CONTINUED FRACTIONS WITH THREE LIMIT POINTS

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1. Introduction

The research described in this paper was motivated by an enigmatic entry in Ramanujan's lost notebook [11, p. 45] in which he claimed, in an unorthodox fashion, that a certain q-continued fraction possesses three limit points. More precisely, he claimed that as n tends to ∞ in the three residue classes modulo 3, the nth partial quotients tend, respectively, to three distinct limits, which he explicitly gives. We think that there is no other example of this kind in the literature, and so we investigated the possibility of further analytic continued fractions having three distinct limit points. The purpose of this paper is to prove Ramanujan's elusive entry, to prove a general theorem giving a class of continued fractions with three limit points, and to explicitly give further examples.

To relate Ramanujan's entry, we first introduce the customary notation

$$(a)_0 := (a;q)_0 := 1, \qquad (a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad \text{if } n \geq 1,$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \qquad |q|<1.$$

Also, set $\omega = e^{2\pi i/3}$. Then, except for the simplification of notation, Ramanujan [11, p. 45] claimed that, for $|q|<1$,

$$\lim_{n\to\infty}\left(\frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^n+a}\right) = -\omega^2\left(\frac{\Omega-\omega^{n+1}}{\Omega-\omega^{n-1}}\right)\cdot\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}, \tag{1.1}$$

where

$$\Omega := \frac{1-a\omega^2}{1-a\omega}\,\frac{(\omega^2 q;q)_\infty}{(\omega q;q)_\infty}. \tag{1.2}$$

After (1.1), Ramanujan appended the note, "Numerators and Denominators can be equated separately."

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Of course, because of the appearance of the limiting variable n on the right side of (1.1), Ramanujan's claim is meaningless as it stands. The authors conjectured that Ramanujan was indicating that there are three distinct limits depending on the congruence class modulo 3 in which $n \to \infty$. To confirm our instincts, we calculated the q-expansions of both sides of (1.1) on MAPLE for $1 \leq n \leq 25$, and found that in each case, the two sides agreed up to $O(q^n)$. In the note after (1.2), Ramanujan is evidently indicating that the limits can be obtained by determining separately the limits of both the partial numerators and denominators.

Ramanujan's claim is very interesting for several reasons.

First, if $a = 0$, the left side of (1.1) is a continued fraction which diverges. We shall prove that the partial quotients tend to the required limits if n is restricted to any one of the three residue classes modulo 3. This is in contrast to the classical result from the general theory of continued fractions, which asserts that if all the elements of a divergent continued fraction are positive, then the even and odd approximants approach distinct limits [9, pp. 96–97].

Second, if $a \neq 0$, we prove that the continued fraction in (1.1) converges "generally" in the sense that when n is confined to any one of the three residue classes modulo 3, the limit of the left side indeed exists and is equal to that claimed on the right side of (1.1) in each of the three cases. The concept of general convergence is due to L. Jacobsen [8] in 1986. See also her book with H. Waadeland [9, pp. 41–44]. For some results of Ramanujan of a different kind on general convergence, see a paper by S.–S. Huang [7]. Thus, we have one further example where Ramanujan had a fundamental concept long ahead of his time, before anyone else ever thought of it.

Third, in his notebooks [10, Vol. 2, p. 290], for $|q| < 1$, Ramanujan offered the continued fraction

$$\begin{aligned}\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} &= \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \cdots \\ &= \frac{1}{1} - \frac{1}{q^{-1}+1} - \frac{1}{q^{-2}+1} - \frac{1}{q^{-3}+1} - \cdots.\end{aligned} \tag{1.3}$$

Thus, when $a = 0$, the continued fraction on the left side of (1.1) is the same as the continued fraction on the far right side of (1.3), but with q replaced by $1/q$. Observe that, remarkably, $(q^2;q^3)_\infty/(q;q^3)_\infty$ also appears in the three limits on the right side of (1.1). In this sense, Ramanujan's result (1.1) is analogous to his theorem on the divergence of the Rogers–Ramanujan continued fraction found on pages 374 and 382 in his third notebook [10], which was first proved by Andrews, Berndt, Jacobsen, and Lamphere [3], [5, p. 30, Entry 11]. In the latter result, Ramanujan explicitly determines the limits of the even and odd indexed approximants of the divergent Rogers–Ramanujan continued fraction for $|q| > 1$ and shows that these limits can be expessed in terms of the Rogers–Ramanujan continued fraction itself, *but at different arguments.*

Define

$$P_0(a) = 0, \qquad P_1(a) = 1, \qquad Q_0(a) = 1, \qquad Q_1(a) = 1, \tag{1.4}$$

and, for $N \geq 2$, set

$$\frac{P_N(a)}{Q_N(a)} = \frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^{N-1}+a}. \tag{1.5}$$

From the general theory of continued fractions [9, p. 9, eq. (1.2.9)], for $N \geq 2$, the partial numerators $P_N(0)$ and $Q_N(0)$ satisfy the recurrence relations

$$P_N(0) = (1+q^{N-1})P_{N-1}(0) - P_{N-2}(0) \quad \text{and} \quad Q_N(0) = (1+q^{N-1})Q_{N-1}(0) - Q_{N-2}(0), \tag{1.6}$$

where $P_0(0), P_1(0), Q_0(0)$, and $Q_1(0)$ are defined by (1.4). After proving (1.1) in Section 2, we examine in Section 3 sequences $\{x_n\}$ satisfying the more general recurrence relation,

$$x_n = (1+a_{n-1})x_{n-1} - x_{n-2}, \qquad n \geq 2, \qquad \text{with } x_0 = v, \quad x_1 = u. \tag{1.7}$$

For a large class of such sequences we show that they have exactly six limit points. We then examine quotients of such sequences and prove that these quotient sequences have exactly three limit points. In Section 4, we apply the theory developed in Section 3 to continued fractions with numerators and denominators satisfying the recurrence relation (1.7) and initial conditions like those in (1.4), and prove that the continued fractions have three limit points. Thus, Ramanujan's example (1.1) is not an isolated one. We conclude Section 4 with two examples to illustrate our theory, one of which we briefly describe here. Since neither example is a q-continued fraction, we use a more conventional letter z to denote a complex variable. If $|z| < 1$, then

$$\frac{1}{1} - \frac{1}{(1-z)^{-1}} - \frac{1}{1-z^2} - \frac{1}{(1-z)^{-1}-z^2} - \frac{1}{(1-z^2)^{-1}} - \frac{1}{1-z^4} - \frac{1}{(1-z^2)^{-1}-z^3}$$
$$- \cdots - \frac{1}{(1-z^n)^{-1}} - \frac{1}{1-z^{2n}} - \frac{1}{(1-z^n)^{-1}-z^{n+1}} - \cdots$$

has three limit points, and they are given by

$$L_0 := \frac{z}{z^2+2z-1}, \qquad L_1 := \frac{1}{1+z}, \qquad L_2 := \frac{1}{2z}.$$

Before recording his claim (1.1), at the top of page 45 in [11], Ramanujan states the special case when $a = \omega$. In Section 5, we use the Bauer–Muir transformation to give a shorter and completely different proof of this special case of (1.1).

2. A Proof of Ramanujan's Formula (1.1)

Throughout the paper, $|q| < 1$.

To prove (1.1), our first task will be to derive explicit formulas for $P_N(0)$ and $Q_N(0)$. To do so, we need to recall the definition of the Gaussian polynomials and two versions of the q-binomial theorem [1, pp. 35–36].

Lemma 2.1. *If* $\begin{bmatrix} n \\ m \end{bmatrix}$ *denotes the Gaussian polynomial defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix} := \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}, & \text{if } 0 \leq m \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

then,

$$(z;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}, \tag{2.1}$$

$$\frac{1}{(z;q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j. \tag{2.2}$$

Lemma 2.2. *Let* $N-1 = 3v + \epsilon$, *where* $\epsilon = 0, \pm 1$. *Then*

$$(-1)^v P_N(0) = \sum_{\substack{n,r=0 \\ n+r \equiv \epsilon \pmod 3}}^{\infty} (-1)^{(\epsilon-n-r)/3} q^{n(n+1)/2+r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} \begin{bmatrix} n + \frac{N-1-(n+r)}{3} \\ n \end{bmatrix}_{q^3}. \tag{2.3}$$

Proof. Recall from (1.6) and (1.4) that $P_N(0)$ satisfies the recurrence relation,

$$P_N(0) = (1+q^{N-1})P_{N-1}(0) - P_{N-2}(0), \qquad N \geq 2, \tag{2.4}$$

and the initial conditions $P_0(0) = 0$ and $P_1(0) = 1$.

