

Nearly Optimal Patchy Feedbacks

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ABSTRACT. The paper is concerned with a general optimization problem for a nonlinear control system, in the presence of a running cost and a terminal cost, with free terminal time. We prove the existence of a patchy feedback whose trajectories are all nearly optimal solutions, with pre-assigned accuracy.

1 - Introduction

Consider a general optimization problem

$$\min_{T, u(\cdot)} \left\{ \psi(x(T)) + \int_0^T L(x(t), u(t)) dt \right\}, \quad (1.1)$$

for a nonlinear control system of the form

$$\dot{x} = f(x, u) \quad u(t) \in \mathbf{U}. \quad (1.2)$$

Here $x \in \mathbf{R}^n$ describes the state of the system, the upper dot denotes a derivative w.r.t. time, and $\mathbf{U} \subset \mathbf{R}^m$ is the set of admissible control values. The minimum is sought over all times $T \geq 0$ and all measurable control functions $u : [0, T] \mapsto \mathbf{U}$.

In the literature, several results are available, which provide the existence of an optimal control $t \mapsto u^{opt}(t)$ in open-loop form [14, 16, 23], for any fixed initial condition

$$x(0) = y \in \mathbf{R}^n. \quad (1.3)$$

On the other hand, the existence and regularity of an optimal control in feedback form is a far more difficult issue. In an ideal situation, one would like to construct a (sufficiently regular) feedback $u = U(x)$ such that all trajectories of the corresponding O.D.E.

$$\dot{x} = f(x, U(x)) \quad (1.4)$$

are optimal w.r.t. the cost criterion (1.1). Only few general results are presently known in this direction [7, 16, 20, 26]. In general, the optimal feedback can be discontinuous, with an extremely complicated structure [8, 17]. Moreover, its performance may not be robust: an arbitrarily small external perturbation may produce trajectories which are far from being optimal [24].

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An alternative strategy, pursued in [3, 18, 19], is to construct sub-optimal feedbacks, trading off the full optimality in favor of a simpler structure of the control and the robustness of the resulting system. This approach also faces difficulties. In some cases, because of topological obstructions it is not possible to construct any continuous asymptotically stabilizing feedback [10, 13, 14, 25], or any continuous near-optimal feedback [9]. Therefore, one needs to work with discontinuous feedback controls [11, 12, 21, 22]. For discontinuous O.D.E's, however, no general result about existence and uniqueness of solutions is available. Carathéodory solutions can be constructed only under additional assumptions on the structure of discontinuities [15].

Following the approach developed in [1,2,3], asymptotic stabilization and optimal control problems can be solved using *patchy feedbacks* as discontinuous controls. We recall that a patchy feedback has a particularly simple structure, since it is a function $u = U(x)$ that is piecewise constant on the state space \mathbf{R}^n . For patchy vector fields, one can prove that Carathéodory solutions forward in time always exist [1]. Moreover, the set of forward solutions is stable w.r.t. small perturbations [2]. The analysis in [3] showed that any minimum time problem can be approximately solved using these patchy feedbacks.

Aim of the present paper is to extend the results in [3] to the general optimization problem (1.1). In addition, we present a construction which greatly simplifies the previous approach, thus clarifying the main lines of the proof.

For convenience, we list here all the basic assumptions.

- (A) The set of admissible control values $U \subset \mathbf{R}^m$ is a compact, the function $f : \mathbf{R}^n \times U \mapsto \mathbf{R}^n$ is continuous w.r.t. both variables, and twice continuously differentiable w.r.t. x . In addition, f satisfies the sub-linear growth condition

$$|f(x, u)| \leq C_f(1 + |x|) \quad \forall u \in U, \quad (1.5)$$

for some constant C_f . Both the terminal cost $\psi : \mathbf{R}^n \mapsto \mathbf{R}$ and the running cost $L : \mathbf{R}^n \times U \mapsto \mathbf{R}$ are continuous and non-negative. Moreover, L is strictly positive:

$$L(x, u) \geq \alpha_0 > 0 \quad \forall x \in \mathbf{R}^n, u \in U. \quad (1.6)$$

