RAMANUJAN'S LOST NOTEBOOK: COMBINATORIAL PROOFS OF IDENTITIES ASSOCIATED WITH HEINE'S TRANSFORMATION OR PARTIAL THETA FUNCTIONS

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Abstract. Combinatorial proofs are given for certain entries in Ramanujan's lost notebook. Bijections of Sylvester, Franklin, and Wright, and applications of Algorithm Z of Zeilberger are employed. A new bijection, involving the new concept of the parity sequence of a partition, is used to prove one of Ramanujan's fascinating identities for a partial theta function.

1. Introduction

In [13], the first and third authors provided bijective proofs for several entries found in Ramanujan's lost notebook [28]. The entries for which combinatorial proofs were given arise from the Rogers–Fine identity and false theta functions, and are found in Chapter 9 of [9]. Although G. E. Andrews [5] had previously devised a combinatorial proof of the Rogers–Fine identity, the combinatorics of each of the identities proved in [13] is substantially different from that in Andrews's proof, so that even what might be considered small or subtle changes in an identity markedly alter the combinatorics. This paper can be considered as a sequel to [13] in that we combinatorially prove further entries from Ramanujan's lost notebook. The entries to be examined in the present paper are connected with either Heine's transformation or partial theta functions. Readers may have difficulty discerning the connections of some of the entries with either Heine's transformation or partial theta functions. To see these relationships, consult the book [10] by Andrews and the first author, where all of the identities established in this paper are proved analytically. The second author, in another paper [22], has combinatorially proved some further identities involving partial theta functions found in the lost notebook.

Algorithm Z of D. Zeilberger and its variant that was established by the third author play an important role. Euler's partition identity and Sylvester's bijective proof of it also play leading roles. We will recall these and other bijections in Section 2. In Section 3, we present combinatorial proofs of some identities arising from Euler's identity. In Section 4, we give bijective proofs of entries that are special cases of the q-Gauss summation formula. The next goal of our paper is to provide combinatorial proofs of entries that are related to Heine's ${}_2\phi_1$ transformation formula. Some of the proofs follow along the lines of Andrews's proof of Heine's ${}_2\phi_1$ transformation formula [4], but others do not. In the final section, we introduce a new class of partitions, namely partitions with the parity sequence. We obtain the generating function of these partitions

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analytically and bijectively. Using this generating function, we give a combinatorial proof of an identity that is related with partial theta functions.

2. Preliminary Results

A *partition* of a positive integer n is a weakly decreasing sequence of positive integers $(\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 + \cdots + \lambda_r = n$, and we shall write $\lambda \vdash n$ (see [6].) We relax our definition of a partition by including 0 as a part, if necessary. We denote the number of parts of a partition λ by $\ell(\lambda)$. As a convention, we denote the partition of 0 by $\emptyset$.

We employ the standard notation

$$(a;q)_0 = 1, \qquad (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n \geq 1,$$

and

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n, \qquad |q| < 1.$$

We recall some familiar bijections that are used in the sequel.

Sylvester's bijection. Sylvester's map for Euler's identity

$$\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty \tag{2.1}$$

and many further contributions of Sylvester have been discussed by Andrews in [8]. We note here that Sylvester's bijection preserves the following statistic [19, 20, 30]:

$$\ell(\lambda) + (\lambda_1 - 1)/2 = \mu_1, \tag{2.2}$$

where λ is a partition into odd parts and μ is the partition into distinct parts associated with λ under Sylvester's bijection.

Franklin's involution. Recall that Franklin's involution provides a bijective proof of Euler's pentagonal number theorem [6, pp. 10–11]

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{2.3}$$

Wright's bijection. Recall that Wright's bijection [31] gives a bijective proof for the Jacobi triple product identity

$$(-zq;q)_\infty(-z^{-1};q)_\infty = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2}. \tag{2.4}$$

Algorithm Z and its application. The following bijection is an application of Algorithm Z discovered by D. Zeilberger [11, 16]. It was first observed by J. T. Joichi and D. Stanton [21] that Algorithm Z can apply in this way to the q-binomial theorem and used by the third author in [32] to establish a combinatorial proof for Ramanujan's ${}_1\psi_1$ summation formula. Recall the q-binomial theorem [6, p. 17]

$$\sum_{n=0}^{\infty} \frac{(-a;q)_n}{(q;q)_n} (zq)^n = \frac{(-azq;q)_\infty}{(zq;q)_\infty}. \tag{2.5}$$

For a positive integer n, let π be a partition into nonnegative distinct parts less than n and σ a partition into exactly n parts. We define μ by

$$\mu_i = \sigma_{n-\pi_i} + \pi_i, \qquad \text{for all } 1 \le i \le \ell(\pi),$$

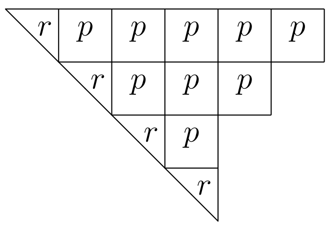
and let ν be the partition consisting of the remaining $n - \ell(\pi)$ parts of σ. It follows from the construction that μ and ν are uniquely determined by π and σ. Furthermore, μ has distinct parts. The left-hand side of (2.5) generates the pairs of partitions (π, σ), and the right-hand side generates the pairs of partitions (μ, ν). Thus this map is a bijection between the two sets of such pairs of partitions.

A variation of Algorithm Z. For a positive integer n, let π be a partition into nonnegative distinct parts less than n and σ a partition into exactly n distinct parts. We define μ by

$$\mu_{\ell(\pi)-i+1} = \sigma_{\pi_i+1} + \pi_i, \qquad \text{for all } 1 \le i \le \ell(\pi),$$

and let ν be the partition consisting of the remaining $n - \ell(\pi)$ parts of σ. It follows from the construction that μ and ν are uniquely determined by π and σ. Furthermore, the parts of μ are greater than or equal to n, since $\sigma_{\pi_i+1} \ge n - \pi_i$.

Modular Ferrers diagram. We introduce a p-modular Ferrers diagram. For a partition λ into parts λ_i congruent to r modulo p, its p-modular Ferrers diagram is the diagram in which the i-th row has $\lceil \lambda_i/p \rceil$ boxes, the boxes in the first column have r, and the other boxes have p. It can easily be seen that the sum of the numbers in the boxes equals the number that λ partitions. If a partition has distinct parts, we can draw its modular Ferrers diagram in the form of a staircase. Moreover, if necessary, we may use triangles for the boxes on the main diagonal. For instance, the following is a p-modular Ferrers diagram in the form of a staircase.



3. Bijective proofs of identities arising from the Euler identity

A combinatorial proof of the following theorem was given by the first and third authors in the process of combinatorially proving another entry from Ramanujan's lost notebook [13, p. 413]. We now provide a shorter proof.

