- \( u(t, x) \) the put option price;
- \( K \) the strike price;
- \( T \) the maturity time;
- \( B(t) \) the exercise boundary.

Setting

\[ L := \frac{1}{2} \sigma^2(t, x) x^2 \frac{\partial^2}{\partial x^2} - r x \frac{\partial}{\partial x} + r, \]  

(3.8)
the free boundary problem becomes, see appendix A in [9]:

\[
\begin{aligned}
& (\partial_t - L)u(t, x) = 0, \quad 0 \leq t \leq T, \ x > B(t), \\
& u(t, x) = K - x, \quad 0 \leq t \leq T, \ x \leq B(t), \\
& \frac{\partial}{\partial x} u(t, B(t)) = -1, \\
& B(0) = K, \\
& u(0, x) = (K - x)^+, \quad x > 0, \\
& \lim_{x \to +\infty} u(t, x) = 0.
\end{aligned}
\]  

(3.9)

We have that the first equation in (3.9) corresponds to equation (3.4), i.e., it is the continuous region, the second equation in (3.9) corresponds to equation (3.3), i.e., it is the exercise region, the third equation in (3.9) corresponds to equation (3.5), i.e., it is the boundary conditions, the fifth equation in (3.9) is the initial condition and, finally, the sixth equation in (3.9) describes how the solution behaves when the value of the stock tends to infinity. Together, it is possible to show (3.9) has a unique solution for the American put price, see [6], [2]. If we put together the first and the second equation in (3.9), then we obtain, see Section 11.2 in [6],

\[
(\partial_t - L)u(t, x) = r K \cdot H(B(t) - x),
\]  

(3.10)

where \( H(x) \) is the Heaviside function defined by:

\[
H(x) := \begin{cases} 
1, & x \geq 0 \\
0, & x < 0 
\end{cases}
\]

We consider now \( t = T - \tau \) where \( T \) is the maturity, \( u(t, x) \) the put option price, where \( x = S_t \) is the stock price as before, and \( \mathcal{D} = \{(x, t) : x \in [0, \infty), t \in [0, T]\} \) the region in which \( u(t, x) \) is defined. In this region the option value \( u(t, x) \) is convex in \( x \) for all \( t \), continuously differentiable in \( t \) for all \( x \) and twice continuously differentiable in \( x \) for all \( t \) a.e., see Appendix in [10]. Let

\[
Z(x, t) := e^{-rt}u(t, x),
\]  

(3.11)

be the discounted put price, then also \( Z(t, x) \) is in \( C^{1,2}([0, T] \times [0, \infty)) \). Therefore, we can apply the Itô-Doeblin formula, see (A.1.4), to obtain:

\[
Z_T = Z_0 + \int_0^T \frac{\partial Z}{\partial x}(t, x) dS_t + \int_0^T \left[ \frac{1}{2} \sigma^2(t, x) x^2 \frac{\partial^2 Z}{\partial x^2}(t, x) + \frac{\partial Z}{\partial t}(t, x) \right] dt.
\]  

(3.12)
CHAPTER 3. THE DYSON-TAYLOR COMMUTATOR METHOD

Using (3.11) we have:

\[
e^{-rT}u(T, x) = u(0, x) + \int_0^T e^{-rt} \frac{\partial u}{\partial x}(t, x) dS_t
+ \int_0^T \left[ e^{-rt} \frac{1}{2} \sigma^2(t, x)x^2 \frac{\partial^2 u}{\partial x^2}(t, x) - re^{-rt}u(t, x) + e^{-rt} \frac{\partial u}{\partial t}(t, x) \right] dt.
\] (3.13)

From equation (1.3), we have that the relation between the stock price \( S_t \) and the discounted stock price \( \tilde{S}_t \) is given by:

\[
e^{-rt} dS_t = d\tilde{S}_t + re^{-rt} S_t dt
= \sigma \tilde{S}_t d\tilde{W}_t + re^{-rt} S_t dt.
\] (3.14)

Therefore, substituting (3.14) in (3.13), we obtain:

\[
e^{-rT}u(T, x) = u(0, x) + \int_0^T \frac{\partial u}{\partial x}(t, x) \sigma \tilde{S}_t d\tilde{W}_t
+ \int_0^T \left[ e^{-rt} (\partial_t + L)u(t, x) \right] dt,
\] (3.15)

and then, using (3.8), we obtain:

\[
e^{-rT}u(T, x) = u(0, x) + \int_0^T \frac{\partial u}{\partial x}(t, x) \sigma \tilde{S}_t d\tilde{W}_t
- \int_0^T \left[ e^{-rt} rK \cdot H(B(t) - x) \right] dt.
\] (3.16)

Since from (3.10) we have that

\[ (\partial_t + L)u(t, x) = -rK \cdot H(B(t) - x), \]

and then:

\[
e^{-rT}u(T, x) = u(0, x) + \int_0^T \frac{\partial u}{\partial x}(t, x) \sigma \tilde{S}_t d\tilde{W}_t
- \int_0^T \left[ e^{-rt} rK \cdot H(B(t) - x) \right] dt.
\] (3.17)

Taking the expectation with refer to the probability measure \( \mathbb{Q} \), under which the discounted stock price \( \tilde{S}_t \) is a martingale, see Section 1.2, we have:

\[
u(T, x) = u_e(T, x) + \int_0^T e^{-rt} E^\mathbb{Q} \left[ rK \cdot H(B(t) - x) \right] dt.
\] (3.18)
where \( u_e(T, x) \) and \( u(T, x) \) denote, respectively, the price of the European and American options at time 0, with maturity \( T \).

Now, recalling from chapter 2 that \( u(t, x) = \int G_t(x, y)f(y)dy \) where the function \( f \) represents the initial data and \( G_t(x, y) \) is the Green’s function for the operator \( \partial_t - L \), we can rewrite (3.18) as, see Appendix in [9]:

\[
\int_{0}^{K} (K - y) \cdot G_t(x, y)dy + rK \int_{0}^{t} \int_{\mathbb{R}^+} H(B(\tau) - y)G_{t-\tau}(x, y)dyd\tau,
\]

that can be rewritten as follows:

\[
u(t, x) = \int_{0}^{K} (K - y) \cdot G_t(x, y)dy + rK \int_{0}^{t} \int_{0}^{B(\tau)} G_{t-\tau}(x, y)dyd\tau. \tag{3.19}\]

In order to solve equation (3.20), we need an other equation, so we can evaluate it using the second equation in (3.9), that we can rewrite as follows:

\[
K - B(t) = \int_{0}^{K} (K - y) \cdot G_t(B(t), y)dy + rK \int_{0}^{t} \int_{0}^{B(\tau)} G_{t-\tau}(B(t), y)dyd\tau. \tag{3.21}\]

Now, in order to evaluate the American put option value, we need first to calculate the boundary \( B(t) \) from equation (3.21), under the initial condition \( B(0) = K \), the fourth equation in (3.9). In order to do this, we exploit the iterative algorithm stated in [9] (see Section 6.4), namely the following one.

**Numerical Algorithm: American Option Pricing**

1. Divide the interval \([0, T]\) in \( n \) subintervals of the same length, and then the nodal points are \( 0 = t_0 < t_1 < t_2 \cdots t_{n-1} < t_n = T \);

2. Notice that \( B(0) = K \) is known. Set \( t = t_1 \) in (3.21), choose a numerical quadrature rule to approximate the integral. And we will get an equation with the unknown \( B(t_1) \). Solve it for \( B(t_1) \) numerically;

3. Assume \( B(t_0), B(t_1), \cdots, B(t_k) \) are known. Estimate the integral at the right hand side of (3.21) by a numerical quadrature at the nodal points \( 0 = t_0, t_1, \cdots, t_k, t_{k+1} \). Solve the resulting equation for \( B(t_{k+1}) \). Eventually, we shall get all values of \( B(i), 0 \leq i \leq n \);

4. Once we have the values \( B(i), 0 \leq i \leq n \), use (3.20) to compute the American put option price by numerical integration.
Then, since we want to use the Dyson-Taylor commutator method introduced in Chapter 2, we will replace the Green’s function in equation (3.21) and in equation (3.20) with its approximation obtained in paragraph (2.1). Later on, we will see how to approximate the Green’s function replacing it with equation (2.47), namely by its approximation at the first order, or exploiting it with its approximation at the second order, as stated by equation (2.48). Obtained numerical results will then compared to show the superiority of the second order approximation, hence according to Theorem 1.2.1 in [9].

