

Extremal selections of multifunctions generating a continuous flow

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Abstract. Let $F : [0, T] \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a continuous multifunction with compact, not necessarily convex values. In this paper, we prove that, if F satisfies the following Lipschitz Selection Property:

(LSP) *For every t, x , every $y \in \overline{\text{co}}F(t, x)$ and $\varepsilon > 0$, there exists a Lipschitz selection ϕ of $\overline{\text{co}}F$, defined on a neighborhood of (t, x) , with $|\phi(t, x) - y| < \varepsilon$,*

then there exists a measurable selection f of $\text{ext } F$ such that, for every x_0 , the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

has a unique Carathéodory solution, depending continuously on x_0 .

We remark that every Lipschitz multifunction with compact values satisfies (LSP). Another interesting class for which (LSP) holds consists of those continuous multifunctions F whose values are compact and have convex closure with nonempty interior.

1. Introduction. Let $F : [0, T] \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a continuous multifunction with compact, not necessarily convex values. If F is Lipschitz continuous, it was shown in [5] that there exists a measurable selection f of F such that, for every x_0 , the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

has a unique Carathéodory solution, depending continuously on x_0 .

In this paper, we prove that the above selection f can be chosen so that $f(t, x) \in \text{ext } F(t, x)$ for all t, x . More generally, the result remains valid if F satisfies the following Lipschitz Selection Property:

(LSP) *For every t, x , every $y \in \overline{\text{co}}F(t, x)$ and $\varepsilon > 0$, there exists a Lipschitz selection ϕ of $\overline{\text{co}}F$, defined on a neighborhood of (t, x) , with $|\phi(t, x) - y| < \varepsilon$.*

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We remark that, by [10, 12], every Lipschitz multifunction with compact values satisfies (LSP). Another interesting class for which (LSP) holds consists of those continuous multifunctions F whose values are compact and have convex closure with nonempty interior. Indeed, for any given t, x, y, ε , choosing $y' \in \text{int } \overline{\text{co}}F(t, x)$ with $|y' - y| < \varepsilon$, the constant function $\phi \equiv y'$ is a local selection from $\overline{\text{co}}F$ satisfying the requirements.

In the following, $\Omega \subseteq \mathbb{R}^n$ is an open set, $\overline{B}(0, M)$ is the closed ball centered at the origin with radius M , $\overline{B}(D, MT)$ is the closed neighborhood of radius MT around the set D , while \mathcal{AC} is the Sobolev space of all absolutely continuous functions $u : [0, T] \rightarrow \mathbb{R}^n$, with norm $\|u\|_{\mathcal{AC}} = \int_0^T (|u(t)| + |\dot{u}(t)|) dt$.

THEOREM 1. *Let $F : [0, T] \times \Omega \rightarrow 2^{\mathbb{R}^n}$ be a bounded continuous multifunction with compact values, satisfying (LSP). Assume that $F(t, x) \subseteq \overline{B}(0, M)$ for all t, x and let D be a compact set such that $\overline{B}(D, MT) \subset \Omega$. Then there exists a measurable function f with*

$$(1.1) \quad f(t, x) \in \text{ext } F(t, x) \quad \forall t, x,$$

such that, for every $(t_0, x_0) \in [0, T] \times D$, the Cauchy problem

$$(1.2) \quad \dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a unique Carathéodory solution $x(\cdot) = x(\cdot, t_0, x_0)$ on $[0, T]$, depending continuously on t_0, x_0 in the norm of \mathcal{AC} .

Moreover, if $\varepsilon_0 > 0$ and a Lipschitz continuous selection f_0 of $\overline{\text{co}}F$ are given, then one can construct f with the following additional property: Denoting by $y(\cdot, t_0, x_0)$ the unique solution of

$$(1.3) \quad \dot{y}(t) = f_0(t, y(t)), \quad y(t_0) = x_0,$$

for every $(t_0, x_0) \in [0, T] \times D$ one has

$$(1.4) \quad |y(t, t_0, x_0) - x(t, t_0, x_0)| \leq \varepsilon_0 \quad \forall t \in [0, T].$$

The proof of the above theorem, given in Section 3, starts with the construction of a sequence f_n of directionally continuous selections from $\overline{\text{co}}F$ which are piecewise Lipschitz continuous in the (t, x) -space. For every $u : [0, T] \rightarrow \mathbb{R}^n$ in a class of Lipschitz continuous functions, we then show that the composed maps $t \rightarrow f_n(t, u(t))$ form a Cauchy sequence in $\mathcal{L}^1([0, T]; \mathbb{R}^n)$ converging pointwise almost everywhere to a map of the form $f(\cdot, u(\cdot))$, taking values within the extreme points of F . This convergence is obtained through an argument which is considerably different from previous works. Indeed, it relies on a careful use of the likelihood functional introduced in [4], interpreted here as a measure of "oscillatory nonconvergence" of a set of derivatives.

Among various corollaries, Theorem 1 yields an extension, valid for the wider class of multifunctions with the property (LSP), of the following results, proved in [7], [5] and [8], respectively.

(i) Existence of selections from the solution set of a differential inclusion, depending continuously on the initial data.

(ii) Existence of selections from a multifunction, which generate a continuous flow.

(iii) Contractibility of the solution sets of $\dot{x} \in F(t, x)$ and $\dot{x} \in \text{ext } F(t, x)$.

These consequences, together with an application to bang-bang feedback controls, are described in Section 4. Topological properties of the set of solutions of nonconvex differential inclusions have been studied in [3, 6] with the technique of directionally continuous selections and in [8, 9, 13] using the method of Baire category.

2. Preliminaries. As customary, \bar{A} and $\overline{\text{co}} A$ denote here the closure and the closed convex hull of A respectively, while $A \setminus B$ indicates a set-theoretic difference. The Lebesgue measure of a set $J \subset \mathbb{R}$ is $m(J)$. The characteristic function of a set A is written as χ_A .