Theorem 3.1. [28, p. 38], [10, Entry 1.6.4] *For each complex number a,*

$$\sum_{n=0}^{\infty} \frac{(-aq)^n}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-a)^n q^{n(n+1)/2}}{(-aq;q)_n}. \tag{3.1}$$

Proof. Replace a by $-a$ in (3.1). Then the left-hand side generates partitions λ into odd parts, and the exponent of a equals $\ell(\lambda)+(\lambda_1-1)/2$. The right-hand side of (3.1) generates partitions into distinct parts, and the exponent of a is the largest part. The identity now follows by Sylvester's bijection and its preserved statistic (2.2). $\square$

Theorem 3.2. [28, p. 31], [10, Entry 6.5.1] *We have*

$$\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_{2n}}=\sum_{n=0}^{\infty}q^{12n^2+n}(1-q^{22n+11})+q\sum_{n=0}^{\infty}q^{12n^2+7n}(1-q^{10n+5}) \tag{3.2}$$

and

$$\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_{2n+1}}=\sum_{n=0}^{\infty}q^{12n^2+5n}(1-q^{14n+7})+q^2\sum_{n=0}^{\infty}q^{12n^2+11n}(1-q^{2n+1}). \tag{3.3}$$

Proof. We prove the first identity. The second one can be proved in a similar way and we omit its proof. Replacing q by q^2 in (3.2), we obtain the identity

$$\sum_{n=0}^{\infty}\frac{q^{2n}}{(-q^2;q^2)_{2n}}=\sum_{n=0}^{\infty}q^{24n^2+2n}(1-q^{44n+22})+q^2\sum_{n=0}^{\infty}q^{24n^2+14n}(1-q^{20n+10}). \tag{3.4}$$

The left-hand side generates partitions λ into an even number of odd parts with weight $(-1)^{(\lambda_1-1)/2}$. Clearly, λ is a partition of an even number $2N$. Thus, we obtain

$$\sum_{n=0}^{\infty}\frac{q^{2n}}{(-q^2;q^2)_{2n}}=\sum_{N=0}^{\infty}\sum_{\lambda\in O(2N)}(-1)^{(\lambda_1-1)/2}q^{2N}, \tag{3.5}$$

where $O(2N)$ is the set of partitions of $2N$ into odd parts. Let $D(2N)$ be the set of partitions of $2N$ into distinct parts. It follows from Euler's identity (2.1) that $O(2N)$ and $D(2N)$ are equinumerous. Let μ be the image of λ under Sylvester's bijection, which is a partition in $D(2N)$. Since λ is a partition of $2N$ into odd parts, $\ell(\lambda)$ is even. Thus we see from (2.2) that

$$(-1)^{(\lambda_1-1)/2}=(-1)^{\mu_1}.$$

It then follows that

$$\sum_{N=0}^{\infty}\sum_{\lambda\in O(2N)}(-1)^{(\lambda_1-1)/2}q^{2N}=\sum_{N=0}^{\infty}\sum_{\mu\in D(2N)}(-1)^{\mu_1}q^{2N}. \tag{3.6}$$

We now apply Franklin's involution for Euler's pentagonal number theorem (2.3), in which we compare the smallest part and the number of consecutive parts including the largest part. Note that in the pentagonal number theorem, partitions π have weight $(-1)^{\ell(\pi)}$. However, the involutive proof still works in our setting, since we move the smallest part to the right of the consecutive parts or subtract 1 from each of the consecutive parts in order to add the number of consecutive parts as a new part. Thus only the partitions of the even pentagonal numbers survive under the involution in our setting, too. Under the involution, only partitions λ of the form $(2n, 2n-1, \ldots, n+1)$ or $(2n-1, 2n-2, \ldots, n)$ survive. That is, $\lambda \vdash n(3n \pm 1)/2$. It is easy to see that

$$\begin{aligned} n(3n+1)/2 &\equiv 0 \pmod 2, \quad \text{if } n \equiv 0, 1 \pmod 4, \\ n(3n-1)/2 &\equiv 0 \pmod 2, \quad \text{if } n \equiv 0, 3 \pmod 4. \end{aligned}$$

When $n \equiv 0, 1 \pmod 4$, the surviving partition of $n(3n+1)/2$ has parts $2n, 2n-1, \ldots, n+1$. The largest part of the partition is even. When $n \equiv 0, 3 \pmod 4$, the largest part of the partition of $n(3n-1)/2$ is odd. Then

$$\begin{aligned} \sum_{N=0}^{\infty} \sum_{\mu \in D(2N)} (-1)^{\mu_1} q^{2N} &= \sum_{\substack{n=0 \\ n(3n+1)/2 \equiv 0 \pmod 2}}^{\infty} q^{n(3n+1)/2} - \sum_{\substack{n=1 \\ n(3n-1)/2 \equiv 0 \pmod 2}}^{\infty} q^{n(3n-1)/2} \\ &= \sum_{n=0}^{\infty} q^{24n^2+2n}(1-q^{44n+22}) + q^2 \sum_{n=0}^{\infty} q^{24n^2+14n}(1-q^{20n+10}). \end{aligned} \tag{3.7}$$

Hence, by (3.5), (3.6) and (3.7), we complete the proof of (3.4) and therefore also of Theorem 3.2. □

M. Monks [23], at about the same time that the present authors gave their proof of Theorem 3.2 above, established an equivalent, combined version of (3.2) and (3.3) by essentially the same methods. We provide her formulation, which is also found in the lost notebook [28, p. 36], [9, p. 235, Entry 9.4.7]. The function on the left-hand side below is one of Ramanujan's mock theta functions.

Theorem 3.3. *Define*

$$\chi_6(n) = \begin{cases} 1, & \text{if } n \equiv 1, 5, 7, 11 \pmod{24}, \\ -1, & \text{if } n \equiv 13, 17, 19, 23 \pmod{24}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q^2; q^2)_n} = \sum_{n=1}^{\infty} \chi_6(n) q^{(n^2-1)/24}.$$

In Theorem 3.2, Ramanujan anticipated a later theorem of N. J. Fine [19], [20, p. 45]. Let $Q_a(n)$ denote the number of partitions of n into distinct parts such that the largest part is $a \pmod 2$, $a = 0, 1$. Also, let $Q_b^*(n)$ denote the number of partitions of n into odd parts such that the largest part is $b \pmod 4$, $b = 1, 3$. Then

$$Q_0(n) - Q_1(n) = (-1)^n (Q_1^*(n) - Q_3^*(n))$$

and

$$Q_0(n) - Q_1(n) = \begin{cases} 1, & \text{if } n = (3k^2+k)/2, \quad k \geq 0, \\ -1, & \text{if } n = (3k^2-k)/2, \quad k > 0, \\ 0, & \text{otherwise.} \end{cases}$$

If we replace q by q^2 in Theorem 3.2, then the two series generate the odd and even parts for $Q_1^*(n) - Q_3^*(n)$. In other words, the left-hand sides of (3.2) and (3.3) are, respectively,

$$\sum_{n=0}^{\infty}\{Q_1^*(2n) - Q_3^*(2n)\}q^{2n}$$

and

$$\sum_{n=0}^{\infty}\{Q_1^*(2n+1) - Q_3^*(2n+1)\}q^{2n+1},$$

and the right-hand sides provide the non-vanishing of the partitions counted on the left-hand sides at the pentagonal numbers, as observed by Fine.

