A BIJECTIVE PROOF OF A LIMITING CASE OF WATSON'S ${}_8\phi_7$ TRANSFORMATION FORMULA

MICHAEL ROWELL AND AE JA YEE¹

Abstract. Andrews gave a combinatorial proof of the Rogers–Fine identity. In this paper, we present a combinatorial proof of a special case of Watson's ${}_8\phi_7$ transformation formula, which is a generalization of Andrews' proof.

1. INTRODUCTION

In his book, *Basic Hypergeometric Series and Applications* [6, p. 15], N. J. Fine thoroughly studies the Rogers–Fine identity

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \tau^n = \sum_{n=0}^{\infty} \frac{(1 - \alpha\tau q^{2n})(\alpha)_n(\alpha\tau q/\beta)_n \beta^n \tau^n q^{n^2-n}}{(\beta)_n(\tau)_{n+1}}, \quad (1.1)$$

where

$$\begin{aligned} (a)_0 &:= (a; q)_0 := 1, \\ (a)_n &:= (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad n \geq 1, \\ (a)_{\infty} &:= (a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| < 1. \end{aligned}$$

As the name suggests, the identity (1.1) was first proved by L. J. Rogers [8] in 1917. However, Fine extensively studied the summation on the left hand side of (1.1) and made several applications of it in his monograph [6]. G. E. Andrews [2, Sect. 4] proved (1.1) combinatorially and gave it the appellation, the Rogers–Fine identity. Subsequently, it was learned that G. W. Starcher [9, p. 803] had also discovered and proved most of (1.1). In 2004, using overpartitions, S. Corteel and J. Lovejoy again proved the Rogers–Fine identity [5]. One year later, using a different method, J. Zeng also proved the identity combinatorially [10].

In basic hypergeometric series, (1.1) can be viewed as a special case of Watson's ${}_8\phi_7$ transformation formula [7]. In this paper, we generalize the idea of Andrews' proof of Rogers–Fine identity and give a combinatorial proof of a special case of Watson's q -analog:

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta\gamma/\alpha\tau)_n \tau^n}{(\beta)_n(\gamma q)_n} = \sum_{n=0}^{\infty} \frac{(1 - \alpha\tau q^{2n})(\alpha)_n(\alpha\tau q/\beta)_n(\alpha\tau/\gamma)_n q^{n(n-1)/2} \left(-\frac{\beta\gamma}{\alpha}\right)^n}{(\beta)_n(\gamma q)_n(\tau)_{n+1}}. \quad (1.2)$$

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Letting γ tend to zero, we see that (1.2) yields to (1.1).

In Section 2, we review the combinatorial proof of Andrews. In Section 3, we state (1.2) in terms of partitions and give a combinatorial proof. We conclude by explaining how our proof includes the proof of Andrews.

1.1. Definitions. In this section, we provide definitions that we will need.

A *partition* of a positive integer n is a weakly decreasing sequence of positive integers $(\lambda_1, \dots, \lambda_r)$ such that $\lambda_1 + \dots + \lambda_r = n$ (see [3].) We denote the number of parts of a partition λ by $\ell(\lambda)$. As a convention, we denote a partition of 0 by $\emptyset$. In later sections we relax our definition of partition to finite sequences which may not be weakly decreasing or have 0 as a part.

We introduce partitions with subscripts. (Some similar notion can be found in the literature, for instance, colored integers [1].) For a variable x , we define a *partition* λ with the *subscript* x , denoted by λ^x , as a partition whose parts have the subscript x . We also write the sum of the parts of λ^x without the subscript x as $|\lambda^x|$ and denote the number of parts by $\ell(\lambda^x)$. If there is no confusion, we drop the subscript x of the parts of λ^x . For partitions λ^x and λ^y , we define their *union* $\lambda^x \cup \lambda^y$ as a partition consisting of the parts of λ^x and λ^y . We define the *natural order* $\succeq$ for the parts of $\lambda^x \cup \lambda^y$ by $a_r \succeq b_s$ if and only if

$$a > b \quad \text{or} \quad (a = b \text{ and } r = s) \quad \text{or} \quad (a = b \text{ and } r = x, s = y).$$

We write the parts of the union of partitions in weakly decreasing natural order $\succeq$. For instance, $(3_x, 3_x, 2_x) \cup (4_y, 3_y, 2_y, 2_y, 1_y)$ is the partition $(4_y, 3_x, 3_x, 3_y, 2_x, 2_y, 2_y, 1_y)$. We define the union of more than two partitions inductively.

Let p be a nonnegative integer and λ^y be a partition. We define an *insertion* $\iota(p, \lambda)$ as follows. Let i be the largest j such that

$$\lambda_j \geq p + j + 1. \tag{1.3}$$

Then we insert the integer p between λ_i and λ_{i+1} . If there does not exist such an i , then we put the p before λ_1 . We denote the resulting partition by $(p) \cup^* \lambda^y$. For instance,

$$\begin{aligned} (6) \cup^* (7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) &= (6_x, 7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y), \\ (5) \cup^* (7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) &= (7_y, 5_x, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y), \end{aligned}$$

where we used the variable x to distinguish p from the parts of λ^y .

