ALDER’S CONJECTURE

AE JA YEE

Abstract. In 1956, Alder conjectured that the number of partitions of \( n \) into parts differing by at least \( d \) is greater than or equal to that of partitions of \( n \) into parts \( \equiv \pm 1 \pmod{d+3} \). The Euler identity, the first Rogers-Ramanujan identity, and a theorem of Schur show that the conjecture is true for \( d = 1, 2, 3 \), respectively. In 1971, Andrews proved that the conjecture holds for \( d = 2^r - 1, r \geq 4 \). In this paper, we prove the conjecture for all \( d \geq 32 \) and \( d = 7 \).

1. Introduction

Most likely, the first theorem in the history of partitions is Euler’s famous discovery that the number of partitions of a positive integer \( n \) into distinct parts equals the number of partitions of \( n \) into odd parts.

In the past century, two of the most celebrated theorems in the theory of partitions have been the Rogers–Ramanujan identities. Usually written in their analytic forms, they were first discovered by L. J. Rogers [11] in 1894 and rediscovered by S. Ramanujan, shortly before he left India for Cambridge in 1914 [9], [10, p. 330]. P. A. MacMahon [8] first observed their partition-theoretic interpretation in 1915. If \( c = 1 \) or \( 2 \), then the number of partitions of \( n \) into parts \( \geq c \) with minimal difference 2 between parts equals the number of partitions of \( n \) into parts \( \equiv \pm c \pmod{5} \).

Strongly motivated by these identities, I. Schur searched for further analogous partition identities. In 1926 [12], he proved that the number of partitions of \( n \) with minimal difference 3 between parts and no consecutive multiples of 3 equals the number of partitions of \( n \) into parts \( \equiv \pm 1 \pmod{6} \).

In 1948, H. L. Alder [1] showed that if further identities exist with difference \( d > 3 \), then more complex conditions than those stated in Schur’s theorem would be required, and noticing a common thread in the theorems of Euler, Rogers and Ramanujan, and Schur, in 1956, he posed the following problems [2]. Let \( q_d(n) \) be the number of partitions of \( n \) into parts differing by at least \( d \); let \( Q_d(n) \) be the number of partitions of \( n \) into parts \( \equiv \pm 1 \pmod{d+3} \); and let \( \Delta_d(n) = q_d(n) - Q_d(n) \). (a) Is \( \Delta_d(n) \) nonnegative for all positive \( d \) and \( n \)? Observe that \( \Delta_1(n) = 0 \) by Euler’s identity given above, \( \Delta_2(n) = 0 \) by the first Rogers–Ramanujan identity, and \( \Delta_3(n) \geq 0 \) by Schur’s theorem, which states that \( \Delta_3(n) \) equals the number of those partitions of \( n \) into parts differing by at least 3 that contain at least one pair of consecutive multiples of 3. (b) If (a) is true, can \( \Delta_3(n) \) be characterized as the number of partitions of \( n \) of a certain restricted type, as is the case for \( d = 3 \)? Alder mentioned these problems again in [3].

\[\text{1The author is an Alfred P. Sloan Research Fellow.}\]
\[\text{2Mathematics Subject Classification 2000: Primary, 05A17; Secondary, 11P81.}\]
After the initial theorems of Euler, Rogers and Ramanujan, and Schur, the only person to have contributed a partial answer to Alder’s problems was G. E. Andrews. In 1971 [5], he showed that $\Delta_d(n) \geq 0$ for all $n > 0$ if $d = 2^r - 1$ and $r \geq 4$.

The following table then summarizes those cases of Alder’s Conjecture (a) that have been proved up to the present time.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\Delta_d(n) \geq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 1$</td>
<td>Euler</td>
</tr>
<tr>
<td>$d = 2$</td>
<td>Rogers–Ramanujan</td>
</tr>
<tr>
<td>$d = 3$</td>
<td>Schur</td>
</tr>
<tr>
<td>$d = 2^r - 1, r \geq 4$</td>
<td>Andrews</td>
</tr>
</tbody>
</table>

In this paper, building on Andrews’ work and constructing an injection, we prove Alder’s conjecture for all $d \geq 32$. In the last section, we will provide a proof of the case when $d = 7$ as well. Therefore, Alder’s conjecture still remains unresolved for $4 \leq d \leq 30$ and $d \neq 7$.

We begin by summarizing Andrews’ ideas. For $d = 2^r - 1, r \geq 4$, Andrews studied the set of partitions of $n$ into distinct parts $\equiv 2^i \pmod{d}$ for $0 \leq i < r$, the size of which is greater than or equal to $Q_d(n)$ for any $n$, and succeeded in finding a set of partitions with difference conditions between parts that are complicated but much stronger than the difference condition for $q_d(n)$, and proving that partitions into distinct parts $\equiv 2^i \pmod{d}$ for $0 \leq i < r$ and partitions with the difference conditions are equinumerous. As a result, he was able to prove Alder’s conjecture in these particular cases.

In the sequel, we assume that $|q| < 1$ and use the customary notation for $q$-series

\[(a)_0 := (a; q)_0 := 1,\]

\[(a)_\infty := (a; q)_\infty := \prod_{k=0}^\infty (1 -aq^k),\]

\[(a)_n := (a; q)_n := \frac{(a; q)_\infty}{(aq^n; q)_\infty} \text{ for any } n.\]

We require modular partitions, which were introduced by MacMahon [6, p. 13]. Let $n$ and $k$ be positive integers. Then there exist $h \geq 0$ and $0 < j \leq k$ such that $n = kh + j$. The modular representation of a partition to the modulus $k$ is a modification of the Ferrers graph so that $n$ is represented by a row of $h$ $k$’s and one $j$. Thus the representation of the partition $5 + 4 + 4 + 3 + 1$ to the modulus 2 is the following.

\[
\begin{array}{ccc}
2 & 2 & 1 \\
2 & & \\
2 & 2 & \\
2 & 1 & \\
& & 1
\end{array}
\]

For convenience, we call the modular representation of a partition the modular Ferrers graph of the partition to the modulus $k$. 
Throughout this paper, we arrange parts of a partition in increasing order and write $a^x$ if the number $a$ occurs $x$ times as a part in a partition. We denote the coefficient of $q^n$ in an infinite series $s(q)$ by $[q^n](s(q))$.

2. The case when $d \neq 2^r - 1$

For a given $d$, uniquely define the integer $r$ by

$$2^r \leq d + 1 < 2^{r+1}.$$ 

To prove that $q_d(n) \geq Q_d(n)$ for $d \geq 32$, not of the from $2^r - 1$, we will divide $n$ into two cases: $n \leq 4d + 2^r$ and $n > 4d + 2^r$. The first case is proved in the following lemma and the other case will be proved after the lemma.

**Lemma 2.1.** For any $d \geq 32$, $q_d(n)$ is greater than or equal to $Q_d(n)$ for $n \leq 4d + 2^r$.

**Proof.** Let

$$U = \{a \mid a \equiv 1, d + 2 \pmod{d + 3}\}.$$ 

Then any partitions counted by $Q_d(n)$ have parts taken from $U$. In particular, for $n \leq 4d + 2^r$, parts $> 5d$ do not occur. Thus we may consider the following set

$$V = \{1, d + 2, d + 4, 2d + 5, 2d + 7, 3d + 8, 3d + 10, 4d + 11, 4d + 13\}.$$ 

For $n \leq d + 1$, it is trivial that $Q_d(n) = q_d(n) = 1$. For $n = d + 2$, $Q_d(n) = q_d(n) = 2$ since there are two partitions $1^{d+2}$ and $d + 2$ counted by $Q_d(n)$ and two partitions $d + 2$ and $1(d + 1)$ counted by $q_d(n)$. Similarly, we can see that $Q_d(n) = q_d(n) = 2$ when $n = d + 3$. For any $n$, $d + 4 \leq n \leq 2d + 3$, $Q_d(n) = 3$ since there are three partitions

$$1^n, 1^{n-d-2}(d + 2), 1^{n-d-4}(d + 4).$$ 

Meanwhile, $q_d(n) \geq 3$ since $q_d(n)$ is an increasing function of $n$ and $q_d(d + 4) = 3$. Thus $q_d(n) \geq Q_d(n)$ for $d + 4 \leq n \leq 2d + 3$. We can also check that $Q_d(4d + 3) = 18$ by thinking of all possible combinations of elements in $V$ whose sum equals $4d + 3$, but we omit the details. On the other hand, $q_d(2d + 4) \geq 19$ since there are at least 19 partitions with at most two parts as follows:

