THE FINITE HEINE TRANSFORMATION AND CONJUGATE DURFEE SQUARES

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Abstract. We introduce the idea of a conjugate Durfee square and use it to answer a combinatorial question regarding a finite form of the Heine transformation posed by G. E. Andrews in a recent paper.

1. Introduction

In a recent publication [3], Andrews gave the following finite version of the Heine transformation:

**Theorem 1.1 (Andrews).** For any $n$, we have

\[
\sum_{k=0}^{n} \frac{(q^{-n})_k(\alpha)_k(\beta)_k}{(q)_k(\gamma)_k(q^{1-n}/\tau)_k} q^k = \frac{(\beta, \alpha\tau)_n}{(\gamma, \tau)_n} \sum_{k} \frac{(q^{-n})_k(\gamma/\beta)_k(\tau)_k}{(q)_k(\alpha\tau)_k(q^{1-n}/\beta)_k} q^k.
\]

(1.1)

In [3] Andrews asked for a combinatorial proof of Theorem 1.1 along the lines of his proof of Heine’s $2\phi_1$ transformation formula when $n$ tends to infinity [1]. This paper provides such a proof.

2. Conjugate Durfee Squares and Preliminary Results

We define a *partition* of a positive integer $n$ as a sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 + \cdots + \lambda_k = n$ with $\lambda_i \geq \lambda_{i+1}$. We relax our definition of a partition which may have 0 as a part if necessary. We refer to each $\lambda_i$ as a part of our partition and denote by $|\lambda|$ the sum of its parts. We denote the number of parts of $\lambda$ as $\ell(\lambda)$. For example, there are 7 partitions of 5, namely

(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).

To each partition we can associate a Ferrers diagram. Each part of the partition is given as a row of boxes, each row aligned and put in descending order. Figure 1 represents the Ferrers diagram of (4,2,1).

For a partition $\lambda$ into at most $m$ parts less than or equal to $n$, we define the $(m,n)$-conjugate Durfee square as the largest square that can fit with the Ferrers diagram of $\lambda$ inside of a $m \times n$ rectangle without the two overlapping. Figure 2 illustrates the $(m,n)$-conjugate Durfee square. It is simple to see that for a given partition, the $(m,n)$-conjugate Durfee square is unique.

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The $q$-binomial coefficient is defined by

$$\binom{n}{k}_q := \binom{n}{k} q^{|\lambda|},$$

where $0 \leq k \leq n$, otherwise,

$$\binom{a}{0} := (a;q)_0 = 1,$$

$$(a)_n := (a;q)_n := (1-a)(1-aq) \cdots (1-aq^{n-1}), \quad n \geq 1.$$

A partition theoretic interpretation of the $q$-binomial coefficient is as follows:

$$\binom{M+N}{M} = \sum_{\lambda} q^{\lambda},$$

where the sum is over all partitions $\lambda$ whose Ferrers diagram can fit inside an $M \times N$ rectangle.

For more information on partitions, Ferrers diagrams or the $q$-binomial coefficient, see [2].

We prove the following lemma combinatorially, which is well known in the literature.

**Lemma 2.1.**

$$\binom{n}{k} \binom{n-k}{j} = \binom{n}{j} \binom{n-j}{k}$$

**Proof.** Note that the $q$-binomial coefficient $\binom{n}{k}$ counts many interesting combinatorial objects including the partitions with Ferrers diagram fitting inside an $(n-k) \times k$ rectangle. Here,
we use inversions of permutations, namely,
\[
\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{w \in \text{Per}(0^k, 1^{n-k})} q^{\text{inv}(w)},
\]
where \( \text{Per}(0^k, 1^{n-k}) \) is the set of permutations of \( k \) 0’s and \( (n-k) \) 1’s, and \( \text{inv}(w) \) is the number of inversions in \( w \). Adopting this interpretation, we see that
\[
\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ j \end{bmatrix} = \sum_{w \in \text{Per}(0^k, 1^{n-k-j}, 2^j)} q^{\text{inv}(w)},
\]
where \( \begin{bmatrix} n \\ k \end{bmatrix} \) accounts for the inversions between \( k \) 0’s and \( (n-k) \) 1’s, and \( \begin{bmatrix} n-k \\ j \end{bmatrix} \) accounts for the inversions between \( (n-k-j) \) 1’s and \( j \) 2’s. By counting the inversions between 2’s and 0 or 1’s first, and then the inversions between 0’s and 1’s, we obtain
\[
\begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix},
\]
which completes the proof.

It should be noted that one can combinatorially interchange from the partition and the permutation interpretations of the \( q \)-binomial coefficient. Suppose we are given the partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) where \( \lambda_1 \leq n-k \) and \( l(\lambda) \leq k \). We can obtain the permutation of \( (0^k, 1^{n-k}) \) by first considering
\[
(0 \cdots 0 \underbrace{1 \cdots 1}_{n-k}).
\]
We move the first 0 to the right past \( \lambda_1 \) 1’s, the next 0 to the right past \( \lambda_2 \) 1’s, and so on. It should be clear that \( |\lambda| = \text{inv}(w) \). We can consider an example with \( n = 8, k = 3 \) and \( \lambda = (4, 2, 1) \). The corresponding permutation is \((10101101)\). More can be found on this correspondence in [2].