In the following, \mathcal{K}_n denotes the family of all nonempty compact convex subsets of \mathbb{R}^n , endowed with Hausdorff metric. A key technical tool used in our proofs will be the function $h : \mathbb{R}^n \times \mathcal{K}_n \rightarrow \mathbb{R} \cup \{-\infty\}$ defined by

$$(2.1) \quad h(y, K) \doteq \sup \left\{ \left(\int_0^1 |w(\xi) - y|^2 d\xi \right)^{1/2}; w : [0, 1] \rightarrow K, \int_0^1 w(\xi) d\xi = y \right\}$$

with the understanding that $h(y, K) = -\infty$ if $y \notin K$. Observe that $h^2(y, K)$ can be interpreted as the maximum variance among all random variables supported inside K whose mean value is y . The following results were proved in [4]:

LEMMA 1. *The map $(y, K) \mapsto h(y, K)$ is upper semicontinuous in both variables; for each fixed $K \in \mathcal{K}_n$ the function $y \mapsto h(y, K)$ is strictly concave down on K . Moreover, one has*

$$(2.2) \quad h(y, K) = 0 \quad \text{if and only if} \quad y \in \text{ext } K,$$

$$(2.3) \quad h^2(y, K) \leq r^2(K) - |y - c(K)|^2,$$

where $c(K)$ and $r(K)$ denote the Chebyshev center and the Chebyshev radius of K , respectively.

Remark 1. By the above lemma, the function h has all the qualitative properties of the Choquet function d_F considered, for example, in [9,

Proposition 2.6]. It could thus be used within any argument based on Baire category. Moreover, the likelihood functional

$$L(u) \doteq \left(\int_0^T h^2(\dot{u}(t), F(t, u(t))) dt \right)^{1/2}$$

provides an upper bound to the distance $\|\dot{v} - \dot{u}\|_{\mathcal{L}^2}$ between derivatives, for solutions of $\dot{v} \in F(t, v)$ which remain close to u uniformly on $[0, T]$. This additional quantitative property of the function h will be a crucial ingredient in our proof.

For the basic theory of multifunctions and differential inclusions we refer to [1]. As in [2], given a map $g : [0, T] \times \Omega \rightarrow \mathbb{R}^n$, we say that g is *directionally continuous* along the directions of the cone $\Gamma^N = \{(s, y) ; |y| \leq Ns\}$ if

$$g(t, x) = \lim_{k \rightarrow \infty} g(t_k, x_k)$$

for every (t, x) and every sequence (t_k, x_k) in the domain of g such that $t_k \rightarrow t$ and $|x_k - x| \leq N(t_k - t)$ for every k . Equivalently, g is Γ^N -continuous iff it is continuous w.r.t. the topology generated by the family of all half-open cones of the form

$$(2.4) \quad \{(s, y) ; \hat{t} \leq s < \hat{t} + \varepsilon, |y - \hat{x}| \leq N(s - t)\}$$

with $(\hat{t}, \hat{x}) \in \mathbb{R} \times \mathbb{R}^n, \varepsilon > 0$. A set of the form (2.4) will be called an N -cone.

Under the assumptions on Ω, D made in Theorem 1, consider the set of Lipschitzean functions

$$Y \doteq \{u : [0, T] \rightarrow \bar{B}(D, MT) ; |u(t) - u(s)| \leq M|t - s| \forall t, s\}.$$

The Picard operator of a map $g : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ is defined as

$$\mathcal{P}^g(u)(t) \doteq \int_0^t g(s, u(s)) ds, \quad u \in Y.$$

The distance between two Picard operators will be measured by

$$(2.5) \quad \|\mathcal{P}^f - \mathcal{P}^g\| = \sup \left\{ \left| \int_0^t [f(s, u(s)) - g(s, u(s))] ds \right| ; t \in [0, T], u \in Y \right\}.$$

The next lemma will be useful in order to prove the uniqueness of solutions of the Cauchy problems (1.2).

LEMMA 2. *Let f be a measurable map from $[0, T] \times \Omega$ into $\bar{B}(0, M)$, with \mathcal{P}^f continuous on Y . Let D be compact, with $\bar{B}(D, MT) \subset \Omega$, and assume that the Cauchy problem*

$$(2.6) \quad \dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \in [0, T],$$

has a unique solution, for each $(t_0, x_0) \in [0, T] \times D$.

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property. If $g : [0, T] \times \Omega \rightarrow \overline{B}(0, M)$ satisfies $\|\mathcal{P}^g - \mathcal{P}^f\| \leq \delta$, then for every $(t_0, x_0) \in [0, T] \times D$, any solution of the Cauchy problem

$$(2.7) \quad \dot{y}(t) = g(t, y(t)), \quad y(t_0) = x_0, \quad t \in [0, T],$$

has distance $< \varepsilon$ from the corresponding solution of (2.6). In particular, the solution set of (2.7) has diameter $\leq 2\varepsilon$ in $C^0([0, T]; \mathbb{R}^n)$.

Proof. If the conclusion fails, then there exist sequences of times t_ν, t'_ν , maps g_ν with $\|\mathcal{P}^{g_\nu} - \mathcal{P}^f\| \rightarrow 0$, and couples of solutions $x_\nu, y_\nu : [0, T] \rightarrow \overline{B}(D, MT)$ of

$$(2.8) \quad \dot{x}_\nu(t) = f(t, x_\nu(t)), \quad \dot{y}_\nu(t) = g_\nu(t, y_\nu(t)), \quad t \in [0, T],$$

with

$$(2.9) \quad x_\nu(t_\nu) = y_\nu(t_\nu) \in D, \quad |x_\nu(t'_\nu) - y_\nu(t'_\nu)| \geq \varepsilon \quad \forall \nu.$$

By taking subsequences, we can assume that $t_\nu \rightarrow t_0, t'_\nu \rightarrow \tau, x_\nu(t_0) \rightarrow x_0$, while $x_\nu \rightarrow x$ and $y_\nu \rightarrow y$ uniformly on $[0, T]$. From (2.8) it follows that

$$(2.10) \quad \left| y(t) - x_0 - \int_{t_0}^t f(s, y(s)) ds \right| \leq |y(t) - y_\nu(t)| + |x_0 - y_\nu(t_0)| \\ + \left| \int_{t_0}^t [f(s, y(s)) - f(s, y_\nu(s))] ds \right| + \left| \int_{t_0}^t [f(s, y_\nu(s)) - g_\nu(s, y_\nu(s))] ds \right|.$$

As $\nu \rightarrow \infty$, the right hand side of (2.10) tends to zero, showing that $y(\cdot)$ is a solution of (2.6). By the continuity of $\mathcal{P}^f, x(\cdot)$ is also a solution of (2.6), distinct from $y(\cdot)$ because

$$|x(\tau) - y(\tau)| = \lim_{\nu \rightarrow \infty} |x_\nu(\tau) - y_\nu(\tau)| = \lim_{\nu \rightarrow \infty} |x_\nu(t'_\nu) - y_\nu(t'_\nu)| \geq \varepsilon.$$

This contradicts the uniqueness assumption, proving the lemma.