$$y(2d + 4 - y),$$ 

where $0 \leq y \leq \lfloor d/2 \rfloor + 2$ and $\lfloor d/2 \rfloor \geq 16$ since $d \geq 32$. Thus $q_d(n) \geq Q_d(n)$ for $2d + 4 \leq n \leq 4d + 3$. Let $4d + 4 \leq n \leq 4d + 2^r$. Then we can obtain $Q_d(4d + 2^r) = 38$ by counting possible combinations of parts taken from $V$. On the other hand, $q_d(4d + 4) \geq d + \lfloor d/2 \rfloor + 2 \geq 50$ since we can generate partitions with at most two parts differing by at least $d$ as follows:

$$y(4d + 4 - y),$$
where $0 \leq y \leq d + \lfloor d/2 \rfloor + 2$ and $\lceil d/2 \rceil \geq 16$. Thus $q_d(n) \geq Q_d(n)$ for $4d + 4 \leq n \leq 4d + 2^r$. The following table shows the values of $q_d(n)$ and $Q_d(n)$ in each interval.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$q_d(n)$</th>
<th>$Q_d(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \leq d + 1$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$d + 2 \leq n \leq d + 3$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$d + 4 \leq n \leq 2d + 3$</td>
<td>$\geq 3$</td>
<td>3</td>
</tr>
<tr>
<td>$2d + 4 \leq n \leq 4d + 3$</td>
<td>$\geq 19$</td>
<td>$\leq 18$</td>
</tr>
<tr>
<td>$4d + 4 \leq n \leq 4d + 2^r$</td>
<td>$\geq 50$</td>
<td>$\leq 38$</td>
</tr>
</tbody>
</table>

Therefore, $q_d(n) \geq Q_d(n)$ for any $n \leq 4d + 2^r$. □

From the definitions of $q_d(n)$ and $Q_d(n)$, the generating functions for $q_d(n)$ and $Q_d(n)$ are, respectively,

$$\sum_{n=0}^{\infty} q_d(n) q^n = \sum_{n=0}^{\infty} \frac{q_d(\frac{n}{2})}{(q)_n},$$

$$\sum_{n=0}^{\infty} Q_d(n) q^n = \frac{1}{(q; q^{d+3})_\infty (q^{d+2}; q^{d+3})_\infty}.$$ 

We define

$$f_d(q) = \sum_{n=0}^{\infty} L_d(n) q^n = (-q; q^{d})_\infty (-q^2; q^{d})_\infty \cdots (-q^{2^{r-1}}; q^{d})_\infty,$$ (2.1)

$$k_d(q) = \sum_{n=0}^{\infty} K_d(n) q^n = \frac{1 - q^{d+1}}{1 - q} (-q^{d+1}; q^{d})_\infty (-q^{d+2}; q^{d})_\infty \cdots (-q^{d+2^{r-1}}; q^{d})_\infty.$$ (2.2)

Comparing (2.1) and (2.2), we can see that $[q^n][(1 + q^{2^r})f_d(q)] \geq [q^n](k_d(q))$, which implies that $L_d(n) + L_d(n - 2^r) \geq K_d(n)$. To prove that Alder’s conjecture holds for $d \geq 32$, $d \neq 2^r - 1$, we will show that for $n > 4d + 2^r$,

$$q_d(n) \geq L_d(n + L_d(n - 2^r),$$

$$K_d(n) \geq Q_d(n).$$

Inequality (2.4) will be shown in Lemmas 2.2 and 2.4, and inequality (2.3) will be shown in Lemma 2.7.

**Lemma 2.2.** For any $d \geq 4$ not of the form $2^r - 1$, let 

$$g_d(q) = \frac{(-q^{d+2^{r-1}}; q^{2d})_\infty}{(q; q^{2d})_\infty (q^{d+2}; q^{2d})_\infty \cdots (q^{d+2^{r-1}}; q^{2d})_\infty}.$$ 

Then $[q^n](k_d(q)) \geq [q^n](g_d(q))$ for any $n \geq 0$. 
Proof. With some algebraic manipulations, we obtain

\[ k_d(q) = \frac{(1 - q^{d+1})(q^{2d+2}; q^{2d})_\infty (q^{2d+4}; q^{2d})_\infty \cdots (q^{2d+2d}; q^{2d})_\infty}{(1 - q)(q^{d+1}; q^d)_\infty (q^{d+2}; q^d)_\infty \cdots (q^{d+2d-1}; q^d)_\infty} \]

\[ = \frac{(q; q^{2d})_\infty (q^{2d+1}; q^{2d})_\infty (q^{d+2}; q^{2d})_\infty \cdots (q^{d+2d-2}; q^{2d})_\infty}{(q; q^{2d})_\infty (q^{2d+1}; q^{2d})_\infty (q^{d+2}; q^{2d})_\infty \cdots (q^{d+2d-2}; q^{2d})_\infty}. \]

Let \( S(n) \) denote the set of partitions of \( n \) generated by \( k_d(q) \), and define the sign of \( \pi \in S(n) \) by

\[ \text{sgn}(\pi) = \begin{cases} 1, & \text{if } \pi \text{ has an even number of parts } \equiv 2r \pmod{4d}, \\ -1, & \text{if } \pi \text{ has an odd number of parts } \equiv 2r \pmod{4d}. \end{cases} \]

Let \( S^+(n) \) and \( S^-(n) \) be the subsets of \( S(n) \) such that partitions have sign 1 and -1, respectively. Then we can deduce that \([q^n](k_d(n)) = |S^+(n)| - |S^-(n)|\). Let \( T(n) \) denote the set of partitions of \( n \) generated by \( g_d(n) \). Note that \( T(n) \) is a subset of \( S^+(n) \). To prove the lemma, we will construct a sign reversing involution \( \varphi(\pi) \) on \( S(n) \) under which partitions in \( T(n) \) are fixed.

In this proof, we write \( \pi_i \in \pi \) if \( \pi_i \) is a part of \( \pi \), and let \( m_i \) be the number of \( i \)'s occurring as parts in \( \pi \). Let \( \alpha = d + 2r - 1 \). For any \( \pi \in S(n) \), let \( x \) be the smallest \( i \) such that \( 2^ix^2 + 2^r \in \pi \), \( y \) be the smallest \( j \) such that \( (2^rj - 1)d + 1 \notin \pi \), and \( z \) be the smallest \( l > y \) such that \( m_{ld+1} \geq 2^r \). If there are no such \( x, y, \) or \( z \), we denote them by \( x = \infty, y = \infty, \) or \( z = \infty \). We define \( \varphi_0(\pi) \) by replacing

\[ \begin{aligned} 2^r xd + 2^r &\rightarrow (2^r x - 1)d + 1, 1^\alpha, & \text{if } x \leq y \text{ and } x < \infty, \\
(2^ry - 1)d + 1, 1^\alpha &\rightarrow 2^ryd + 2^r, & \text{if } x > y, y < \infty, \text{and } m_1 \geq \alpha, \\
2^r xd + 2^r &\rightarrow (xd + 1)^2, & \text{if } x > y, x < \infty, m_1 < \alpha, \text{ and } x \leq z, \\
zd + 1 \rightarrow 2^r zd + 2^r, & \text{if } x > y, m_1 < \alpha, \text{ and } x > z, \end{aligned} \]

and define \( \varphi_0(\pi) = \pi \) otherwise. Since the part \( 2^d + 2^r \) is always mapped to \( (2^r - 1)d, 1^\alpha \) under \( \varphi(\pi) \), not to \( (d + 1)^2 \), the map \( \varphi_0 \) is well defined. From the definition of \( \varphi_0 \), we see that

\[ \varphi_0^2(\pi) = \pi, \quad \text{and} \quad \text{sgn}(\pi)\text{sgn}(\varphi_0(\pi)) = -1, \text{ if } \varphi_0(\pi) \neq \pi. \]

Let

\[ S_0(n) = \{ \pi \in S(n) \mid \varphi_0(\pi) \neq \pi \}, \]

\[ T_0(n) = S(n) \setminus S_0(n). \]

Note that for any \( \pi \in T_0(n) \),

1. \( \pi \) has no part \( \equiv 2^r \pmod{2^d} \), and
2. \( m_1 < \alpha \) and \( z = \infty \) if \( y < \infty \), i.e., \( \pi \) has a part \( \equiv (2^r - 1)d + 1 \pmod{2^d} \).