We generalize the insertion ι . Let λ^x be a partition into distinct nonnegative parts and λ^y be an ordinary partition. We successively insert the parts of λ^x into λ^y from the largest as follows. Suppose the first i parts of λ^x have been inserted into λ^y . Let λ be the resulting partition. Then we do comparison (1.3) with $p = \lambda_{i+1}^x$ and only the parts of λ with the subscript y , that is, let

$$m = \max\{j \mid \lambda_j \geq \lambda_{i+1}^x + j + 1 \text{ for } \lambda_j \text{ with the subscript } y\}.$$

We insert λ_{i+1}^x between λ_m and λ_{m+1} . If there is no such m , then we insert λ_{i+1}^x right before the largest part with the subscript y . We denote the resulting partition by $\lambda^x \cup^* \lambda^y$. For instance, let

$$\lambda^x = (6_x, 5_x, 3_x, 0_x), \quad \lambda^y = (7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y).$$

Then

$$\begin{aligned}
(6_x) \cup^* (7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) &= (6_x, 7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) \\
(5_x) \cup^* (6_x, 7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) &= (6_x, 5_x, 7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) \\
(3_x) \cup^* (6_x, 5_x, 7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) &= (6_x, 5_x, 7_y, 3_x, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) \\
(0_x) \cup^* (6_x, 5_x, 7_y, 3_x, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) &= (6_x, 5_x, 7_y, 3_x, 6_y, 0_x, 6_y, 3_y, 3_y, 2_y, 1_y).
\end{aligned}$$

Thus $(6_x, 5_x, 3_x, 0_x) \cup^* (7_y, 6_y, 6_y, 3_y, 3_y, 2_y, 1_y) = (6_x, 5_x, 7_y, 3_x, 6_y, 0_x, 6_y, 3_y, 3_y, 2_y, 1_y)$.

Throughout this paper, we define the weight of a partition λ with subscripts $x, y, z, \dots$ as

$$Wt(\lambda) = x^{n(x)} y^{n(y)} z^{n(z)} \dots q^{|\lambda|},$$

where $n(t)$ is the number of parts of λ with the subscript t , and we define the weighted generating function as

$$\sum_{\lambda} Wt(\lambda),$$

where the sum is over all partitions λ .

The l -Durfee rectangle of a partition is the largest rectangle with width minus height equal to l that fits inside of the Ferrers diagram of the partition [4]. The 0-Durfee rectangle of a partition is simply the Durfee square of the partition.

2. THE ROGERS-FINE IDENTITY

In this section, we sketch the combinatorial proof of (1.1) given by Andrews. First, we make some changes of variables in (1.1). Substituting $-aq$, bq , and t for α , β , and τ , respectively, we have

$$\sum_{n=0}^{\infty} \frac{(-aq)_n}{(bq)_n} t^n = \sum_{n=0}^{\infty} \frac{(1 + atq^{2n+1})(-aq)_n(-atq/b)_n b^n t^n q^{n^2}}{(bq)_n (t)_{n+1}}. \quad (2.1)$$

Note that $(-aq)_n$ generates partitions into distinct parts less than or equal to n and $1/(bq)_n$ generates partitions into parts less than or equal to n . Thus, the summation on the left hand side of (2.1) can be rewritten as

$$\frac{1}{1-t} \sum_{\lambda = \lambda^b \cup \lambda^a} Wt(\lambda) t^r,$$

where λ^b is an ordinary partition, λ^a is a partition into distinct parts, and r is the largest part of λ . We see that (2.1) is equivalent to

$$\sum_{\lambda = \lambda^b \cup \lambda^a} Wt(\lambda) t^r = \sum_{n=0}^{\infty} \frac{(1 + atq^{2n+1})(-aq)_n(-atq/b)_n b^n t^n q^{n^2}}{(bq)_n (tq)_n}. \quad (2.2)$$

We now show that the right hand side of (2.2) generates the same pairs of partitions. For a given $\lambda = \lambda^b \cup \lambda^a$, we form its Ferrers graph and break it down in the following way. Let d be the side of the Durfee square. We consider two cases depending on whether or not the λ_d equals d_a .

Case 1: $\lambda_d \succ d_a$, i.e., $\lambda_d > d$ or $\lambda_d = d_b$. Then the parts below the Durfee square are less than or equal to d . Thus $(-aq)_d/(bq)_d$ generates the parts below the Durfee square. For the parts to the right of the Durfee square, we take its conjugate. Then the parts of the conjugate cannot be greater than d . Thus $(-atq/b)_d/(tq)_d$ generates parts less than or equal

to d . Combining these with the Durfee square, we see that the generating function for such partitions is

$$\frac{(-aq)_d(-atq/b)_d(bt)^d q^{d^2}}{(bq)_d(tq)_d}.$$

Case 2: $\lambda_d = d_a$. Then the parts below the Durfee square or to the right of it are less than d . Peeling off the last row and column in the Durfee square, we obtain a partition that satisfies the condition for Case 1. We multiply the generating function for the resulting partitions by $(at)q^{2d-1}$ that accounts for the last row and column removed from the Durfee square. Then we see that the generating function for such partitions is

$$\frac{(-aq)_{d-1}(-atq/b)_{d-1}(at)(bt)^d q^{d^2}}{(bq)_{d-1}(tq)_{d-1}}.$$

Combining our two generating functions and summing over d gives the desired identity.