We review a bijection that was first introduced by the second author in [5] to establish a combinatorial proof for Ramanujan’s \( 1\psi_1 \) summation formula. Recall the \( q \)-binomial theorem [4]:
\[
\sum_{n=0}^{\infty} \frac{(-a; q)_n}{(q; q)_n} (zq)^n = \frac{(-aq; q)_\infty}{(zq; q)_\infty}.
\]

**Yee’s bijection.** For a positive integer \( n \), let \( \pi \) be a partition into nonnegative distinct parts less than \( n \) and \( \sigma \) a partition into exactly \( n \) parts. We define \( \mu \) by
\[
\mu_i = \sigma_{n-\pi_i} + \pi_i, \quad \text{for all } 1 \leq i \leq \ell(\pi),
\]
and let \( \nu \) be the partition consisting of the remaining \( n - \ell(\pi) \) parts of \( \sigma \). Then, it can be easily seen that \( \mu \) has distinct parts. It also follows from the construction that \( \mu \) and \( \nu \) are uniquely determined by \( \pi \) and \( \sigma \). Thus, this map is reversible. The left-hand side of (2.2)
generates the pairs of \((\pi, \sigma)\) and the right-hand side generates the pairs of \((\mu, \nu)\). The map is a bijection between the two sets of such pairs of partitions.

3. The Finite Heine Transformation

In this section, we will demonstrate a combinatorial proof of Theorem 1.1 along the lines of Andrews’s proof of Heine’s \(2_{\phi 1}\) transformation formula. We start by proving a special case of Theorem 1.1. By replacing \(\alpha, \tau, \gamma\) by \(-\alpha, \tau q, \gamma \beta\), respectively, and letting \(\beta\) approach 0 in (1.1), we obtain the following lemma.

**Lemma 3.1.** We have
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(-\alpha)^k}{(\tau q^{n-k+1})_k} (\tau q)^k = \frac{(-\alpha \tau q)^n}{(\tau q)_n}. \quad (3.1)
\]

**Proof.** Let \(\mu\) be a partition into distinct parts less than or equal to \(n\) and \(\nu\) be a partition into parts less than or equal to \(n\). Then the right-hand side of (3.1) generates such pairs of partitions, namely
\[
\frac{(-\alpha \tau q)^n}{(\tau q)_n} = \sum_{\mu, \nu} \tau^{\ell(\mu) + \ell(\nu)} \alpha^{\ell(\mu)} q^{\mu + |\nu|}.
\]
Let \(m = \ell(\mu) + \ell(\nu)\). We apply the reverse map of Yee’s bijection to \(\mu\) and \(\nu\) and denote the resulting partitions by \(\pi\) and \(\sigma\), where \(\pi\) is a partition into \(\ell(\mu)\) nonnegative distinct parts and \(\sigma\) is a partition into exactly \(m\) parts less than or equal to \(n\). We find the \((m, n+1)\)-conjugate Durfee square of \(\sigma\) and denote its side as \(k\). Figure 3 illustrates the conjugate Durfee square of \(\sigma\). Note that the \(k\) parts of \(\sigma\) below the dashed line are less than or equal to \(n - k + 1\); the other parts above the dashed line are larger than or equal to \(n - k + 1\), and less than or equal to \(n\). Thus, the generating function of \(\sigma\) is
\[
\sum_{\sigma} \tau^{\ell(\sigma)} q^{\sigma} = \binom{n}{k} \frac{(\tau q)^k}{(\tau q^{n-k+1})_k}.
\]
Furthermore, our process ensures that \(\pi\) has no part exceeding \(k - 1\). Suppose that \(\pi_1 \geq k\). Then, by Yee’s bijection (2.3), we see that
\[
\mu_1 = \sigma_{m-\pi_1} + \pi_1 \geq \sigma_{m-k} + k \geq n + 1 - k + k = n + 1,
\]
which is a contradiction to the fact that \( \mu \) has parts less than or equal to \( n \). Thus, the generating function of \( \pi \) is \( (-\alpha)_k \). Therefore, summing over all possible values of \( \pi \) and \( \sigma \), we obtain

\[
\sum_{k=0}^{n} \sum_{\pi,\sigma} x^{\ell(\sigma)} x^{\ell(\pi)} q^{|\pi|+|\sigma|} = \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha)_k}{(\tau q^{n-k+1})_k} (\beta q^k),
\]

which completes the proof. \( \square \)