We now define an involution on \( T_0(n) \). For any \( \pi \in T_0(n) \), let \( u \) be the smallest integer \( p \) such that \( (2^p - 1)d + 1 \in \pi \), \( x \) be the smallest odd integer \( i \) such that \( 2^{-1}i + 2^r \in \pi \), \( y \) be the smallest odd integer \( j \) such that \( (2^{-1}j - 1)d + 1 \in \pi \), \( w \) be the smallest odd integer \( l \) such that \( m_{ld+2} \geq 2^r-1 \), and \( z \) be the smallest odd integer \( l > y \) such that
\[ m_{id+2} \geq 2^{r-1}. \] If there are no such \( u, x, y, w, \) or \( z, \) we denote them by \( u = \infty, x = \infty, y = \infty, w = \infty, \) or \( z = \infty. \) We define \( \varphi_1(\pi) \) by replacing

\[
\begin{align*}
2^{-r+1}xd + 2^r &\rightarrow (xd + 2^{r+1})^2, & &\text{if } u < \infty, x < \infty, \text{ and } x \leq w, \\
(wd + 2)^{2-1} &\rightarrow 2^{-r-1}wd + 2^r, & &\text{if } u < \infty \text{ and } x > w, \\
2^{-r+1}xd + 2^r &\rightarrow (2^{-r+1}x - 1)d + 1, 1^a, & &\text{if } u = \infty, x \leq y, \text{ and } x < \infty, \\
(2^{-r+1}y - 1)d + 1, 1^a &\rightarrow 2^{-r+1}yd + 2^r, & &\text{if } u = \infty, x > y, y < \infty, \text{ and } m_1 \geq \alpha, \\
2^{-r+1}xd + 2^r &\rightarrow (xd + 2^{r+1})^2, & &\text{if } u = \infty, x > y, x < \infty, m_1 < \alpha, \text{ and } x \leq z, \\
(zd + 2)^{2-1} &\rightarrow 2^{-r-1}zd + 2^r, & &\text{if } u = \infty, x > y, m_1 < \alpha, \text{ and } x > z,
\end{align*}
\]

and define \( \varphi_1(\pi) = \pi \) otherwise. From the definition of \( \varphi_1, \) we see that

\[ \varphi_1^2(\pi) = \pi, \quad \text{and} \quad \text{sgn}(\pi)\text{sgn}(\varphi_1(\pi)) = -1, \quad \text{if } \varphi_1(\pi) \neq \pi. \]

Let

\[
S_t(n) = \{ \pi \in T_{t-1}(n) \mid \varphi_t(\pi) \neq \pi \}, \\
T_t(n) = T_{t-1}(n) \setminus S_t(n).
\]

Note that for any \( \pi \in T_1(n), \)

1. \( \pi \) has no part \( \equiv 2^r \pmod{2^{r-1}d} \), and
2. \( m_1 < \alpha \) if \( \pi \) has a part \( \equiv (2^{r-1} - 1)d + 1 \pmod{2^{r-1}d}. \)

Now, we recursively define \( \varphi_t, S_t, \) and \( T_t \) for \( t \) with \( 2 \leq t \leq r - 2 \) as follows. For any \( \pi \in T_{t-1}(n), \) let \( u \) be the smallest integer \( p \) such that \( (2^{r-1+t}p - 1)d + 1 \in \pi, \) let \( x \) be the smallest odd integer \( i \) such that \( 2^{-i}i + 2^r \in \pi, \) let \( y \) be the smallest odd integer \( j \) such that \( (2^{-i}j - 1)d + 1 \in \pi, \) let \( w \) be the smallest odd integer \( l \) such that \( m_{id+2^r} \geq 2^{-l}, \) and let \( z \) be the smallest odd integer \( l > y \) such that \( m_{id+2^r} \geq 2^{-l}. \) If there are no such \( u, x, y, w, \) or \( z, \) we set \( u = \infty, x = \infty, y = \infty, w = \infty, \) or \( z = \infty. \) We define \( \varphi_t(\pi) \) by replacing

\[
\begin{align*}
2^{-r+1}xd + 2^r &\rightarrow (xd + 2^{r+1})^2, & &\text{if } u < \infty, x < \infty, \text{ and } x \leq w, \\
(wd + 2)^{2-1} &\rightarrow 2^{-r-1}wd + 2^r, & &\text{if } u < \infty \text{ and } x > w, \\
2^{-r+1}xd + 2^r &\rightarrow (2^{-r+1}x - 1)d + 1, 1^a, & &\text{if } u = \infty, x \leq y, \text{ and } x < \infty, \\
(2^{-r+1}y - 1)d + 1, 1^a &\rightarrow 2^{-r+1}yd + 2^r, & &\text{if } u = \infty, x > y, y < \infty, m_1 \geq \alpha, \\
2^{-r+1}xd + 2^r &\rightarrow (xd + 2^{r+1})^2, & &\text{if } u = \infty, x > y, x < \infty, m_1 < \alpha, \text{ and } x \leq z, \\
(zd + 2)^{2-1} &\rightarrow 2^{-r-1}zd + 2^r, & &\text{if } u = \infty, x > y, m_1 < \alpha, \text{ and } x > z,
\end{align*}
\]

and define \( \varphi_t(\pi) = \pi \) otherwise. From the definition of \( \varphi_t, \) we see that

\[ \varphi_t^2(\pi) = \pi, \quad \text{and} \quad \text{sgn}(\pi)\text{sgn}(\varphi_t(\pi)) = -1, \quad \text{if } \varphi_t(\pi) \neq \pi. \]

Let

\[
S_t(n) = \{ \pi \in T_{t-1}(n) \mid \varphi_t(\pi) \neq \pi \}, \\
T_t(n) = T_{t-1}(n) \setminus S_t(n).
\]

From the definition of \( \varphi_t, \) we see that any partitions in \( S_t(n) \) have parts \( \equiv 2^r \) or \( \equiv (2^{r-1} - 1)d + 1 \pmod{2^{r-1}d}. \) Thus \( T(n) \subset T_t(n). \)
Since any partitions in $T\equiv 2^{r} (\mod 4d)$, we see that the sign of partitions in $T_{r-2}(n)$ is $1$. Thus $T_{r-2}(n) \subset S^+(n)$. Note that

$$S(n) = S_0(n) \cup S_1(n) \cup S_2(n) \cup \cdots \cup S_{r-2}(n) \cup T_{r-2}(n)$$

with $S_i(n) \cap S_j(n) = \emptyset$ for $0 \leq i < j \leq r - 2$ and $S_i \cap T_{r-2}(n) = \emptyset$ for $0 \leq i \leq r - 2$. We define $\varphi$ on $S(n)$ by

$$\varphi(\pi) = \begin{cases} \varphi_i(\pi), & \text{if } \pi \in S_i(n) \text{ for some } t, 0 \leq t \leq r - 2, \\ \pi, & \text{otherwise.} \end{cases}$$

From the definition of $\varphi_i$, we see that $\varphi$ is a sign reversing involution. Since partitions in $T(n)$ have no parts $\equiv 2^{r}$ or $3d + 1 \pmod{2d}$, we see that $T(n) \subset T_{r-2}(n)$. Thus all partitions in $T(n)$ are fixed under $\varphi$. This shows that $|T(n)| \leq |S^+(n)| - |S^-(n)|$ for any $n \geq 0$, which completes the proof.

We need a theorem of Andrews [5], for which we give a combinatorial proof.

**Theorem 2.3.** Let $S = \{a_i\}_{i=1}^{\infty}$ and $T = \{b_i\}_{i=1}^{\infty}$ be two strictly increasing sequences of positive integers such that $b_1 = 1$ and $a_i \geq b_i$ for all $i$. Let $\rho(S;n)$ and $\rho(T;n)$ denote the numbers of partitions of $n$ into parts taken from $S$ and $T$, respectively. Then, for $n \geq 1$,

$$\rho(T;n) \geq \rho(S;n).$$

**Proof.** We prove the theorem by constructing an injection $\phi$ as follows. Let $a_i$ and $b_i$ be the $i$th smallest elements of $S$ and $T$, respectively. For any partition $\lambda$ counted by $\rho(S;n)$, let $\{e_i\}_{i \geq 1}$ be a sequence of the numbers of occurrences of $a_i$ in $\lambda$. We define a sequence $\{m_i\}_{i \geq 1}$ by

$$m_i = \begin{cases} e_i + \sum_{j=1}^{\infty} (a_j - b_j)e_j, & \text{if } i = 1, \\ e_i, & \text{if } i > 1. \end{cases}$$

Let $\phi(\lambda)$ be a partition where $b_i$ occurs as parts $m_i$ times. Then it is clear that $\phi$ is injective. Therefore, the theorem holds for $n \geq 1$.