Remark 3.3.1. The Dyson-Taylor method is inherently a short-time approximation. For European option, a bootstrap argument using at least the second-order Green’s function extend the method to arbitrary time (see the discussion in [5]). It is not clear how to extend the bootstrapping to the case of an American option, due to presence of the free-boundary.
3.4 The Implicit Euler method applied to American Option pricing

In the previous paragraph we have introduced how to use the Dyson-Taylor commutator method in order to obtain the price of an American Put option. Since in the literature there is no exact solution for the American Option problem, we need to use other methods that already exist and that are already used in literature to find the price of an American option, in order to compare our approach with this methods. So, we briefly introduced one of the comparison methods used in this thesis, the Implicit Euler Method. (see [11], [12]).

In order to do this, before we need to explain the finite difference approximations (see [13]).

3.4.1 Finite-difference Approximations

The main idea is to replace the partial derivatives present in a partial differential equation by an approximation based on the Taylor expansion, see Section 8.2 in [3].

We suppose to have a function of one variable \( u(x) \) with \( x \in \mathbb{R} \), assumed to be smooth. We want to approximate the derivative \( u' \) by the finite difference method, namely, for any \( h \geq 0 \), we can exploit one of the following three solutions:

1. **Backward differences**:
   \[
   u'(x) \approx \frac{u(x) - u(x - h)}{h}
   \]

2. **Forward differences**:
   \[
   u'(x) \approx \frac{u(x + h) - u(x)}{h}
   \]

3. **Centered differences**:
   \[
   u'(x) \approx \frac{u(x + h) - u(x - h)}{2h}
   \]

Considering the Taylor expansion, for the first case we have:
\[
u(x - h) = u(x) + u'(x)(-h) + \frac{1}{2} u''(x)(-h)^2 + \cdots \]
and hence
\[ \frac{u(x) - u(x - h)}{h} = \frac{u(x) - [u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 + \cdots]}{h} = u'(x) - \frac{1}{2}u''(x)h + \cdots = u'(x) + O(h). \]

For the Forward differences we have:
\[ u(x + h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \cdots \]

and hence
\[ \frac{u(x + h) - u(x)}{h} = \frac{[u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \cdots] - u(x)}{h} = u'(x) + \frac{1}{2}u''(x)h + \cdots = u'(x) + O(h). \]

For Centered differences:
\[ \frac{u(x + h) - u(x - h)}{2h} = \frac{[u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \cdots] - [u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + \cdots]}{2h} = u'(x) + \frac{1}{6}u'''(x)h^2 + \cdots = u'(x) + O(h^2). \]

Let us now recall the Implicit Euler method, particularly covering the simplest case of a European Put option. We assume to have a stock price that evolves according to the following stochastic differential equation:
\[ dS_t = \mu S_t dt + \sigma(t, S_t) dW_t. \]

Then, as we have just seen in the previous paragraph, with a change of coordinates \( t = T - \tau \) where \( T \) is the maturity time, that the option value \( u(t, x) \), with \( x = S_t \), satisfies the following differential parabolic equation, see equation (3.7):
\[ \frac{\partial u(t, x)}{\partial t} - \frac{1}{2} \sigma^2(t, x)x^2 \frac{\partial^2 u(t, x)}{\partial x^2} - rx \frac{\partial u(t, x)}{\partial x} + ru(t, x) = 0. \] (3.22)

Let:
\[ L := \frac{1}{2} \sigma^2(t, x)x^2 \frac{\partial^2}{\partial x^2} + rx \frac{\partial}{\partial x} - r, \]
that can be rewritten in a simpler form, in view of next calculations, as follows:

\[ L(t, x) := \frac{1}{2} a(t, x)^2 \partial_x^2 + b(t, x) \partial_x + c(t, x), \]

where \( a(t, x) = \sigma(t, x)x, b(t, x) = rx, c(t, x) = -r \). Then, in a more readable form, setting also \( u := u(t, x) \), equation (3.22) becomes:

\[ \partial_t u = Lu. \] (3.23)

In order to implement the Implicit Euler method, we need to create a space-time grid. We determine it by dividing the interval \([0, T]\), where \( T \) is the maturity time of the option, in \( N \) subintervals of equal length \( dt := T/N \), and dividing the interval (space) \([0, S_{max}]\) in \( M \) subintervals of equal length \( dx := S_{max}/M \). \( S_{max} \) is the upper bound defined so that the end conditions are met, for example for a European Put option we must have that its value in \( S_{max} \) has to be 0 for all this life, see Section 19.8 in [12]. We will call these, respectively, discretization in time and space. This discretization in time and space defines a grid consisting on \((N + 1)(M + 1)\) points. We need this grid in order to calculate the value of option when the time is \(0, dt, 2dt, \cdots T\), and when the stock is \(0, dx, 2dx, \cdots S_{max}\). In the grid, the point \((i, j)\) is the point that corresponds to time \(idt\) and stock \(jdx\). The value of the option in this point is denoted by \(u_{i,j}\).

Now we explain how the Implicit method works, using the finite-difference approximations. We can approximate the partial derivatives as, see Section 19.8 in [12],

\[
\frac{\partial u}{\partial t} \approx \frac{u_{i+1,j} - u_{i,j}}{dt},
\]

\[
\frac{\partial u}{\partial x} \approx \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2dx},
\]

\[
\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}}{(dx)^2}.
\]

Substituting these in (3.23), we obtain:

\[
\frac{u_{i+1,j} - u_{i,j}}{dt} = \frac{1}{2} a_{i+1,j} \left( \frac{u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}}{(dx)^2} \right) + b_{i+1,j} \left( \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2dx} \right) + c_{i+1,j}.
\]
Rearranging the terms, we have that, for the points $u_{i,j}$ of the grid:

$$u_{i,j} = u_{i+1,j-1} \left( -\frac{1}{2} a_{i+1,j}^2 \frac{dt}{dx^2} + \frac{1}{2} b_{i+1,j} \frac{dt}{dx} \right) + u_{i+1,j} \left( 1 + a_{i+1,j}^2 \frac{dt}{dx^2} - c_{i+1,j} dt \right) + u_{i+1,j+1} \left( -\frac{1}{2} a_{i+1,j+1}^2 \frac{dt}{dx^2} - \frac{1}{2} b_{i+1,j+1} \frac{dt}{dx} \right).$$

We also have to take into account that the value of the put option at time $T$ is $\max(K - S_T)$, where $S_T$ is the stock price at time $T$, it is $K$ when the stock price is zero, and it is zero when $S = S_{max}$, see Section 19.8 in [12]. It follows that, to evaluate an American put option, we have to consider also the early exercise. This can be incorporated by checking at each point in the grid if the option value $u_{i,j}$ is lower than the payoff from early exercise $K - jdx$. If this happens, then the value $u_{i,j}$ is replaced by $K - jdx$.

The Implicit Euler scheme is unconditionally stable, i.e., it is not required a relation between $dt$ and $dx$ in order that the scheme could be converge, see Theorem 12.15 in [6]. If we had considered the Explicit Euler scheme, we should impose a stability condition between $dt$ and $dx$ in order to have convergence, for more details see always Theorem 12.15 in [6]. This method is of order one with respect to $t$, namely $O(dt)$, and of order two with respect to $x$, namely $O(dx^2)$, see Section 3.1.3 in [11].