Throughout this paper, V denotes the value function for the optimization problem (1.1)-(1.2), namely

$$V(y) \doteq \inf_{T, x(\cdot, u)} \left\{ \psi(x(T)) + \int_0^T L(x(t), u(t)) dt \right\}, \quad (1.7)$$

where the minimization is taken over all $T \geq 0$ and all solutions of $t \mapsto x(t, u)$, corresponding to a measurable control $u : [0, T] \mapsto U$. Our main result can be stated as follows.

Theorem 1. *Let the functions ψ, L, f in (1.1)-(1.2) satisfy the assumptions (A). Let $\varepsilon > 0$ and a compact set $K \subset \mathbf{R}^n$ be given. Then there exist a closed terminal set $S \subseteq \mathbf{R}^n$ and a patchy feedback $u = U(x)$ defined on the complement $\mathbf{R}^n \setminus S$ such that the following holds. For each $y \in K$, every Carathéodory solution of*

$$\dot{x} = f(x, U(x)), \quad x(0) = y \quad (1.8)$$

reaches the set S within finite time. Calling $\tau \doteq \inf \{t; x(t) \in S\}$ the first time where the trajectory reaches S , we have

$$\psi(x(\tau)) + \int_0^\tau L(x(t), U(x(t))) dt \leq V(y) + \varepsilon. \quad (1.9)$$

We recall that, by well known properties of patchy vector fields, for every initial point $y \in \mathbf{R}^n \setminus S$ the O.D.E. (1.8) has at least one forward Carathéodory solution. According to (1.9), all of the solutions starting from the compact set K are nearly optimal, for the cost (1.1).

In the remainder of the paper, Section 2 contains a brief review of the main definitions and properties of patchy feedbacks and patchy vector fields. The proof of Theorem 1 is then worked out in Section 3.

2 - Review of patchy feedbacks

The following definitions were introduced in [1].

Definition 1. By a **patch** we mean a pair (Ω, g) where $\Omega \subset \mathbf{R}^n$ is an open domain with smooth boundary $\partial\Omega$, and g is a Lipschitz continuous vector field defined on a neighborhood of the closure $\bar{\Omega}$ of Ω , which points strictly inward at each boundary point $x \in \partial\Omega$.

Calling $\mathbf{n}(x)$ the outer normal at the boundary point x , and denoting the inner product by a dot, we thus require

$$\mathbf{n}(x) \cdot g(x) < 0 \quad \forall x \in \partial\Omega. \quad (2.1)$$

Definition 2. We say that $g : \Omega \mapsto \mathbf{R}^n$ is a **patchy vector field** on the open domain Ω if there exists a family of patches $\{(\Omega_\alpha, g_\alpha); \alpha \in \mathcal{A}\}$ such that

- \mathcal{A} is a totally ordered set of indices,
- the open sets Ω_α form a locally finite covering of Ω ,
- the vector field g can be written in the form

$$g(x) = g_\alpha(x) \quad \text{if} \quad x \in \Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta. \quad (2.2)$$

We shall occasionally adopt the longer notation $(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha \in \mathcal{A}})$ to indicate a patchy vector field, specifying both the domain and the single patches.