In the following lemma, we will prove that $[q^r](g_d(q)) \geq Q_d(n)$ using Theorem 2.3

**Lemma 2.4.** For any $d \geq 32$ not of the form $2^r - 1$, $[q^n](g_d(q)) \geq Q_d(n)$ for $n \geq 0$.

**Proof.** Let

$$S = \{a \mid a \equiv 1, d + 2 \pmod{d + 3}\},$$

$$T = \{b \mid b \equiv 1, d + 2, \ldots, d + 2^{r-2} \pmod{2d}\}.$$
Let \( a_i \) and \( b_i \) be the \( i \)th smallest elements of \( S \) and \( T \), respectively. The first four elements of \( S \) and \( T \) are listed below.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( a_i )</th>
<th>( b_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( d + 2 )</td>
<td>( d + 2 )</td>
</tr>
<tr>
<td>3</td>
<td>( d + 4 )</td>
<td>( d + 4 )</td>
</tr>
<tr>
<td>4</td>
<td>( 2d + 5 )</td>
<td>( d + 8 )</td>
</tr>
</tbody>
</table>

By comparing the parts of \( S \) and \( T \), we see that \( b_1 = 1 \) and \( a_i \geq b_i \), for \( i \geq 1 \). Therefore, by Theorem 2.3, we obtain

\[
[q^n](g_d(q)) = \rho(T; n) \geq \rho(S; n) = Q_d(n),
\]

which completes the proof. \( \square \)

It follows from Lemmas 2.2 and 2.4 that

\[ K_d(n) \geq Q_d(n). \]

To prove inequality (2.3), we require some terminology. For given \( d \) and \( r \), let

\[ A_d = \{ a \mid a \equiv 2^i \pmod{d}, \ 0 \leq i \leq r - 1 \}, \]

and

\[ A'_d = \{ a \mid a \equiv i \pmod{d}, \ 1 \leq i \leq 2r - 1 \}. \]

Let \( \beta_d(a) \) denote the least positive residue of \( a \) modulo \( d \). For \( a \in A'_d \), let \( b(a) \) be the number of terms appearing in the binary representation of \( a \), and let \( v(a) \) denote the least \( 2^i \) in this representation. We need a theorem of Andrews [4], which we state without proof in the following theorem.

**Theorem 2.5.** Let \( D(A_d; n) \) denote the number of partitions of \( n \) into distinct parts taken from \( A_d \), and let \( E(A'_d; n) \) denote the number of partitions of \( n \) into parts taken from \( A'_d \) of the form \( n = \lambda_1 + \lambda_2 + \cdots + \lambda_s \), such that

\[
\lambda_i - \lambda_{i+1} \geq d \cdot b(\beta_d(\lambda_{i+1})) + v(\beta_d(\lambda_{i+1})) - \beta_d(\lambda_{i+1}). \tag{2.5}
\]

Then \( D(A_d; n) = E(A'_d; n) \).

In 1991 [7], Andrews and J. B. Olsson found another set of partitions that is equinumerous to the set of partitions counted by \( D(A_d; n) \), which we state in the following theorem without proof.

**Theorem 2.6.** Let \( P(A_d; n) \) denote the number of partitions of \( n \) into parts \( \equiv 0 \pmod{d} \) or parts taken from \( A_d \), with the restrictions that only parts divisible by \( d \) may be repeated, the smallest part is \( < d \), and the difference between successive parts is at most \( d \) and strictly less than \( d \) if either part is divisible by \( d \). Then for each \( n \geq 0 \), \( P(A_d; n) = D(A_d; n) \).
Andrews and Olsson proved Theorem 2.6 by showing that the conjugate of a partition counted by \( P(A_d; n) \) is a partition counted by \( E(A'_d; n) \). The description of \( P(A_d; n) \) makes it much better to understand the structure of partitions counted by \( E(A'_d; n) \). We will employ the conditions for \( P(A_d; n) \) if needed. Figure 1 shows the conjugate of the Ferrers graph of \( \lambda \) counted by \( P(A_d; n) \), where we arrange parts in increasing order.

\[
\begin{array}{cccccccc}
2^{i_1} & 2^{i_2} & \cdots & 2^{i_j} & d & d & \cdots & d \\
d & d & \cdots & d & d & \cdots & d & 2^{h_k} & 2^{h_{k-1}} & \cdots & 2^{h_1} \\
\vdots & \vdots & & & & & & & & & \vdots \\
\end{array}
\]

**Figure 1.** Conjugate of the modular Ferrers graph of \( \lambda \)

Note that \( D(A_d; n) = L_d(n) \) from the definition of \( f_d(q) \). Thus \( E(A'_d; n) = L_d(n) \). In the following lemma, we will prove inequality (2.3) for \( n \geq 4d + 2r \) by constructing an injection from the set of partitions counted by \( E(A'_d; n) \) or \( E(A'_d; n - 2r) \) to the set of partitions counted by \( q_d(n) \).

**Lemma 2.7.** For any \( d \geq 32 \) not of the form \( 2^r - 1 \), \( q_d(n) \geq L_d(n) + L_d(n - 2^r) \) for \( n \geq 4d + 2^r \).

**Proof.** Note that \( d \geq 2^r \) since \( d \neq 2^r - 1 \). From the definition of \( f_d(q) \), this lemma holds if

\[
q_d(n) \geq E(A'_d; n) + E(A'_d; n - 2^r).
\]

Let \( X_n \) and \( Y_n \) be the sets of partitions of \( n \) counted by \( q_d(n) \) and \( E(A'_d; n) \), respectively. Since every partition in \( Y_n \) has parts differing by at least \( d \),

\[
X_n \supset Y_n.
\]

To prove \( q_d(n) - E(A'_d; n) \geq E(A'_d; n - 2^r) \), we will construct an injection \( \psi \) from \( Y_{n-2^r} \) to \( X_n \) \( \setminus Y_n \) for any \( n \geq 4d + 2^r \).

In this proof, as we noted in Section 1, we arrange parts \( \lambda_i \) of a partition \( \lambda \) in increasing order, and let \( \ell(\lambda) \) be the number of parts of \( \lambda \) and \( \lambda_{\ell(\lambda)+1} = \infty \). For a partition \( \lambda \) in \( Y_n \), let

\[
r_i = \begin{cases} 
\beta(\lambda_i), & \text{if } i \leq \ell(\lambda), \\
\infty, & \text{if } i = \ell(\lambda) + 1,
\end{cases}
\]

and

\[
u_i = \begin{cases} 
\lambda_i - r_i, & \text{if } i \leq \ell(\lambda), \\
\infty, & \text{if } i = \ell(\lambda) + 1.
\end{cases}
\]
Note that \( r_i < 2^r \) or \( r_i = \infty \) for any partition in \( Y_n \) from Theorems 2.5 and 2.6.

Let \( Z_{n-2^r} \) be a subset of \( Y_{n-2^r} \) whose partitions \( \lambda \) satisfy

\[
\lambda_{i+1} - \lambda_i \geq 2d \quad \text{and} \quad r_i + 2^r \leq d
\]

(2.6) for some \( i \). For a partition \( \lambda \in Z_{n-2^r} \), let \( s \) be the smallest \( i \) satisfying (2.6). We define \( \mu := \psi(\lambda) \) by

\[
\mu_j = \begin{cases} 
\lambda_j + 2^r, & \text{if } j = s, \\
\lambda_j, & \text{if } j \neq s.
\end{cases}
\]

The resulting partition \( \mu \) is in \( X_n \setminus Y_n \) since \( \lambda \) satisfies (2.6), \( r_s + 2^r > 2^r \), and there are no partitions with \( r_i > 2^r \) in \( Y_n \). Furthermore, it is clear that \( \psi \) is injective since we took the smallest \( s \). Let \( \bar{Z}_n \) be \( \psi(Z_{n-2^r}) \). We subtract \( Z_{n-2^r} \) and \( \bar{Z}_n \) from \( Y_n \setminus Y_n \) and \( X_n \setminus Y_n \), respectively. Then we are left with \( Y_{n-2^r} \setminus Z_{n-2^r} \) and \( (X_n \setminus Y_n) \setminus \bar{Z}_n \). We will define \( \psi \) from \( Y_{n-2^r} \setminus Z_{n-2^r} \) to \( (X_n \setminus Y_n) \setminus \bar{Z}_n \) so that images under \( \psi \) do not satisfy the difference condition (2.5) for partitions in \( Y_n \) and have parts with positive residues \( \leq 2^r \). We partition \( Y_{n-2^r} \setminus Z_{n-2^r} \) into three cases when \( \lambda_1 \geq 2d, d \leq \lambda_1 < 2d, \) and \( \lambda_1 < d \).

Case I: \( \lambda_1 \geq 2d \). We define \( \mu := \psi(\lambda) \) by

\[
\mu_j = \begin{cases} 
2^r, & \text{if } j = 1, \\
\lambda_{j-1}, & \text{if } j \geq 2.
\end{cases}
\]

Figure 2 shows the modular Ferrers graph of \( \mu \) to the modulus \( d \). Let \( W_1 \) be the set of such \( \mu \). Since \( \lambda_1 \geq 2d \) and \( d \geq 2^r \), \( \mu \) satisfies the difference condition for \( q_d(n) \), and it is clear that \( \mu \notin Y_n \cup \bar{Z}_n \) since there are no partitions with \( r_1 = 2^r \) in \( Y_n \cup \bar{Z}_n \). Furthermore, it is clear that \( \psi \) is injective since we just added \( 2^r \) as a part.