Define

$$F(t) := \sum_{N=1}^{\infty} P_N(0) t^N.$$

Multiplying the recurrence relation (2.4) by t^N and summing over $N \geq 2$, we obtain

$$F(t) - t = tF(t) + tF(tq) - t^2 F(t).$$

So,

$$F(t) = \frac{t}{1-t+t^2} + \frac{t}{1-t+t^2} F(tq).$$

Iterating and noting that $F(0) = 0$, we find that, by (2.1) and (2.2),

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} \frac{t^{n+1} q^{n(n+1)/2}}{\prod_{j=0}^{n}(1 - tq^j + t^2 q^{2j})} \\ &= \sum_{n=0}^{\infty} t^{n+1} q^{n(n+1)/2} \frac{(-t;q)_{n+1}}{(-t^3;q^3)_{n+1}} \\ &= \sum_{n,r,s=0}^{\infty} (-1)^s t^{n+1+r+3s} q^{n(n+1)/2+r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} \begin{bmatrix} n+s \\ s \end{bmatrix}_{q^3}. \end{aligned}$$

Now we choose the terms t^N by setting $s=(N-1-n-r)/3$. Hence, equating the coefficients of t^N on both sides, we find that

$$P_N(0)=\sum_{\substack{n,r=0\\ n+r\equiv N-1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{(N-1-n-r)/3}q^{n(n+1)/2+r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\begin{bmatrix}n+\frac{N-1-(n+r)}{3}\\ n\end{bmatrix}_{q^3},$$

or, with $N-1=3v+\epsilon$,

$$(-1)^vP_N(0)=\sum_{\substack{n,r=0\\ n+r\equiv \epsilon\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{(\epsilon-n-r)/3}q^{n(n+1)/2+r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\begin{bmatrix}n+\frac{N-1-(n+r)}{3}\\ n\end{bmatrix}_{q^3},$$

as required. □

Lemma 2.3. *Let $N-1=3v+\epsilon$, where $\epsilon=0,\pm1$. Then*

$$\begin{aligned}(-1)^vQ_N(0)=&\sum_{\substack{n,r=0\\ n+r\equiv \epsilon\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{(\epsilon-n-r)/3}q^{n(n+1)/2+r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\begin{bmatrix}n+\frac{N-1-(n+r)}{3}\\ n\end{bmatrix}_{q^3}\\ &-\sum_{\substack{n,r=0\\ n+r\equiv \epsilon-1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{(\epsilon-1-n-r)/3}q^{n(n+3)/2+r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\begin{bmatrix}n+\frac{N-2-(n+r)}{3}\\ n\end{bmatrix}_{q^3}.\end{aligned}\tag{2.5}$$

Proof. Recall from (1.6) and (1.4) that $Q_N(0)$ satisfies the recurrence relation,

$$Q_N(0)=(1+q^{N-1})Q_{N-1}(0)-Q_{N-2}(0),\qquad N\geq 2,\tag{2.6}$$

and the initial conditions $Q_0(0)=1$ and $Q_1(0)=1$.

Define

$$G(t):=\sum_{N=1}^{\infty}Q_N(0)t^N.$$

Multiplying the recurrence relation (2.6) by t^N and summing over $N\geq 2$, we obtain

$$G(t)-t=tG(t)+tG(tq)-t^2G(t)-t^2.$$

So,

$$G(t)=\frac{t-t^2}{1-t+t^2}+\frac{t}{1-t+t^2}G(tq).$$

Iterating and noting that $G(0)=0$, we arrive at, by (2.1) and (2.2),

$$\begin{aligned}G(t)&=\sum_{n=0}^{\infty}\frac{t^{n+1}(1-tq^n)q^{n(n+1)/2}}{\prod_{j=0}^{n}(1-tq^j+t^2q^{2j})}\\ &=\sum_{n=0}^{\infty}t^{n+1}(1-tq^n)q^{n(n+1)/2}\frac{(-t;q)_{n+1}}{(-t^3;q^3)_{n+1}}\\ &=\sum_{n,r,s=0}^{\infty}(-1)^st^{n+1+r+3s}(1-tq^n)q^{n(n+1)/2+r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\begin{bmatrix}n+s\\ s\end{bmatrix}_{q^3}.\end{aligned}$$

Separating the sum above into two parts, we set $s = (N-1-n-r)/3$ and $s = (N-2-n-r)/3$, respectively, in the two sums. Hence, equating coefficients of t^N on both sides, we find that

$$\begin{aligned}Q_N(0) = &\sum_{\substack{n,r=0\\ n+r\equiv N-1\,(\mathrm{mod}\,3)}}^{\infty} (-1)^{(N-1-n-r)/3} q^{n(n+1)/2+r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} \begin{bmatrix} n+\frac{N-1-(n+r)}{3} \\ n \end{bmatrix}_{q^3} \\ &- \sum_{\substack{n,r=0\\ n+r\equiv N-2\,(\mathrm{mod}\,3)}}^{\infty} (-1)^{(N-2-n-r)/3} q^{n(n+1)/2+n+r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} \begin{bmatrix} n+\frac{N-2-(n+r)}{3} \\ n \end{bmatrix}_{q^3}.\end{aligned}$$

If $N-1 = 3v+\epsilon$, then,

$$\begin{aligned}(-1)^v Q_N(0) = &\sum_{\substack{n,r=0\\ n+r\equiv \epsilon\,(\mathrm{mod}\,3)}}^{\infty} (-1)^{(\epsilon-n-r)/3} q^{n(n+1)/2+r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} \begin{bmatrix} n+\frac{N-1-(n+r)}{3} \\ n \end{bmatrix}_{q^3} \\ &- \sum_{\substack{n,r=0\\ n+r\equiv \epsilon-1\,(\mathrm{mod}\,3)}}^{\infty} (-1)^{(\epsilon-1-n-r)/3} q^{n(n+1)/2+n+r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} \begin{bmatrix} n+\frac{N-2-(n+r)}{3} \\ n \end{bmatrix}_{q^3},\end{aligned}$$

as required. □

The previous two lemmas are actually special cases of a theorem due to M. D. Hirschhorn [6], who used a more difficult proof.

To calculate the limits of $P_N(0)$ and $Q_N(0)$ as $N \to \infty$ in the three residue classes modulo 3, we need the following result from Ramanujan's lost notebook, which was first proved by Andrews [2].

Lemma 2.4. *Let* $\omega = e^{2\pi i/3}$. *Then*

$$\sum_{n=0}^{\infty} \frac{(-\omega)^n q^{n(n+1)/2} (\omega q)_n}{(q^3;q^3)_n} = (\omega q)_\infty (q^2;q^3)_\infty.$$

Note that, by conjugation, Lemma 2.4 also holds if ω is replaced by ω^2.

Lemma 2.5. *Let* $N-1 = 3v+\epsilon$, *where* $\epsilon = 0, \pm 1$. *Then*

$$\lim_{v\to\infty} (-1)^v P_N(0) = \frac{1}{3}(-\omega)^\epsilon (1-\omega^2)\left(\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right)(\omega q)_\infty (q^2;q^3)_\infty. \tag{2.7}$$