Figure 1 describes two cases examined in the proof of the Rogers-Fine identity.

FIGURE 1. The Rogers-Fine Identity.

3. WATSON'S TRANSFORMATION FORMULA FOR ${}_8\phi_7$ SERIES

We introduce Watson's transformation formula for ${}_8\phi_7$ series [7, Appendix III.17]

$$\begin{aligned} & {}_8\phi_7 \left(a, qa^{1/2}, -qa^{1/2}, b, c, d, e, f; a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq/f; q; \frac{a^2 q^2}{bcdef} \right) \\ &= \frac{(aq)_\infty (aq/de)_\infty (aq/df)_\infty (aq/ef)_\infty}{(aq/d)_\infty (aq/e)_\infty (aq/f)_\infty (aq/def)_\infty} {}_4\phi_3(aq/bc, d, e, f; aq/b, aq/c, def/a; q; q), \end{aligned} \quad (3.1)$$

whenever the ${}_8\phi_7$ series converges and the ${}_4\phi_3$ series terminates, where

$${}_{r+1}\phi_r(a_1, a_2, \dots, a_{r+1}; b_1, b_2, \dots, b_r; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n.$$

Specializing a, b, c, d, e, f to $\alpha\tau, \alpha\tau q/\beta, \alpha\tau/\gamma, \alpha, q^{-N}, q$ and letting N go to infinity, we can obtain (1.2). For our purpose, however, we substitute $-\alpha\tau q, -\alpha\tau q/\beta, -\alpha\tau q/\gamma, -\alpha q, q^{-N}$, and q for a, b, c, d, e , and f , respectively. We then let N approach infinity and multiply each side by $(1 + \alpha\tau q)/(1 - \tau)$.

Theorem 3.1 (A special case of Watson's ${}_8\phi_7$ transformation formula).

$$\sum_{n=0}^{\infty} \frac{\left(-\frac{\beta\gamma}{\alpha\tau}\right)_n (-\alpha q)_n \tau^n}{(\beta q)_n (\gamma q)_n} = \sum_{n=0}^{\infty} \frac{(1 + \alpha\tau q^{2n+1})(-\alpha q)_n \left(-\frac{\alpha\tau q}{\beta}\right)_n \left(-\frac{\alpha\tau q}{\gamma}\right)_n q^{n(n-1)/2} \left(\frac{\beta\gamma}{\alpha}\right)^n}{(\beta q)_n (\gamma q)_n (\tau)_{n+1}}. \quad (3.2)$$

We now give a bijective proof of Theorem 3.1.

Proof. We first write the left hand side in terms of partition generating functions as follows:

$$\sum_{n=0}^{\infty} \sum_{\lambda^{\beta\gamma/\alpha\tau}, \lambda^\alpha, \lambda^\beta, \lambda^\gamma} \tau^n \left(\frac{\beta\gamma}{\alpha\tau}\right)^{\ell(\lambda^{\beta\gamma/\alpha\tau})} \alpha^{\ell(\lambda^\alpha)} \beta^{\ell(\lambda^\beta)} \gamma^{\ell(\lambda^\gamma)} q^{|\lambda^{\beta\gamma/\alpha\tau}| + |\lambda^\alpha| + |\lambda^\beta| + |\lambda^\gamma|}, \quad (3.3)$$

where $\lambda^{\beta\gamma/\alpha\tau}$ is a partition into distinct nonnegative parts less than n , λ^α is a partition into distinct positive parts less than or equal to n , and $\lambda^\beta, \lambda^\gamma$ are partitions into positive parts less than or equal to n . We will define a bijection from a quadruple $(\lambda^{\beta\gamma/\alpha\tau}, \lambda^\alpha, \lambda^\beta, \lambda^\gamma)$ of these four partitions to a septuple $(\lambda^{\beta\gamma/\alpha}, \lambda^{\alpha\tau/\beta}, \lambda^{\alpha\tau/\gamma}, \lambda^\tau, \nu^\alpha, \nu^\beta, \nu^\gamma)$ of partitions, the first four from our later defined a and the other three from our later defined ν . We will then see that the right hand side of (3.2) is the generating function for these 7-tuples $(\lambda^{\beta\gamma/\alpha}, \lambda^{\alpha\tau/\beta}, \lambda^{\alpha\tau/\gamma}, \lambda^\tau, \nu^\alpha, \nu^\beta, \nu^\gamma)$ of partitions with the last one a staircase.