3. Proof of the main theorem. Observing that $\text{ext } F(t, x) = \text{ext } \overline{\text{co}} F(t, x)$ for every compact set $F(t, x)$, it is clearly not restrictive to prove Theorem 1 under the additional assumption that all values of F are convex. Moreover, the bounds on F and D imply that no solution of the Cauchy problem

$$\dot{x}(t) \in F(t, x(t)), \quad x(t_0) = x_0, \quad t \in [0, T],$$

with $x_0 \in D$, can escape from the set $\overline{B}(D, MT)$. Therefore, it suffices to construct the selection f on the compact set $\Omega^\dagger \doteq [0, T] \times \overline{B}(D, MT)$. Finally, since every convex-valued multifunction satisfying (LSP) admits a globally defined Lipschitz selection, it suffices to prove the second part of the theorem, with f_0 and $\varepsilon_0 > 0$ assigned.

We shall define a sequence of directionally continuous selections of F , converging a.e. to a selection from $\text{ext } F$. The basic step of our constructive procedure will be provided by the next lemma.

LEMMA 3. Fix any $\varepsilon > 0$. Let S be a compact subset of $[0, T] \times \Omega$ and let $\phi : S \rightarrow \mathbb{R}^n$ be a continuous selection of F such that

$$(3.1) \quad h(\phi(t, x), F(t, x)) < \eta \quad \forall (t, x) \in S,$$

with h as in (2.1). Then there exists a piecewise Lipschitz selection $g : S \rightarrow \mathbb{R}^n$ of F with the following properties:

(i) There exists a finite covering $\{\Gamma_i\}_{i=1, \dots, \nu}$, consisting of Γ^{M+1} -cones, such that, if we define the pairwise disjoint sets $\Delta^i \doteq \Gamma_i \setminus \bigcup_{l < i} \Gamma_l$, then on each Δ^i the following holds:

(a) There exist Lipschitzian selections $\psi_j^i : \overline{\Delta^i} \rightarrow \mathbb{R}^n$, $j = 0, \dots, n$, such that

$$(3.2) \quad g|_{\Delta^i} = \sum_{j=0}^n \psi_j^i \chi_{A_j^i},$$

where each A_j^i is a finite union of strips of the form $([t', t''] \times \mathbb{R}^n) \cap \Delta^i$.

(b) For every $j = 0, \dots, n$ there exists an affine map $\varphi_j^i(\cdot) = \langle a_j^i, \cdot \rangle + b_j^i$ such that

$$(3.3) \quad \varphi_j^i(\psi_j^i(t, x)) \leq \varepsilon, \quad \varphi_j^i(z) \geq h(z, F(t, x)), \quad \forall (t, x) \in \overline{\Delta^i}, \quad z \in F(t, x).$$

(ii) For every $u \in Y$ and every interval $[\tau, \tau']$ such that $(s, u(s)) \in S$ for $\tau \leq s < \tau'$, the following estimates hold:

$$(3.4) \quad \left| \int_{\tau}^{\tau'} [\phi(s, u(s)) - g(s, u(s))] ds \right| \leq \varepsilon,$$

$$(3.5) \quad \int_{\tau}^{\tau'} |\phi(s, u(s)) - g(s, u(s))| ds \leq \varepsilon + \eta(\tau' - \tau).$$

REMARK 2. Thinking of $h(y, K)$ as a measure of the distance of y from the extreme points of K , the above lemma can be interpreted as follows. Given any selection ϕ of F , one can find a Γ^{M+1} -continuous selection g whose values lie close to the extreme points of F and whose Picard operator \mathcal{P}^g , by (3.4), is close to \mathcal{P}^ϕ . Moreover, if the values of ϕ are near the extreme points of F , i.e. if η in (3.1) is small, then g can be chosen close to ϕ . The estimate (3.5) will be a direct consequence of the definition (2.1) of h and of Hölder's inequality.

Remark 3. Since h is only upper semicontinuous, the two assumptions $y_\nu \rightarrow y$ and $h(y_\nu, K) \rightarrow 0$ do not necessarily imply $h(y, K) = 0$. As a consequence, the a.e. limit of a convergent sequence of approximately extremal selections f_ν of F need not take values inside $\text{ext } F$. To overcome this difficulty, the estimates in (3.3) provide upper bounds for h in terms of the affine maps φ_j^i . Since each φ_j^i is continuous, limits of the form $\varphi_j^i(y_\nu) \rightarrow \varphi_j^i(y)$ will be straightforward.