### 3.5 The Monte Carlo method applied to American Option pricing

Another method used to calculate the value of an America option is the Monte Carlo method. Since in the following chapter we will use also this method in order to compare the results given by the algorithm in (3.3), we give a brief explanation of it. Since we are considering local-volatility models in which the dynamics of the underlying asset is given by

$$dS_t = rS_t dt + \sigma(t, S_t) dW_t,$$

in order to use the Monte Carlo method, we first need some realizations of $S_T$. Then, as proposed in Section 12.4.1 of [6], we use the Euler-Maruyama scheme (for high-order schemes see Section 4 of [19]) to simulate the simple paths and then we can use Monte Carlo. In this case the discretization error of the SDE must be added to the error of the Monte Carlo method (see always [6]).

We briefly introduced the Euler-Maruyama scheme, see Section 4.A in [19] and Section 12.2 in [6].
Consider an SDE as equation (A.8):

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0, T]$$

in the same hypothesis of Theorem A.1.5.

We need to divide the interval $[0, T]$ into $N$ intervals of the same length $\delta := \frac{T}{N}$ and we define the discretized process $X_\delta$. Consider, for $t = t_n$:

$$X_{tn+1}^\delta = X_{tn} + b(t_n, X_{tn})\delta + \sigma(t_n, X_{tn})(W_{tn+1} - W_{tn}),$$

and, recursively, we can obtain the discretized process $X_\delta$, starting from the initial point $X_0^\delta = x_0$. We have that the order of strong convergence of this Euler-Maruyama scheme is $\frac{1}{2}$, see Theorem 12.9 in [6]. So, generically, the procedure to price a European option consists of the following step:

1. Simulate $M$ sample paths using the Euler-Maruyama scheme just explained;

2. Evaluate the discounted value of the option on each sample path;

3. Take the average over the values found in step 2.

Since Monte Carlo method has a slow convergence rate (the error is the order of $\frac{1}{\sqrt{M}}$, see Section 12.4 in [6]), we need $M >> 1$. Then, we want to evaluate an American Put option. So we need also the determination of the optimal exercise time. Thus, since it depends on an average over future events, Monte Carlo simulation for an American option has a "Monte Carlo on Monte Carlo" feature that makes it computationally complex. This means that for each simulation of the simple path at time $t_i$ (we recall that the number of simple path are $M$), we need $M$ simple path at time $t_{i+1}$, and then, if we have $N$ time steps, in total we have $M(M^N - 1)/(M - 1)$ simulations. For example, if $M = 1000$ and $N = 5$, we have $10^{15}$ simulations! For this reason, we prefer to use the Least Square Monte Carlo method (see [20]), proposed by Longstaff and Schwartz in [14], for the Longstaff-Schwartz Algorithm we refer also to [17].
Chapter 4

Numerical Implementations

As anticipated in Section 3.3 in this chapter we want to see how the Dyson-Taylor Commutator method performs in the case of American put option pricing. In order to do this, we consider the algorithm proposed in (3.3) used to find the price of an American put option, substituting in equations (3.21) and (3.20) the approximate Green’s function found in Section 2.1. More precisely: equation (2.47) if we want to consider the approximation of the Green’s function only at the first order, or equation (2.48) if we want to consider the more precise approximation (according to Theorem 2.0.1) at the second order. We begin by considering to price an American put option under the Black and Scholes model, i.e., when the underlying asset follows this dynamic:

\[
dS_t = S_t(\mu dt + \sigma dB_t),
\]

where \(\mu\) and \(\sigma\) are two constants and \(B_t\) is the standard Brownian motion, as we have already seen in Section 1.2. We begin from this model since for this case it is possible to calculate the explicit form of the Green’s function, denoted by \(G_t^{BSM}(x, y)\) and given by (see Section 1.3.2 of [9]),

\[
G_t^{BSM}(x, y) = \frac{e^{-rt}}{y \sqrt{2\pi \sigma^2 t}} \cdot e^{-\left(\frac{\ln(x/y) + (r - \frac{\sigma^2}{2}) t}{2 \sigma^2 t}\right)^2},
\]

(4.1)

where \(r\) is the interest rate as we have always seen in Section 1.2. Then, if we use the algorithm proposed in (3.3) using equation (4.1), we can find the exact solution (the only significant error introduced is given by the quadrature rule that we have to use in the algorithm) of an American put option price under the Black and Scholes model, and then we can compare it with the approximations given by the Dyson-Taylor commutator method, i.e., substituting in equations (3.21) and (3.20) the approximate Green’s functions calculated in equations (2.47), (2.48).
CHAPTER 4. BLACK AND SCHOLES MODEL

4.1 Pricing an American put option under the Black and Scholes model

Let us start considering the Black and Scholes model. At first, we compute the solution for an American put price using the algorithm proposed in Section 3.3, using in equations (3.21) and (3.20) the exact Green’s formula for the Black and Scholes model, given by equation (4.1). Since in literature an explicit formula for the American put price does not exist, we consider this solution as the exact solution for the Black and Scholes case, since the only error is given by the numerical quadrature rule used to approximate the integral (choosing a time discretization with time step \( dt = 10^{-3} \), we have an error of order \( O(10^{-6}) \)). We start by choosing the parameters \( K = 15, \sigma = 0.3, r = 0.1, T = 0.1 \), and comparing the exact solution found as we have just said, with the price found using now in (3.21) and (3.20) the approximate Green’s function at the first and second order. We show that the first order approximation is less accurate than the second order approximation (we recall that the global error that we do replacing the exact Green’s function \( G_t \) with the n-th order approximate Green’s function \( G_t^{[n]} \) is of order \( t^{(n+1)/2} \), according to Theorem 2.0.1 for \( n = 2 \), and Theorem 1.2.1 in [9] for \( n \in \mathbb{N} \), see also Section 3.6 in [9]). We plot the exact and the approximate solutions for \( 8 < x < 26 \).

![Comparison between the American put price found by the explicit B-S Green’s function and its approximation at the first order. Parameters: \( K = 15, \sigma = 0.3, r = 0.1, T = 0.1 \).](image)

1We have taken these parameters from literature, see, for example, [5]. We consider the dimensionless maturity time \( T = 1 \) as one year, and then the other parameters are calculated over this period, i.e., the volatility \( \sigma \) in the annual volatility and the interest rate \( r \) is the annual interest rate.
Figure 4.2: Comparison between the American put price found by the explicit B-S Green’s function and its approximation at the second order. Parameters: $K = 15, \sigma = 0.3, r = 0.1, T = 0.1$

| max $|\text{Ex-Ap1}|$ | max $|\text{Ex-Ap2}|$ |
|----------------|----------------|
| $3.727 \cdot 10^{-2}$ | $7.552 \cdot 10^{-3}$ |

Table 4.1: Distance in $l^\infty$ between the exact solution (Ex) and the first order approximation (Ap1) and second order approximation (Ap2) respectively

In the previous table, the distance among the methods is calculated in $l^\infty$ (it is a discrete norm), i.e., we take the maximum of the modulus of the difference between the option values found by the different methods for some values of $x$ (the same for each method). The error in this norm is the maximum error that can be done. Then, as we can seen from table 4.1, our numerical tests show that the second order approximation of the Green’s function is more accurate than the first order approximation, according to Theorem 1.2.1 in [9]. Then, from now on, we use the second order approximation.

Now, we want to compare the accuracy and the efficiency among the Dyson-Taylor commutator method, the Implicit Euler method and the Monte Carlo method, choosing as the benchmark the same exact solution as before. In order to do this, we compare these methods for different maturity times $T = 0.1; 0.2; 0.5$. 
We begin from the Dyson-Taylor commutator method.