Case II: \( d \leq \lambda_1 < 2d \). Since \( n \geq 4d + 2^r \), we see that \( \ell(\lambda) \geq 2 \). We also see that \( \lambda_1 = d + r_1 \) since \( r_1 < 2^r \leq d \) and \( \lambda_i \equiv 0 \pmod{d} \) for any \( i \) from the definition of \( Y_{n-2^r} \). Let \( s \) be the smallest integer \( i \) such that

\[
u_i + 2d \leq \nu_{i+1}.
\]
Note that such an \( i \) always exists since \( u_{\ell(\lambda)+1} = \infty \). We define a partition \( \mu := \psi(\lambda) \) by

\[
\mu_j = \begin{cases} 
  r_1, & \text{if } j = 1, \\
  u_{j-1} + r_j, & \text{if } 2 \leq j \leq s, \\
  u_{j-1} + 2^r, & \text{if } j = s + 1, \\
  \lambda_{j-1}, & \text{if } j \geq s + 2.
\end{cases}
\]

Figure 3 shows the modular Ferrers graph of \( \mu \) to the modulus \( d \). Let \( W_2 \) be the set of such \( \mu \). Since \( u_j + d = u_{j+1} \) for \( j < s \), we see that \( r_j \leq r_{j+1} < d \). Thus, \( \mu_j + d \leq \mu_{j+1} \) for \( j < s \). Since \( r_s < 2^r \), we see that \( \mu_s + d \leq \mu_{s+1} \). Meanwhile, \( \mu_{s+1} + d \leq \mu_{s+2} \) since \( u_s + 2d \leq u_{s+1} \) and \( 2^r \leq d \). Therefore, \( \mu \) satisfies the difference condition for \( q_d(n) \). It is clear that \( \mu \notin Y_n \cup \bar{Z}_n \) since there are no partitions with \( r_i = 2^r, i > 1 \), in \( Y_n \cup \bar{Z}_n \). It is also clear that \( \psi \) is injective since \( s \) is the smallest \( i \). Moreover, \( W_1 \cap W_2 = \emptyset \) since \( r_1 = 2^r \) in Case I and \( r_s = 2^r, s > 1 \), in Case II.

Case III: \( \lambda_1 < d \). Note that \( \lambda_1 = r_1 \) and \( u_2 \geq d \) since \( \lambda_1 < d \) and \( n \geq 4d + 2^r \). We partition Case III into five subcases according to the size of \( u_2 \): \( u_2 \geq 6d, u_2 = 4d \) or \( 5d, u_2 = 3d, u_2 = 2d \), and \( u_2 = d \). Recall that \( b(m) \) is the length of the binary representation of \( m \). Thus, \( b(2^r - 1) = r \) since \( 2^r - 1 = 1 + 2 + \cdots + 2^{r-1} \). Throughout this proof, we denote \( 2^r - 1 \) by \( \alpha_r \).

Case (i): \( u_2 \geq 6d \). We define \( \mu := \psi(\lambda) \) by

\[
\mu_j = \begin{cases} 
  2^r - 1, & \text{if } j = 1, \\
  2d + r_1, & \text{if } j = 2, \\
  \lambda_{j-1} - 2d + 1, & \text{if } j = 3, \\
  \lambda_{j-1}, & \text{if } j \geq 4.
\end{cases}
\]

Figure 4 shows the modular Ferrers graph of \( \mu \) to the modulus \( d \), where \( \tilde{r}_2 = r_2 + 1 \).
Figure 4. Modular Ferrers graph of \( \mu \) in Case (i)

Let \( W_{3,1} \) be the set of such \( \mu \). Since \( d \geq 2^r > r_1 > 0 \) and \( \lambda_2 > u_2 = 6d \), we obtain

\[
\begin{align*}
\mu_1 + d &= 2r - 1 + d < 2d < 2d + r_1 = \mu_2, \\
\mu_2 + d &= 3d + r_1 < 4d < 4d + r_2 + 1 \leq \mu_3.
\end{align*}
\]

Since \( \lambda_j + d \leq \lambda_{j+1} \) for \( j \geq 1 \), we obtain

\[
\mu_j + d \leq \mu_{j+1}
\]

for \( j \geq 3 \). Thus \( \mu \) satisfies the difference condition for \( q_d(n) \). It is clear that \( \mu \notin Y_n \cup \bar{Z}_n \) and \( W_{3,1} \cap W_i = \emptyset \) for \( i = 1, 2 \) since \( b(2^r - 1) = r \geq 5 \) but \( \mu_2 < 3d \). Let \( \lambda \) and \( \nu \) be partitions in Case (i). Then, by comparing the parts of \( \psi(\lambda) \) and \( \psi(\nu) \), we see from the definition of \( \psi \) that \( \psi(\lambda) \) and \( \psi(\nu) \) are different. Therefore, \( \psi \) is injective.

Case (ii): \( u_2 = 4d \) or \( 5d \). We define \( \mu := \psi(\lambda) \) by

\[
\mu_j = \begin{cases} 
    r_1, & \text{if } j = 1, \\
    d + 2^r - 1, & \text{if } j = 2, \\
    \lambda_{j-1} - d + 1, & \text{if } j = 3, \\
    \lambda_{j-1}, & \text{if } j \geq 4.
\end{cases}
\]

Figure 5 shows the modular Ferrers graph of \( \mu \) to the modulus \( d \), where \( \tilde{r}_2 = r_2 + 1 \).

Figure 5. Modular Ferrers graph of \( \mu \) in Case (ii)
Let $W_{3,2}$ be the set of such $\mu$. Since $d \geq 2^r > r_1$, we obtain

\[ \mu_1 + d = r_1 + d \leq 2^r - 1 + d = \mu_2, \]
\[ \mu_2 + d = 2d + 2^r - 1 < 3d + r_2 + 1 \leq \mu_3. \]

Since $\lambda_j + d \leq \lambda_{j+1}$ for $j \geq 1$,
\[ \mu_j + d \leq \mu_{j+1} \]
for $j \geq 3$. Thus $\mu$ satisfies the difference condition for $q_d(n)$. It is clear that $\mu \notin Y_n \cup \bar{Z}_n$ and $W_{3,2} \cap W_i = \emptyset$ for $i = 1, 2$ since $b(2^r - 1) = r \geq 5$, but $\mu_3 \leq 5d$ and $\mu_2 > d$. We see that $W_{3,2} \cap W_{3,1} = \emptyset$ since $\mu_2 < 2d$ for $\mu \in W_{3,2}$, while $\tilde{\mu}_2 > 2d$ for any partition $\tilde{\mu} \in W_{3,1}$. Let $\lambda$ and $\nu$ be partitions in Case (ii). Then we see from the definition of $\psi$ that their images under $\psi$ are different. Therefore, $\psi$ is injective.

Case (iii): $u_2 = 3d$. We define $\mu := \psi(\lambda)$ by

\[ \mu_j = \begin{cases} 3, & \text{if } j = 1, \\ d + r_1 + 2, & \text{if } j = 2, \\ \lambda_{j-1} - d + 2^r - 5, & \text{if } j = 3, \\ \lambda_{j-1}, & \text{if } j = 4, \\ \lambda_{j-1}, & \text{if } j \geq 5. \end{cases} \]

Figure 6 shows the modular Ferrers graph of $\mu$ to the modulus $d$, where $\tilde{r}_1 = r_1 + 2$ and $\tilde{r}_2 = r_2 + 2^r - 5$. Let $W_{3,3}$ be the set of such $\mu$. From the definition of $\mu$, we find

\[ \mu_1 + d = 3 + d < d + r_1 + 2 = \mu_2 \]

since $r_1 > 0$. Since $u_2 = 3d$, we see that

\[ r_1 \leq 2^{r-1} + 2^{r-2} + 2^{r-3}. \]

If $r_1 = 2^{r-1} + 2^{r-2} + 2^t$ for some $t \leq r - 3$, then $r_2 \geq 2^t$ since $\lambda \in Y_{n-2^r}$. Thus, since $2^{r-1} + 2^{r-2} = 2^r - 2^{r-2}$ and $2^{r-2} - 2 > 5$,

\[ \mu_2 + d = 2d + r_1 + 2 = 2d + 2^r - 2^{r-2} + 2^t + 2 \leq 2d + 2^r - 5 + r_2 = \mu_3. \]