In the formulation of Ramanujan's next two identities, it will be convenient to use the notation for Ramanujan's theta functions, namely,

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab| < 1.$$

Theorem 3.4. [28, p. 31], [10, Entry 6.5.2] *We have*

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_{2n}} = \frac{f(q^5,q^3)}{(q;q)_\infty} \tag{3.8}$$

and

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_{2n+1}} = \frac{f(q^7,q)}{(q;q)_\infty}. \tag{3.9}$$

Proof. We prove the first identity. The second one can be proved in a similar way. In (3.8), replace q by q^2. Then we obtain

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_{2n}} = \frac{f(q^{10},q^6)}{(q^2;q^2)_\infty}. \tag{3.10}$$

The left-hand side generates partitions into an even number of odd parts. Equivalently, it generates partitions of an even number into odd parts. Thus, we obtain

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_{2n}} = \sum_{N=0}^{\infty}\sum_{\lambda\in O(2N)} q^{2N} = \sum_{N=0}^{\infty}\sum_{\mu\in D(2N)} q^{2N},$$

where the second equality follows from Sylvester's bijection. By decomposing the parts of μ into even parts and odd parts, we obtain

$$\sum_{N=0}^{\infty}\sum_{\mu\in D(2N)} q^{2N} = (-q^2;q^2)_\infty \sum_{n=0}^{\infty}\sum_{\nu\in DO(2n)} q^{2n},$$

where $DO(2n)$ is the set of partitions on $2n$ into distinct odd parts. Let ν^1 and ν^3 be the partitions consisting of parts of ν congruent to 1 and 3 modulo 4, respectively. Note that since ν is a partition of $2n$, the number of parts of ν is even. Thus $\ell(\nu^1) \equiv \ell(\nu^3) \pmod 2$. We now use staircase 4-modular Ferrers diagrams for the partitions ν^1 and ν^3, in which the triangles on the main diagonal have the residue 1 or 3 and the remaining boxes have 4. We then apply Wright's bijection to the pair (ν^1, ν^3). Since $\ell(\nu^1) \equiv \ell(\nu^3) \pmod 2$, we collect only even powers of z from the summation on the right-hand side of the Jacobi triple product identity (2.4). By substituting q^{-1} and q^4 for z and q, respectively, we obtain

$$\sum_{n=0}^{\infty} \sum_{\nu \in DO(2n)} q^{2n} = \frac{1}{(q^4;q^4)_\infty} \sum_{k=-\infty}^{\infty} q^{8k^2+2k}.$$

Thus it follows that

$$(-q^2;q^2)_\infty \sum_{n=0}^{\infty} \sum_{\nu \in DO(2n)} q^{2n} = \frac{(-q^2;q^2)_\infty}{(q^4;q^4)_\infty} \sum_{k=-\infty}^{\infty} q^{8k^2+2k} = \frac{1}{(q^2;q^2)_\infty} f(q^{10}, q^6).$$

This completes our bijective proof of (3.10). □

Corollary 3.5. [28, p. 35], [10, Entry 1.7.7] *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)/2}}{(q)_n (1-q^{2n+1})} = q f(q, q^7).$$

Proof. By Theorem 3.4, it suffices to show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)/2}}{(q)_n (1-q^{2n+1})} = q(q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q)_{2m+1}} = \sum_{m=0}^{\infty} q^{m+1} (q^{2m+2})_\infty.$$

Let λ be a partition arising from $(q^{2m+2})_\infty$. Then the parts of λ are distinct and larger than $2m+1$. Let $n = \ell(\lambda)$. Detach $2m$ from each of the n parts. By combining this with m from q^{m+1}, we have $(2n+1)m$, which is generated by $1/(1-q^{2n+1})$. The resulting parts of λ form a partition into distinct parts that are larger than 1 with weight $(-1)^n$. Such partitions are generated by

$$\frac{(-1)^n q^{2+3+\cdots+(n+1)}}{(q)_n}.$$

Combining them with q that was left from q^{m+1}, we arrive at

$$\frac{(-1)^n q^{(n+1)(n+2)/2}}{(q)_n}.$$

This completes the proof. □

The following corollary can be proved by a similar argument, and so we omit the proof.

Corollary 3.6. [28, p. 35], [10, Entry 1.7.9] *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n (1-q^{2n+1})} = f(q^3, q^5).$$

4. Bijective proofs of identities arising from the q-Gauss summation formula

Recall that the q-Gauss summation theorem is given by [6, p. 20, Corollary 2.4]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}\left(\frac{c}{ab}\right)^n = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/(ab);q)_\infty}, \tag{4.1}$$

where $|c/(ab)| < 1$. Bijective proofs of (4.1) have been given by the first and third authors [13], S. Corteel [17], Corteel and J. Lovejoy [18], and the third author [**?**]. Here, we prove a special case of (4.1).

Theorem 4.1. [28, p. 370], [10, Entry 1.3.2] *For arbitrary complex numbers a, b,*

$$\sum_{n=0}^{\infty} \frac{(-b/a)_n a^n q^{n(n+1)/2}}{(q)_n(bq)_n} = \frac{(-aq)_\infty}{(bq)_\infty}. \tag{4.2}$$

Proof. In (4.1), we replace b and c by bq and $-b/d$, respectively. We then let a go to infinity to obtain

$$\sum_{n=0}^{\infty} \frac{(-b/d)_n d^n q^{n(n+1)/2}}{(q)_n(bq)_n} = \frac{(-dq)_\infty}{(bq)_\infty},$$

whose combinatorial proof just follows from the proof of the q-Gauss summation. □

Proof. We observe that $(-b/a)_n$ generates partitions into nonnegative distinct parts $< n$, and $a^n q^{n(n+1)/2}/(q)_n$ generates partitions into n distinct parts. Let π and σ be such partitions, respectively. We apply the variation of Algorithm Z to π and σ. Let μ and ν denote the resulting partitions, namely μ is a partition into parts $\geq n$ and ν is a partition into distinct parts. Let ω be a partition generated by $1/(bq)_n$, and denote by $\mu \cup \omega$ the partition consisting of the parts of μ and ω. We have thus obtained the pair $(\nu, \mu \cup \omega)$ of a partition into distinct parts and an ordinary partition. Note that the exponent of a counts $n - \ell(\pi)$, which is equal to $\ell(\nu)$, and the exponent of b counts $\ell(\mu \cup \omega)$. Thus ν is generated by $(-aq)_\infty$, and $\mu \cup \omega$ is generated by $1/(bq)_\infty$.