Figure 4.3: Comparison between the American put price found by the explicit B-S Green’s function and its approximation at the second order. Parameters: $K = 15, \sigma = 0.3, r = 0.1$, for, starting from the figure at the top left, $T = 0.1, T = 0.2, T = 0.5$.

For the Implicit Euler method we have:

Figure 4.4: Comparison between the American put price found by the explicit B-S Green’s function and the Implicit Euler method (time step $dt = 10^{-3}$, $M = 10000$). Parameters: $K = 15, \sigma = 0.3, r = 0.1$, for, starting from the figure at the top left, $T = 0.1, T = 0.2, T = 0.5$. 
For the Monte Carlo method we have:

![Figure 4.5: Comparison between the American put price found by the explicit B-S Green’s function and the Monte Carlo method (10000 sample paths). Parameters: $K = 15, \sigma = 0.3, r = 0.1$, for, starting from the figure at the top left, $T = 0.1, T = 0.2, T = 0.5$.](image)

From these graphics we can see that the distance is visually very small (eye-ball norm), and for this reason, we tabular the distance in $l^\infty$. Then, in the following table, we report the distance in $l^\infty$ (the maximum distance) between this three methods and the exact solution, for different maturity time $T = 0.1, T = 0.2, T = 0.5$.

| $T$  | $\max |\text{Ex}-\text{Ap2}|$ | $\max |\text{Ex}-\text{EI}|$ | $\max |\text{Ex}-\text{MC}|$ |
|------|---------------------|---------------------|---------------------|
| 0.1  | $7.552 \cdot 10^{-3}$ | $3.277 \cdot 10^{-3}$ | $2.009 \cdot 10^{-2}$ |
| 0.2  | $1.043 \cdot 10^{-2}$ | $3.300 \cdot 10^{-3}$ | $1.912 \cdot 10^{-2}$ |
| 0.5  | $3.235 \cdot 10^{-2}$ | $3.976 \cdot 10^{-3}$ | $2.640 \cdot 10^{-2}$ |

Table 4.2: Distance in $l^\infty$ between the exact solution (Ex) and the second order approximation (Ap2), the Implicit Euler method (EI) and the Monte Carlo method (MC), respectively, for different maturity time $T = 0.1, T = 0.2, T = 0.5$.

From table 4.2, we can see that if $T$ increases, the distance between the Implicit Euler method and the exact solution does not increase, and the same holds for the Monte Carlo method. Instead, if we consider the Dyson-Taylor Commutator method, we can see that the distance increases when the maturity time $T$ increases, according to Theorem 2.0.1. Indeed, the Dyson-Taylor commutator method gives an asymptotic expansion of the Green’s function.
when $T \to 0$ and its accuracy is principally limited to times $T$ relatively small. If time increases, as we see in this table and according to Theorem 2.0.1, the error is bigger. (In Section 4.4 in [9] it is introduced a bootstrap strategy to obtain accurate approximate solutions over bigger times and it is used in European option pricing, as we have already advanced in Remark 3.3.1. It could be a good starting point to apply the same strategy to our case in a future research).

Now, in the following table, we can see the time spent by each method for each maturity time $T = 0.1$, $T = 0.2$, $T = 0.5$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$t_e$ (DT)</th>
<th>$t_e$ (EI)</th>
<th>$t_e$ (MC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>6.036437</td>
<td>23.589263</td>
<td>96.779504</td>
</tr>
<tr>
<td>0.2</td>
<td>15.089274</td>
<td>45.571336</td>
<td>175.108720</td>
</tr>
<tr>
<td>0.5</td>
<td>66.562284</td>
<td>116.281414</td>
<td>406.971220</td>
</tr>
</tbody>
</table>

Table 4.3: Time employed ($t_e$, in seconds) by each method: by the Dyson-Taylor commutator method (DT), by the Implicit Euler method (EI) and by the Monte Carlo method (MC), respectively, for different maturity time $T = 0.1$, $T = 0.2$, $T = 0.5$.

In this table we can see that the Dyson-Taylor commutator method is the more efficient, since the time employed by it is the smallest time for each maturity time $T$. We recall that for the Implicit Euler method we have considered 10001 spatial node, referring to Section 3.4 it is like saying that $M = 10000$, and for the Monte Carlo method we have used 10000 sample paths.

In these previous numerical tests we have chosen a time discretization with $dt = 10^{-2}$ for the Dyson-Taylor commutator method, and a time discretization with $dt = 10^{-3}$ for the Implicit Euler method. We want to show that, if we choose the same $dt = 10^{-2}$ for each method and $T \to 0$, the efficiency between the methods are more or less the same, but the results given by the Implicit Euler method are less precise than those given by the Dyson Taylor commutator method. We plot the results for $T = 0.1$ and $8 < x < 26$. 
Figure 4.6: Comparison among the American put price found by the explicit B-S Green’s function, the Dyson-Taylor commutator method and the Implicit Euler method ($M = 10000$). Parameters: $K = 15, \sigma = 0.3, r = 0.1, T = 0.1$.

Now, in the following tables, we report the distance in $l^\infty$ between this two methods with the same $dt = 10^{-2}$ and the exact solution, and the time employed by each method, for maturity time $T = 0.1$.

| $T=0.1$ | $\max|\text{Ex-DT}|$ | $\max|\text{Ex-EI}|$ |
|---------|----------------|----------------|
|         | $7.552 \cdot 10^{-3}$ | $1.208 \cdot 10^{-2}$ |

Table 4.4: Distance in $l^\infty$ between the exact solution (Ex) and the Dyson-Taylor commutator method (DT), and the Implicit Euler method (EI) respectively, for maturity time $T = 0.1$

<table>
<thead>
<tr>
<th>$T=0.1$</th>
<th>$t_e(\text{DT})$</th>
<th>$t_e(\text{EI})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6.036437</td>
<td>4.235564</td>
</tr>
</tbody>
</table>

Table 4.5: Time employed ($t_e$, in seconds) by each method: by the Dyson-Taylor commutator method (DT), by the Implicit Euler method (EI), for maturity time $T = 0.1$. 
4.2 Pricing an American put option under the CEV model

We now consider numerical tests on the CEV model explained in Remark 1.2.4, choosing $\beta = 2/3$. Unlike the Black and Scholes model, in this case there is not a simple explicit form as equation 4.1 for the Green’s function (we can find some closed-form expression, but are difficult to evaluate, for example see [25], [26] and there are not closed-form formulas for $\beta \neq 2/3$, $0 < \beta < 1$). Then, we can compare the Dyson-Taylor commutator method only with the Implicit Euler method (time step $dt = 10^{-3}$, $M = 10000$) and the Monte Carlo method (10000 sample paths). We choose the same parameters as the previous paragraph, i.e., $K = 15$, $\sigma = 0.3$, $r = 0.1$, and we see the behavior of these methods for different maturity times $T = 0.1, T = 0.2, T = 0.5, T = 0.7$.

We begin plotting the results for $T = 0.1$ and $8 < x < 26$.

![Comparison among the American put price found by Dyson-Taylor commutator method, the Implicit Euler method and the Monte Carlo method. Parameters: $K = 15, \sigma = 0.3, r = 0.1, T = 0.1$](figure4.7.png)
Now we plot the results for $T = 0.2$ and $8 < x < 26$.

Figure 4.8: Comparison among the American put price found by Dyson-Taylor commutator method, the Implicit Euler method and the Monte Carlo method. Parameters: $K = 15, \sigma = 0.3, r = 0.1, T = 0.2$

Now we plot the results for $T = 0.5$ and $8 < x < 26$.

Figure 4.9: Comparison among the American put price found by Dyson-Taylor commutator method, the Implicit Euler method and the Monte Carlo method. Parameters: $K = 15, \sigma = 0.3, r = 0.1, T = 0.5$
Now we plot the results for $T = 0.7$ and $8 < x < 26$.