Proof. Let $N \to \infty$ through values such that $N-1 \equiv \epsilon \,(\mathrm{mod}\, 3)$. Then, from (2.3),

$$\begin{aligned}\lim_{v\to\infty} (-1)^v P_N(0) &= \sum_{\substack{n,r=0\\ n+r\equiv \epsilon\,(\mathrm{mod}\,3)}}^{\infty} (-1)^{(\epsilon-n-r)/3} q^{n(n+1)/2+r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} \frac{1}{(q^3;q^3)_n} \\ &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q^3;q^3)_n} \sum_{\substack{r=0\\ r\equiv \epsilon-n\,(\mathrm{mod}\,3)}}^{\infty} q^{r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} \rho^{\epsilon-n-r},\end{aligned}$$

where $\rho = e^{\pi i/3}$. Recall that $\omega = \rho^2$. Using the elementary fact,

$$\frac{1+\omega^a+\bar{\omega}^a}{3} = \begin{cases} 1, & \text{if } a \equiv 0 \pmod 3, \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

we find, by (2.1), Lemma 2.4, and Lemma 2.4 with ω replaced by ω^2, that

$$\begin{aligned}
&\lim_{v\to\infty}(-1)^v P_N(0) \\
=&\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^3;q^3)_n}\sum_{r=0}^{\infty}q^{r(r-1)/2}\begin{bmatrix} n+1 \\ r\end{bmatrix}\rho^{\epsilon-n-r}\frac{1+\omega^{\epsilon-n-r}+\bar{\omega}^{\epsilon-n-r}}{3} \\
=&\frac{1}{3}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^3;q^3)_n}\left\{\rho^{\epsilon-n}(-\bar{\rho};q)_{n+1}+(-1)^{\epsilon-n}(1;q)_{n+1}+\rho^{n-\epsilon}(-\rho;q)_{n+1}\right\} \\
=&\frac{1}{3}(-\omega^2)^{\epsilon}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega)^n}{(q^3;q^3)_n}(\omega;q)_{n+1}+\frac{1}{3}(-\omega)^{\epsilon}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega^2)^n}{(q^3;q^3)_n}(\omega^2;q)_{n+1} \\
=&\frac{1}{3}(-\omega^2)^{\epsilon}(1-\omega)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega)^n}{(q^3;q^3)_n}(\omega q;q)_n \\
&+\frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega^2)^n}{(q^3;q^3)_n}(\omega^2 q;q)_n \\
=&\frac{1}{3}(-\omega^2)^{\epsilon}(1-\omega)(\omega q)_\infty(q^2;q^3)_\infty+\frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)(\omega^2 q)_\infty(q^2;q^3)_\infty \\
=&\frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty}-\omega^{\epsilon+1}\right\}(\omega q)_\infty(q^2;q^3)_\infty.
\end{aligned}$$

□

To establish the corresponding lemma for $Q_N(0)$, we need an analogue of Lemma 2.4, which was established in [4, Lemma 2.3].

Lemma 2.6. *Let* $\omega = e^{2\pi i/3}$. *Then*

$$-\frac{\omega}{(\omega q)_\infty}\left\{\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n}+\omega\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2+n}}{(q)_n(\omega^2 q)_n}\right\} = (q;q^3)_\infty. \tag{2.9}$$

Note that, by conjugation, (2.9) is also valid with ω replaced by ω^2.

Lemma 2.7. *Let* $N-1 = 3v+\epsilon$, *where* $\epsilon = 0, \pm 1$. *Then*

$$\lim_{v\to\infty}(-1)^v Q_N(0) = \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left(\frac{(\omega^2 q)_\infty}{(\omega q)_\infty}-\omega^{\epsilon-1}\right)(\omega q)_\infty(q;q^3)_\infty. \tag{2.10}$$

Proof. Since the details are similar to those in the proof of Lemma 2.5, we suppress some of them.

Let $N\to\infty$ through values such that $N-1\equiv\epsilon\,(\mathrm{mod}\,3)$. Then, from (2.5),

$$\lim_{v\to\infty}(-1)^vQ_N(0)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^3;q^3)_n}\sum_{\substack{r=0\\ r\equiv\epsilon-n\,(\mathrm{mod}\,3)}}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-n-r}$$
$$-\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2+n}}{(q^3;q^3)_n}\sum_{\substack{r=0\\ r\equiv\epsilon-n-1\,(\mathrm{mod}\,3)}}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-1-n-r},$$

where $\rho=e^{\pi i/3}$. By (2.8), (2.1), (2.9), the remark following (2.9), and calculations analogous to whose used in the proof of Lemma 2.5,

$$\begin{aligned}
&\lim_{v\to\infty}(-1)^vQ_N(0)\\
=&\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^3;q^3)_n}\sum_{r=0}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-n-r}\frac{1+\omega^{\epsilon-n-r}+\bar{\omega}^{\epsilon-n-r}}{3}\\
&-\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2+n}}{(q^3;q^3)_n}\sum_{r=0}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-1-n-r}\frac{1+\omega^{\epsilon-1-n-r}+\bar{\omega}^{\epsilon-1-n-r}}{3}\\
=&\frac{1}{3}(-\omega^2)^{\epsilon}(1-\omega)\left\{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega)^n}{(q^3;q^3)_n}(\omega q;q)_n+\omega\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2+n}(-\omega)^n}{(q^3;q^3)_n}(\omega q;q)_n\right\}\\
&+\frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega^2)^n}{(q^3;q^3)_n}(\omega^2q;q)_n\right.\\
&\left.+\omega^2\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2+n}(-\omega^2)^n}{(q^3;q^3)_n}(\omega^2q;q)_n\right\}\\
=&\frac{1}{3}(-\omega^2)^{\epsilon+1}(1-\omega)(\omega q)_\infty(q;q^3)_\infty+\frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)(\omega^2q)_\infty(q;q^3)_\infty\\
=&\frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{(\omega^2q)_\infty}{(\omega q)_\infty}-\omega^{\epsilon-1}\right\}(\omega q)_\infty(q;q^3)_\infty.
\end{aligned}$$

□

Theorem 2.8. *Let $N-1=3v+\epsilon$, where $\epsilon=0$ or ±1. Then*

$$\lim_{N\to\infty}\left(\frac{1}{1}-\frac{1}{1+q}-\frac{1}{1+q^2}-\cdots-\frac{1}{1+q^{N-1}}\right)=-\omega^2\frac{\dfrac{(\omega^2q;q)_\infty}{(\omega q;q)_\infty}-\omega^{\epsilon+1}}{\dfrac{(\omega^2q;q)_\infty}{(\omega q;q)_\infty}-\omega^{\epsilon-1}}\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}. \tag{2.11}$$

Proof. The result follows immediately from (1.5) and Lemmas 2.5 and 2.7. □

Theorem 2.9. *Let $N-1=3v+\epsilon$, where $\epsilon=0$ or ±1. Then*

$$\lim_{N\to\infty}\left(\frac{1}{1}-\frac{1}{1+q}-\frac{1}{1+q^2}-\cdots-\frac{1}{1+q^{N-1}+a}\right)=-\omega^2\frac{\Omega-\omega^{\epsilon+1}}{\Omega-\omega^{\epsilon-1}}\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}, \tag{2.12}$$

where

$$\Omega = \frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty}.$$

Proof. Recall that the partial numerators $P_N(a)$ and partial denominators $Q_N(a)$ are defined in (1.4) and (1.5). It is easily shown by induction that, for $N \geq 2$,

$$\begin{aligned} P_N(a) &= P_N(0) + aP_{N-1}(0), \\ Q_N(a) &= Q_N(0) + aQ_{N-1}(0). \end{aligned}$$

For example, see [9, p. 8]. Hence

$$\begin{aligned} \lim_{N\to\infty} P_N(a) &= \lim_{N\to\infty} P_N(0) + a \lim_{N\to\infty} P_{N-1}(0), \\ \lim_{N\to\infty} Q_N(a) &= \lim_{N\to\infty} Q_N(0) + a \lim_{N\to\infty} Q_{N-1}(0). \end{aligned}$$

Let $N = 3v + \epsilon + 1$, where $\epsilon = 0, \pm 1$; we consider two cases: $\epsilon = 0, 1$ and $\epsilon = -1$.

Suppose that $\epsilon = 0$ or 1. From Lemma 2.5,

$$\begin{aligned} \lim_{N\to\infty} (-1)^v P_N(a) &= \lim_{N\to\infty} (-1)^v P_N(0) + a \lim_{N\to\infty} (-1)^v P_{N-1}(0) \\ &= \frac{1}{3}(-\omega)^\epsilon (1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}(\omega q)_\infty (q^2;q^3)_\infty \\ &\quad + a\frac{1}{3}(-\omega)^{\epsilon-1}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon}\right\}(\omega q)_\infty (q^2;q^3)_\infty \\ &= \frac{1}{3}(-\omega)^\epsilon (1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1} - a\omega^2\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} + a\omega^{\epsilon-1}\right\}(\omega q)_\infty (q^2;q^3)_\infty \\ &= \frac{1}{3}(-\omega)^\epsilon (1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}(1-a\omega)(\omega q)_\infty (q^2;q^3)_\infty. \end{aligned}$$