Proof of Lemma 3. For every $(t, x) \in S$ there exist values $y_j(t, x) \in F(t, x)$ and coefficients $\theta_j(t, x) \geq 0$ with

$$\begin{aligned} \phi(t, x) &= \sum_{j=0}^n \theta_j(t, x) y_j(t, x), & \sum_{j=0}^n \theta_j(t, x) &= 1, \\ & & h(y_j(t, x), F(t, x)) &< \varepsilon/2. \end{aligned}$$

By the concavity and the upper semicontinuity of h , for every $j = 0, \dots, n$ there exists an affine function $\varphi_j^{(t,x)}(\cdot) = \langle a_j^{(t,x)}, \cdot \rangle + b_j^{(t,x)}$ such that

$$\begin{aligned} \varphi_j^{(t,x)}(y_j(t, x)) &< h(y_j(t, x), F(t, x)) + \varepsilon/2 < \varepsilon, \\ \varphi_j^{(t,x)}(z) &> h(z, F(t, x)) \quad \forall z \in F(t, x). \end{aligned}$$

By (LSP) and the continuity of each $\varphi_j^{(t,x)}$, there exists a neighborhood \mathcal{U} of (t, x) together with Lipschitzian selections $\psi_j^{(t,x)} : \mathcal{U} \rightarrow \mathbb{R}^n$ such that, for every j and every $(s, y) \in \mathcal{U}$,

$$(3.6) \quad |\psi_j^{(t,x)}(s, y) - y_j(t, x)| < \frac{\varepsilon}{4T},$$

$$(3.7) \quad \varphi_j^{(t,x)}(\psi_j^{(t,x)}(s, y)) < \varepsilon.$$

Using again the upper semicontinuity of h , we can find a neighborhood \mathcal{U}' of (t, x) such that

$$(3.8) \quad \varphi_j^{(t,x)}(z) \geq h(z, F(s, y)) \quad \forall z \in F(s, y), (s, y) \in \mathcal{U}', j = 0, \dots, n.$$

Choose a neighborhood $\Gamma_{t,x}$ of (t, x) , contained in $\mathcal{U} \cap \mathcal{U}'$, such that, for every point (s, y) in the closure $\bar{\Gamma}_{t,x}$, one has

$$(3.9) \quad |\phi(s, y) - \phi(t, x)| < \frac{\varepsilon}{4T}.$$

It is not restrictive to assume that $\Gamma_{t,x}$ is an $(M + 1)$ -cone, i.e. it has the form (2.4) with $N = M + 1$. By the compactness of S we can extract a finite subcovering $\{\Gamma^i ; 1 \leq i \leq \nu\}$, with $\Gamma_i \doteq \Gamma_{t_i, x_i}$. Define $\Delta^i \doteq \Gamma_i \setminus \bigcup_{j < i} \Gamma_j$ and set $\theta_j^i = \theta_j(t_i, x_i)$, $y_j^i = y_j(t_i, x_i)$, $\psi_j^i = \psi_j^{(t_i, x_i)}$, $\varphi_j^i = \varphi_j^{(t_i, x_i)}$. Choose

an integer N such that

$$(3.10) \quad N > \frac{8M\nu^2T}{\varepsilon}$$

and divide $[0, T]$ into N equal subintervals J_1, \dots, J_N , with

$$(3.11) \quad J_k = [t_{k-1}, t_k), \quad t_k = \frac{kT}{N}.$$

For each i, k such that $(J_k \times \mathbb{R}^n) \cap \Delta^i \neq \emptyset$, we then split J_k into $n+1$ subintervals $J_{k,0}^i, \dots, J_{k,n}^i$ with lengths proportional to $\theta_0^i, \dots, \theta_n^i$, by setting

$$J_{k,j}^i = [t_{k,j-1}, t_{k,j}), \quad t_{k,j} = \frac{T}{N} \left(k + \sum_{l=0}^j \theta_l^i \right), \quad t_{k,-1} = \frac{Tk}{N}.$$

For any point $(t, x) \in \overline{\Delta^i}$ we now set

$$(3.12) \quad \begin{cases} g^i(t, x) \doteq \psi_j^i(t, x) \\ \bar{g}^i(t, x) = y_j^i \end{cases} \quad \text{if } t \in \bigcup_{k=1}^N J_{k,j}^i.$$

The piecewise Lipschitz selection g and a piecewise constant approximation \bar{g} of g can now be defined as

$$(3.13) \quad g = \sum_{i=1}^{\nu} g^i \chi_{\Delta^i}, \quad \bar{g} = \sum_{i=1}^{\nu} \bar{g}^i \chi_{\Delta^i}.$$

By construction, recalling (3.7) and (3.8), the conditions (a), (b) in (i) clearly hold.

It remains to show that the estimates in (ii) hold as well. Let $\tau, \tau' \in [0, T]$ and $u \in Y$ be such that $(t, u(t)) \in S$ for every $t \in [\tau, \tau']$, and define

$$E^i = \{t \in I; (t, u(t)) \in \Delta^i\}, \quad i = 1, \dots, \nu.$$

From our previous definition $\Delta^i \doteq \Gamma_i \setminus \bigcup_{j < i} \Gamma_j$, where each Γ_j is an $(M+1)$ -cone, it follows that every E^i is the union of at most i disjoint intervals. We can thus write

$$E^i = \left(\bigcup_{J_k \subset E^i} J_k \right) \cup \widehat{E}^i,$$

with J_k given by (3.11) and

$$(3.14) \quad m(\widehat{E}^i) \leq \frac{2iT}{N} \leq \frac{2\nu T}{N}.$$

Since

$$(3.15) \quad \phi(t_i, x_i) = \sum_{j=0}^n \theta_j^i y_j^i,$$

the definition of \bar{g} in (3.12), (3.13) implies

$$\int_{J_k} [\phi(t_i, x_i) - \bar{g}(s, u(s))] ds = m(J_k) \left[\phi(t_i, x_i) - \sum_{j=0}^n \theta_j^i y_j^i \right] = 0.$$

Therefore, from (3.9) and (3.6) it follows that

$$\begin{aligned} & \left| \int_{J_k} [\phi(s, u(s)) - g(s, u(s))] ds \right| \\ & \leq \left| \int_{J_k} [\phi(s, u(s)) - \phi(t_i, x_i)] ds \right| \\ & \quad + \left| \int_{J_k} [\phi(t_i, x_i) - \bar{g}(s, u(s))] ds \right| + \left| \int_{J_k} [\bar{g}(s, u(s)) - g(s, u(s))] ds \right| \\ & \leq m(J_k) \left[\frac{\varepsilon}{4T} + 0 + \frac{\varepsilon}{4T} \right] = m(J_k) \frac{\varepsilon}{2T}. \end{aligned}$$

The choice of N in (3.10) and the bound (3.14) thus imply

$$\begin{aligned} \left| \int_{\tau}^{\tau'} [\phi(s, u(s)) - g(s, u(s))] ds \right| & \leq 2Mm \left(\bigcup_{i=1}^{\nu} \widehat{E}^i \right) + (\tau' - \tau) \frac{\varepsilon}{2T} \\ & \leq 2M\nu \frac{2\nu T}{N} + \frac{\varepsilon}{2} \leq \varepsilon, \end{aligned}$$

proving (3.4).