Figure 4.10: Comparison among the American put price found by Dyson-Taylor commutator method, the Implicit Euler method and the Monte Carlo method. Parameters: $K = 15, \sigma = 0.3, r = 0.1, T = 0.7$

As before, since the distance is visually almost indistinguishable, we report in the following table the distance in $l^\infty$ among this three methods, for different maturity time $T = 0.1, T = 0.2, T = 0.5, T = 0.7$.

| $T$  | $\max |\text{EI-DT}|$  | $\max |\text{MC-DT}|$  | $\max |\text{EI-MC}|$  |
|------|-----------------|-----------------|-----------------|
| $0.1$ | $1.080 \cdot 10^{-2}$ | $1.152 \cdot 10^{-2}$ | $1.986 \cdot 10^{-2}$ |
| $0.2$ | $1.309 \cdot 10^{-2}$ | $8.867 \cdot 10^{-3}$ | $2.0939 \cdot 10^{-2}$ |
| $0.5$ | $3.365 \cdot 10^{-2}$ | $1.729 \cdot 10^{-2}$ | $1.974 \cdot 10^{-2}$ |
| $0.7$ | $5.782 \cdot 10^{-2}$ | $4.161 \cdot 10^{-2}$ | $2.044 \cdot 10^{-2}$ |

Table 4.6: Distance in $l^\infty$ among the Dyson-Taylor commutator method (DT), the Implicit Euler method (EI) and the Monte Carlo method (MC), respectively, for different maturity time $T = 0.1, T = 0.2, T = 0.5, T = 0.7$

From the previous table we can see that the distance between the Implicit Euler method and the Monte Carlo method is of the same order for different maturity time $T$. Instead, the distance between the Dyson Taylor commutator method with refer to the other two methods increases when $T$ increases. This is according to the numerical tests made in the previous paragraph, even if we can not refer to Theorem 2.0.1 since, as we have already seen in Remark 2.0.1, in this case the assumptions of this theorem are not satisfied.
Now, in the following table, we can see the time spent by each method for each maturity time $T = 0.1$, $T = 0.2$, $T = 0.5$, $T = 0.7$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$t_e$ (DT)</th>
<th>$t_e$ (EI)</th>
<th>$t_e$ (MC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>7.092803</td>
<td>23.756920</td>
<td>104.909528</td>
</tr>
<tr>
<td>0.2</td>
<td>17.773944</td>
<td>53.934817</td>
<td>189.658776</td>
</tr>
<tr>
<td>0.5</td>
<td>68.687057</td>
<td>116.438919</td>
<td>433.921906</td>
</tr>
<tr>
<td>0.7</td>
<td>122.445732</td>
<td>170.778401</td>
<td>606.658777</td>
</tr>
</tbody>
</table>

Table 4.7: Time employed ($t_e$, in seconds) by each method: by the Dyson-Taylor commutator method (DT), by the Implicit Euler method (EI) and by the Monte Carlo method (MC), respectively, for different maturity time $T = 0.1$, $T = 0.2$, $T = 0.5$, $T = 0.7$

In this table we can see that the Dyson-Taylor commutator method is the more efficient, since the time employed by it is the smallest time for each maturity time $T$. We recall that for the Implicit Euler method we have considered 10001 spatial node, referring to Section 3.4 it is like saying that $M = 10000$, and for the Monte Carlo method we have used 10000 sample paths, as in the Black and Scholes case.

In these previous numerical tests we have chosen a time discretization with $dt = 10^{-2}$ for the Dyson-Taylor commutator method, and a time discretization with $dt = 10^{-3}$ for the Implicit Euler method. We want to show that, for $T \to 0$, if we choose a smaller $dt$ for each method, in this case $dt = 5 \cdot 10^{-3}$ and $dt = 5 \cdot 10^{-4}$ respectively, and then the two methods are more precise, the distance between them decreases, even if the time employed by the two methods increases. We see this only for the maturity time $T = 0.1$. We begin plotting the results for $T = 0.1$ and $8 < x < 26$. 
CHAPTER 4. CEV MODEL

Figure 4.11: Comparison between the American put price found by Dyson-Taylor commutator method and the Implicit Euler method. Parameters: $K = 15, \sigma = 0.3, r = 0.1, T = 0.1$

In the following tables we report the distance in $l^\infty$ between these two methods with the new $dt$, and the time employed by each one for maturity time $T = 0.1$

| $T=0.1$ | $\max |EI-DT|$ |
|--------|----------------|
|        | $7.053 \cdot 10^{-3}$ |

Table 4.8: Distance in $L^\infty$ between the Dyson-Taylor commutator method (DT), the Implicit Euler method (EI) for maturity time $T = 0.1$

<table>
<thead>
<tr>
<th>$T=0.1$</th>
<th>$t_e(DT)$</th>
<th>$t_e(EI)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>18.664986</td>
<td>47.464576</td>
</tr>
</tbody>
</table>

Table 4.9: Time employed ($t_e$, in seconds) by each method: by the Dyson-Taylor commutator method (DT), by the Implicit Euler method (EI), for maturity time $T = 0.1$.

As we have mentioned earlier, we can see that the distance between the two methods decreases (see $4.6$), but the time employed increases (more or less the double with refer to the previous results given by $4.7$). The method more efficient is always the Dyson-Taylor commutator method, since the time employed is the smallest, even if we don’t know if it converges.

From all these numerical results we can only conjecture that the Dyson-Taylor commutator method for the CEV model converges for $T \to 0$ as in
the previous case of Black and Scholes model, where the hypothesis of theorem 2.0.1 are satisfied, but here we are not supported by theoretical results (see, however, [4] for remarks in the case of local volatility models of the form $\sigma(t,S)S^2 \partial_S^2 - \beta(t,S)S \partial_S - \gamma(t,S)$, with $\sigma, \beta, \gamma$ uniformly bounded).

We give now two other example of how the Dyson-Taylor method behaves for the CEV model, taking as maturity time two extreme $T$. We want to see what happens if the maturity time $T$ tends to zero or when it is $\geq 1$.

We begin from the case when $T = 1$. Recall that we compare the Dyson-Taylor commutator method (time step $dt = 10^{-2}$) with the Implicit Euler method (time step $dt = 10^{-3}$, $M = 10000$) and the Monte Carlo method (10000 sample paths). We choose always the same parameters, i.e., $K = 15, \sigma = 0.3, r = 0.1$. We begin plotting the results for $T = 1$ and $8 < x < 26$.

![Figure 4.12](image.png)

Figure 4.12: Comparison among the American put price found by Dyson-Taylor commutator method, the Implicit Euler method and the Monte Carlo method. Parameters: $K = 15, \sigma = 0.3, r = 0.1, T = 1$

In the following tables we report the distance in $l^\infty$ among these three methods, for maturity time $T = 1$.

<table>
<thead>
<tr>
<th></th>
<th>max</th>
<th>EI-DT</th>
<th>max</th>
<th>MC-DT</th>
<th>max</th>
<th>EI-MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T=1$</td>
<td>1.120 $\cdot 10^{-1}$</td>
<td>9.503 $\cdot 10^{-2}$</td>
<td>1.919 $\cdot 10^{-2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.10: Distance in $l^\infty$ among the Dyson-Taylor commutator method (DT), the Implicit Euler method (EI) and the Monte Carlo method (MC) for maturity time $T = 1$. 

We continue now with the second case, and we choose a maturity time $T = 0.01$. Since the maturity time is so small, we have to use different $dt$ from before. We recall that before we have used $dt = 10^{-3}$ for the Implicit Euler method and a $dt = 10^{-2}$ for the Dyson-Taylor commutator method and for the Monte Carlo method. Now we will use $dt = 10^{-4}$ for the Implicit Euler method and $dt = 10^{-3}$ for the others two methods ($M$ is always 10000 both for the Implicit Euler method and the Monte Carlo method).