If $r_1 \leq 2^{r-1} + 2^{r-2}$, then

\[ \mu_2 + d = 2d + r_1 + 2 \leq 2d + 2^r - 2^{r-2} + 2 < 2d + 2^r - 5 + r_2 = \mu_3, \]
since \( r_2 > 0, 2^{r-1} + 2^{r-2} = 2^r - 2^{r-2} \), and \( 2^{r-2} - 2 > 5 \). Since \( \lambda_2 = 3d + r_2, d \geq 2^r \), and \( \lambda_2 + d \leq \lambda_3 \), we obtain
\[
\mu_3 + d = 3d + r_2 + 2^r - 5 < 4d + r_2 \leq \mu_4.
\]
Since \( \lambda_j + d \leq \lambda_{j+1} \) for \( j \geq 1 \) and \( \mu_j = \lambda_{j-1} \) for \( j \geq 4 \), we see that
\[
\mu_j + d \leq \mu_{j+1}
\]
for \( j \geq 4 \). Thus \( \mu \) satisfies the difference condition for \( q_d(n) \). Note that
\[
\mu_2 = d + r_1 + 2 \leq d + 2^{r-1} + 2^{r-2} + 2^{r-3} + 2 < d + 2^r - 1 < 2d,
\]
since \( d \geq 2^r \), while \( b(\mu_1) = 2 \). Thus it is clear that \( \mu \notin \overline{\bar{Y}_n} \cup Z_n \) and \( W_{3,3} \cap W_i = \emptyset \) for \( i = 1, 2 \). We can show that \( W_{3,3} \cap W_{3,1} = \emptyset \) since \( \mu_1 = 3 \), while \( \mu_1 = 2^r - 1 \) for any partition \( \bar{\mu} \in W_{3,1} \), and \( W_{3,3} \cap W_{3,2} = \emptyset \) since \( \mu_2 = d + r_1 + 2 < d + 2^r - 1 \), while \( \bar{\mu}_2 = d + 2^r - 1 \) for any partition \( \bar{\mu} \in W_{3,2} \). Let \( \lambda \) and \( \nu \) be partitions in Case (iii). Then we see from the definition of \( \psi \) that their images under \( \psi \) are different. Therefore, \( \psi \) is injective.

Case (iv): \( u_2 = 2d \). Note that \( \ell(\lambda) \geq 3 \) since \( n \geq 4d + 2^r \). We define \( \mu := \psi(\lambda) \) by
\[
\mu_j = \begin{cases} 
2^r - r_1 - 1, & \text{if } j = 1, \\
\lambda_j, & \text{if } 2 \leq j < \ell(\lambda), \\
\lambda_j + 2r_1 + 1, & \text{if } j = \ell(\lambda).
\end{cases}
\]
Figure 7 shows the modular Ferrers graph of \( \mu \) to the modulus \( d \), where \( \tilde{r}_1 = 2^r - r_1 - 1 \) and \( \tilde{r}_{\ell(\lambda)} = r_{\ell(\lambda)} + 2r_1 + 1 \). Let \( W_{3,4} \) be the set of such \( \mu \). We can easily show that \( \mu \)

\[
\begin{array}{cccc}
\mu_1 & \tilde{r}_1 \\
\mu_2 & d & d & r_2 \\
\mu_3 \\
\vdots \\
\mu_{\ell(\lambda)} & \tilde{r}_{\ell(\lambda)}
\end{array}
\]

**Figure 7.** Modular Ferrers graph of \( \mu \) in Case (iv)
satisfies the difference condition for \( q_d(n) \) by comparing the parts of \( \mu \), but we omit the details. Since \( u_2 = 2d \), we see that \( b(r_1) \leq 2 \). Thus \( b(\mu_1) = r - 2 \geq 3 \). Meanwhile,
\[
\mu_2 = 2d + r_2 < 3d.
\]
Thus \( \mu \notin \overline{\bar{Y}_n} \cup \overline{\bar{Z}_n} \cup W_1 \cup W_2 \). We can show that \( W_{3,4} \cap W_{3,i} = \emptyset \) for \( i = 1, 2, 3 \) by comparing \( \mu_1 \) and \( \mu_2 \), but we omit the details. Let \( \lambda \) and \( \nu \) be partitions in Case (iv). Then we see from the definition of \( \psi \) that their images under \( \psi \) are different. Therefore, \( \psi \) is injective.

Case (v): \( u_2 = d \). Let \( s \) be the smallest integer \( i \) such that
\[
u_i + 2d \leq \nu_{i+1}.
\]
Note that such an $i$ always exists since $u_{\ell(\lambda)+1} = \infty$, and $i > 1$ since $u_2 = d$. Note that for $j < s$, $b(r_j) = 1$ since $u_{j+1} - u_j = d$. Let

$$x = \begin{cases} 5, & \text{if } r_{s-1} \neq 1 \text{ and } 4, \\ 10, & \text{if } r_{s-1} = 1 \text{ or } 4. \end{cases}$$

If $u_{s+1} \neq \infty$, then we define $\mu := \psi(\lambda)$ by

$$\mu_j = \begin{cases} \lambda_j, & \text{if } j < s - 1, \\ \lambda_j + x, & \text{if } j = s - 1, s, \\ \lambda_j, & \text{if } s < j < \ell(\lambda), \\ \lambda_j + 2r - 2x, & \text{if } j = \ell(\lambda). \end{cases}$$

Figure 8 shows the modular Ferrers graph of $\mu$ to the modulus $d$, where $\tilde{r}_{s-1} = r_{s-1} + x$, $\tilde{r}_s = r_s + x$, and $\tilde{r}_{\ell(\lambda)} = r_{\ell(\lambda)} + 2r - 2x$. Let $W_{3,5}$ be the set of such $\mu$. Since we added

the same number $x$ to $\lambda_{s-1}$ and $\lambda_s$,

$$\mu_s - \mu_{s-1} \geq d.$$ 

Now, we check the difference between $\mu_s$ and $\mu_{s+1}$. If $u_{s+1} - u_s \geq 3d$, then

$$\mu_{s+1} - \mu_s = u_{s+1} + r_{s+1} - (u_s + r_s + x) \geq d$$

since $r_s + x < 2d$. Suppose that $u_{s+1} - u_s = 2d$. Then $b(r_s) \leq 2$. If $b(r_s) = 1$, then

$$r_s + x \leq d.$$ 

Thus we find that

$$\mu_{s+1} - \mu_s = u_{s+1} + r_{s+1} - (u_s + r_s + x) \geq d.$$ 

If $b(r_s) = 2$, namely $r_s = 2s_1 + 2s_2$, where $s_1 > s_2$, then,

$$r_s + x = 2s_1 + 2s_2 + x \leq 2r + 2s_2 \leq d + 2s_2.$$ 

On the other hand, $r_{s+1} \geq 2s_2$ since $\lambda$ is an element of $Y_{n-2r}$. Thus we obtain

$$\mu_{s+1} - \mu_s = u_{s+1} + r_{s+1} - (u_s + r_s + x) \geq d.$$
Therefore, $\mu$ satisfies the difference condition for $q_d(n)$. Since $b(r_{s-1}) = 1$, we see that $\beta(\mu_{s-1}) = r_{s-1} + x$, which implies that $b(\beta(\mu_{s-1})) = 3$.

Meanwhile,

$$\mu_s - u_{s-1} = d + r_s + x \leq d + 2r - 1 + x < 3d$$

since $r_s \leq 2r - 1$. Therefore, $\mu \notin Y_n \cup \bar{Z}_n$.

Now, we consider the case when $u_{s+1} = \infty$, i.e., $s = \ell(\lambda)$. Note that $\ell(\lambda) \geq 3$ since $n \geq 4d + 2r$. If $u_{s+1} = \infty$, then we define $\mu := \psi(\lambda)$ by

$$\mu_j = \begin{cases} 
\lambda_j, & \text{if } j \leq s - 2, \\
\lambda_j + x, & \text{if } j = s - 1, \\
\lambda_j + 2r - x, & \text{if } j = s.
\end{cases}$$

Figure 9 shows the modular Ferrers graph of $\mu$ to the modulus $d$, where $\tilde{r}_{s-1} = r_{s-1} + x$ and $\tilde{r}_s = r_s + 2r - x$. Let $W_{3,6}$ be the set of such $\mu$. We can show that $\mu$ satisfies the difference condition for $q_d(n)$ by comparing adjacent parts of $\mu$, but we omit the details. Since $b(r_{s-1}) = 1$, we see that $\beta(\mu_{s-1}) = r_{s-1} + x$. Thus, we obtain $b(\beta(\mu_{s-1})) = 3$.

Meanwhile,

$$\mu_s - u_{s-1} = u_s + r_s + 2r - x - u_{s-1} = d + r_s + 2r - x < 3d$$

Therefore $\mu \notin Y_n \cup \bar{Z}_n$.