We next consider (3.5). For a fixed $i \in \{1, \dots, \nu\}$, let E^i be as before and define

$$\xi_{-1} = 0, \quad \xi_j = \sum_{l=0}^j \theta_l^i, \quad w^i(\xi) = \sum_{j=0}^n y_j^i \chi_{[\xi_{j-1}, \xi_j]}.$$

Recalling (3.15), the definition of h at (2.1) and Hölder's inequality together imply

$$\begin{aligned} h(\phi(t_i, x_i), F(t_i, x_i)) & \geq \left(\int_0^1 |\phi(t_i, x_i) - w^i(\xi)|^2 d\xi \right)^{1/2} \\ & \geq \int_0^1 |\phi(t_i, x_i) - w^i(\xi)| d\xi \\ & = \sum_{j=0}^n \theta_j^i |\phi(t_i, x_i) - y_j^i|. \end{aligned}$$

Using this inequality we obtain

$$\begin{aligned} \int_{J_k} |\phi(t_i, x_i) - \bar{g}(s, u(s))| ds &= m(J_k) \sum_{j=0}^n \theta_j^i |\phi(t_i, x_i) - y_j^i| \\ &\leq m(J_k) \cdot h(\phi(t_i, x_i), F(t_i, x_i)) \leq \eta m(J_k), \end{aligned}$$

and therefore, by (3.9) and (3.6),

$$\begin{aligned} &\int_{J_k} |\phi(s, u(s)) - g(s, u(s))| ds \\ &\leq \int_{J_k} |\phi(s, u(s)) - \phi(t_i, x_i)| ds + \int_{J_k} |\bar{g}(s, u(s)) - g(s, u(s))| ds \\ &\quad + \int_{J_k} |\phi(t_i, x_i) - \bar{g}(s, u(s))|, \\ &\leq m(J_k) \left[\frac{\varepsilon}{4T} + \frac{\varepsilon}{4T} + \eta \right] = m(J_k) \left(\frac{\varepsilon}{2T} + \eta \right). \end{aligned}$$

Using again (3.14) and (3.10), we conclude that

$$\begin{aligned} \int_{\tau}^{\tau'} |\phi(s, u(s)) - g(s, u(s))| ds &\leq (\tau' - \tau) \left(\frac{\varepsilon}{2T} + \eta \right) + 2M\nu \frac{2\nu T}{N} \\ &\leq \varepsilon + (\tau' - \tau)\eta. \end{aligned}$$

which finishes the proof of Lemma 3.

Using Lemma 3, given any continuous selection \tilde{f} of F on Ω^\dagger , and any sequence $(\varepsilon_k)_{k \geq 1}$ of strictly positive numbers, we can generate a sequence $(f_k)_{k \geq 1}$ of selections from F as follows.

To construct f_1 , we apply the lemma with $S = \Omega^\dagger$, $\phi = f_0$, $\varepsilon = \varepsilon_1$. This yields a partition $\{A_1^i; i = 1, \dots, \nu_1\}$ of Ω^\dagger and a piecewise Lipschitz selection f_1 of F of the form

$$f_1 = \sum_{i=1}^{\nu_1} f_1^i \chi_{A_1^i}.$$

In general, at the beginning of the k th step we are given a partition of Ω^\dagger , say $\{A_k^i; i = 1, \dots, \nu_k\}$, and a selection

$$f_k = \sum_{i=1}^{\nu_k} f_k^i \chi_{A_k^i},$$

where each f_k^i is Lipschitz continuous and satisfies

$$h(f_k(t, x), F(t, x)) \leq \varepsilon_k \quad \forall (t, x) \in \overline{A_k^i}.$$

We then apply Lemma 3 separately to each A_k^i , choosing $S = \overline{A_k^i}$, $\varepsilon = \varepsilon_k$, $\phi = f_k^i$. This yields a partition $\{A_{k+1}^i; i = 1, \dots, \nu_{k+1}\}$ of Ω^\dagger and functions of the form

$$f_{k+1} = \sum_{i=1}^{\nu_{k+1}} f_{k+1}^i \chi_{A_{k+1}^i}, \quad \varphi_{k+1}^i(\cdot) = \langle a_{k+1}^i, \cdot \rangle + b_{k+1}^i,$$

where each $f_{k+1}^i : \overline{A_{k+1}^i} \rightarrow \mathbb{R}^n$ is a Lipschitz continuous selection from F , satisfying the following estimates:

$$(3.16) \quad \varphi_{k+1}^i(z) > h(z, F(t, x)) \quad \forall (t, x) \in A_{k+1}^i,$$

$$(3.17) \quad \varphi_{k+1}^i(f_{k+1}^i(t, x)) \leq \varepsilon_{k+1} \quad \forall (t, x) \in A_{k+1}^i,$$

$$(3.18) \quad \left| \int_{\tau}^{\tau'} [f_{k+1}(s, u(s)) - f_k(s, u(s))] ds \right| \leq \varepsilon_{k+1},$$

$$(3.19) \quad \int_{\tau}^{\tau'} |f_{k+1}(s, u(s)) - f_k(s, u(s))| ds \leq \varepsilon_{k+1} + \varepsilon_k(\tau' - \tau),$$

for every $u \in Y$ and every τ, τ' , as long as the values $(s, u(s))$ remain inside a single set A_k^i , for $s \in [\tau, \tau']$.