We plot the results for $T = 0.01$ and $8 < x < 26$.

![Figure 4.13: Comparison among the American put price found by Dyson-Taylor commutator method, the Implicit Euler method and the Monte Carlo method. Parameters: $K = 15, \sigma = 0.3, r = 0.1, T = 0.01$](image)

Now, in the following table, we can see the distance in $l^\infty$ among this three methods, with maturity time $T = 0.01$.

|      | $\max |EI-DT|$ | $\max |MC-DT|$ | $\max |EI-MC|$ |
|------|---------|---------|---------|
| $T=0.01$ | $3.571 \cdot 10^{-3}$ | $2.057 \cdot 10^{-3}$ | $5.335 \cdot 10^{-3}$ |

Table 4.11: Distance in $l^\infty$ among the Dyson-Taylor commutator method (DT), the Implicit Euler method (EI) and the Monte Carlo method (MC), respectively, for maturity time $T = 0.01$. 
As we have already seen in table 4.6 the distance between the Implicit Euler method and the Monte Carlo method is more or less the same also for these extreme maturity time \( T \). Instead, the distance between the Dyson Taylor commutator method with refer to the other two methods increase when \( T = 1 \) and it decreases when \( T = 0.01 \). It is very small when \( T = 0.01 \) and it is big when \( T = 1 \). As we have already said, we can not refer to Theorem 2.0.1 since the assumptions of this theorem are not satisfied for the CEV model. We can only conjecture from these numerical results that the method converges for \( T \to 0 \), but we are not supported by theoretical results.

**Conclusions**

We can conclude that, if we have to price an American option under models that respects the hypothesis of Theorem 2.0.1 for example the Black and Scholes model as we have seen in Section 4.1, the Dyson-Taylor commutator method is the most efficient among the methods that we have used, but it is accurate only for small value of \( T \), since it is gives an asymptotic expansion of the Green’s function when \( T \to 0 \). In Remark 3.3.1 we have introduced that there exist a bootstrap strategy using at least the second-order Green’s function, that permits to extend the method to arbitrary time. This technique is used in European option pricing, see [5], and it is not yet clear if it can be used in American option pricing, because of the presence of the free-boundary (this can be a good starting point for future research).

In the case of pricing an American put option under the CEV model, instead, as we have already seen in Section 4.2 we can not use Theorem 2.0.1 since the hypothesis are not satisfied. We have only made a numerical study, comparing the results given by the Dyson-Taylor commutator method with the results given by the other two methods considered, the Implicit Euler method and the Monte Carlo method, already known and used in the literature. From these numerical tests we can see that the Dyson commutator method seems to be always the most efficient, but we can only conjecture that it converge for \( T \to 0 \) as the Black and Scholes case, since we are not supported by theoretical results.
Appendix A

Preliminary notions

A.1 Introduction to stochastic calculus

In this paragraph we recall some basic notions on stochastic calculus that are been explained through the present work. The definitions and the results that will be presented are known in the literature, see for example [1] and [2], to which we shall refer in the treatment that follows.

Definition A.1.1. (Probability space). Let $\Omega$ be a nonempty set, and let $\mathcal{A}$ be a $\sigma$-algebra of subsets of $\Omega$. A probability measure $\mathbb{P}$ is a function that, to every set $A \in \mathcal{A}$, assigns a number in $[0,1]$, called the probability of $A$, that are indicated by $\mathbb{P}(A)$ and such that the following hold

(i) $\mathbb{P}(\Omega) = 1$;

(ii) (countable additive) whenever $A_1, A_2, ...$ is a sequence of disjoint sets in $\mathcal{A}$, then

$$\mathbb{P} \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

The triple $(\Omega, \mathcal{A}, \mathbb{P})$ is called a probability space.

Definition A.1.2. (Random variable). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A function $X : \Omega \rightarrow \mathbb{R}$, is called a random variable if, for every Borel subset $B \in \mathbb{R}$, we have that $X^{-1}(B) \in \mathcal{A}$, i.e, $X$ is $\mathcal{A}$-measurable with refer to $\mathbb{P}$.

Definition A.1.3. (Continuous-time stochastic process). Let $(E, \mathcal{E})$ be a measurable space. A continuous-time stochastic process with state space $(E, \mathcal{E})$ is a family $(X_t)_{t \in \mathbb{R}^+}$ of random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $(E, \mathcal{E})$.

Definition A.1.4. (Filtration). A filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is an increasing family $(\mathcal{F}_t)_{t \geq 0}$ of $\sigma$-algebras included in $\mathcal{A}$.
The $\sigma$-algebra $(\mathcal{F}_t)_{t \geq 0}$ represents the information available at time $t$. We say that a process $(X_t)_{t \geq 0}$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$, if, for any $t$, $X_t$ is $\mathcal{F}_t$ measurable.

**Definition A.1.5.** (Brownian motion). A Brownian motion is a real-valued, continuous stochastic process $(X_t)_{t \geq 0}$, with independent and stationary increments, i.e:

- continuity: $\mathbb{P}$ a.s the map $s \mapsto X_s(w)$ is continuous;
- independent increments: if $s \leq t$, $X_t - X_s$ is independent of $\mathcal{F}_s = \sigma(X_u, u \leq s)$;
- stationary increments: if $s \leq t$, $X_t - X_s$ and $X_{t-s} - X_0$ have the same probability law.

**Definition A.1.6.** (Standard Brownian Motion). A Brownian motion is said to be standard if

$$X_0 = 0, \quad \mathbb{P}\text{-a.s., } \mathbb{E}(X_t) = 0, \quad \mathbb{E}(X_t^2) = t.$$ 

From now on, a Brownian motion is assumed to be standard if nothing else is mentioned. In that case, the density function of $X_t$ reads as follow:

$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}} dx$$

where $dx$ is the Lebesgue measure on $\mathbb{R}$.

**Definition A.1.7.** (Brownian motion w.r.t the filtration $(\mathcal{F}_t) = (\mathcal{F}_t)_{t \geq 0}$)

A real-valued, continuous stochastic process is an $(\mathcal{F}_t)$-Brownian motion if it satisfies:

- For any $t \geq 0$, $X_t$ is $(\mathcal{F}_t)$-measurable;
- If $s \leq t$, $X_t - X_s$ is independent of the $\sigma$-algebra $(\mathcal{F}_s)$;
- If $s \leq t$, $X_t - X_s$ and $X_{t-s} - X_0$ have the same law.

**Remark A.1.1.** From the first point of this definition we see that $\sigma(X_u, u \leq t) \subset \mathcal{F}_t$. Therefore it is straightforward that an $\mathcal{F}_t$-Brownian motion is also a Brownian motion with refer to its natural filtration.

**Definition A.1.8.** (Martingale, supermartingale, submartingale) Let us consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t \geq 0}$ on this space. An adapted family $(M_t)_{t \geq 0}$ of integrable random variables, i.e., $\mathbb{E}(|M_t|) < +\infty$, for any $t$ is:
• a martingale if, for any $s \leq t$, $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$;
• a supermartingale if, for any $s \leq t$, $\mathbb{E}(M_t|\mathcal{F}_s) \leq M_s$;
• a submartingale if, for any $s \leq t$, $\mathbb{E}(M_t|\mathcal{F}_s) \geq M_s$;

Remark A.1.2. From definition follows that, if $(M_t)_{t \geq 0}$ is a martingale, then $\mathbb{E}(M_t) = \mathbb{E}(M_0)$ for any $t$.