In both cases when $s < \ell(\lambda)$ and $s = \ell(\lambda)$, by comparing parts, we can show that $W_{3,h} \cap W_i = \emptyset$ and $W_{3,h} \cap W_{3,j} = \emptyset$ for $h = 5, 6$, $i = 1, 2$, and $j = 1, 2, 3, 4$. Moreover, $W_{3,5} \cap W_{3,6} = \emptyset$ since $s < \ell(\mu)$ in Case (iv), while $s = \ell(\mu)$ in Case (v). Let $\lambda$ and $\nu$ be partitions in Case (v). Then we see from the definition of $\psi$ that their images under $\psi$ are different. Therefore, $\psi$ is injective.

As we have shown in each case, $\psi$ is a well defined injection from $Y_{n-2r}$ to $X_n \setminus Y_n$ for $n \geq 4d + 2r$. Thus, $q_d(n) \geq [q^n][(1 + q^{2r})f_d(q)]$ for $n \geq 4d + 2r$. $\square$

From Lemmas 2.1, 2.2, 2.4, and 2.7, and Andrews’ result, we obtain the following theorem.
**Theorem 2.8.** For \( d \geq 31 \),
\[
q_d(n) \geq Q_d(n), \quad n \geq 1.
\]

### 3. The Case When \( d = 7 \)

In the case when \( d = 2^r - 1 \), the Alder conjecture has been proved for \( r \geq 1 \) except for \( r = 3 \) by the identities of Euler and Schur, and by Andrews’ result. In this section, we will provide a proof of the conjecture for \( r = 3 \).

**Theorem 3.1.** For any \( n \geq 1 \),
\[
q_7(n) \geq Q_7(n).
\]

**Proof.** For \( 1 \leq n \leq 30 \), we can show \( q_7(n) \geq Q_7(n) \) by comparing the coefficients of their generating functions, but we omit the details.

To prove this theorem for \( n > 30 \), we consider
\[
f_7(q) = \frac{1}{(q; q^{14})_\infty(q^9; q^{14})_\infty(q^{11}; q^{14})_\infty}.
\]
By Theorem 2.5, we see that for any \( n \geq 1 \),
\[
q_7(n) \geq L_7(n).
\]
Therefore, we complete the proof if we can prove
\[
L_7(n) \geq Q_7(n), \quad n > 30. \tag{3.1}
\]
To prove inequality (3.1), we compare possible parts that compose partitions counted by \( L_7(n) \) and \( Q_7(n) \) following Andrews’ approach [5]. Let
\[
S = \{ a \mid a \equiv 1, 9 \pmod{10} \},
\]
\[
T = \{ b \mid b \equiv 1, 9, 11 \pmod{14} \}.
\]
All elements of \( S \) and \( T \) in residue classes modulo 6 are listed below.

<table>
<thead>
<tr>
<th>elements</th>
<th>( S )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6k + 1st</td>
<td>30k + 1</td>
<td>28k + 1</td>
</tr>
<tr>
<td>6k + 2nd</td>
<td>30k + 9</td>
<td>28k + 9</td>
</tr>
<tr>
<td>6k + 3rd</td>
<td>30k + 11</td>
<td>28k + 11</td>
</tr>
<tr>
<td>6k + 4th</td>
<td>30k + 19</td>
<td>28k + 15</td>
</tr>
<tr>
<td>6k + 5th</td>
<td>30k + 21</td>
<td>28k + 23</td>
</tr>
<tr>
<td>6k + 6th</td>
<td>30k + 29</td>
<td>28k + 25</td>
</tr>
</tbody>
</table>

Let \( a_i \) and \( b_i \) denote the \( i \)th smallest elements of \( S \) and \( T \), respectively. Then \( b_1 = 1 \) and \( a_i \geq b_i \) for any \( i \) except \( i = 5 \). Let
\[
S_1 = S \setminus \{ 21 \},
\]
\[
T_1 = T \setminus \{ 23 \}.
\]
Let $\rho(S_1; n)$ and $\rho(T_1; n)$ be the number of partitions of $n$ into parts taken from $S_1$ and the number of partitions of $n$ into parts taken from $T_1$, respectively. Then it follows from Theorem 2.3 that $\rho(T_1; n) \geq \rho(S_1; n)$ for all $n$.

Let $\mathcal{P}(S; n)$ and $\mathcal{P}(T; n)$ be the sets of partitions of $n$ into parts taken from $S$ and $T$, respectively. For $n > 30$, we will define an injection $\varphi$ from $\mathcal{P}(S; n)$ to $\mathcal{P}(T; n)$ by partitioning $\mathcal{P}(S; n)$ into three subsets. For a partition $\nu \in \mathcal{P}(S; n)$, let $p_1$ and $q_i$ be the numbers of occurrences of $a_i$'s and $b_i$'s in $\nu$ and $\varphi(\nu)$, respectively. Note that 21 is the 5th element of $S$, and $p_5$ is the number of occurrences of 21 in $\nu$. We consider the following cases.

Case I: $p_1 \geq 2p_5$. We replace $21p_5$ and $13p_5$ by $23p_5$, and then we replace the remaining parts $a_i$ of $\nu$ by $b_i$ and $1^{a_i-b_i}$. Then we see that $\varphi(\nu)$ has parts from $T$ and

$$q_i = \begin{cases} 
    p_1 + \sum_{j=1}^{\infty} (a_j - b_j)p_j, & \text{if } i = 1, \\
    p_i, & \text{if } i > 1.
\end{cases}$$

Case II: $p_1 < 2p_5$ and $p_5 > 1$. Let $5x + y = p_5 - \lfloor p_1/2 \rfloor$, where $x \geq 0$ and $y = 1, 2, 3, 4, 5$. If $5x + y > 1$, then we replace $21^{\lfloor p_1/2 \rfloor}$ and $12^{\lfloor p_1/2 \rfloor}$ by $23^{\lfloor p_1/2 \rfloor}$, and then replace the remaining 21's as follows:

$$21^{5x+y} \rightarrow 15^{7x+2y-3}, 9^{5-y}.$$ 

Now, replace the remaining parts $a_i$ of $\nu$ by $b_i$ and $1^{a_i-b_i}$. Then,

$$q_i = \begin{cases} 
    p_1 - 2\lfloor p_1/2 \rfloor + \sum_{j=1}^{\infty} (a_j - b_j)p_j, & \text{if } i = 1, \\
    p_i + 5 - y, & \text{if } i = 2, \\
    p_i, & \text{if } i = 3, \\
    p_i + 7x + 2y - 3, & \text{if } i = 4, \\
    \lfloor p_1/2 \rfloor, & \text{if } i = 5, \\
    p_i, & \text{if } i \geq 6.
\end{cases}$$

If $5x + y = 1$, then we replace $21^{\lfloor p_1/2 \rfloor - 1}$ and $12^{\lfloor p_1/2 \rfloor - 1}$ by $23^{\lfloor p_1/2 \rfloor - 1}$, and the remaining $21^2$ by $15^1$ and $9^3$, and then we replace the remaining parts $a_i$ of $\nu$ by $b_i$ and $1^{a_i-b_i}$. Then we see that $\varphi(\nu)$ has parts from $T$ and

$$q_i = \begin{cases} 
    p_1 - 2(\lfloor p_1/2 \rfloor - 1) + \sum_{j=1}^{\infty} (a_j - b_j)p_j, & \text{if } i = 1, \\
    p_i + 3, & \text{if } i = 2, \\
    p_i, & \text{if } i = 3, \\
    p_i + 1, & \text{if } i = 4, \\
    \lfloor p_1/2 \rfloor - 1, & \text{if } i = 5, \\
    p_i, & \text{if } i \geq 6.
\end{cases}$$

Case III: $p_1 < 2p_5$ and $p_5 = 1$. Since $n > 30$, $p_1 < 2$, and $p_5 = 1$, there must be a part $> 1$ taken from $S_1$. Let $a_k$ be the smallest part $> 1$ of $\nu$ that is taken from $S_1$. If
\( a_k \geq 19 \), then we replace 21 and \( a_k \) as follows:

\[
21, a_k \rightarrow 23, b_k, 1^{a_k-b_k-2}.
\]

Since \( a_i - b_i \geq 2 \) for \( i = 4 \) or \( i \geq 6 \), the replacement is well defined. If \( a_k < 19 \), then replace 21 and \( a_k \) as follows:

\[
21, 9 \rightarrow 15^2, \quad \text{if } a_k = 9,
\]

\[
21, 11 \rightarrow 15^2, 1^2, \quad \text{if } a_k = 11.
\]