Given four partitions $\lambda^{\beta\gamma/\alpha\tau}$, $\lambda^\beta, \lambda^\gamma$, and λ^α , we let $\lambda = \lambda^\beta \cup \lambda^\gamma \cup \lambda^\alpha$ and $\ell(\lambda^{\beta\gamma/\alpha\tau}) = l$. We take the l -Durfee rectangle of λ . Let m be the width of the Durfee rectangle. We define μ as the partition made up of the $(m - l)$ largest parts of λ and ν as the partition made up of all parts below the l -Durfee rectangle of λ . We let a be $\lambda^{\beta\gamma/\alpha\tau} \cup^* \mu$. For instance, given

$$\lambda^{\beta\gamma/\alpha\tau} = (15, 9, 1) \quad \text{and} \quad \lambda = (13_\beta, 13_\alpha, 12_\alpha, 9_\gamma, 7_\beta, 6_\beta, 6_\gamma, 6_\alpha, 4_\alpha, 1_\beta, 1_\beta, 1_\gamma, 1_\alpha),$$

we see that $m = 7$ and

$$\begin{aligned} \mu &= (13_\beta, 13_\alpha, 12_\alpha, 9_\gamma), \\ \nu &= (7_\beta, 6_\beta, 6_\gamma, 6_\alpha, 4_\alpha, 1_\beta, 1_\beta, 1_\gamma, 1_\alpha). \end{aligned}$$

We also see that

$$a = \lambda^{\beta\gamma/\alpha\tau} \cup^* \mu = (15_{\beta\gamma/\alpha\tau}, 13_\beta, 13_\alpha, 9_{\beta\gamma/\alpha\tau}, 12_\alpha, 9_\gamma, 1_{\beta\gamma/\alpha\tau}).$$

We now divide a into three cases: Case 1: $\mu_{\ell(\mu)} \succeq m_\gamma$, Case 2: $\mu_{\ell(\mu)} = m_\alpha$, $\lambda^{\beta\gamma/\alpha\tau}$ has a part of size 0, and Case 3: $\mu_{\ell(\mu)} = m_\alpha$, $\lambda^{\beta\gamma/\alpha\tau}$ has no part of size 0.

Case 1: $\mu_{\ell(\mu)} \succeq m_\gamma$. For each a_i with the subscript γ or α , define $b_i = i$; otherwise define $b_i = 0$. For all a_i with the subscript β , define $c_i = i$; otherwise $c_i = 0$. So there exists d_i such that

$$a_i = (m - i) + b_i + c_i + d_i. \quad (3.4)$$

From $a = (15_{\beta\gamma/\alpha\tau}, 13_\beta, 13_\alpha, 9_{\beta\gamma/\alpha\tau}, 12_\alpha, 9_\gamma, 1_{\beta\gamma/\alpha\tau})$ above, we find that

$$\begin{aligned}
15_{\beta\gamma/\alpha\tau} &= 6 + 0 + 0 + 9, \\
13_\beta &= 5 + 0 + 2 + 6, \\
13_\alpha &= 4 + 3 + 0 + 6, \\
9_{\beta\gamma/\alpha\tau} &= 3 + 0 + 0 + 6, \\
12_\alpha &= 2 + 5 + 0 + 5, \\
9_\gamma &= 1 + 6 + 0 + 2, \\
1_{\beta\gamma/\alpha\tau} &= 0 + 0 + 0 + 1.
\end{aligned}$$

Note that $(d_1, \dots, d_m)$ is a weakly decreasing nonnegative sequence due to the definition of $\cup^*$. Also, note that $d_i > 0$ for each a_i with the subscript α since $\mu_{\ell(\mu)} \succ m_\alpha$. For each a_i with the subscript α (starting with the largest i), let h_i be the largest j such that $d_j = d_i$. The h_i is uniquely determined. We then subtract 1 from each of the first h_i d_j 's, and with the extra h_i nodes replace c_{h_i} (previously 0) with h_i . Note that our d_i 's could be different after possibly having to subtract 1's from d_j for each part a_i with the subscript α , but the resulting $(d_1, \dots, d_m)$ is still a weakly decreasing nonnegative sequence. For our example, we have

a_i	$(m-i)$	b_i	c_i	d_i	$\rightarrow$	b_i	c_i	d_i	$\rightarrow$	b_i	c_i	d_i
$15_{\beta\gamma/\alpha\tau}$	6	0	0	9		0	0	8		0	0	7
13_β	5	0	2	6		0	2	5		0	2	4
13_α	4	3	0	6		3	0	5		3	0	4
$9_{\beta\gamma/\alpha\tau}$	3	0	0	6		0	0	5		0	4	4
12_α	2	5	0	5		5	5	4		5	5	4
9_γ	1	6	0	2		6	0	2		6	0	2
$1_{\beta\gamma/\alpha\tau}$	0	0	0	1		0	0	1		0	0	1

The final b_i, c_i, d_i are uniquely determined for a given a . Recall the decomposition of the parts of a : $a_i = (m-i) + b_i + c_i + d_i$, which is uniquely determined. After the manipulation, the resulting c_i and d_i could be different from the original ones. However, for each a_i with the subscript α from the largest i , we took the largest h_i for which $d_{h_i} = d_i$. This implies that the resulting c_i and d_i are uniquely determined. Note that the b_i and c_i are nonnegative integers less than or equal to n .

We now show that the process is reversible. Given b_i, c_i , and d_i , we let

$$a_i = (m-i) + b_i + c_i + d_i$$

and mark the a_i with the subscript γ if $b_i > 0$. We now change c_i (starting from the smallest i) as follows. If $c_i = i$ and there exists a $j \leq i$ such that $b_j = j$ and $d_j = d_i$, then we define $c_i = 0$, add one to each of the first i parts of d , and change the subscript of a_j from γ to α . We now mark the unmarked parts of the resulting a_i as follows. If $b_i = c_i = 0$, then mark a_i with the subscript $\beta\gamma/\alpha\tau$; if $b_i = 0$ and $c_i > 0$, then mark a_i with the subscript β . We now show that this process is the inverse of our original process of decomposing a_i . In our original process, for each b_i from a_i with the subscript α , we took the largest h_i for which $d_{h_i} = d_i$. Thus after the manipulation for each b_i from a_i with subscript α , we see that $h_i \geq i$, $c_{h_i} = h_i$, $b_i = i$, and $d_i = d_{h_i}$. By applying the inverse process to the resulting a_i , we find that $c_{h_i} = 0$ and each d_j for $j \leq h_i$ is increased by 1, which shows that the process is invertible.