We now prove Theorem 1.1 combinatorially. We first make some change of variables. Allowing \( \alpha, \tau, \gamma \to -\alpha, \tau q, -\gamma/\beta \) followed by \( \beta \to \beta q \) in Theorem 1.1 yields the equivalent identity,

\[
\sum_{k=0}^{n} \sum_{\pi,\sigma} x^{\ell(\sigma)} x^{\ell(\pi)} q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\beta q^k), \quad (3.2)
\]

**Theorem 3.2.** Equation (3.2) is valid.

**Proof.** We start by interpreting the left hand side of (3.2). We will show that the left and right hand side of (3.2) generate 7-tuples of partitions. We first note that the term

\[
\frac{(-\gamma \beta q^{k+1})_{n-k}}{(\beta q^{k+1})_{n-k}}
\]

on the left hand side of (3.2) can be interpreted as a strict partition \( \mu \) with all parts exceeding \( k \) and no part exceeding \( n \), and a partition \( \nu \) with all parts exceeding \( k \) and no part exceeding \( n \). As we did in the proof of Lemma 3.1, we apply the reverse map of Yee’s bijection to \( \mu \) and \( \nu \) to obtain a pair of partitions \( \pi \) and \( \sigma \), where \( \pi \) is a partition with nonnegative distinct parts and \( \sigma \) is a partition with all parts exceeding \( k \) and no part exceeding \( n \). Let \( j \) be the side of the \((\ell(\sigma))\)-conjugate Durfee square of \( \sigma \). Then, all the parts of \( \pi \) are less than \( j \). Thus, using Lemma 3.1, we can see that

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\beta q^k) \frac{(-\gamma \beta q^{k+1})_{n-k}}{(\beta q^{k+1})_{n-k}} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-\gamma)_k}{(\beta q^{n-k+1})_k} (\beta q^k) \frac{(-\alpha \tau q^{k+1})_{n-k}}{\tau q^{k+1})_{n-k}}.
\]

The interpretation is the same for the right hand side of (3.2), namely

\[
\sum_{j=0}^{n} \binom{n}{j} \frac{(-\gamma)_j}{(\beta q^{n-j+1})_j} (\beta q^j) \frac{(-\alpha \tau q^{j+1})_{n-j}}{\tau q^{j+1})_{n-j}} = \sum_{j=0}^{n} \binom{n}{j} \frac{(-\gamma)_j}{(\beta q^{n-j+1})_j} (\beta q^j) \frac{(-\alpha)_j}{(\tau q^{n-j+1})_j} (\tau q^j)_j.
\]
We can now see that the left hand side of (3.2) generates 7-tuples of partitions \((\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)\),
where \(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\) and \(\lambda_7\) are generated by
\[
\left\lfloor \frac{n}{k} \right\rfloor, \left(\tau q\right)^k (\beta q^{k+1})^j, \left(\frac{(-\alpha)}{k}\right), \frac{1}{(\tau q^{n-k+1})_k},
\]
\[
\left\lfloor \frac{n-k}{j} \right\rfloor, \left(\frac{-\gamma}{j}\right), \frac{1}{(\beta q^{n-j+1})_j},
\] respectively; while the right hand side generates 7-tuples of partitions \((\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7)\), where \(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\) and \(\mu_7\) are generated by
\[
\left\lfloor \frac{n}{j} \right\rfloor, \left(\beta q^j\right)^k (\tau q^{j+1})^k, \left(\frac{(-\gamma)}{j}\right), \frac{1}{(\beta q^{n-j+1})_j}
\] respectively.

To show (3.2), given \(\lambda_3, \lambda_4, \lambda_6\) and \(\lambda_7\), we take \(\mu_3 = \lambda_6\), \(\mu_4 = \lambda_7\), \(\mu_6 = \lambda_3\) and \(\mu_7 = \lambda_4\). To construct a bijection between \((\lambda_1, \lambda_5)\) and \((\mu_1, \mu_5)\) we apply Lemma 2.1, noting that we can combinatorially interchange between partitions and permutations as seen in the Section 2. Lastly, we must construct the bijection between \(\lambda_2\) and \(\mu_2\). We note that \(\lambda_2\) is a partition with \(k\) 1’s each marked with a \(\tau\) and \(j\) \(k+1\)’s each marked with a \(\beta\). We subtract \(k\) from each of the \(j\) parts of size \(k+1\) and add \(j\) to each of the \(k\) parts of size 1. Thus, we have a partition with \(j\) 1’s each marked with a \(\beta\) and \(k\) \(j+1\)’s each marked with a \(\tau\). It is easy to see this is \(\mu_2\). \(\Box\)

4. Conclusion

We see as \(n \to \infty\) that our conjugate Durfee square gets pushed further to the right, eliminating all of the parts which lie above it and reducing our proof down to a proof similar to Andrews. In terms of Ferrers diagrams, the integral part of Andrews proof of the Heine transformation is removing a rectangle, flipping it on its diagonal and reinserting it. We can see this in our proof when we show the bijection from \(\lambda_2\) to \(\mu_2\).

REFERENCES


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