Observe that, according to Lemma 3, each A_k^i is closed-open in the finer topology generated by all $(M + 1)$ -cones. Therefore, each f_k is Γ^{M+1} -continuous. By Theorem 2 in [2], the substitution operator $S^{f_k} : u(\cdot) \mapsto f_k(\cdot, u(\cdot))$ is continuous from the set Y defined in (2.5) into $\mathcal{L}^1([0, T]; \mathbb{R}^n)$. The Picard map \mathcal{P}^{f_k} is thus continuous as well.

Furthermore, there exists an integer N_k with the following property. Given any $u \in Y$, there exists a finite partition of $[0, T]$ with nodes $0 = \tau_0 < \tau_1 < \dots < \tau_{n(u)} = T$, with $n(u) \leq N_k$, such that, as t ranges in any $[\tau_{l-1}, \tau_l]$, the point $(t, u(t))$ remains inside one single set A_k^i . Otherwise stated, the number of times the curve $t \mapsto (t, u(t))$ crosses a boundary between two distinct sets A_k^i, A_k^j is smaller than N_k , for every $u \in Y$. The construction of the A_k^i in terms of $(M + 1)$ -cones implies that all these crossings are transversal. Since the restriction of f_k to each A_k^i is Lipschitz continuous, it is clear that every Cauchy problem

$$\dot{x}(t) = f_k(t, x(t)), \quad x(t_0) = x_0,$$

has a unique solution, depending continuously on the initial data $(t_0, x_0) \in [0, T] \times D$.

From (3.18), (3.19) and the property of N_k it follows that

$$(3.20) \quad \left| \int_0^t [f_{k+1}(s, u(s)) - f_k(s, u(s))] ds \right| \leq \sum_{l=1}^L \left| \int_{\tau_{l-1}}^{\tau_l} [f_{k+1}(s, u(s)) - f_k(s, u(s))] ds \right| \leq N_k \varepsilon_{k+1},$$

where $0 = \tau_0 < \tau_1 < \dots < \tau_L = t$ are the times at which the map $s \mapsto (s, u(s))$ crosses a boundary between two distinct sets A_k^i, A_k^j . Since (3.20) holds for every $t \in [0, T]$, we conclude that

$$(3.21) \quad \|\mathcal{P}f_{k+1} - \mathcal{P}f_k\| \leq N_k \varepsilon_{k+1}.$$

Similarly, for every $u \in Y$ one has

$$(3.22) \quad \left\| f_{k+1}(\cdot, u(\cdot)) - f_k(\cdot, u(\cdot)) \right\|_{\mathcal{L}^1([0, T]; \mathbb{R}^n)} \\ \leq \sum_{l=1}^{n(u)} \int_{\tau_{l-1}}^{\tau_l} |f_{k+1}(s, u(s)) - f_k(s, u(s))| ds \\ \leq \sum_{l=1}^{n(u)} [\varepsilon_{k+1} + \varepsilon_k(\tau_l - \tau_{l-1})] \leq N_k \varepsilon_{k+1} + \varepsilon_k T.$$

Now consider the functions $\varphi_k : \mathbb{R}^n \times \Omega^\dagger \rightarrow \mathbb{R}$ with

$$(3.23) \quad \varphi_k(y, t, x) \doteq \langle a_k^i, y \rangle + b_k^i \quad \text{if } (t, x) \in A_k^i.$$

From (3.16), (3.17) it follows that

$$(3.24) \quad \varphi_k(y, t, x) \geq h(y, F(t, x)) \quad \forall (t, x) \in \Omega^\dagger, y \in F(t, x),$$

$$(3.25) \quad \varphi_k(f_k(t, x), t, x) \leq \varepsilon_k \quad \forall (t, x) \in \Omega^\dagger.$$

For every $u \in Y$, (3.18) and the linearity of φ_k in y imply

$$(3.26) \quad \left| \int_0^T [\varphi_k(f_{k+1}(s, u(s)), s, u(s)) - \varphi_k(f_k(s, u(s)), s, u(s))] ds \right| \\ \leq \sum_{l=1}^{n(u)} \max\{|a_k^1|, \dots, |a_k^{\nu_k}|\} \left| \int_{\tau_{l-1}}^{\tau_l} [f_{k+1}(s, u(s)) - f_k(s, u(s))] ds \right| \\ \leq N_k \max\{|a_k^1|, \dots, |a_k^{\nu_k}|\} \varepsilon_{k+1}.$$

Moreover, for every $l \geq k$, from (3.19) it follows that

$$(3.27) \quad \int_0^T \left| \varphi_k(f_{l+1}(s, u(s)), s, u(s)) - \varphi_k(f_l(s, u(s)), s, u(s)) \right| ds \\ \leq \max\{|a_k^1|, \dots, |a_k^{\nu_k}|\} \int_0^T |f_{l+1}(s, u(s)) - f_l(s, u(s))| ds \\ \leq \max\{|a_k^1|, \dots, |a_k^{\nu_k}|\} \cdot (N_l \varepsilon_{l+1} + \varepsilon_l T).$$

Observe that all of the above estimates hold regardless of the choice of the ε_k . We now introduce an inductive procedure for choosing the constants ε_k ,

which will yield the convergence of the sequence f_k to a function f with the desired properties.

Given f_0 and ε_0 , by Lemma 2 there exists $\delta_0 > 0$ such that, if $g : \Omega^\dagger \rightarrow \overline{B}(0, M)$ and $\|\mathcal{P}^g - \mathcal{P}^{f_0}\| \leq \delta_0$, then, for each $(t_0, x_0) \in [0, T] \times D$, every solution of (2.7) remains ε_0 -close to the unique solution of (1.3). We then choose $\varepsilon_1 = \delta_0/2$.