On the other hand, if $\epsilon = -1$, then

$$\begin{aligned} \lim_{N\to\infty} (-1)^v P_N(a) &= \lim_{N\to\infty} (-1)^v P_N(0) + a \lim_{N\to\infty} (-1)^v P_{N-1}(0) \\ &= \frac{1}{3}(-\omega)^\epsilon (1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}(\omega q)_\infty (q^2;q^3)_\infty \\ &\quad - a\frac{1}{3}(-\omega)^{\epsilon+2}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+3}\right\}(\omega q)_\infty (q^2;q^3)_\infty \\ &= \frac{1}{3}(-\omega)^\epsilon (1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1} - a\omega^2\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} + a\omega^{\epsilon+2}\right\}(\omega q)_\infty (q^2;q^3)_\infty \\ &= \frac{1}{3}(-\omega)^\epsilon (1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}(1-a\omega)(\omega q)_\infty (q^2;q^3)_\infty. \end{aligned}$$

Therefore, in both cases,

$$\lim_{N\to\infty} (-1)^v P_N(a) = \frac{1}{3}(-\omega)^\epsilon (1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}(1-a\omega)(\omega q)_\infty (q^2;q^3)_\infty. \tag{2.13}$$

Similarly, we can determine the limits of the denominator $Q_N(a)$. Suppose that $\epsilon = 0$ or 1. Then, from Lemma 2.7,

$$\begin{aligned}\lim_{N\to\infty}(-1)^v Q_N(a) &= \lim_{N\to\infty}(-1)^v Q_N(0) + a\lim_{N\to\infty}(-1)^v Q_{N-1}(0)\\ &= \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1} - a\omega^2\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} + a\omega^\epsilon\right\}(\omega q)_\infty (q;q^3)_\infty\\ &= \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1}\right\}(1-a\omega)(\omega q)_\infty (q;q^3)_\infty.\end{aligned}$$

On the other hand, if $\epsilon = -1$, then

$$\begin{aligned}\lim_{N\to\infty}(-1)^v Q_N(a) &= \lim_{N\to\infty}(-1)^v Q_N(0) + a\lim_{N\to\infty}(-1)^v Q_{N-1}(0)\\ &= \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1} - a\omega^2\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} + a\omega^\epsilon\right\}(\omega q)_\infty (q;q^3)_\infty\\ &= \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1}\right\}(1-a\omega)(\omega q)_\infty (q;q^3)_\infty.\end{aligned}$$

Therefore, in both cases,

$$\lim_{N\to\infty}(-1)^v Q_N(a) = \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1}\right\}(1-a\omega)(\omega q)_\infty (q;q^3)_\infty. \tag{2.14}$$

Combining (2.13) and (2.14) with (1.5), we complete the proof. □

Observe that our proof of Theorem 2.9 justifies the addendum made by Ramanujan after his statement of (1.1).

3. A Class of Sequences with Three Distinct Limit Points

Let u and v be complex numbers, and let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers such that

$$\sum_{n=1}^{\infty}|a_n| < \infty. \tag{3.1}$$

Generalizing the recurrence relations (1.6), we study the limit points of the sequence $\{x_n\}_{n\geq 0}$ given by

$$x_0 = v, \qquad x_1 = u, \tag{3.2}$$

and

$$x_n = (1 + a_{n-1})x_{n-1} - x_{n-2}. \tag{3.3}$$

We show that any such sequence has at most six limit points. Precisely, we have the following theorem.

Theorem 3.1. *Let u and v be complex numbers not both zero, and let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers satisfying* (3.1). *Then:*

(i) *For any $j \in \{0, \ldots, 5\}$ the sequence $\{x_{6n+j}\}_{n\geq 0}$ is convergent.*

(ii) *If we denote $l_j = \lim_{n\to\infty} x_{6n+j}$, $0 \leq j \leq 5$, then*
$$l_2 = l_1 - l_0,\ l_3 = -l_0,\ l_4 = -l_1,\ l_5 = -l_2.$$

(iii) *At most one of the numbers l_0, l_1, l_2 is zero.*

Proof. If we assume (i), then (ii) follows imediately, since the equality (3.3) gives, as $n = 6m + j \to \infty$,

$$l_j = l_{j-1} - l_{j-2}. \tag{3.4}$$

Let us now prove (i). We first show that the sequence $\{x_n\}_{n\geq 0}$ is bounded. In order to do this, denote

$$M_n := \max\{|x_0|, |x_1|, \dots, |x_n|\} \tag{3.5}$$

for any $n \geq 0$. By applying (3.3) twice we obtain

$$\begin{aligned} x_n =&(1 + a_{n-1})x_{n-1} - x_{n-2} \\ =&(1 + a_{n-1}) ((1 + a_{n-2})x_{n-2} - x_{n-3}) - x_{n-2} \\ =&(a_{n-1} + a_{n-2} + a_{n-1}a_{n-2})x_{n-2} - (1 + a_{n-1})x_{n-3}. \end{aligned} \tag{3.6}$$

The sequence $\{M_n\}_{n\geq 0}$ is increasing, and we want to show that it does not tend to infinity. Since $|x_{n-2}|$ and $|x_{n-3}|$ are bounded by M_{n-2}, from (3.6), we deduce that

$$|x_n| \leq (1 + 2|a_{n-1}| + |a_{n-2}| + |a_{n-1}a_{n-2}|)M_{n-2}. \tag{3.7}$$

This implies that

$$M_n \leq (1 + 2|a_{n-1}| + |a_{n-2}| + |a_{n-1}a_{n-2}|)M_{n-1}. \tag{3.8}$$

Define $b_n := 2|a_n| + |a_{n-1}| + |a_n a_{n-1}|$. For large n it follows that $b_n \leq 2(|a_n| + |a_{n-1}|)$. Thus,

$$\sum_{n=2}^{\infty} |b_n| < \infty, \tag{3.9}$$

by (3.1). Relation (3.9) further implies

$$\prod_{n=2}^{\infty}(1 + |b_n|) < \infty. \tag{3.10}$$

We conclude from (3.8) and (3.10) that the sequence $\{M_n\}_{n\geq 0}$ is bounded.

Choose $M > 0$ so that $|x_n| \leq M$ for any n. We write (3.6) in the form

$$x_n + x_{n-3} = (a_{n-1} + a_{n-2} + a_{n-1}a_{n-2})x_{n-2} - a_{n-1}x_{n-3}. \tag{3.11}$$

It follows that

$$|x_n + x_{n-3}| \leq (2|a_{n-1}| + |a_{n-2}| + |a_{n-1}a_{n-2}|)M = b_{n-1}M. \tag{3.12}$$

Using (3.12) twice, we find that

$$|x_n - x_{n-6}| \leq |x_n + x_{n-3}| + |x_{n-3} + x_{n-6}| \leq (b_{n-1} + b_{n-4})M. \tag{3.13}$$

Since $\sum_n (b_{n-1} + b_{n-4})M < \infty$, it follows that for any $j \in \{0, 1, \dots, 5\}$ the sequence $\{x_{6n+j}\}_{n\geq 0}$ is a Cauchy sequence, and hence it converges. This proves (i).

It remains to prove (iii). If more than one of the numbers l_0, l_1, l_2 is zero, then from (ii) we deduce that all six numbers $l_0, l_1, \dots, l_5$ are zero. In other words, the sequence $\{x_n\}_{n\geq 0}$ is convergent and has limit zero. We will show that this cannot happen under our assumption that $(u, v) \neq (0, 0)$. Let us first remark that no two consecutive elements in the sequence $\{x_n\}_{n\geq 0}$ are zero. Indeed, if $x_{n-1} = x_n = 0$ for some n, then, using (3.3) repeatedly, we see that $0 = x_{n-2} = x_{n-3} = \cdots = u = v$, which contradicts the above assumption on u, v. Thus, for any n, at least one of x_{n-1}, x_n is not zero.