We now define $\lambda^{\beta\gamma/\alpha}$ to be $((m-1), (m-2), \dots, 1, 0)$, $\lambda^{\alpha\tau/\beta}$ to be made up of the nonzero b_i 's, $\lambda^{\alpha\tau/\gamma}$ to be made up of the nonzero c_i 's, λ^τ to be the conjugate of the partition made up of the nonzero d_i 's. Then it can be easily seen that the generating function of $(\lambda^{\beta\gamma/\alpha}, \lambda^{\alpha\tau/\beta}, \lambda^{\alpha\tau/\gamma}, \lambda^\tau)$ is

$$\frac{q^{m(m-1)/2} \left(\frac{\beta\gamma}{\alpha}\right)^m \left(-\frac{\alpha\tau q}{\beta}\right)_m \left(-\frac{\alpha\tau q}{\gamma}\right)_m}{(\tau q)_m}. \quad (3.5)$$

We now consider the multiplication of the subscripts of $\lambda^{\beta\gamma/\alpha}, \lambda^{\alpha\tau/\beta}, \lambda^{\alpha\tau/\gamma}, \lambda^\tau$. For convenience, we denote the original d_1 by e . Then, $m+e$ is less than or equal to n since the parts of $\lambda^{\beta\gamma/\alpha\tau}$ are less than n and the parts of $\lambda^\alpha, \lambda^\beta, \lambda^\gamma$ are less than or equal to n . The multiplication of the subscripts of the parts of $\lambda^{\beta\gamma/\alpha}, \lambda^{\alpha\tau/\beta}, \lambda^{\alpha\tau/\gamma}, \lambda^\tau$ yields τ^{m+e} multiplied by the subscripts of a_i . For our example,

$$\left(\frac{\beta\gamma}{\alpha}\right)^7 \left(\frac{\alpha\tau}{\beta}\right)^3 \left(\frac{\alpha\tau}{\gamma}\right)^3 \tau^{11} = \alpha^{-1} \beta^4 \gamma^4 \tau^{17} = \left(\frac{\beta\gamma}{\alpha\tau}\right)^3 \alpha^2 \beta \gamma \tau^{7+13}.$$

To show the equality of the multiplications of the subscripts, we compare the exponents of α, β, γ , and τ . We first decomposed a_i as a sum of $(m-i), b_i, c_i, d_i$:

$$a_i = (m-i) + b_i + c_i + d_i.$$

Then, if the subscript of a_i is α , we create a nonzero c_i . So we can think that the subscripts of a_i are decomposed as follows:

$$\begin{aligned} \frac{\beta\gamma}{\alpha\tau} &= \frac{\beta\gamma}{\alpha} \times \frac{1}{\tau}, \\ \beta &= \frac{\beta\gamma}{\alpha} \times \frac{\alpha\tau}{\gamma} \times \frac{1}{\tau}, \\ \gamma &= \frac{\beta\gamma}{\alpha} \times \frac{\alpha\tau}{\beta} \times \frac{1}{\tau}, \\ \alpha &= \frac{\beta\gamma}{\alpha} \times \frac{\alpha\tau}{\beta} \times \frac{\alpha\tau}{\gamma} \times \frac{1}{\tau^2}, \end{aligned}$$

which shows that the exponents of α, β, γ are well preserved during the process. For the exponent of τ , we first denote the number of a_i with the subscript α, β , and γ by s_α, s_β , and s_γ , respectively. Note that the final d_1 is less than e by s_α since we subtracted 1 from d_i when we have a_i with the subscript α . We also note that $\ell(\lambda^\tau)$ equals the final d_1 since we took the conjugate. Thus $e = \ell(\lambda^\tau) + s_\alpha$. Hence

$$\begin{aligned} m+e - \ell(\lambda^{\beta\gamma/\alpha\tau}) &= \ell(a) + \ell(\lambda^\tau) + s_\alpha - \ell(\lambda^{\beta\gamma/\alpha\tau}) \\ &= 2s_\alpha + s_\beta + s_\gamma + \ell(\lambda^\tau) \\ &= \ell(\lambda^{\alpha\tau/\gamma}) + \ell(\lambda^{\alpha\tau/\beta}) + \ell(\lambda^\tau), \end{aligned}$$

where the last equality follows from the decompositions of α, β, γ shown above. We now define ν^α, ν^β , and ν^γ to be made up of all parts of ν with the subscripts α, β , and γ , respectively,

$$\begin{aligned}\lambda^{\beta\gamma/\alpha} &= (6, 5, 4, 3, 2, 1) \\ \lambda^{\alpha\tau/\beta} &= (6, 5, 3) \\ \lambda^{\alpha\tau/\gamma} &= (5, 4, 2) \\ \lambda^\tau &= (7, 6, 5, 5, 1, 1, 1) \\ \nu^\alpha &= (6, 4, 1) \\ \nu^\beta &= (7, 6, 1, 1) \\ \nu^\gamma &= (6, 1)\end{aligned}$$