By induction on k , assume that the functions f_1, \dots, f_k have been constructed, together with the linear functions $\varphi_l^i(\cdot) = \langle a_l^i, \cdot \rangle + b_l^i$ and the integers N_l , $l = 1, \dots, k$. Let the values $\delta_0, \delta_1, \dots, \delta_k > 0$ be inductively chosen, satisfying

$$(3.28) \quad \delta_l \leq \delta_{l-1}/2, \quad l = 1, \dots, k,$$

and such that $\|\mathcal{P}^g - \mathcal{P}^{f_l}\| \leq \delta_l$ implies that for every $(t_0, x_0) \in [0, T] \times D$ the solution set of (2.7) has diameter $\leq 2^{-l}$, for $l = 1, \dots, k$. This is possible again because of Lemma 2. For $k \geq 1$ we then choose

$$(3.29) \quad \varepsilon_{k+1} \doteq \min \left\{ \frac{\delta_k}{2N_k}, \frac{2^{-k}}{N_k}, \frac{2^{-k}}{N_k \max\{|a_l^i|; 1 \leq l \leq k, 1 \leq i \leq \nu_l\}} \right\}.$$

Using (3.28), (3.29) in (3.21), with $N_0 \doteq 1$, we now obtain

$$(3.30) \quad \sum_{k=p}^{\infty} \|\mathcal{P}^{f_{k+1}} - \mathcal{P}^{f_k}\| \leq \sum_{k=p}^{\infty} N_k \frac{\delta_k}{2N_k} \leq \sum_{k=p}^{\infty} \frac{2^{p-k} \delta_p}{2} \leq \delta_p$$

for every $p \geq 0$. From (3.22) and (3.29) we further obtain

$$(3.31) \quad \begin{aligned} \sum_{k=1}^{\infty} \|f_{k+1}(\cdot, u(\cdot)) - f_k(\cdot, u(\cdot))\|_{\mathcal{L}^1} &\leq \sum_{k=1}^{\infty} \left(N_k \frac{2^{-k}}{N_k} + \frac{2^{1-k}T}{N_k} \right) \\ &\leq \sum_{k=1}^{\infty} (2^{-k} + 2^{1-k}T) \leq 1 + 2T. \end{aligned}$$

Define

$$(3.32) \quad f(t, x) \doteq \lim_{k \rightarrow \infty} f_k(t, x)$$

for all $(t, x) \in \Omega^\dagger$ at which the sequence f_k converges. By (3.31), for every $u \in Y$ the sequence $f_k(\cdot, u(\cdot))$ converges in $\mathcal{L}^1([0, T]; \mathbb{R}^n)$ and a.e. on $[0, T]$. In particular, considering the constant functions $u \equiv x \in \overline{B}(D, MT)$, by Fubini's theorem we conclude that f is defined a.e. on Ω^\dagger . Moreover, the substitution operators $\mathcal{S}^{f_k} : u(\cdot) \mapsto f_k(\cdot, u(\cdot))$ converge to the operator $\mathcal{S}^f : u(\cdot) \mapsto f(\cdot, u(\cdot))$ uniformly on Y . Since each \mathcal{S}^{f_k} is continuous, \mathcal{S}^f is also continuous. Clearly, the Picard map \mathcal{P}^f is continuous as well. By (3.30)

we have

$$\|\mathcal{P}^f - \mathcal{P}^{f_k}\| \leq \sum_{k=p}^{\infty} \|\mathcal{P}^{f_{k+1}} - \mathcal{P}^{f_k}\| \leq \delta_p \quad \forall p \geq 1.$$

Recalling the property of δ_p , this implies that, for every p , the solution set of (2.7) has diameter $\leq 2^{-p}$. Since p is arbitrary, for every $(t_0, x_0) \in [0, T] \times D$ the Cauchy problem can have at most one solution. On the other hand, the existence of such a solution is guaranteed by Schauder's theorem. The continuous dependence of this solution on the initial data t_0, x_0 , in the norm of \mathcal{AC} , is now an immediate consequence of uniqueness and of the continuity of the operators $\mathcal{S}^f, \mathcal{P}^f$. Furthermore, for $p = 0$, (3.30) yields $\|\mathcal{P}^f - \mathcal{P}^{f_0}\| \leq \delta_0$. The choice of δ_0 thus implies (1.4).

It now remains to prove (1.1). Since every set $F(t, x)$ is closed, it is clear that $f(t, x) \in F(t, x)$. For every $u \in Y$ and $k \geq 1$, by (3.24)–(3.27) the choices of ε_k at (3.29) yield

$$\begin{aligned} (3.33) \quad & \int_0^T h(f(s, u(s)), F(s, u(s))) \, ds \\ & \leq \int_0^T \varphi_k(f(s, u(s)), s, u(s)) \, ds \\ & \leq \int_0^T \varphi_k(f_k(s, u(s)), s, u(s)) \, ds \\ & \quad + \left| \int_0^T [\varphi_k(f_{k+1}(s, u(s)), s, u(s)) - \varphi_k(f_k(s, u(s)), s, u(s))] \, ds \right| \\ & \quad + \sum_{l=k+1}^{\infty} \int_0^T |\varphi_k(f_{l+1}(s, u(s)), s, u(s)) - \varphi_k(f_l(s, u(s)), s, u(s))| \, ds \\ & \leq 2^{1-k}T + 2^{-k} + \sum_{l=k+1}^{\infty} (2^{-l} + 2^{1-l}T). \end{aligned}$$

Observing that the right hand side of (3.33) approaches zero as $k \rightarrow \infty$, we conclude that

$$\int_0^T h(f(t, u(t)), F(t, u(t))) \, dt = 0.$$

By (2.2), given any $u \in Y$, this implies $f(t, u(t)) \in \text{ext } F(t, u(t))$ for almost every $t \in [0, T]$. By possibly redefining f on a set of measure zero, this yields (1.1).

4. Applications. Throughout this section we make the following assumptions:

- (H) $F : [0, T] \times \Omega \rightarrow \overline{B}(0, M)$ is a bounded continuous multifunction with compact values satisfying (LSP), while D is a compact set such that $\overline{B}(D, MT) \subset \Omega$.

An immediate consequence of Theorem 1 is

COROLLARY 1. *Let the hypotheses (H) hold. Then there exists a continuous map $(t_0, x_0) \mapsto x(\cdot, t_0, x_0)$ from $[0, T] \times D$ into \mathcal{AC} such that*

$$\begin{cases} \dot{x}(t, t_0, x_0) \in \text{ext } F(t, x(t, t_0, x_0)) & \forall t \in [0, T], \\ x(t_0, t_0, x_0) = x_0 & \forall t_0, x_0. \end{cases}$$

Another consequence of Theorem 1 is the contractibility of the sets of solutions of certain differential inclusions. We recall here that a metric space X is contractible if there exist a point $\tilde{u} \in X$ and a continuous mapping $\Phi : X \times [0, 1] \rightarrow X$ such that

$$\Phi(v, 0) = \tilde{u}, \quad \Phi(v, 1) = v, \quad \forall v \in X.$$

The map Φ is then called a *null homotopy* of X .