Since the variation of Algorithm Z is reversible, it suffices to show that μ can be uniquely determined when ν and $\mu \cup \omega$ are given. Since the parts of μ are larger than or equal to n and those of ω are less than or equal to n, namely,

$$\mu_{\ell(\mu)} \geq n = \ell(\nu) + \ell(\mu) \geq \omega_1, \tag{4.3}$$

we need to determine how many largest parts of $\mu \cup \omega$ came from μ. Consider the following inequality

$$(\mu \cup \omega)_k \geq \ell(\nu) + k$$

for a positive integer k, where $(\mu \cup \omega)_k$ denotes the k-th part of $\mu \cup \omega$. If there is no such k, then we define μ to be the empty partition. Otherwise, we take the largest k and define μ by the partition consisting of the largest k parts of $\mu \cup \omega$. This μ is indeed the original μ. It follows from the choice of k that

$$(\mu \cup \omega)_{k+1} < \ell(\nu) + k + 1.$$

That is, k is uniquely determined and $(\mu\cup\omega)_k \geq \ell(\nu)+k \geq (\mu\cup\omega)_{k+1}$. Thus it follows from (4.3) that $\ell(\nu)+k$ and the first k parts of $\mu\cup\omega$ came from the original μ. ☐

Theorem 4.2. [28, p. 41], [10, Entry 4.2.6] *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n^2} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{4.4}$$

Proof. We replace q with $-q$ in (4.4) to obtain the equivalent identity

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n^2} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty},$$

which is the case of (4.2) with a, b, and q replaced by q^{-1}, 1, and q^2, respectively. Therefore, the theorem follows. ☐

5. Bijective Proofs of Identities Arising from Heine's Transformation

The identities in this section are proved in [10, Chapter 1] by appealing to Heine's transformation or some variant or generalization thereof.

Theorem 5.1. [28, p. 16], [10, Entry 1.4.8] *For arbitrary complex numbers a, b,*

$$\begin{aligned}\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty} \frac{(aq;q)_n b^n q^{n^2}}{(q^2;q^2)_n} &= (-bq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(aq)^{2n}}{(q;q)_{2n}(-bq;q^2)_n} \\ &\quad + (-bq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(aq)^{2n+1}}{(q;q)_{2n+1}(-bq^2;q^2)_n}.\end{aligned} \tag{5.1}$$

Proof. Rewrite the left-hand side of (5.1) as

$$\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty} \frac{(aq;q)_n b^n q^{n^2}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{b^n q^{n^2}}{(aq^{n+1};q)_\infty (q^2;q^2)_n}. \tag{5.2}$$

The right-hand side is a generating function for vector partitions (π,ν) such that π is a partition into parts that are strictly larger than n, and ν is a partition into n distinct odd parts. We examine these partitions in two cases.

Case 1: π has an even number of parts. Let $2k$ be the number of parts in π. Detach n from each part of π and attach $2k$ to each part of ν. Denote the resulting partitions by σ and λ, respectively. It is clear that σ is a partition into $2k$ parts, and λ is a partition into distinct odd parts that are greater than or equal to $2k+1$. These are generated by

$$\sum_{k=0}^{\infty} \frac{(aq)^{2k}}{(q;q)_{2k}}(-bq^{2k+1};q^2)_\infty. \tag{5.3}$$

Case 2: π has an odd number of parts. Let $2k+1$ be the number of parts in π. Detach $2k+1$ from each part of π and attach $2k+1$ to each part of ν. By reasoning similar to that above, we can see that the resulting partition pairs are generated by

$$\sum_{k=0}^{\infty} \frac{(aq)^{2k+1}}{(q;q)_{2k+1}}(-bq^{2k+2};q^2)_\infty. \tag{5.4}$$

Combining the two generating functions (5.3) and (5.4) together with (5.2), we complete the proof. ☐

Theorem 5.2. [28, p. 10], [10, Entry 1.4.9] *We have*

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n^2} = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{5.5}$$

Proof. Multiplying both sides of (5.5) by $(q)_\infty$, we obtain the equivalent identity

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n}(q^{n+1};q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n}(-q^{n+1};q)_\infty, \tag{5.6}$$

since $(q^2;q^2)_\infty = (-q;q)_\infty(q;q)_\infty$. The left side of (5.6) is a generating function for the pair of partitions (π,ν), such that π is a partition into n distinct parts and ν is a partition into distinct parts that are strictly larger than n, and where the exponent of (-1) is the number of parts in ν. For a given partition pair $(\pi,\ \nu)$ generated by the left side of (5.6), let k be the number of parts in ν. Detach n from each part of ν and attach k to each part of π. Then we obtain partition pairs $(\sigma,\ \lambda)$, such that σ is a partition into k distinct parts and λ is a partition into distinct parts that are strictly larger than k, and the exponent of (-1) is the number of parts in σ. These partitions are generated by the right side of (5.6). Since this process is easily reversible, our proof is complete. ☐

The identity in Theorem 5.2 is connected with the theory of gradual stacks with summits [7].

Theorem 5.3. [28, p. 10], [10, Entry 1.4.12] *For each $n > 0$,*

$$\sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}}{(q)_m}(-bq^{nm+n};q^n)_\infty = \sum_{m=0}^{\infty} \frac{b^m q^{nm(m+1)/2}}{(q^n;q^n)_m}(-aq^{nm+1};q)_\infty.$$

Proof. First observe that $\frac{a^m q^{m(m+1)/2}}{(q)_m}$ generates partitions into m distinct parts, where the exponent of a is the number of parts. Second, $(-bq^{nm+n};q^n)_\infty$ generates partitions into distinct parts, where each part is at least $nm+n$, each part is a multiple of n, and the exponent of b equals the number of parts. Let $(\pi,\ \nu)$ be the partition pair generated by $\frac{a^m q^{m(m+1)/2}}{(q)_m}$ and $(-bq^{nm+n};q^n)_\infty$, respectively. Detach nm from each part of ν. The remaining partition is generated by $\frac{b^k q^{nk(k+1)/2}}{(q^n;q^n)_k}$. Attach mk to each part of π. Then the resulting partition is a partition into distinct parts that are greater than or equal to $nk+1$. Since this process is reversible, we are finished with the proof. ☐

Theorem 5.4. [28, p. 30], [10, Entry 1.4.17] *For each* $n > 0$,

$$(-aq)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q)_m(-aq)_{nm}} = (-bq)_\infty \sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}}{(q)_m(-bq)_{nm}}. \tag{5.7}$$