Figure 2 shows how $\lambda^{\beta\gamma/\alpha\tau} \cup^* (\lambda^\beta \cup \lambda^\gamma \cup \lambda^\alpha)$ decomposes into seven partitions $\lambda^{\beta\gamma/\alpha}, \lambda^{\alpha\tau/\beta}, \lambda^{\alpha\tau/\gamma}, \lambda^\tau, \nu^\alpha, \nu^\beta, \nu^\gamma$. The empty circles do not contribute the weight of $\lambda^{\beta\gamma/\alpha\tau} \cup^* (\lambda^\beta \cup \lambda^\gamma \cup \lambda^\alpha)$.

FIGURE 2. A Ferrers graph of $\lambda^{\beta\gamma/\alpha\tau} \cup^* (\lambda^\beta \cup \lambda^\gamma \cup \lambda^\alpha)$ from our example and the expansion into the seven partitions $\lambda^{\beta\gamma/\alpha}, \lambda^{\alpha\tau/\beta}, \lambda^{\alpha\tau/\gamma}, \lambda^\tau, \nu^\alpha, \nu^\beta, \nu^\gamma$.

Case 2: $\mu_{\ell(\mu)} = m_\alpha$ and $\lambda^{\beta\gamma/\alpha\tau}$ has a part of size 0. In this case, it follows from the definition of $\cup^*$ that the last part of a is 0. We delete the last part of a and change the subscript α of the part m_α of a to γ . We then subtract one from all the remaining nonzero parts from a and add a new largest part $m-1$ with the subscript β to ν . We now continue as though this were Case 1 with the width of the Durfee rectangle equal to $m-1$. For example, consider

$$\lambda^{\beta\gamma/\alpha\tau} = (5, 4, 3, 0) \quad \text{and} \quad \lambda = \lambda^\beta \cup \lambda^\gamma \cup \lambda^\alpha = (6_\gamma, 6_\alpha, 4_\alpha, 3_\beta, 3_\alpha, 2_\beta, 2_\gamma, 1_\beta).$$

Then

$$\begin{aligned}\mu &= (6_\gamma, 6_\alpha) \\ \nu &= (4_\alpha, 3_\beta, 3_\alpha, 2_\beta, 2_\gamma, 1_\beta) \\ a &= (5_{\beta\gamma/\alpha\tau}, 4_{\beta\gamma/\alpha\tau}, 3_{\beta\gamma/\alpha\tau}, 6_\gamma, 6_\alpha, 0_{\beta\gamma/\alpha\tau}).\end{aligned}$$

After our defined manipulation we have

$$\begin{aligned}\nu &= (5_\beta, 4_\alpha, 3_\beta, 3_\alpha, 2_\beta, 2_\gamma, 1_\beta), \\ a &= (4_{\beta\gamma/\alpha\tau}, 3_{\beta\gamma/\alpha\tau}, 2_{\beta\gamma/\alpha\tau}, 5_\gamma, 5_\gamma),\end{aligned}$$

and

$$\begin{aligned}
4_{\beta\gamma/\alpha\tau} &= 4 + 0 + 0 + 0 \\
3_{\beta\gamma/\alpha\tau} &= 3 + 0 + 0 + 0 \\
2_{\beta\gamma/\alpha\tau} &= 2 + 0 + 0 + 0 \\
5_\gamma &= 1 + 4 + 0 + 0 \\
5_\gamma &= 0 + 5 + 0 + 0.
\end{aligned}$$

Applying the manipulation for Case 1 to the resulting a , we get

$$\begin{aligned}
\lambda^{\beta\gamma/\alpha} &= (4, 3, 2, 1) \\
\lambda^{\alpha\tau/\beta} &= (5, 4) \\
\lambda^{\alpha\tau/\gamma} &= \emptyset \\
\lambda^\tau &= \emptyset \\
\nu^\alpha &= (4, 3) \\
\nu^\beta &= (5, 3, 2, 1) \\
\nu^\gamma &= (2)
\end{aligned}$$

We now show that the multiplication of the subscripts is well preserved after the defined manipulation. Without loss of generality, we consider only the subscripts of the parts changed through the manipulation. Note that deleting one from the largest part of a results in a loss of a τ . Since

$$\frac{\beta\gamma}{\alpha\tau} \times \alpha \times \tau = \beta \times \gamma,$$

the multiplication of the subscripts is preserved.