We now look at the maximum of any three consecutive elements in the sequence. The point is to show that this maximum does not approach zero. More precisely, for any $n \geq 2$, we first set

$$D_n := \max\{|x_n|, |x_{n-1}|, |x_{n-2}|\}. \tag{3.14}$$

By the remark above, we know that $D_n > 0$ for any n. Let us compare D_n and D_{n-1} for large n. By definition,

$$D_{n-1} = \max\{|x_{n-1}|, |x_{n-2}|, |x_{n-3}|\}. \tag{3.15}$$

If $|x_{n-3}|$ is not the largest of the three numbers on the right side of (3.15), then obviously

$$D_n \geq D_{n-1}. \tag{3.16}$$

Assume now that $|x_{n-3}|$ is the largest of the three numbers, so that

$$|x_{n-1}|, |x_{n-2}| \leq |x_{n-3}| = D_{n-1}. \tag{3.17}$$

Write x_{n-3} in the form $x_{n-3} = D_{n-1}e^{it}$ for some real number t. Then from (3.11) it follows that

$$\begin{aligned} |e^{-it}x_n + D_{n-1}| =&|x_n + e^{it}D_{n-1}| = |x_n + x_{n-3}| \\ \leq&(|a_{n-1}| + |a_{n-2}| + |a_{n-1}a_{n-2}|)|x_{n-2}| + |a_{n-1}|D_{n-1} \leq b_{n-1}D_{n-1}. \end{aligned} \tag{3.18}$$

We deduce that

$$|x_n| = |e^{-it}x_n| \geq D_{n-1} - b_{n-1}D_{n-1} = (1 - b_{n-1})D_{n-1}. \tag{3.19}$$

By (3.16) and (3.19), in all cases,

$$D_n \geq (1 - b_{n-1})D_{n-1}. \tag{3.20}$$

Of course, the inequality (3.20) is only nontrivial when $b_{n-1} < 1$. Choose n_0 so that $b_n < 1$ for all $n \geq n_0$. We know that $D_{n_0} > 0$. Also, from (3.20) we know that for any $n > n_0$,

$$D_n \geq (1 - b_{n-1})(1 - b_{n-2}) \cdots (1 - b_{n_0})D_{n_0}. \tag{3.21}$$

Now the product $\prod_{n>n_0}(1 - b_n)$ is convergent to a *nonzero* number D, since $1 - b_n > 0$ for $n \geq n_0$ and $\sum_{n>n_0} b_n < \infty$. Thus $D_n \geq DD_{n_0}$ for any $n > n_0$, and so

$$\max\{|l_0|, |l_1|, |l_2|\} \geq DD_{n_0} > 0, \tag{3.22}$$

which completes the proof of the theorem. □

Next we derive some consequences of the above result. First, if we fix the sequence $\{a_n\}_{n\geq 1}$ satisfying (3.1) and allow u, v to vary in $\mathbf{C}$, we obtain a map from $\mathbf{C}^2$ to $\mathbf{C}^6$ which sends the pair (u, v) to the 6-tuple $(l_0, l_1, \ldots, l_5)$. Clearly this map is linear, so it is given by a matrix A which depends on the sequence $\{a_n\}_{n\geq 1}$. By (iii) we know moreover that this linear map is injective, and hence rank $A = 2$. We summarize these observations in the next corollary.

Corollary 3.2. *For any sequence of complex numbers* $\{a_n\}_{n\geq 1}$ *satisfying* (3.1), *there is a* 6×2 *matrix* A *such that for any* $u, v \in \mathbf{C}$,

$$\lim_{n\to\infty} \begin{pmatrix} x_{6n} \\ x_{6n+1} \\ \vdots \\ x_{6n+5} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.23}$$

where the sequence $\{x_n\}_{n\geq 0}$ *is given by* (3.2) *and* (3.3). *Moreover, the rank of* A *equals* 2.

Given such a sequence $\{a_n\}_{n\geq 1}$ and the associated matrix A, consider the 6×2 matrix B given by

$$B = A \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.24}$$

Thus the first column of B consists of the six limit points $b_{01}, b_{11}, \ldots, b_{51}$ of the sequence $\{P_n\}_{n\geq 0}$ defined by

$$P_n = (1 + a_{n-1})P_{n-1} - P_{n-2} \tag{3.25}$$

and

$$P_0 = 0, \qquad P_1 = 1. \tag{3.26}$$

The second column of B consists of the six limit points $b_{02}, b_{12}, \ldots, b_{52}$ of the sequence $\{Q_n\}_{n\geq 0}$ defined by

$$Q_n = (1 + a_{n-1})Q_{n-1} - Q_{n-2} \tag{3.27}$$

and

$$Q_0 = 1, \qquad Q_1 = 1. \tag{3.28}$$

We are interested in the limit points of the sequence $\{P_n/Q_n\}_{n\geq 0}$, and for this we examine the quotients

$$b_j := \frac{b_{j1}}{b_{j2}}, \qquad 0 \leq j \leq 5. \tag{3.29}$$

Since $b_{(j+3),1} = -b_{j,1}$ and $b_{(j+3),2} = -b_{j,2}$ for $j = 0, 1, 2$, it follows that $b_3 = b_0$, $b_4 = b_1$, and $b_5 = b_2$. This is not enough to conclude that the sequence $\{P_n/Q_n\}_{n\geq 0}$ has at most three limit points in $\mathbf{C} \cup \{\infty\}$. The reason is that each of the two columns of the matrix B might have a component equal to zero for some $j \in \{0, 1, 2\}$. And if $b_{j1} = b_{j2} = 0$, the sequence $\{P_{6n+j}/Q_{6n+j}\}_{n\geq 0}$ may very well have infinitely many limit points. We now show that this cannot happen, that is, there is no j for which $b_{j1} = b_{j2} = 0$. Indeed, let us assume that for some $j \in \{0, 1, 2\}$ one has $b_{j1} = b_{j2} = 0$. Then from (iii), with $(u, v) = (1, 0)$ and, respectively, $(u, v) = (1, 1)$, we know that, except for b_{j1} and b_{j2}, the other four numbers in the set $\{b_{01}, b_{11}, b_{21}, b_{02}, b_{12}, b_{22}\}$ are nonzero. If we choose $i \in \{0, 1, 2\}$, $i \neq j$, and let $\lambda \in \mathbf{C}$ be such that $b_{i1} = \lambda b_{i2}$, then the following happens. The sequence $\{x_n\}_{n\geq 0}$ associated with the vector

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.30}$$

and the same $\{a_n\}_{n\geq 1}$ will have two of its limit points, namely l_j and l_i, equal to zero, which contradicts (iii). In conclusion, if both columns of B have components equal to zero, these components belong to different rows. Therefore, the sequence $\{P_n/Q_n\}_{n\geq 0}$

has at most three limit points in $\mathbf{C}\cup\{\infty\}$. Moreover, since at most one of b_{02}, b_{12}, b_{22} is zero, it follows that at most one of b_0, b_1, b_2 equals ∞. Similarly, at most one of b_0, b_1, b_2 is zero. We claim that the numbers b_0, b_1, b_2 are distinct. If not, say $b_1 = b_2 = \lambda$. Thus, $\lambda \neq \infty$. Again if we define u, v by (3.30), then the associated sequence $\{x_n\}_{n\geq 0}$ will have two limit points l_1, l_2 equal to zero, contradicting (iii). We have therefore proved the following theorem.

Theorem 3.3. *For any sequence of complex numbers* $\{a_n\}_{n\geq 1}$ *satisfying* (3.1), *the sequence* $\{P_n/Q_n\}_{n\geq 0}$, *with* P_n *and* Q_n *given by* (3.25)–(3.28), *has exactly three distinct limit points in* $\mathbf{C}\cup\{\infty\}$.

4. Continued Fractions with Three Distinct Limit Points

In this section, we establish a general class of continued fractions which have exactly three limit points.

Let F and G be two meromorphic functions which are defined on the unit disc, $U := \{z \in \mathbf{C} : |z| < 1\}$, are analytic at the origin, and satisfy the functional equation,

$$F(z) + G(z) + zF(z)G(z) = 1. \tag{4.1}$$

For instance, we may take any analytic function F on U and set

$$G(z) := \frac{1 - F(z)}{1 + zF(z)}. \tag{4.2}$$

Consider the continued fraction

$$\frac{1}{1} - \frac{1}{1 + zF(z)} - \frac{1}{1 + zG(z)} - \frac{1}{F(z) + G(z^2)} - \frac{1}{1 + z^2F(z^2)} - \frac{1}{1 + z^2G(z^2)} \tag{4.3}$$
$$- \frac{1}{F(z^2) + G(z^3)} - \cdots - \frac{1}{1 + z^nF(z^n)} - \frac{1}{1 + z^nG(z^n)} - \frac{1}{F(z^n) + G(z^{n+1})} - \cdots.$$

So that the values $F(z), F(z^2), \dots, F(z^n), \dots$ and $G(z), G(z^2), \dots, G(z^n), \dots$ are defined, we need to assume that $z^n, n \geq 1$, is not a pole of either F or G. Our main theorem is as follows.