COROLLARY 2. *Let the assumptions (H) hold. Then, for any $\bar{x} \in D$, the sets $\mathcal{M}, \mathcal{M}^{\text{ext}}$ of solutions of*

$$\begin{aligned} \dot{x}(t) &\in F(t, x(t)), & t \in [0, T], \\ x(0) &= \bar{x}, & \dot{x} \in \text{ext } F(t, x(t)), & t \in [0, T], \end{aligned}$$

are both contractible in \mathcal{AC} .

Proof. Let f be a selection from $\text{ext } F$ with the properties stated in Theorem 1. As usual, we denote by $x(\cdot, t_0, x_0)$ the unique solution of the Cauchy problem (1.2). Define the null homotopy $\Phi : \mathcal{M} \times [0, 1] \rightarrow \mathcal{M}$ by

$$\Phi(v, \lambda)(t) \doteq \begin{cases} v(t) & \text{if } t \in [0, \lambda T], \\ x(t, \lambda T, v(\lambda T)) & \text{if } t \in [\lambda T, T]. \end{cases}$$

By Theorem 1, Φ is continuous. Moreover, setting $\tilde{u}(\cdot) \doteq u(\cdot, 0, \bar{x})$, we obtain

$$\Phi(v, 0) = \tilde{u}, \quad \Phi(v, 1) = v, \quad \Phi(v, \lambda) \in \mathcal{M} \quad \forall v \in \mathcal{M},$$

proving that \mathcal{M} is contractible. We now observe that, if $v \in \mathcal{M}^{\text{ext}}$, then $\Phi(v, \lambda) \in \mathcal{M}^{\text{ext}}$ for every λ . Therefore, \mathcal{M}^{ext} is contractible as well.

Our last application is concerned with feedback controls. Let $\Omega \subseteq \mathbb{R}^n$ be open, $U \subset \mathbb{R}^m$ compact, and let $g : [0, T] \times \Omega \times U \rightarrow \mathbb{R}^n$ be a continuous function. By a well-known theorem of Filippov [11], the solutions of the control system

$$(4.1) \quad \dot{x} = g(t, x, u), \quad u \in U,$$

correspond to the trajectories of the differential inclusion

$$(4.2) \quad \dot{x} \in F(t, x) \doteq \{g(t, x, \omega); \omega \in U\}.$$

In connection with (4.1), one can consider the “relaxed” system

$$(4.3) \quad \dot{x} = g^\#(t, x, u^\#), \quad u^\# \in U^\#,$$

whose trajectories are precisely those of the differential inclusion

$$\dot{x} \in F^\#(t, x) \doteq \overline{\text{co}}F(t, x).$$

The control system (4.3) is obtained by defining the compact set

$$U^\# \doteq U \times \dots \times U \times \Delta_n = U^{n+1} \times \Delta_n,$$

where

$$\Delta_n \doteq \left\{ \theta = (\theta_0, \dots, \theta_n) ; \sum_{i=0}^n \theta_i = 1, \theta_i \geq 0 \forall i \right\}$$

is the standard simplex in \mathbb{R}^{n+1} , and by setting

$$g^\#(t, x, u^\#) = g^\#(t, x, (u_0, \dots, u_n, (\theta_0, \dots, \theta_n))) \doteq \sum_{i=0}^n \theta_i f(t, x, u_i).$$

Generalized controls of the form $u^\# = (u_0, \dots, u_n, \theta)$ taking values in the set $U^{n+1} \times \Delta_n$ are called *chattering controls*.

COROLLARY 3. *Consider the control system (4.1), with $g : [0, T] \times \Omega \times U \rightarrow \overline{B}(0, M)$ Lipschitz continuous. Let D be a compact set with $\overline{B}(D; MT) \subset \Omega$. Let $u^\#(t, x) \in U^\#$ be a chattering feedback control such that the mapping*

$$(t, x) \mapsto g^\#(t, x, u^\#(t, x)) \doteq f_0(t, x)$$

is Lipschitz continuous.

Then for every $\varepsilon_0 > 0$ there exists a measurable feedback control $\bar{u} = \bar{u}(t, x)$ with the following properties:

- (a) *for every (t, x) , one has $g(t, x, \bar{u}(t, x)) \in \text{ext } F(t, x)$, with F as in (4.2),*
- (b) *for every $(t_0, x_0) \in [0, T] \times D$, the Cauchy problem*

$$\dot{x}(t) = g(t, x(t), \bar{u}(t, x(t))), \quad x(t_0) = x_0,$$

has a unique solution $x(\cdot, t_0, x_0)$,

- (c) *if $y(\cdot, t_0, x_0)$ denotes the (unique) solution of the Cauchy problem*

$$\dot{y} = f_0(t, y(t)), \quad y(t_0) = x_0,$$

then for every (t_0, x_0) one has

$$|x(t, t_0, x_0) - y(t, t_0, x_0)| < \varepsilon_0 \quad \forall t \in [0, T].$$

Proof. The Lipschitz continuity of g implies that the multifunction F in (4.2) is Lipschitz continuous in the Hausdorff metric, hence it satisfies

(LSP). We can thus apply Theorem 1, and obtain a suitable selection f of $\text{ext } F$, in connection with f_0, ε_0 . For every (t, x) , the set

$$W(t, x) \doteq \{\omega \in U; g(t, x, \omega) = f(t, x)\} \subset \mathbb{R}^m$$

is a compact nonempty subset of U . Let $\bar{u}(t, x) \in W(t, x)$ be the lexicographic selection. Then the feedback control \bar{u} is measurable, and it is trivial to check that \bar{u} has all the required properties.

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