Lastly, replace the remaining \( a_i \) by \( b_i \) and \( 1^{a_i-b_i} \). Then, we see that if \( a_k \geq 19 \), then

\[
q_i = \begin{cases} 
p_1 + \sum_{j=1}^{\infty} (a_j - b_j)p_j, & \text{if } i = 1, \\
p_i, & \text{if } i \geq 2,
\end{cases}
\]

and if \( a_k < 19 \), then

\[
q_1 = \begin{cases} 
p_1 + \sum_{j=1}^{\infty} (a_j - b_j)p_j, & \text{if } a_k = 9, \\
p_1 + 2 + \sum_{j=1}^{\infty} (a_j - b_j)p_j, & \text{if } a_k = 11,
\end{cases}
\]

\[
q_2 = \begin{cases} 
p_2 - 1, & \text{if } a_k = 9, \\
p_2, & \text{if } a_k = 11,
\end{cases}
\]

\[
q_3 = \begin{cases} 
p_3, & \text{if } a_k = 9, \\
p_2 - 1, & \text{if } a_k = 11,
\end{cases}
\]

\[
q_4 = p_4 + 2,
\]

\[
q_5 = p_5 - 1,
\]

\[
q_i = p_i, \quad \text{if } i \geq 6.
\]

We will show that \( \varphi \) is injective. We take two partitions \( \nu \) and \( \tilde{\nu} \) from \( \mathcal{P}(S; n) \). Let \( \{p_i\} \) and \( \{\tilde{p}_i\} \) be sequences of the numbers of occurrences of \( a_i \) in \( \nu \) and \( \tilde{\nu} \), respectively. Let \( \{q_i\} \) and \( \{\tilde{q}_i\} \) be sequences of the numbers of occurrences of \( b_i \) in \( \varphi(\nu) \) and \( \varphi(\tilde{\nu}) \), respectively. If \( p_i \neq \tilde{p}_i \) for some \( i \geq 6 \), then \( \varphi(\nu) \) and \( \varphi(\tilde{\nu}) \) are different since \( q_i = p_i \) and \( \tilde{q}_i = \tilde{p}_i \) for \( i \geq 6 \). Thus without loss of generality, we assume that \( p_i = \tilde{p}_i = 0 \) for \( i \geq 6 \). If both \( \nu \) and \( \tilde{\nu} \) are in either Case I, or Case II, or Case III, then we can show that \( \varphi(\nu) \) and \( \varphi(\tilde{\nu}) \) are distinct by comparing \( q_i \) and \( \tilde{q}_i \), but we omit the details. We consider the case when \( \nu \) and \( \tilde{\nu} \) are in different cases each other. Note that \( a_i = b_i \) for \( i = 1, 2, 3 \).

Let \( \nu \) be in Case I and \( \tilde{\nu} \) be in Case II. Then,

\[
q_1 = p_1 + 4p_4 - 2p_5,
\]

while

\[
\tilde{q}_1 = \begin{cases} 
\tilde{p}_1 - 2[\tilde{p}_1/2] + 4\tilde{p}_4, & \text{if } 5x + y > 1, \\
\tilde{p}_1 - 2([\tilde{p}_1/2] - 1) + 4\tilde{p}_4, & \text{if } 5x + y = 1.
\end{cases}
\]
If \( q_1 \neq \tilde{q}_1 \), then \( \varphi(\nu) \) and \( \varphi(\tilde{\nu}) \) are different. If \( q_1 = \tilde{q}_1 \), then
\[
q_1 = \tilde{q}_1 \leq \tilde{p}_1 - 2|\tilde{p}_1/2| + 2 + 4\tilde{p}_4 < 4 + 4\tilde{p}_4,
\]
which implies that \( \tilde{p}_4 \geq q_4 \) since \( q_1 \geq 4p_4 \) and \( p_4 = q_4 \). However, \( \tilde{q}_4 > \tilde{p}_4 \) from the definition of \( \varphi \). Thus \( \tilde{q}_4 > q_4 \), which implies \( \varphi(\nu) \) and \( \varphi(\tilde{\nu}) \) are different.

We consider the case when one of them is in Case I and the other is in Case III. Let \( \nu \) be in Case I. If \( a_k \geq 19 \), then we can prove that \( \varphi(\nu) \) and \( \varphi(\tilde{\nu}) \) are distinct by comparing \( q_i \) and \( \tilde{q}_i \), but we omit the details. We consider the case when \( a_k < 19 \). Then,
\[
q_1 = p_1 + 4p_4 - 2p_5,
\]
while
\[
\tilde{q}_1 = \begin{cases} 
\tilde{p}_1 + 4\tilde{p}_4, & \text{if } a_k = 9, \\
\tilde{p}_1 + 2 + 4\tilde{p}_4, & \text{if } a_k = 11.
\end{cases}
\]
If \( q_1 \neq \tilde{q}_1 \), then \( \varphi(\nu) \) and \( \varphi(\tilde{\nu}) \) are different. If \( q_1 = \tilde{q}_1 \), then
\[
q_1 = \tilde{q}_1 \leq \tilde{p}_1 + 2 + 4\tilde{p}_4 < 4 + 4\tilde{p}_4,
\]
which implies that \( \tilde{p}_4 \geq q_4 \) since \( q_1 \geq 4p_4 \) and \( p_4 = q_4 \). Meanwhile, \( \tilde{q}_4 > \tilde{p}_4 \) from the definition of \( \varphi \). Thus \( \varphi(\nu) \) and \( \varphi(\tilde{\nu}) \) are different.

Lastly, we consider the case when one of \( \nu \) and \( \tilde{\nu} \) is in Case II and the other is in Case III. Let \( \nu \) be in Case II. If \( a_k \geq 11 \), then 9 does not occur in \( \varphi(\tilde{\nu}) \), while 9 occurs in \( \varphi(\nu) \) except for the case when \( y = 5 \). If \( y = 5 \), then
\[
q_1 = p_1 - 2|p_1/2| + 4p_4.
\]
Meanwhile,
\[
\tilde{q}_1 = \begin{cases} 
\tilde{p}_1 + 2 + 4\tilde{p}_4, & \text{if } a_k = 11, \\
\tilde{p}_1 + 4\tilde{p}_4 - 2, & \text{if } a_k \geq 19.
\end{cases}
\]
Assume that \( q_1 = \tilde{q}_1 \). If \( a_k \geq 19 \), then
\[
p_1 - 2|p_1/2| + 4p_4 = \tilde{p}_1 + 4\tilde{p}_4 - 2,
\]
which contradicts the fact that \( p_4 \) and \( \tilde{p}_4 \) are integers since \( 0 \leq p_1 - 2|p_1/2| \leq 1 \) and \( 0 \leq \tilde{p}_1 \leq 1 \). If \( a_k = 11 \), then
\[
p_1 - 2|p_1/2| + 4p_4 = \tilde{p}_1 + 2 + 4\tilde{p}_4,
\]
which contradicts the fact that \( p_4 \) and \( \tilde{p}_4 \) are integers since \( 0 \leq p_1 - 2|p_1/2| \leq 1 \) and \( 0 \leq \tilde{p}_1 \leq 1 \). Therefore, in any case, \( \varphi(\nu) \) and \( \varphi(\tilde{\nu}) \) are different.

Now, we will examine the case when \( a_k = 9 \). If \( a_k = 9 \), then
\[
q_1 = \begin{cases} 
p_1 - 2|p_1/2| + 4p_4, & \text{if } 5x + y > 1, \\
p_1 - 2|p_1/2| + 2 + 4p_4, & \text{if } 5x + y = 1,
\end{cases}
\]
\[
\tilde{q}_1 = \tilde{p}_1 + 4\tilde{p}_4.
\]
Thus, if \( 5x + y = 1 \), then
\[
p_1 - 2|p_1/2| + 2 + 4p_4 = \tilde{p}_1 + 4\tilde{p}_4,
\]
which contradicts the fact that \( p_4 \) and \( \tilde{p}_4 \) are integers, since \( 0 \leq p_1 - 2\lfloor p_1/2 \rfloor \leq 1 \) and \( 0 \leq \tilde{p}_1 \leq 1 \). If \( 5x + y > 1 \), then
\[
p_1 - 2\lfloor p_1/2 \rfloor + 4p_4 = \tilde{p}_1 + 4\tilde{p}_4,
\]
which contradicts the fact that \( p_4 \) and \( \tilde{p}_4 \) are integers if \( p_1 - 2\lfloor p_1/2 \rfloor \neq \tilde{p}_1 \). If \( p_1 - 2\lfloor p_1/2 \rfloor = \tilde{p}_1 \), then \( p_4 = \tilde{p}_4 \). However,
\[
q_4 = p_4 + 7x + z,
\]
\[
\tilde{q}_4 = \tilde{p}_4 + 2,
\]
where \( z = 1, 3, 5, 6, 7 \). Thus \( q_4 \neq \tilde{q}_4 \). Therefore, the images of \( \nu \) and \( \tilde{\nu} \) are different. \( \square \)

**Acknowledgment.** The author thanks George E. Andrews for suggesting that she work on this project and Bruce C. Berndt for his comments and advice.

**References**


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