Remark A.1.3. If $(X_t)_{t \geq 0}$ is a standard $\mathcal{F}_t$-Brownian motion, then:
• $X_t$ is an $\mathcal{F}$-martingale;
• $X_t^2 - t$ is an $\mathcal{F}$-martingale;

A.1.1 Quadratic Variation

Definition A.1.9. (Quadratic Variation). Let $f$ be a real-valued function on $[0, T]$, where $T \in \mathbb{R}^+, T < \infty$. Then its quadratic variation up to time $T$ is defined by

$$[f, f](T) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2 \quad (A.1)$$

where $\Pi = \{t_0, t_1, \ldots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = T$.

Remark A.1.4. Suppose that the function $f$ has a continuous derivative. Then

$$\sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2 = \sum_{j=0}^{n-1} |f'(t_j^*)|^2(t_{j+1} - t_j)^2 \leq ||\Pi|| \cdot \sum_{j=0}^{n-1} |f'(t_j^*)|^2(t_{j+1} - t_j),$$

and thus

$$[f, f](T) \leq \lim_{||\Pi|| \to 0} \left[ ||\Pi|| \cdot \sum_{j=0}^{n-1} |f'(t_j^*)|^2(t_{j+1} - t_j) \right]$$

$$= \lim_{||\Pi|| \to 0} ||\Pi|| \cdot \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|^2(t_{j+1} - t_j)$$

$$= \lim_{||\Pi|| \to 0} ||\Pi|| \cdot \int_0^T |f'(t)|^2dt = 0.$$

For a Brownian motion $W_t$, there is no value of $t$ for which $\frac{d}{dt}W(t)$ is defined (see pag. 102 and figure 3.2.2 in [2]), so Remark A.1.4 can not be applied. So

Theorem A.1.1. Let $(W_t)_{t \geq 0}$ be a Brownian motion. Then $[W, W]_t = t$ for all $t \geq 0$ almost surely.
Proof. See Theorem 3.4.3 in [2].

Remark A.1.5. Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$ (i.e., $0 = t_0 < t_1 < \cdots < t_n = T$). We have just seen that the quadratic variation of Brownian motion is

$$[W, W](T) = \lim_{||\Pi|| \to 0} n^{-1} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = T. \quad (A.2)$$

We can also compute the cross variation of $W(t)$ with $t$ and the quadratic variation of $t$ with itself, which are:

$$\lim_{||\Pi|| \to 0} n^{-1} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))(t_{j+1} - t_j) = 0, \quad (A.3)$$

$$\lim_{||\Pi|| \to 0} n^{-1} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 = 0. \quad (A.4)$$

For (A.2) we informally write

$$dW(t)dW(t) = dt,$$

while for (A.3) and (A.4) we write, respectively,

$$dW(t)dt = 0, \quad dtdt = 0.$$

(for more details see Section 3.4 in [2]).

A.1.2 Construction of the stochastic integral and Itô calculus

Suppose that $(W_t)_{t \geq 0}$ is a standard $\mathcal{F}_t$-Brownian motion defined on a filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}).$$

We want to give a meaning to the expression

$$\int_0^t H_s(\omega)dW_s$$

for a certain class of process $H_s(\omega)$ adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$. We start constructing this stochastic integral for a set of processes called simple processes.

Definition A.1.10. (Simple process). $(H_t)_{0 \leq t \leq T}$ is called a simple process if it can be written as

$$H_t(\omega) = \sum_{i=1}^N \phi_i(\omega)1_{(t_{i-1}, t_i]}(t)$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ and $\phi_i$ is $\mathcal{F}_{t_{i-1}}$-measurable and bounded function.
Definition A.1.11. (Stochastic integral for simple processes). Let \((H_t)_{0 \leq t \leq T}\) be a simple process. The stochastic integral for this process with respect to a Brownian motion \(W_t\) is defined as:
\[
\int_0^T H_t(\omega) dW_t(\omega) := \sum_{i=0}^{N-1} \phi_i(\omega) [W(t_{i+1}) - W(t_i)].
\]

Let us extend the concept outlined in Definition \((A.1.11)\) to the following larger class of adapted processes \(\mathcal{H}\):
\[
\mathcal{H} = \{(H_t)_{0 \leq t \leq T}, (\mathcal{F}_t)_{t \geq 0}\}-adapted process, \mathbb{E} \left( \int_0^T H_s^2 ds \right) < +\infty \}
\]
We shall use the fact that if \((H_s)_{s \leq T}\) is in \(\mathcal{H}\), then there exists a sequence \((H^n_s)_{s \leq T}\) of simple processes such that
\[
\lim_{n \to +\infty} \mathbb{E} \left( \int_0^T |H_s - H^n_s|^2 \right) = 0.
\]
i.e. the simple processes are dense in the space of square integrable processes (see pag. 133-134 in [2]).

Then the Itô integral for \(H_s\) is defined as, see Section 4.3 in [2],
\[
\int_0^T H_s dW_s = \lim_{n \to +\infty} \int_0^T H^n_s dW_s
\]

**Theorem A.1.2. (Itô Isometry)**

Let \((H_t)_{0 \leq t \leq T} \in \mathcal{H}\), then we have
\[
\mathbb{E} \left[ \left( \int_0^t H_s dW_s \right)^2 \right] = \mathbb{E} \left[ \int_0^t H_s^2 ds \right]
\]

**Theorem A.1.3.** Let \((H_t)_{0 \leq t \leq T} \in \mathcal{H}\), then
\[
\left( \int_0^t H_s dW_s \right)_{0 \leq t \leq T}
\]
is a continuous \(\mathcal{F}_t\)-martingale.

**Definition A.1.12. (Itô process).** Let \((\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space and \((W_t)_{t \geq 0}\) an \(\mathcal{F}_t\) Brownian motion. \((X_t)_{0 \leq t \leq T}\) is an \(\mathbb{R}\)-valued Itô process if it can be written as
\[
\mathbb{P} \text{ a.s. } \forall t \leq T \quad X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s, \quad (A.6)
\]
where
• $X_0$ is $\mathcal{F}_0$ measurable;
• $(K_t)_{0 \leq t \leq T}$ and $(H_t)_{0 \leq t \leq T}$ are $\mathcal{F}_t$-adapted process;
• $\int_0^T |K_s| \, ds < +\infty$ $\mathbb{P}$ a.s.
• $\int_0^T |H_s|^2 \, ds < +\infty$ $\mathbb{P}$ a.s.

Lemma A.1.1. The quadratic variation of the Itô process (A.6) is

$$[X, X]_t = \int_0^t H_s^2 \, ds \quad \text{(A.7)}$$

Theorem A.1.4. (Itô-Doeblin formula for an Itô process)
Let $(X_t)_{0 \leq t \leq T}$ be an Itô process, namely

$$X_t = X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s,$$

and $f$ be a twice continuously differentiable function. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \, dX_s + \frac{1}{2} \int_0^t f''(X_s) \, d[X, X]_s$$

where $[X, X]_t$ is the quadratic variation given by (A.7), and

$$\int_0^t f'(X_s) \, dX_s = \int_0^t f'(X_s) K_s \, ds + \int_0^t f'(X_s) H_s \, dW_s.$$  

Likewise, if $(t, x) \to f(t, x)$ is a function that is twice differentiable with respect to $x$ and once with respect to $t$, and if these partial derivatives are continuous with respect to $(t, x)$ (i.e., $f$ is a function of class $C^{1,2}$), the Itô-Doeblin formula becomes

$$f(t, X_t) = f(0, X_0) + \int_0^t f'_x(s, X_s) \, ds + \int_0^t f'_x(s, X_s) \, dX_s + \frac{1}{2} \int_0^t f''_{xx}(s, X_s) \, d[X, X]_s.$$  

[See Theorem 1.1.4 in [1]].

Proof. See Theorem 3.3 in [18].