Case 3: $\mu_{\ell(\mu)} = m_\alpha$ and $\lambda^{\beta\gamma/\alpha\tau}$ has no part of size 0. In this case, all parts of a are positive and all parts of ν are less than m . We remove the part m_α and subtract one from each of the remaining parts of a . Then the sum of the parts of the resulting a is $2m - 1$ less than that of the original a . We now see that the resulting partition falls in Cases 1 or 2 with the width of the Durfee rectangle replaced with $m - 1$. For example,

$$\lambda^{\beta\gamma/\alpha\tau} = (5, 4, 3, 1) \quad \text{and} \quad \lambda = (6_\gamma, 6_\alpha, 4_\alpha, 3_\beta, 3_\alpha, 2_\beta, 2_\gamma, 1_\beta).$$

Then $m = 6$ and

$$\begin{aligned}
\mu &= (6_\gamma, 6_\alpha) \\
\nu &= (4_\alpha, 3_\beta, 3_\alpha, 2_\beta, 2_\gamma, 1_\beta) \\
a &= (5_{\beta\gamma/\alpha\tau}, 4_{\beta\gamma/\alpha\tau}, 3_{\beta\gamma/\alpha\tau}, 6_\gamma, 1_{\beta\gamma/\alpha\tau}, 6_\alpha).
\end{aligned}$$

So we take out $q^{2 \times 6 - 1}$ and are left with

$$a = (4_{\beta\gamma/\alpha\tau}, 3_{\beta\gamma/\alpha\tau}, 2_{\beta\gamma/\alpha\tau}, 5_\gamma, 0_{\beta\gamma/\alpha\tau}),$$

from which we get

$$\begin{aligned}
\lambda^{\beta\gamma/\alpha\tau} &= (4_{\beta\gamma/\alpha\tau}, 3_{\beta\gamma/\alpha\tau}, 2_{\beta\gamma/\alpha\tau}, 0_{\beta\gamma/\alpha\tau}) \\
\mu &= (5_\gamma)
\end{aligned}$$

Notice that the μ_1 is 5_γ . So we continue as in Case 1.

We once again show that the multiplication of the subscripts is well preserved after the defined manipulation. Note that deleting one from the largest part of a results in a loss of a τ . Since

$$\alpha \times \tau = \alpha\tau$$

the multiplication of the subscripts is preserved.

We now show that the generating function of $(\lambda^{\beta\gamma/\alpha\tau}, \lambda^\alpha, \lambda^\beta, \lambda^\gamma)$ is the right hand side of (3.2), namely

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{(\lambda^{\beta\gamma/\alpha\tau}, \lambda^\alpha, \lambda^\beta, \lambda^\gamma)} \tau^n \left(\frac{\beta\gamma}{\alpha\tau} \right)^{\ell(\lambda^{\beta\gamma/\alpha\tau})} \alpha^{\ell(\lambda^\alpha)} \beta^{\ell(\lambda^\beta)} \gamma^{\ell(\lambda^\gamma)} q^{|\lambda^{\beta\gamma/\alpha\tau}| + |\lambda^\alpha| + |\lambda^\beta| + |\lambda^\gamma|} \\ &= \frac{1}{1-\tau} \sum_{n=0}^{\infty} \frac{(1 + \alpha\tau q^{2n+1})(-\alpha q)_n \left(-\frac{\alpha\tau q}{\beta} \right)_n \left(-\frac{\alpha\tau q}{\gamma} \right)_n q^{n(n-1)/2} \left(\frac{\beta\gamma}{\alpha} \right)^n}{(\beta q)_n (\gamma q)_n (\tau q)_n}. \end{aligned} \quad (3.6)$$

Recall that the parts of $\lambda^{\beta\gamma/\alpha\tau}$ are less than n and the parts of $\lambda^\alpha, \lambda^\beta, \lambda^\gamma$ are less than or equal to n . Thus $m+e$ is less than or equal to n , where e is the original d_1 . We saw that the multiplication of the subscripts of the parts of $\lambda^{\beta\gamma/\alpha\tau}, \lambda^{\alpha\tau/\beta}, \lambda^{\alpha\tau/\gamma}, \lambda^\tau$ yields τ^{m+e} multiplied by the subscripts of a_i , from which it follows that only $\tau^{m+e-\ell(\lambda^{\beta\gamma/\alpha\tau})}$ on the left hand side of (3.6) is preserved. We see that $1/(1-\tau)$ on the right hand side of (3.6) accounts for τ^{n-m-e} . Therefore, it suffices to show that the generating function of (a, ν) is

$$\sum_{n=0}^{\infty} \frac{(1 + \alpha\tau q^{2n+1})(-\alpha q)_n \left(-\frac{\alpha\tau q}{\beta} \right)_n \left(-\frac{\alpha\tau q}{\gamma} \right)_n q^{n(n-1)/2} \left(\frac{\beta\gamma}{\alpha} \right)^n}{(\beta q)_n (\gamma q)_n (\tau q)_n}.$$

Suppose the generating function for Cases 1 and 2 is

$$\sum_{n=0}^{\infty} \frac{(-\alpha q)_n \left(-\frac{\alpha\tau q}{\beta} \right)_n \left(-\frac{\alpha\tau q}{\gamma} \right)_n q^{n(n-1)/2} \left(\frac{\beta\gamma}{\alpha} \right)^n}{(\beta q)_n (\gamma q)_n (\tau q)_n}. \quad (3.7)$$

Then, since q^{2m-1} is taken out for given $(\lambda^{\beta\gamma/\alpha\tau}, \lambda^\alpha, \lambda^\beta, \lambda^\gamma)$ in Case 3 and the resulting partitions fall in Cases 1 or 2, the generating function of Case 3 becomes

$$\sum_{m=1}^{\infty} \alpha\tau q^{2m-1} \frac{(-\alpha q)_{m-1} \left(-\frac{\alpha\tau q}{\beta} \right)_{m-1} \left(-\frac{\alpha\tau q}{\gamma} \right)_{m-1} q^{(m-1)(m-2)/2} \left(\frac{\beta\gamma}{\alpha} \right)^{m-1}}{(\beta q)_{m-1} (\gamma q)_{m-1} (\tau q)_{m-1}}.$$

Combining this with (3.7), we obtain the left hand side of (3.2). Thus it suffices to show the generating function of Cases 1 and 2 is (3.7).