Proof. Rewrite the left-hand side of (5.7) in the form

$$(-aq)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q)_m(-aq)_{nm}} = \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q)_m}(-aq^{mn+1})_\infty. \tag{5.8}$$

First, $\frac{b^m q^{m(m+1)/2}}{(q)_m}$ generates partitions into m distinct parts with the exponent of b keeping track of the number of parts. Second, $(-aq^{mn+1})_\infty$ generates partitions into distinct parts, each strictly larger than mn. Let (σ, ν) denote a pair of partitions generated by $\frac{b^m q^{m(m+1)/2}}{(q)_m}$ and $(-aq^{mn+1})_\infty$, respectively. Let k denote the number of parts in ν. Detach mn from each part of ν and denote the resulting partition by ν'. Attach kn to each part of σ and denote the resulting partition by σ'. Then ν' is a partition into k distinct parts, and σ' is a partition into distinct parts, each strictly larger than kn. Such partitions are generated by the right side of (5.8). Since the process is reversible, the proof is complete. $\square$

Theorem 5.4 provides a generalization of a certain *Duality* that was utilized by D. M. Bressoud [15] in connecting the well-known identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{1}{(-q^2;q^2)_\infty (q;q^5)_\infty (q^4;q^5)_\infty}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n} = \frac{1}{(-q^2;q^2)_\infty (q^2;q^5)_\infty (q^3;q^5)_\infty}$$

of L. J. Rogers [29] with the Rogers–Ramanujan identities. In particular, if we consider the case $n = 1$ in Theorem 5.4,

$$\sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}(-aq^{m+1})_\infty}{(q)_m} = \sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}(-bq^{m+1})_\infty}{(q)_m}, \tag{5.9}$$

and replace q by q^2 and a by a/q in (5.9) we obtain the identity

$$F(a,b) := \sum_{m=0}^{\infty} \frac{a^m q^{m^2}(-bq^{2m+2};q^2)_\infty}{(q^2;q^2)_m} = \sum_{m=0}^{\infty} \frac{b^m q^{m^2+m}(-aq^{2m+1};q^2)_\infty}{(q^2;q^2)_m} = F(bq, a/q). \tag{5.10}$$

Note that the transformation T defined by

$$T(F(a,b)) = F(bq, aq^{-1})$$

is an involution. Thus (5.10) is a fixed point under this involution.

Bressoud [15] does not state this Duality explicitly but uses the underlying combinatorics in his paper [15]. K. Alladi [2] observed the involution (5.10) as Bressoud's Duality and used it to connect six identities of Rogers [29] with the Rogers–Ramanujan identities via the modified convergence of a certain continued fraction of Ramanujan, A. Selberg, and B. Gordon.

Similarly Theorem 5.3 is also a generalization of Bressoud's Duality.

Theorem 5.5. [28, p. 42], [10, Entry 1.5.1] *We have*

$$\sum_{n=0}^{\infty}\frac{a^nq^{n^2}}{(q)_n}=(-aq^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{a^nq^{n^2}}{(q^2;q^2)_n(-aq^2;q^2)_n} \tag{5.11}$$

$$=(-aq;q^2)_\infty\sum_{n=0}^{\infty}\frac{a^nq^{n^2+n}}{(q^2;q^2)_n(-aq;q^2)_n}. \tag{5.12}$$

Proof. We prove (5.11). Moving $(-aq^2;q^2)_\infty$ inside the summation sign and using a corollary of the q-binomial theorem [6, p. 19, Eq. (2.2.6)], namely,

$$(-aq^{2n+2};q^2)_\infty=\sum_{m=0}^{\infty}\frac{a^mq^{m^2+m+2mn}}{(q^2;q^2)_m},$$

we find that it suffices to show that

$$\sum_{k=0}^{\infty}\frac{a^kq^{k^2}}{(q)_k}=\sum_{m,n=0}^{\infty}\frac{a^{m+n}q^{n^2+m^2+m+2mn}}{(q^2;q^2)_m(q^2;q^2)_n}. \tag{5.13}$$

Let us interpret the right side of (5.13). Consider a Durfee square of side $m+n$. Attach 1 to each of the first m rows. Append the 2-modular diagram of a partition generated by $\frac{1}{(q^2;q^2)_m}$ to the first m rows. Finally append the 2-modular diagram of a partition generated by $\frac{1}{(q^2;q^2)_n}$ to the next n rows. Then, it is clear that the resulting partition is generated by the sum on the left side of (5.13). For the reverse process, let π be a partition generated by the left side of (5.13). Then π has a Durfee square of side k, and below the Durfee square there are no parts. Let π_r be a partition to the right of the Durfee square in π. Let m be the number of odd parts in π_r. Rearrange the order of π_r so that the first m parts are odd. Detach 1 from each part of the first m parts of π_r. Then the first m parts are generated by $\frac{1}{(q^2;q^2)_m}$, and the remaining parts are generated by $\frac{1}{(q^2;q^2)_{k-m}}$. Setting $n=k-m$, we are done.

Since the proof of (5.12) is similar, we omit it. □

The next identity is technically not in Ramanujan's lost notebook [28] but is the lone entry on a page published with the lost notebook. In fact, this identity is from the years prior to Ramanujan's departure for England, since it can be found as Entry 9 of Chapter 16 in Ramanujan's second notebook [27]. Prior to the proof given in [1] and [12, p. 52], proofs were given by V. Ramamani [25] and Ramamani and K. Venkatachaliengar [26]. S. Bhargava and C. Adiga [14] have established a generalization.

Theorem 5.6. [28, p. 362], [10, Entry 1.6.1] *For $a\neq 0$,*

$$(aq)_\infty\sum_{n=0}^{\infty}\frac{b^nq^{n^2}}{(q)_n(aq)_n}=\sum_{n=0}^{\infty}\frac{(-1)^n(b/a)_na^nq^{n(n+1)/2}}{(q)_n}. \tag{5.14}$$

Proof. Replace a by $-a$ in (5.14) and move $(-aq)_\infty$ inside the summation on the left side to obtain the equivalent identity

$$\sum_{n=0}^{\infty} \frac{b^n q^{n^2}}{(q)_n}(-aq^{n+1})_\infty = \sum_{n=0}^{\infty} \frac{(-b/a)_n a^n q^{n(n+1)/2}}{(q)_n}. \tag{5.15}$$

Then, the right-hand side of (5.15) is a generating function for the partition pair (π, ν), where π is a partition into k distinct nonnegative parts that are less than n, and where ν is a partition into n distinct parts. Let us define σ to be the partition such that

$$\sigma_i = \pi_{k+1-i} + \nu_{\pi_{k+1-i}+1}, \qquad 1 \le i \le k.$$

Note that each part of σ is greater than or equal to n. Let λ be a partition consisting of the remaining $n-k$ parts of ν. Detach $n-k$ from each part of σ and attach k to each part of λ. Then the resulting partition pairs (σ',λ') have the property that σ' is a partition into k parts that are greater than or equal to k, and λ' is a partition into distinct parts that are strictly larger than k as desired. Since this process is reversible by Algorithm Z, the proof is complete, except for checking the exponents of a and b.