A.1.3 Stochastic differential equations
Let us consider the following equation given in integral form:

$$X_t = x_0 + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s, \quad \text{(A.8)}$$

which is called stochastic differential equation (SDE) whose solution is called a diffusion.
Definition A.1.13. (Solution of a SDE). We consider a probability space \((\Omega, \mathcal{F}, P)\) equipped with a filtration \((\mathcal{F}_t)_{t \geq 0}\). We also have a function \(b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\), \(\sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\), an \(\mathcal{F}_0\)-measurable random variable \(x_0\) and finally a standard \(\mathcal{F}_t\)-Brownian motion \((W_t)_{t \geq 0}\). A solution of equation \((A.8)\) is an \(\mathcal{F}_t\)-adapted continuous stochastic process \((X_t)_{t \geq 0}\) that satisfies:

1. For any \(t \geq 0\), the integrals \(\int_0^t b(s, X_s)ds\) and \(\int_0^t \sigma(s, X_s)dW_s\) exist:
   \[\int_0^t |b(s, X_s)|ds < +\infty \text{ and } \int_0^t |\sigma(s, X_s)|^2 ds < +\infty \text{ P a.s.}\]

2. \((X_t)_{t \geq 0}\) satisfies \((A.8)\), i.e.
   \[
   \forall t \geq 0 \text{ P a.s. } X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s
   \]

Remark A.1.6. We can write \((A.8)\) in differential form as
\[
\begin{cases}
   dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\
   X_0 = x_0
\end{cases}
\]

Theorem A.1.5. (Existence and Uniqueness of a solution of \((A.8)\), see Theorem 3.5.3 in [1])

If \(b\) and \(\sigma\) (defined as in \(A.8)\) are continuous functions, and if there exists a constant \(K < +\infty\) such that:

1. \(|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K|x - y|

2. \(|b(t, x)| + |\sigma(t, x)| \leq K(1 + |x|)

3. \(\mathbb{E}(x_0^2) < +\infty\)

Then, for any \(T \geq 0\), \((A.8)\) admits a unique solution in the interval \([0, T]\). Moreover, this solution \((X_s)_{0 \leq s \leq T}\) satisfies
\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} |X_s|^2 \right) < +\infty.
\]

The uniqueness of the solution means that if \((X_t)_{0 \leq t \leq T}\) and \((Y_t)_{0 \leq t \leq T}\) are two solutions of \((A.8)\), then \(\mathbb{P}\) a.s. \(\forall 0 \leq t \leq T, X_t = Y_t\)

Proof. See Theorem 3.5.3 in [1]. \(\square\)
A.1.4 Markov property

**Definition A.1.14.** (Markov process). Let \((\Omega, \mathcal{A}, \mathbb{P})\) a probability space equipped with a filtration \((\mathcal{F}_t)_{t \geq 0}\). An \((\mathcal{F}_t)\)-adapted process \((X_t)_{t \geq 0}\) is a Markov process if, for any Borel set \(B \in \mathcal{B}(\mathbb{R}^n)\) and for any \(s, t \in \mathbb{R}^+\) such that \(s \leq t\), we have

\[
\mathbb{P}(X_t \in B | \mathcal{F}_s) = \mathbb{P}(X_t \in B | X_s).
\]

Or, in an equivalent formulation, for any bounded Borel function \(f\) and for any \(s \leq t\), we have

\[
\mathbb{E}(f(X_t) | \mathcal{F}_s) = \mathbb{E}(f(X_t) | X_s).
\]

The meaning of the Markov property is that the future behavior of \((X_t)_{t \geq 0}\) after \(t\) depends only on the present value \(X_t\) and it is not influenced by the whole history before \(t\), i.e. the process is memoryless.

We are going to state this property for a solution of \((A.8)\). We denote by \((X_t^{s,x}, t \geq s)\) the solution of equation \((A.8)\) starting from \(x\) at time \(s\) and by \(X^x = X^{0,x}\) the solution starting from \(x\) at time 0. For \(t \geq s\), \(X_t^{s,x}\) satisfies

\[
X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x}) du + \int_s^t \sigma(u, X_u^{s,x}) dW_u.
\]

**Theorem A.1.6.** Let \((X_t)_{t \geq 0}\) be a solution of \((A.8)\). It is a Markov process w.r.t the filtration \((\mathcal{F}_t)_{t \geq 0}\). Moreover, for any bounded Borel function \(f\), we have

\[
\mathbb{P} \text{ a.s. } \mathbb{E}(f(X_t) | \mathcal{F}_s) = \mathbb{E}(f(X_t^{s,x}) | x = X_s).
\]

A.1.5 Change of probability and Girsanov theorem

**Definition A.1.15.** Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space. A probability measure \(Q\) on \((\Omega, \mathcal{A})\) is absolutely continuous with respect to \(\mathbb{P}\) if and only if

\[
\forall A \in \mathcal{A}, \quad \mathbb{P}(A) = 0 \Rightarrow Q(A) = 0.
\]

**Theorem A.1.7.** A probability measure \(Q\) is absolutely continuous with respect to \(\mathbb{P}\) if and only if there exists a non-negative random variable \(Z\) on \((\Omega, \mathcal{A})\) such that

\[
\forall A \in \mathcal{A}, \quad Q(A) = \int_A Z(\omega) d\mathbb{P}(\omega).
\]

\(Z\) is called the density of \(Q\) with respect to \(\mathbb{P}\) and denoted by \(dQ/d\mathbb{P}\).

The probabilities \(\mathbb{P}\) and \(Q\) are equivalent if each of them is absolutely continuous with respect to the other. Note that if \(Q\) is absolutely continuous with respect to \(\mathbb{P}\), with density \(Z\), then \(\mathbb{P}\) and \(Q\) are equivalent if and only if \(\mathbb{P}(Z > 0) = 1\).
Theorem A.1.8. (*Girsanov theorem*)

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) be a filtered probability space and \((B_t)_{0 \leq t \leq T}\) an \((\mathcal{F}_t)\)-standard Brownian motion. Let \((\theta_t)_{0 \leq t \leq T}\) be an adapted process satisfying \(\int_0^T \theta_s^2 \, ds < \infty\) a.s. and such that the process \((L_t)_{0 \leq t \leq T}\) defined by

\[
L_t = \exp \left( - \int_0^t \theta_s \, dB_s - \frac{1}{2} \int_0^t \theta_s^2 \, ds \right)
\]

is a martingale. Then, under the probability \(\mathbb{P}(L)\) with density \(L_T\) with respect to \(\mathbb{P}\), the process \((W_t)_{0 \leq t \leq T}\) defined by \(W_t = B_t + \int_0^t \theta_s \, ds\) is an \((\mathcal{F}_t)\)-standard Brownian motion. [see Theorem 4.2.2 in [1]].

*Proof.* See Theorem 5.1 in [18]. \(\square\)

**Remark A.1.7. (Novikov's criterion)**

A sufficient condition for \((L_t)_{0 \leq t \leq T}\) to be a martingale is that the following inequality holds

\[
\mathbb{E} \left( \exp \left( \frac{1}{2} \int_0^T \theta_t^2 \right) \right) < \infty.
\]

Theorem A.1.9. (*Feynman-Kac Theorem*)

Consider the stochastic differential equation

\[
dX_t = b(t, X_t) \, dt + \sigma(t, X_t) \, dW_t.
\]

Let \(h(y)\) be a Borel-measurable function. Let \(t \in [0, T]\) be given, \(T > 0\). Define the function

\[
g(t, x) = \mathbb{E}^{t,x} h(X_T).
\]

(We assume that \(\mathbb{E}^{t,x} |h(X_T)| < \infty\) for all \(t, x\).) Then \(g(t, x)\) satisfies the following partial differential equation

\[
\frac{\partial g}{\partial t}(t, x) + b(t, x) \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 g}{\partial x^2}(t, x) = 0,
\]

and the terminal condition

\[
g(T, x) = h(x) \quad \forall x.
\]

[see Theorem 6.4.1 in [2]].

*Proof.* See Proof of Theorem 6.4.1 in [2]. \(\square\)
Bibliography