Theorem 4.1. *Let* F *and* G *denote meromorphic functions which are defined on* U, *as given above, which are analytic at the origin, and which satisfy the condition* (4.1). *Then the continued fraction* (4.3) *has exactly three limit points,* L_0, L_1, *and* L_2. *Moreover,*

$$L_0 = \frac{z}{2z - zG(z) - 1}, \tag{4.4}$$

$$L_1 = \frac{z + zG(0) - 1}{(z + zG(0) - 1)(1 - G(z)) + (z - 1)G(0)}, \tag{4.5}$$

$$L_2 = \frac{1 - zG(0)}{(1 - zG(0))(1 - G(z)) + (z - 1)(1 - G(0))}. \tag{4.6}$$

Proof. The continued fraction (4.3) is of the type considered in Section 3, with the coefficients a_n given by

$$a_{3n-2} = z^n F(z^n), \tag{4.7}$$
$$a_{3n-1} = z^n G(z^n), \tag{4.8}$$
$$a_{3n} = F(z^n) + G(z^{n+1}) - 1, \tag{4.9}$$

for each $n \geq 1$. Clearly, the sub-sequences $\{a_{3n-2}\}$ and $\{a_{3n-1}\}$ are exponentially decreasing. As for a_{3n}, it has the Taylor expansion about $z = 0$,

$$a_{3n} = F(0) + G(0) - 1 + F'(0)z^n + G'(0)z^{n+1} + \cdots .$$

Since, by (4.1), $F(0)+G(0) = 1$, it follows that $a_{3n} = O(z^n)$, as $n \to \infty$. Hence, (3.1) is satisfied. We may then apply Theorem 3.3 to conclude that the continued fraction (4.3) has exactly three distinct limit points, L_0, L_1, and L_2. We also know from Theorem 3.1 that the sequences $\{P_{6n+j}\}$ and $\{Q_{6n+j}\}$, defined by (3.25) and (3.27), respectively, with initial conditions (3.26) and (3.28), respectively, are convergent, where a_n is defined by (4.7)–(4.9). For each $j, 0 \leq j \leq 5$, let

$$l_j = \lim_{n\to\infty} P_{6n+j} \qquad \text{and} \qquad l'_j = \lim_{n\to\infty} Q_{6n+j}. \tag{4.10}$$

By Theorem 3.1, we also know that

$$\begin{cases} l_2 = l_1 - l_0, \ l_3 = -l_0, \ l_4 = -l_1, \ l_5 = -l_2, \\ l'_2 = l'_1 - l'_0, \ l'_3 = -l'_0, \ l'_4 = -l'_1, \ l'_5 = -l'_2. \end{cases} \tag{4.11}$$

¿From the proof of Theorem 3.3, the three limit points of the continued fraction (4.3) are given by

$$L_0 = \frac{l_0}{l'_0}, \qquad L_1 = \frac{l_1}{l'_1}, \qquad \text{and} \qquad L_2 = \frac{l_2}{l'_2}. \tag{4.12}$$

The key idea is to use the recurrence relations (3.25) and (3.27) to explicitly calculate P_n and Q_n along a subsequence and then subsequently for any $n \geq 0$.

Consider a general sequence $\{x_n\}_{n\geq 0}$ satisfying the recurrence relation

$$x_n = (1 + a_{n-1})x_{n-1} - x_{n-2}, \qquad n \geq 2, \tag{4.13}$$

where the coefficients a_n are given by (4.7)–(4.9). By successively applying (4.13), we may write x_n as a linear combination of any two previous members of the sequence $\{x_n\}_{n\geq 0}$. We shall write x_n in terms of x_{n-3} and x_{n-6}. Applying (4.13), we find that

$$x_n = (a_{n-1} + a_{n-2} + a_{n-1}a_{n-2})\, x_{n-2} - (1 + a_{n-1})x_{n-3}. \tag{4.14}$$

Next, from the equalities

$$x_{n-6} = (1 + a_{n-5})x_{n-5} - x_{n-4} \qquad \text{and} \qquad x_{n-5} = (1 + a_{n-4})x_{n-4} - x_{n-3},$$

we deduce that

$$\begin{aligned} x_{n-6} =& (1 + a_{n-5})\{(1 + a_{n-4})x_{n-4} - x_{n-3}\} - x_{n-4} \\ =& (a_{n-4} + a_{n-5} + a_{n-4}a_{n-5})x_{n-4} - (1 + a_{n-5})x_{n-3}. \end{aligned} \tag{4.15}$$

Multiply both sides of (4.15) by

$$\frac{a_{n-1}+a_{n-2}+a_{n-1}a_{n-2}}{a_{n-4}+a_{n-5}+a_{n-4}a_{n-5}}$$

to obtain

$$\begin{aligned}0 =&(a_{n-1}+a_{n-2}+a_{n-1}a_{n-2})x_{n-4}-\frac{(a_{n-1}+a_{n-2}+a_{n-1}a_{n-2})(1+a_{n-5})}{a_{n-4}+a_{n-5}+a_{n-4}a_{n-5}}x_{n-3}\\ &-\frac{a_{n-1}+a_{n-2}+a_{n-1}a_{n-2}}{a_{n-4}+a_{n-5}+a_{n-4}a_{n-5}}x_{n-6}.\end{aligned}\tag{4.16}$$

We now add (4.14) and (4.16) and use the equality $x_{n-2}+x_{n-4}=(1+a_{n-3})x_{n-3}$ to deduce that

$$\begin{aligned}x_n =&(a_{n-1}+a_{n-2}+a_{n-1}a_{n-2})(1+a_{n-3})x_{n-3}-\frac{(a_{n-1}+a_{n-2}+a_{n-1}a_{n-2})(1+a_{n-5})}{a_{n-4}+a_{n-5}+a_{n-4}a_{n-5}}x_{n-3}\\ &-(1+a_{n-1})x_{n-3}-\frac{a_{n-1}+a_{n-2}+a_{n-1}a_{n-2}}{a_{n-4}+a_{n-5}+a_{n-4}a_{n-5}}x_{n-6}.\end{aligned}\tag{4.17}$$

This is the general recurrence for x_n in terms of x_{n-3} and x_{n-6} which we sought. Set now $y_0=x_0, y_1=-x_3, y_2=x_6$, and, more generally, $y_k=(-1)^k x_{3k}$, for each nonnegative integer k. We know from our discussion after (4.9) that the sequence $\{y_k\}$ converges. Furthermore, from (4.17),

$$\begin{aligned}y_k =&\left\{\frac{(a_{3k-1}+a_{3k-2}+a_{3k-1}a_{3k-2})(1+a_{3k-5})}{a_{3k-4}+a_{3k-5}+a_{3k-4}a_{3k-5}}\right.\\ &\left.-(a_{3k-1}+a_{3k-2}+a_{3k-1}a_{3k-2})(1+a_{3k-3})+1+a_{3k-1}\right\}y_{k-1}\\ &-\frac{a_{3k-1}+a_{3k-2}+a_{3k-1}a_{3k-2}}{a_{3k-4}+a_{3k-5}+a_{3k-4}a_{3k-5}}y_{k-2}.\end{aligned}\tag{4.18}$$

For $k\geq 1$, by (4.7) and (4.8),

$$a_{3k-1}+a_{3k-2}+a_{3k-1}a_{3k-2}=z^kG(z^k)+z^kF(z^k)+z^{2k}F(z^k)G(z^k)=z^k,\tag{4.19}$$

by (4.1). Using (4.19) with k replaced by $k-1$, we also have, for $k\geq 2$,

$$a_{3k-4}+a_{3k-5}+a_{3k-4}a_{3k-5}=z^{k-1}.\tag{4.20}$$

Thus, for $k\geq 2$, (4.18) reduces to

$$y_k=\left(1+a_{3k-1}+z(1+a_{3k-5})-z^k(1+a_{3k-3})\right)y_{k-1}-zy_{k-2}.\tag{4.21}$$

Also, for $k\geq 2$, by (4.7)–(4.9), we find that

$$a_{3k-5}=z^{k-1}F(z^{k-1}),\quad a_{3k-3}=F(z^{k-1})+G(z^k)-1,\quad a_{3k-1}=z^kG(z^k).\tag{4.22}$$

Hence,

$$\begin{aligned}1+a_{3k-1}+z(1+a_{3k-5})-z^k(1+a_{3k-3}) =&1+z^kG(z^k)+z\left(1+z^{k-1}F(z^{k-1})\right)\\ &-z^k\left(1+F(z^{k-1})+G(z^k)-1\right)\\ =&1+z.\end{aligned}$$