On the right-hand side of (5.15), the power of b equals the number of parts k in π. The power of a is $n-k$, the number of parts of ν minus the number of parts of π. In the partition pair (σ,λ), note that k is the number of parts in σ and $n-k$ is the number of parts in λ. Observe that in the last portion of the process, the number of parts is not changed. This then completes the proof. □

Theorem 5.7. [28, p. 38], [10, Entry 1.6.5] *If a is any complex number, then*

$$\sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)}}{(q^2;q^2)_m(1+aq^{2m+1})} = (-aq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-a)^n q^{n(n+1)/2}}{(-aq;q)_n}. \tag{5.16}$$

Proof. By Theorem 3.1, the identity (5.16) can be written in the equivalent form

$$\sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)}}{(q^2;q^2)_m(1+aq^{2m+1})} = \sum_{n=0}^{\infty} (-aq)^n(-aq^{2n+2};q^2)_\infty.$$

Note that $(-aq^{2n+2};q^2)_\infty$ generates partitions into distinct even parts, each greater than or equal to $2n+2$, with the exponent of a denoting the number of parts. Let m be the number of parts generated by a partition arising from $(-aq^{2n+2};q^2)_\infty$. Detach $2n$ from each of the m parts. Combining this with $(-aq)^n$, we obtain $(-aq^{2m+1})^n$. However, note that, for $n \ge 0$, all of these odd parts are generated by $1/(1+aq^{2m+1})$, and each part is weighted by $-a$. The remaining parts, which are even, are generated by

$$\sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)}}{(q^2;q^2)_m}.$$

For these partitions into m distinct even parts, the exponent of a again denotes the number of parts. □

6. Partitions with a parity sequence

Let D_n be the set of partitions into n distinct parts less than $2n$ such that the smallest part of each partition is 1, and if $2k-1$ is the largest odd part, then all odd positive integers less than $2k-1$ occur as parts. For a partition $\lambda \in D_n$, we define the *parity sequence* as the longest sequence of decreasing consecutive numbers *containing* the largest odd part and denote its length by $c(\lambda)$. Thus, the largest part of the parity sequence might be even. For instance, when $n=5$,

$$\begin{aligned} c((5,4,3,2,1)) &= 5, \\ c((8,6,5,4,3,1)) &= 4, \\ c((9,7,6,5,3,1)) &= 1. \end{aligned}$$

Let

$$\lambda = (\lambda_1, \ldots, \lambda_s, \underline{\lambda_{s+1}, \ldots, \lambda_{s+c}}, \lambda_{s+c+1}, \ldots, \lambda_n) \in D_n,$$

where its parity sequence is underlined. By the definition of a parity sequence, we see that

(P1) $\lambda_1, \ldots, \lambda_s$ are even;
(P2) all the positive odd integers less than or equal to λ_{s+1} occur in λ;
(P3) λ_{s+c} is odd and $\lambda_{s+c} = \lambda_{s+c+1} + 2$.

We now compute the generating function of D_n. For a partition $\lambda \in D_n$, let k be the number of odd parts of λ. Then it follows from the definition of D_n that the odd integers $1, 3, \ldots, 2k-1$ occur in λ and the other $n-k$ parts are distinct even numbers. Note that the generating function of partitions into m distinct even parts less than $2n$ is [6, pp. 33–35]

$$q^{m(m+1)} \begin{bmatrix} n-1 \\ m \end{bmatrix}_{q^2},$$

as the q-binomial coefficient $\begin{bmatrix} a \\ b \end{bmatrix}_q$ generates partitions into at most b parts $\leq (a-b)$ for $0 \leq b \leq a$, where

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_a}{(q;q)_b (q;q)_{a-b}}, & \text{if } 0 \leq b \leq a, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{\lambda \in D_n} q^{\lambda_1 + \cdots + \lambda_n} = \sum_{k=0}^{n-1} q^{(n-k)^2 + k^2 + k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2}. \tag{6.1}$$

Lemma 6.1. *For any positive integer n,*

$$\sum_{k=0}^{n-1} q^{(n-k)^2 + k^2 + k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} = (-q;q)_{n-1} q^{n(n+1)/2}. \tag{6.2}$$

Proof. Let $f_n(q) = (-q;q)_{n-1} q^{n(n+1)/2}$. Then, for $n \geq 1$,

$$f_{n+1}(q) = (q^{n+1} + q^{2n+1}) f_n(q).$$

We prove the lemma by showing that the left-hand side of (6.2) satisfies the same recurrence as $f_n(q)$. First of all, when $n = 1$, (6.2) holds true. For $n \geq 1$, using a familiar recurrence for $\begin{bmatrix} n \\ k \end{bmatrix}_{q^2}$ [6, Eq. (3.3.4)], we find that

$$\begin{aligned}
&\sum_{k=0}^{n} q^{(n+1-k)^2+k^2+k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \\
&= q^{(n+1)^2} + \sum_{k=1}^{n-1} q^{(n+1-k)^2+k^2+k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} + q^{n^2+n+1} \\
&= q^{(n+1)^2} + \sum_{k=1}^{n-1} q^{(n+1-k)^2+k^2+k} \left(q^{2k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q^2} \right) + q^{n^2+n+1} \\
&= q^{(n+1)^2} + \sum_{k=1}^{n-1} q^{(n-k)^2+k^2+k+2n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \sum_{k=0}^{n-2} q^{(n-k)^2+(k+1)^2+k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + q^{n^2+n+1} \\
&= \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k+2n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+3k+2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} \\
&= \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k+2n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \sum_{k=0}^{n-1} q^{(k+1)^2+(n-k-1)^2+3(n-k-1)+2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} \\
&= \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k+2n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k+n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} \\
&= (q^{2n+1} + q^{n+1}) \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2},
\end{aligned}$$

which completes the proof. □

We can prove the following theorem using (6.1) and (6.2). However, we provide a combinatorial proof.

Theorem 6.2. *For any positive integer n, the generating function of D_n is*

$$(-q;q)_{n-1} q^{n(n+1)/2}.$$

Proof. For a positive integer n, let $\tau = (n, n-1, \ldots, 2, 1)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be a partition into distinct parts less than n. We insert the parts μ_i in decreasing order into τ as follows.

Insertion: Let π be τ and begin with $i = 1$.

(1) If $\pi_i + \mu_1$ is even, then add μ_1 to π_i, i.e., add μ_1 horizontally to π, and add 1 to i; if $\pi_i + \mu_1$ is odd, then add 1 to each of the $\pi_i, \dots, \pi_{i+\mu_1-1}$, i.e., add μ_1 vertically down starting from π_i, and the i remains the same.
(2) By an abuse of notation, let us denote the resulting partition by π.
(3) Repeat the process with $\mu_2, \dots, \mu_\ell$, i.e., until the parts of μ are depleted.