We now consider the generating function of Case 2. In this case, we deleted $0_{\alpha\tau/\beta\gamma}$ and changed m_α to m_γ in μ . We then subtracted one from all the remaining nonzero parts of $\lambda^{\alpha\tau/\beta\gamma}$ and μ , and added a new largest part $(m-1)_\beta$ to ν . Note that $\nu_i \preceq (m-1)_\beta$. We continued as though this had been Case 1 with the Durfee rectangle of width $m-1$. Thus the generating function becomes

$$\sum_{m=1}^{\infty} \frac{(-\alpha q)_m \beta q^m}{(\beta q)_m (\gamma q)_m} \sum_{\substack{a=\lambda^{\beta\gamma/\alpha\tau} \cup^* \mu \\ \mu_{\ell(\mu)}=m_\gamma}} \text{Wt}(a).$$

Recall the condition for Case 1: $\mu_{\ell(\mu)} \succeq m_\gamma$. We divide partitions in Case 1 into two groups: $\mu_{\ell(\mu)} = m_\gamma$ and $\mu_{\ell(\mu)} \neq m_\gamma$. If $\mu_{\ell(\mu)} = m_\gamma$, then $\nu_i \preceq m_\gamma$. Thus the generating function of ν becomes

$$\frac{(-\alpha q)_m}{(\beta q)_{m-1}(\gamma q)_m},$$

from which it follows that generating function of the first group is

$$\sum_{m=1}^{\infty} \frac{(-\alpha q)_m}{(\beta q)_{m-1}(\gamma q)_m} \sum_{\substack{a=\lambda^{\beta\gamma/\alpha\tau} \cup^* \mu \\ \mu_{\ell(\mu)}=m_\gamma}} Wt(a).$$

Combining this with the generating function of Case 2, we obtain

$$\sum_{m=1}^{\infty} \frac{(-\alpha q)_m}{(\beta q)_m(\gamma q)_m} \sum_{\substack{a=\lambda^{\beta\gamma/\alpha\tau} \cup^* \mu \\ \mu_{\ell(\mu)}=m_\gamma}} Wt(a).$$

On the other hand, if $\mu_{\ell(\mu)} \neq m_\gamma$, then $\nu_i \preceq m_\beta$. Thus the generating function of the second group becomes

$$\sum_{m=0}^{\infty} \frac{(-\alpha q)_m}{(\beta q)_m(\gamma q)_m} \sum_{\substack{a=\lambda^{\beta\gamma/\alpha\tau} \cup^* \mu \\ \mu_{\ell(\mu)} \neq m_\gamma}} Wt(a).$$

Therefore, the generating function of Cases 1 and 2 is

$$\sum_{m=0}^{\infty} \frac{(-\alpha q)_m}{(\beta q)_m(\gamma q)_m} \sum_{\substack{a=\lambda^{\beta\gamma/\alpha\tau} \cup^* \mu \\ \mu_{\ell(\mu)} \succeq m_\gamma}} Wt(a).$$

To complete the proof, we need only show that

$$\sum_{\substack{a=\lambda^{\beta\gamma/\alpha\tau} \cup^* \mu \\ \mu_{\ell(\mu)} \succeq m_\gamma}} Wt(a) = \frac{q^{m(m-1)/2} \left(\frac{\beta\gamma}{\alpha}\right)^m \left(-\frac{\alpha\tau q}{\beta}\right)_m \left(-\frac{\alpha\tau q}{\gamma}\right)_m}{(\tau q)_m} \quad (3.8)$$

for $m \geq 0$, which was shown in (3.5). $\square$

4. CONCLUSION

We show that the proof given in Section 3 is a generalization of the proof of the Roger-Fine identity by Andrews. We let γ tend to 0, which means that our $\lambda^{\beta\gamma/\alpha\tau} = \lambda^\gamma = \emptyset$. Since $\lambda^{\beta\gamma/\alpha\tau} = \emptyset$, the Durfee rectangle becomes the Durfee square and $\lambda^\beta \cup \lambda^\alpha$ falls in Cases 1 or 3. It can easily be checked that each case is identical with each of Cases 1 and 2 given in the proof in Section 2.

One may ask if our proof can be generalized to obtain Watson's transformation formula (3.1). In order to generalize the aforementioned combinatorial approach to the Rogers-Fine identity and prove a special case of Watson's formula, we introduced the partition operation $\cup^*$ to handle a combination of partitions with four variables. As Watson's theorem has more variables, one needs to generalize $\cup^*$, for which a new idea seems to be needed.

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DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802
E-mail address: rowell@math.psu.edu

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802
E-mail address: yee@math.psu.edu