Thus, (4.21) simplifies to

$$y_k = (1+z)y_{k-1} - zy_{k-2}, \qquad k \geq 2. \tag{4.23}$$

The roots of the associated characteristic equation

$$T^2 - (1+z)T + z = 0 \tag{4.24}$$

are $T = 1, z$. Thus, with c_1 and c_2 complex numbers, the general solution of (4.24) is

$$y_k = c_1 + c_2 z^k, \qquad k \geq 0. \tag{4.25}$$

We now apply our theory to the sequences $\{P_n\}$ and $\{Q_n\}$, defined by (3.25) and (3.27), respectively, with initial conditions (3.26) and (3.28), respectively. By (4.25), for $n \geq 0$, for some constants, c_1, c_2, c_1', c_2',

$$(-1)^n P_{3n} = c_1 + c_2 z^n \qquad \text{and} \qquad (-1)^n Q_{3n} = c_1' + c_2' z^n. \tag{4.26}$$

¿From (3.26) and (3.28), respectively, $c_1 + c_2 = P_0 = 0$ and $c_1' + c_2' = Q_0 = 1$. Thus, $c_2 = -c_1 =: -c$ and $c_2' = 1 - c_1' =: 1 - c'$. Hence, from (4.26), for $n \geq 0$,

$$(-1)^n P_{3n} = c(1 - z^n) \qquad \text{and} \qquad (-1)^n Q_{3n} = c'(1 - z^n) + z^n. \tag{4.27}$$

We can determine c and c' by setting $n = 1$ in (4.27). By (4.27), (3.25), and (3.26),

$$c(1-z) = -P_3 = -(1+a_2)P_2 + P_1 = -(1+a_2)((1+a_1)P_1 - P_0) + P_1 = -(1+a_2)(1+a_1) + 1.$$

Hence, using (4.7), (4.8), and (4.1), we find that

$$\begin{aligned} c =& \frac{1-(1+a_1)(1+a_2)}{1-z} = \frac{1-(1+zF(z))(1+zG(z))}{1-z} \\ =& -\frac{z(F(z)+G(z)+zF(z)G(z))}{1-z} = -\frac{z}{1-z}. \end{aligned} \tag{4.28}$$

Next, by (4.27), (3.27), and (3.28),

$$c'(1-z) + z = -Q_3 = -(1+a_2)Q_2 + Q_1 = -(1+a_2)((1+a_1)Q_1 - Q_0) + 1 = -(1+a_2)a_1 + 1,$$

or, by (4.7), (4.8), and (4.1),

$$\begin{aligned} c'(1-z) + z =& 1 - (1+zG(z))zF(z) \\ =& 1 - (zF(z) + z^2F(z)G(z)) = 1 - (z - zG(z)). \end{aligned}$$

Hence,

$$c' = \frac{1 - 2z + zG(z)}{1-z}. \tag{4.29}$$

It follows from (4.10) and (4.27)–(4.29) that

$$l_0 = \lim_{n\to\infty} P_{6n} = c = \frac{z}{z-1}, \tag{4.30}$$

$$l_0' = \lim_{n\to\infty} Q_{6n} = c' = \frac{2z - zG(z) - 1}{z-1}, \tag{4.31}$$

and from (4.12) we further deduce that

$$L_0 = \frac{l_0}{l_0'} = \frac{z}{2z - zG(z) - 1}, \tag{4.32}$$

as claimed in Theorem 4.1.

Next, we compute l_1, l_2, l_1', and l_2'. To find l_1, first return to (4.14) and set $n = 6k+3$ to obtain

$$P_{6k+3} = (a_{6k+2} + a_{6k+1} + a_{6k+2}a_{6k+1})P_{6k+1} - (1 + a_{6k+2})P_{6k}. \tag{4.33}$$

Since, by (4.27), $P_{6k} = c(1 - z^{2k})$ and $P_{6k+3} = -c(1 - z^{2k+1})$, since, by (4.8),

$$a_{6k+2} = z^{2k+1}G(z^{2k+1}), \tag{4.34}$$

and since, by (4.19),

$$a_{6k+2} + a_{6k+1} + a_{6k+2}a_{6k+1} = z^{2k+1}, \tag{4.35}$$

we find that, by (4.33),

$$\begin{aligned} P_{6k+1} =& \frac{-c(1 - z^{2k+1}) + (1 + z^{2k+1}G(z^{2k+1}))c(1 - z^{2k})}{z^{2k+1}} \\ =& c\left(1 + G(z^{2k+1}) - \frac{1}{z} - z^{2k}G(z^{2k+1})\right). \end{aligned}$$

It follows from (4.10) and (4.28) that

$$l_1 = \lim_{k\to\infty} P_{6k+1} = c\left(1 + G(0) - \frac{1}{z}\right) = \frac{z + zG(0) - 1}{z - 1}. \tag{4.36}$$

By (4.27), $Q_{6k} = c'(1-z^{2k}) + z^{2k}$ and $Q_{6k+3} = -c'(1-z^{2k+1}) - z^{2k+1}$. Using also (4.34) and (4.35) in (4.33), with P replaced by Q, we find that

$$\begin{aligned} Q_{6k+1} =& \frac{-c'(1 - z^{2k+1}) - z^{2k+1} + (1 + z^{2k+1}G(z^{2k+1}))(c'(1 - z^{2k}) + z^{2k})}{z^{2k+1}} \\ =& c' - 1 - \frac{c'}{z} + \frac{1}{z} + c'G(z^{2k+1}) + O(z^{2k}). \end{aligned}$$

It follows from (4.10) that

$$\begin{aligned} l_1' = \lim_{k\to\infty} Q_{6k+1} &= c' - 1 - \frac{c'}{z} + \frac{1}{z} + c'G(0) \\ =& \frac{(c' - 1)(z - 1)}{z} + c'G(0) \\ =& 1 - G(z) + \frac{1 - 2z + zG(z)}{1 - z}G(0), \end{aligned} \tag{4.37}$$

by two applications of (4.29). Hence, by (4.36) and (4.37), we deduce, by (4.12), that

$$L_1 = \frac{l_1}{l_1'} = \frac{z + zG(0) - 1}{z - 1} \cdot \frac{1 - z}{(1 - G(z))(1 - z) + G(0)(1 - 2z + zG(z))},$$

which, upon simplification, yields (4.5).

Finally, we calculate L_2. By (4.11), (4.36), and (4.30),

$$l_2 = l_1 - l_0 = \frac{z + zG(0) - 1}{z - 1} - \frac{z}{z - 1} = \frac{zG(0) - 1}{z - 1}, \tag{4.38}$$

and by (4.11), (4.37), and (4.31),

$$\begin{aligned} l_2' = l_1' - l_0' =& 1 - G(z) + \frac{1-2z+zG(z)}{1-z}G(0) - \frac{2z - zG(z) - 1}{z-1} \\ =& \frac{G(0)(1-2z+zG(z)) + z - G(z)}{1-z}, \end{aligned} \tag{4.39}$$

upon simplification. Using (4.38) and (4.39) in conjunction with (4.12), we conclude that

$$L_2 = \frac{l_2}{l_2'} = \frac{zG(0)-1}{z-1} \cdot \frac{1-z}{G(0)(1-2z+zG(z)) + z - G(z)},$$

which, after simplification, yields (4.6).

This then completes the proof of Theorem 4.1. □

Under certain conditions, the formulas of Theorem 4.1 greatly simplify. If $G(0) = 0$, then

$$L_1 = \frac{1}{1-G(z)}, \tag{4.40}$$

while, if $F(0) = 0$, then, by (4.1), $G(0) = 1$ and

$$L_2 = \frac{1}{1-G(z)}. \tag{4.41}$$

Also, if $G(0) = 0$, then

$$L_2 = \frac{1}{z-G(z)}, \tag{4.42}$$

while, if $G(0) = -1$, then

$$L_1 = \frac{1}{z-G(z)}.$$

We conclude this section by offering two interesting examples to illustrate Theorem 4.1.