Figure 1 illustrates our insertion with an example.

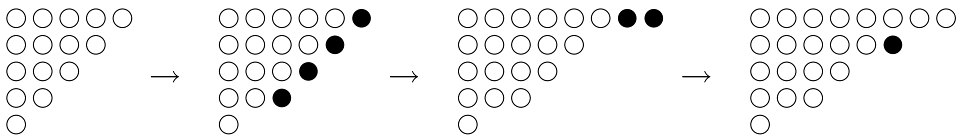


FIGURE 1. Insertion of $\mu = (4, 2, 1)$ into $\tau = (5, 4, 3, 2, 1)$.

Throughout the proof, we assume that $\pi_0 = \infty$. We first show that the final π is a partition in D_n with parity sequence $(\pi_{s+1}, \dots, \pi_{s+c})$ such that if μ_ℓ was inserted horizontally, then

$$c \geq \mu_\ell \quad \text{and} \quad \pi_s - \pi_{s+1} - 1 = \mu_\ell; \tag{6.3}$$

and if μ_ℓ was inserted vertically, then

$$c = \mu_\ell \quad \text{and} \quad \pi_s - \pi_{s+1} - 1 > \mu_\ell. \tag{6.4}$$

We use induction on ℓ. If $\ell = 1$, then

$$\pi = \begin{cases} (n + \mu_1, \underline{n-1, n-2, \dots, 2, 1}), & \text{if } n + \mu_1 \text{ is even,} \\ (\underline{n+1, n, \dots, n - \mu_1 + 2}, n - \mu_1, \dots, 2, 1), & \text{if } n + \mu_1 \text{ is odd,} \end{cases}$$

where in each case the parity sequence is underlined. Since $\mu_1 < n$, we see that $\pi \in D_n$ and the conditions in (6.3) and (6.4) are satisfied. Given $\tau = (n, n-1, \dots, 1)$ and $\mu = (\mu_1, \dots, \mu_\ell)$, suppose that the partition π resulting from the insertion of $\mu_1, \dots, \mu_{\ell-1}$ satisfies either (6.3) or (6.4). We denote

$$\pi = (\pi_1, \dots, \pi_s, \underline{\pi_{s+1}, \dots, \pi_{s+c}}, \pi_{s+c+1}, \dots, \pi_n) \in D_n,$$

where its parity sequence is underlined. By (P1), we see that π_s is even. Since $\mu_j > 1$ for any $j < \ell$, it follows from the definition of insertion that the last horizontal insertion happened at the s-th part. Thus, in order to insert μ_ℓ, we need to examine the parity of $\pi_{s+1} + \mu_\ell$ by (P1). If $\pi_{s+1} + \mu_\ell$ is even, then we make a horizontal insertion; namely, the resulting partition is

$$\pi' = (\pi_1, \dots, \pi_s, \pi_{s+1} + \mu_\ell, \pi_{s+2}, \dots, \pi_{s+c}, \pi_{s+c+1}, \dots, \pi_n).$$

Since $\pi \in D_n$, all odd positive integers $\leq \pi_{s+1}$ occur in π, from which it follows that all odd positive integers $\leq \pi_{s+2}$ occur in π'. Also, since $\pi_s - \pi_{s+1} > \mu_{\ell-1}$ by (6.3) and (6.4), we see that

$$\pi'_s - \pi'_{s+1} = \pi_s - (\pi_{s+1} + \mu_\ell) > \mu_{\ell-1} - \mu_\ell \geq 1.$$

Thus $\pi' \in D_n$. We now show that π' satisfies (6.3). Since $c \geq \mu_{\ell-1}$ by (6.3) and (6.4), and $\mu_{\ell-1} > \mu_\ell$, we see that the parity sequence of π' is $(\pi_{s+2}, \ldots, \pi_{s+c})$, which has length $c - 1 \geq \mu_\ell$. Also, since $\pi_{s+1} = \pi_{s+2} + 1$,

$$\pi'_{s+1} - \pi'_{s+2} = \pi_{s+1} + \mu_\ell - \pi_{s+2} = \mu_\ell + 1.$$

Therefore, π' is a partition in D_n satisfying (6.3). If $\pi_{s+1} + \mu_\ell$ is odd, then we make a vertical insertion; namely, the resulting partition is

$$\pi' = (\pi_1, \ldots, \pi_s, \pi_{s+1} + 1, \ldots, \pi_{s+\mu_\ell} + 1, \pi_{s+\mu_\ell+1}, \ldots, \pi_n).$$

Since $c \geq \mu_{\ell-1}$ by (6.3) and (6.4), and $\mu_{\ell-1} > \mu_\ell$, we see that the parity sequence of π' is

$$(\pi_{s+1} + 1, \ldots, \pi_{s+\mu_{\ell+1}} + 1),$$

whose length is μ_ℓ. Also, since $\pi_s - \pi_{s+1} > \mu_{\ell-1} > \mu_\ell$,

$$\pi'_s - \pi'_{s+1} = \pi_s - (\pi_{s+1} + 1) > \mu_{\ell-1} - 1 \geq \mu_\ell.$$

Thus π' satisfies (6.4). We now show that $\pi' \in D_n$. Since $\pi_{s+1} + \mu_\ell$ is odd, we see that $\pi_{s+\mu_\ell}$ is even, so $\pi_{s+\mu_\ell} + 1$ and $\pi_{s+\mu_\ell+1}$ are consecutive odd integers. Since $\pi \in D_n$, all odd positive integers $\leq \pi_{s+1}$ occur in π, from which it follows that all odd positive integers $\leq \pi_{s+1} + 1$ occur in π'. Therefore, π' is a partition in D_n satisfying (6.4).

We now show that the map is bijective by defining its inverse. Let

$$\lambda = (\lambda_1, \ldots, \lambda_s, \underline{\lambda_{s+1}, \ldots, \lambda_{s+c}}, \lambda_{s+c+1}, \ldots, \lambda_n) \in D_n,$$

where its parity sequence is underlined.

Deletion: We now compare c and $(\lambda_s - \lambda_{s+1} - 1)$.

(1) If there is no λ_s or $c < (\lambda_s - \lambda_{s+1} - 1)$, then we let $\sigma_1 = c$ and subtract 1 from each of $\lambda_{s+1}, \ldots, \lambda_{s+c}$, i.e., subtract σ_1 vertically from λ; if $c \geq (\lambda_s - \lambda_{s+1} - 1)$, then we let $\sigma_1 = \lambda_s - \lambda_{s+1} - 1$ and subtract $(\lambda_s - \lambda_{s+1} - 1)$ from λ_{s-1}, i.e., subtract σ_1 horizontally from λ.
(2) By an abuse of notation, let us denote the resulting partition by λ.
(3) Repeat the process until we arrive at $\lambda = (n, n-1, \ldots, 1)$; we record the amount we subtract in the i-th step as σ_i.