By setting

$$\alpha^*(x) \doteq \max \{ \alpha \in \mathcal{A}; x \in \Omega_\alpha \}, \quad (2.3)$$

we can write (2.2) in the equivalent form

$$g(x) = g_{\alpha^*(x)}(x) \quad \forall x \in \Omega. \quad (2.4)$$

Remark 1. It is important to observe that the patches $(\Omega_\alpha, g_\alpha)$ are not uniquely determined by a patchy vector field (Ω, g) . Indeed, whenever $\alpha < \beta$, by (2.2) the values of g_α on the set $\Omega_\alpha \cap \Omega_\beta$ are irrelevant. Of course, the values of g_α for x outside the domain Ω don't matter either. Therefore, if the open sets Ω_α form a locally finite covering of Ω and if for each $\alpha \in \mathcal{A}$ the vector field g_α satisfies

$$\mathbf{n}_\alpha(x) \cdot g_\alpha(x) < 0 \quad \forall x \in \Omega \cap \partial\Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta, \quad (2.5)$$

then the vector field g in (2.2) is still a patchy vector field. Indeed, without changing the function g , one can suitably redefine the values of each g_α on the set $\bigcup_{\beta > \alpha} \Omega_\beta$, or outside Ω , and achieve the strict inequality

$$\mathbf{n}_\alpha(x) \cdot g_\alpha(x) < 0 \quad \forall x \in \partial\Omega_\alpha.$$

Remark 2. For convenience, we are always assuming that the single patches Ω_α are open, while the vector fields g_α are defined on the closure $\overline{\Omega}_\alpha$. In certain situations, it would be natural to choose patches of the form

$$\Omega_1 \doteq \{x \in \Omega; \mathbf{n} \cdot x < c\}, \quad \Omega_2 \doteq \{x \in \Omega; \mathbf{n} \cdot x > c\},$$

for some unit vector \mathbf{n} . In this way, however, the union $\Omega_1 \cup \Omega_2$ does not cover all of Ω , because it does not contain points where $\mathbf{n} \cdot x = c$. This situation is easily fixed, replacing Ω_1 by a slightly larger open set which contains also these boundary points. The resulting vector field

$$g(x) = \begin{cases} g_1(x) & \text{if } \mathbf{n} \cdot x \leq c, \\ g_2(x) & \text{if } \mathbf{n} \cdot x > c, \end{cases}$$

can still be written in patchy form.

If g is a patchy vector field, the differential equation

$$\dot{x} = g(x) \quad (2.6)$$

has several useful properties. There are collected in the following theorem, proved in [1].

Theorem 2. *Let g be a patchy vector field. Then the set of Carathéodory solutions of (2.6) is closed (in the topology of uniform convergence) but possibly not connected. For each Carathéodory solution $t \mapsto x(t)$, the map $t \mapsto \alpha^*(x(t))$ defined by (2.3) is left-continuous and non-decreasing.*

Given an initial condition

$$x(t_0) = x_0, \quad (2.7)$$

the Cauchy problem (2.6)-(2.7) has at least one forward solution and at most one backward solution, in the Carathéodory sense.

Remark 3. In some situations it is convenient to adopt a more general definition of patchy vector field than the one formulated above. Indeed, one can consider

patches $(\Omega_\alpha, g_\alpha)$ where the boundary of the domain Ω_α is only piecewise smooth. For example, Ω_α could be a polytope, or the intersection between a ball and finitely many half-spaces. In this more general case, the inward-pointing condition (2.1) can be reformulated by asking that, for each boundary point $x \in \partial\Omega_\alpha$, the vector $g_\alpha(x)$ lies in the interior of the tangent cone to Ω_α at the point x . Namely

$$g_\alpha(x) \in \text{int } T_{\Omega_\alpha}(x). \quad (2.8)$$

As in [4], this tangent cone is defined by

$$T_{\Omega_\alpha}(x) \doteq \left\{ v \in \mathbf{R}^n : \liminf_{t \downarrow 0} \frac{d(x + tv, \Omega_\alpha)}{t} = 0 \right\}.$$

One can easily check that all the results concerning patchy vector fields stated in Theorem 2 remain valid with this more general formulation. Indeed, for any initial data \bar{x} in the closure $\overline{\Omega_\alpha}$, the corresponding solution of

$$\dot{x} = g_\alpha(x) \quad x(0) = \bar{x}$$

is well defined and lies strictly in the interior of Ω_α , for every $t > 0$.