Corollary 4.2. *For any complex number z, with $|z| < 1$, the continued fraction*

$$\frac{1}{1} - \frac{1}{(1-z)^{-1}} - \frac{1}{1-z^2} - \frac{1}{(1-z)^{-1} - z^2} - \frac{1}{(1-z^2)^{-1}} - \frac{1}{1-z^4} - \frac{1}{(1-z^2)^{-1} - z^3}$$
$$- \cdots - \frac{1}{(1-z^n)^{-1}} - \frac{1}{1-z^{2n}} - \frac{1}{(1-z^n)^{-1} - z^{n+1}} - \cdots$$

has three limit points, and they are given by

$$L_0 = \frac{z}{z^2+2z-1}, \quad L_1 = \frac{1}{1+z}, \quad L_2 = \frac{1}{2z}.$$

Proof. Set

$$F(z) = \frac{1}{1-z} \qquad \text{and} \qquad G(z) = -z.$$

It is easily checked that F and G satisfy (4.1) and the conditions of Theorem 4.1, and so the desired result readily follows. □

Corollary 4.3. *For any complex number z, with $|z|<1$, the continued fraction*

$$\frac{1}{1}-\frac{1}{\sqrt{1+z}}-\frac{1}{\sqrt{1+z}}-\frac{1}{z^{-1}\left(\sqrt{1+z}-1\right)+z^{-2}\left(\sqrt{1+z^2}-1\right)}$$
$$-\frac{1}{\sqrt{1+z^2}}-\frac{1}{\sqrt{1+z^2}}-\frac{1}{z^{-2}\left(\sqrt{1+z^2}-1\right)+z^{-3}\left(\sqrt{1+z^3}-1\right)}$$
$$-\cdots-\frac{1}{\sqrt{1+z^n}}-\frac{1}{\sqrt{1+z^n}}-\frac{1}{z^{-n}\left(\sqrt{1+z^n}-1\right)+z^{-n-1}\left(\sqrt{1+z^{n+1}}-1\right)}-\cdots$$

has the three limit points

$$L_0=\frac{z}{2z-\sqrt{1+z}},\qquad L_1=\frac{z(3z-2)}{(3z-2)\left(1+z-\sqrt{1+z}\right)+z(z-1)},$$

and

$$L_2=\frac{z(2-z)}{(2-z)\left(1+z-\sqrt{1+z}\right)+z(z-1)}.$$

Proof. In Theorem 4.1, take

$$F(z)=G(z)=\frac{\sqrt{1+z}-1}{z}.$$

The functions F and G clearly satisfy the condition (4.1), as well as the other conditions of the theorem. The desired result now readily follows. □

5. The Special Case $a=\omega$ of (1.1)

It is interesting to note that in this special case $\Omega=0$, and so the three limits in (1.1) are identical.

A Bauer–Muir transformation [9, pp. 76–77] of a continued fraction $b_0+\mathbf{K}(a_n/b_n)$ is a (new) continued fraction whose approximants have the values

$$S_k(w_k):=b_0+\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_k}{b_k+w_k},\quad k=0,1,2,\dots. \tag{5.1}$$

Such a transformation exists if

$$\lambda_k:=a_k-w_{k-1}(b_k+w_k)\neq 0,\quad k\geq 1, \tag{5.2}$$

and it is given by

$$b_0+w_0+\frac{\lambda_1}{b_1+w_1}+\frac{a_1\lambda_2/\lambda_1}{b_2+w_2-w_0\lambda_2/\lambda_1}+\frac{a_2\lambda_3/\lambda_2}{b_3+w_3-w_1\lambda_3/\lambda_2}+\cdots. \tag{5.3}$$

Theorem 5.1. *For a cube root of unity ω,*

$$\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{1+q}-\frac{1}{1+q^2}-\cdots-\frac{1}{1+q^n+\omega}\right)=-\omega\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}. \tag{5.4}$$

Proof. Let L_n denote the reciprocal of the continued fraction on the left hand side of (5.4), and if we employ the notation of (5.1), then $b_0=1, a_n=-1$, and $b_n=1+q^n$, for $n\geq 1$, and $\omega_n=\omega$, for $n\geq 0$.

If $q = 0$, then (5.4) reduces to a tautology. Hence assume that $q \neq 0$. Then, from (5.2), for $n \geq 1$,

$$\lambda_n = -1 - \omega(1 + q^n + \omega) = -1 - \omega - \omega^2 - \omega q^n = -\omega q^n \neq 0.$$

Thus, by (5.3),

$$\begin{aligned} L_n &= 1 + \omega + \frac{-\omega q}{1 + q + \omega} + \frac{-q}{1 + q^2 + \omega - \omega q} + \frac{-q}{1 + q^3 + \omega - \omega q} + \cdots \\ &= -\omega^2 + \frac{\omega^2 q}{1 - \omega q} + \frac{\omega q}{1 + q^2 + \omega - \omega q} + \frac{-q}{1 + q^3 + \omega - \omega q} + \cdots \\ &=: -\omega^2 + \frac{\omega^2 q}{C_1}, \end{aligned} \tag{5.5}$$

after using an equivalence transformation for the continued fraction.

For the continued fraction C_1, in the notation of (5.1),

$$b_0 = 1 - \omega q, \quad a_1 = \omega q, \quad a_n = -q, \quad n \geq 2, \quad \text{and} \quad b_n = 1 + q^{n+1} + \omega - \omega q, \quad n \geq 1.$$

We apply the Bauer–Muir transformation a second time. Set $\omega_0 = -\omega^2 q$ and $\omega_i = \omega q$, for $i \geq 1$. A brief calculation shows that, by (5.2),

$$\lambda_1 = q^3\omega^2 \neq 0 \quad \text{and} \quad \lambda_k = -q^{k+2}\omega \neq 0 \quad \text{for} \quad k \geq 2.$$

Hence, from (5.3), after applying the Bauer–Muir transformation to C_1, we have

$$\begin{aligned} C_1 &= 1 + q + \frac{\omega^2 q^3}{1 + q^2 + \omega} + \frac{-q^2}{1 + q^3 + \omega - \omega q^2} + \frac{-q^2}{1 + q^4 + \omega - \omega q^2} + \cdots \\ &= 1 + q + \frac{-q^3}{1 - \omega q^2} + \frac{\omega q^2}{1 + q^3 + \omega - \omega q^2} + \frac{-q^2}{1 + q^4 + \omega - \omega q^2} + \cdots, \end{aligned} \tag{5.6}$$

after applying an equivalence transformation. Combining (5.5) and (5.6), we have

$$\begin{aligned} L_n &= -\omega^2 + \frac{\omega^2 q}{1 + q} + \frac{-q^3}{1 - \omega q^2} + \frac{\omega q^2}{1 + q^3 + \omega - \omega q^2} + \frac{-q^2}{1 + q^4 + \omega - \omega q^2} + \cdots \\ &=: -\omega^2 + \frac{\omega^2 q}{1 + q} + \frac{-q^3}{C_2}. \end{aligned} \tag{5.7}$$

Applying the Bauer–Muir transformation to C_2 and proceeding as in the two previous applications, we find that if $\omega_0 = -\omega^2 q^2$ and $\omega_i = \omega q^2$, for $i \geq 1$, then

$$\lambda_1 = \omega^2 q^5 \neq 0 \quad \text{and} \quad \lambda_k = -q^{k+4}\omega \neq 0, \quad k \geq 2.$$

Thus, $b_0 + \omega_0 = 1 + q^2$, $b_1 + \omega_1 = 1 + q^3 + \omega$, $b_n + \omega_n - \omega_0\lambda_n/\lambda_{n-1} = 1 + q^{n+2} + \omega - q^3\omega$, and $a_n\lambda_{n+1}/\lambda_n = -q^3$, for $n \geq 1$. Hence, from (5.7), after using an equivalence transformation, we have

$$\begin{aligned}L_n &= -\omega^2 + \frac{\omega^2 q}{1+q} + \frac{-q^3}{1+q^2} + \frac{\omega^2 q^5}{1+q^3+\omega} + \frac{-q^3}{1+q^4+\omega-\omega q^3} + \frac{-q^3}{1+q^5+\omega-\omega q^3} + \cdots \\ &= -\omega^2 + \frac{\omega^2 q}{1+q} + \frac{-q^3}{1+q^2} + \frac{-q^5}{1-\omega q^3} + \frac{\omega q^3}{1+q^4+\omega-\omega q^3} + \frac{-q^3}{1+q^5+\omega-\omega q^3} + \cdots \\ &= \cdots \\ &= -\omega^2 + \frac{\omega^2 q}{1+q} + \frac{-q^3}{1+q^2} + \frac{-q^5}{1+q^3} + \cdots + \frac{-q^{2n-1}}{C_n},\end{aligned} \tag{5.8}$$

where

$$C_n = 1-\omega q^n + \frac{\omega q^n}{1+q^{n+1}+\omega-\omega q^n} + \frac{-q^n}{1+q^{n+2}+\omega-\omega q^n} + \frac{-q^n}{1+q^{n+3}+\omega-\omega q^n} + \cdots$$

after an easy inductive argument on n with $\omega_0 = -\omega^2 q^n$ and $\omega_i = \omega q^n$, for $i \geq 1$ after the n-th step. Upon taking the reciprocal in (5.8), letting n tend to ∞, and using (1.3), we deduce (5.4). □

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