We now show that this process is well-defined, i.e., the resulting partition in each step is still in D_n and the sequence $\sigma_1, \sigma_2, \ldots$ is strictly increasing with each part less than n. If σ_1 was subtracted vertically, then the resulting partition is

$$(\lambda_1, \ldots, \lambda_s, \lambda_{s+1} - 1, \ldots, \lambda_{s+c} - 1, \lambda_{s+c+1}, \ldots, \lambda_n). \tag{6.5}$$

It follows from (P1), (P2), and (P3) that all the positive odd integers less than the largest odd part occur. If σ_1 was subtracted horizontally, then the resulting partition is

$$(\lambda_1, \ldots, \lambda_{s-1}, \lambda_{s+1} + 1, \lambda_{s+1}, \ldots, \lambda_{s+c}, \lambda_{s+c+1} \ldots, \lambda_n). \tag{6.6}$$

The largest odd part of the resulting partition is either $\lambda_{s+1} + 1$ or λ_{s+1}. Again, by (P1), (P2), and (P3), the resulting partition is in D_n.

We now show that the sequence $\sigma_1, \sigma_2, \ldots$ is strictly increasing with each part less than n. First of all, note that if $\lambda \neq (n, n-1, \ldots, 1)$, then $c < n$. Thus we can easily see that $\sigma_1 < n$ since $\sigma_1 \leq c$. It now suffices to show that $\sigma_i > \sigma_{i+1}$ for $i = 1, 2, \ldots$. Suppose that σ_1 was subtracted vertically from λ. Then, in (6.5), the length c^* of the parity sequence of the resulting partition is larger than c. Also,

$$\lambda_s - (\lambda_{s+1} - 1) - 1 = \lambda_s - \lambda_{s+1} > c.$$

Since σ_2 is the minimum of c^* and $\lambda_s - (\lambda_{s+1} - 1) - 1$, we see that $\sigma_2 > \sigma_1$. Suppose that σ_1 was subtracted horizontally from λ. Then, in (6.6), the length c^* of the parity sequence of the resulting partition is larger than c, which is larger than or equal to $(\lambda_s - \lambda_{s+1} - 1)$. Also,

$$\lambda_{s-1} - (\lambda_{s+1} + 1) - 1 = \lambda_{s-1} - \lambda_{s+1} - 2 \geq \lambda_s + 2 - \lambda_{s+1} - 2 > \lambda_s - \lambda_{s+1} - 1 = \sigma_1,$$

where the first inequality follows from (P1). Since σ_2 is the minimum of c^* and $\lambda_{s-1} - (\lambda_{s+1} + 1) - 1$, we see that $\sigma_2 > \sigma_1$.

We now show that the deletion map defined above is the inverse process of our insertion map. Let π be the partition resulting from the insertion of $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ into τ, namely

$$\pi = (\pi_1, \ldots, \pi_s, \underline{\pi_{s+1}, \ldots, \pi_{s+c}}, \pi_{s+c+1}, \ldots, \pi_n) \in D_n.$$

If μ_ℓ was inserted horizontally, then we see that

$$c \geq \mu_\ell = \pi_s - \pi_{s+1} - 1,$$

by (6.3). Thus, by the map, we have to subtract μ_ℓ horizontally. If μ_ℓ was inserted vertically, then we see that

$$c = \mu_\ell \leq \pi_s - \pi_{s+1} - 1,$$

by (6.4). Thus, by the map, we have to subtract μ_ℓ vertically. □

Theorem 6.3. [28, p. 28], [10, Entry 1.6.2] *For any complex number a,*

$$\sum_{n=0}^{\infty} a^n q^{n^2} = \sum_{n=0}^{\infty} \frac{(-q;q)_{n-1} a^n q^{n(n+1)/2}}{(-aq^2;q^2)_n}. \tag{6.7}$$

Proof. Let E_n be the set of partitions into even parts less than or equal to $2n$. By Theorem 6.2, the right-hand side of (6.7) generates pairs of partitions (π, σ) with $\pi \in D_n$ and $\sigma \in E_n$, where the exponent of a denotes the number of parts of π plus the number of parts of σ, with the sign $(-1)^{\ell(\sigma)}$. Let π_e (resp. σ_e) be the largest even part in π (resp. σ). For convenience, we define $\pi_e = 0$ (resp. $\sigma_e = 0$) if there is no even part in π (resp. σ). Note that by the definition of D_n, the following are equivalent:

(i) $\pi = (2n-1, 2n-3, \ldots, 3, 1)$;
(ii) $\pi_e = 0$;
(iii) $\pi_1 = 2n - 1$.

We now compare π_e and σ_e.

Case 1: If $\pi_e > 0$ and $\pi_e \geq \sigma_e$, then move π_e to σ. We denote by (π', σ') the resulting partition pair. Since $\pi \in D_n$ and $\pi_e > 0$, π has n parts $\leq 2n-2$. Thus, π' has $n-1$

parts $< 2n-2$ and σ'_e is still less than or equal to $2n-2$, from which it follows that $\pi' \in D_{n-1}$ and $\sigma' \in E_{n-1}$. The pair (π', σ') is generated by the right-hand side of (6.7), and it has the opposite sign.

Case 2: If $\sigma_e > 0$ and $\sigma_e > \pi_e$, then move σ_e to π. We denote by (π', σ') the resulting partition pair. Since $\pi \in D_n$, π has n parts $< 2n$. Also, since $\sigma \in E_n$, $\sigma_e \leq 2n$. Thus, π' has $n+1$ parts $\leq 2n$, from which it follows that $\pi' \in D_{n+1}$ and $\sigma' \in E_{n+1}$. The pair (π', σ') is generated by the right-hand side of (6.7), and it has the opposite sign.

Therefore, the partition pairs (π, σ) with $\pi_e > 0$ or $\sigma_e > 0$ are cancelled, and there remain only $\pi = (2n-1, 2n-3, \ldots, 1)$ and $\sigma = \emptyset$, which are generated by the left-hand side of (6.7). $\square$

Alladi [3] has devised a completely different proof of Theorem 6.3 and has also provided a number-theoretic interpretation of Theorem 6.3 as a weighted partition theorem. Although we have given a bijective proof of Theorem 6.3, we do not interpret Theorem 6.3 number-theoretically. On the other hand, even though Alladi interpreted Theorem 6.3 number-theoretically, his proof of Theorem 6.3 is q-theoretic. It would be worthwhile to see how our bijective proof of Theorem 6.3 translates into a combinatorial proof of Alladi's weighted partition theorem.

Recently, the third author [33] found another combinatorial proof of Theorem 6.3.

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