Definition 3. Let $(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha \in \mathcal{A}})$ be a patchy vector field. Assume that there exist control values $v_\alpha \in \mathbf{U}$ such that, for each $\alpha \in \mathcal{A}$, there holds

$$g_\alpha(x) = f(x, v_\alpha) \quad \forall x \in \Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta. \quad (2.9)$$

Then the piecewise constant map

$$U(x) \doteq v_\alpha \quad \text{if} \quad x \in \Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta. \quad (2.10)$$

is called a **patchy feedback** control on Ω .

Recalling (2.3), the patchy feedback control can thus be written on the form

$$U(x) = v_{\alpha^*(x)}.$$

3 - Proof of the theorem

The proof of Theorem 1 will be given in several steps.

1. Various reductions can be performed. By a smooth approximation, we can assume that $\psi \in \mathcal{C}^\infty$. Moreover, approximating the cost function L by a more regular function, it is not restrictive to assume that L is twice continuously differentiable w.r.t. x . Recalling that $L(x, u) \geq \alpha_0 > 0$, we can now replace $f(x, u)$ by

$$g(x, u) \doteq \frac{f(x, u)}{L(x, u)}, \quad (3.1)$$

and consider the equivalent problem

$$\inf_{\tau, u(\cdot)} \left\{ \tau + \psi(x(\tau)) \right\}, \quad (3.2)$$

with dynamics

$$\dot{x} = g(x, u), \quad x(0) = y.$$

Notice that the function g in (3.1) is continuous w.r.t. both variables x, u , and twice continuously differentiable w.r.t. x . Moreover it satisfies the growth condition

$$|g(x, u)| \leq \frac{C_f}{\alpha_0} (1 + |x|) \quad \forall u \in \mathbf{U}.$$

In the following, we thus assume without loss of generality that the running cost is simply $L(x, u) \equiv 1$, so that the minimization problem (1.1) reduces to (3.2).

2. Choose a constant M such that

$$M \geq 1, \quad M \geq \max_{x \in K} \psi(x). \quad (3.3)$$

To fix the ideas, throughout the following we assume that $0 < \varepsilon < 1/8$ and that the compact set K is contained in the open ball B_ρ centered at the origin with radius ρ . Because of the sub-linear growth condition (1.5), for $\tau \leq 2M$, every trajectory of the system (1.2) starting at a point $y \in K \subset B_\rho$ will satisfy the a priori bound

$$|x(t)| < \bar{\rho} \quad \text{for all } t \in [0, \tau] \subseteq [0, 2M], \quad (3.4)$$

where

$$\bar{\rho} \doteq e^{C_f \cdot 2M} (\rho + 1). \quad (3.5)$$

3. Let $V = V(y)$ be the value function for the optimization problem (3.2), with dynamics (1.2). We claim that V is semi-concave. More precisely, there exists a constant κ such that, for any $y, y' \in B_{\bar{\rho}}$, one has

$$V(y') \leq V(y) + \mathbf{w} \cdot (y' - y) + \kappa \frac{|y' - y|^2}{2}. \quad (3.6)$$

for some vector $\mathbf{w} \in D^+V(y)$ in the upper gradient of V at the point y .

Indeed, from the theory of optimal control [6] it is well known that the optimization problem (3.2), (1.2) with initial data $x(0) = y$ has at least one solution, within the class of chattering controls. Let $t \mapsto x(t) = x(t; y, \tilde{u}, \tilde{\theta})$ be an optimal chattering trajectory, with

$$x(0) = y, \quad \dot{x}(t) = \sum_{i=0}^n \theta_i(t) f(x(t), u_i(t)) \quad t \in [0, \tau], \quad (3.7)$$

for some measurable functions $(\tilde{u}, \tilde{\theta}) = (u_0, \dots, u_n, \theta_0, \dots, \theta_n)$ satisfying

$$u_i : [0, \tau] \mapsto \mathbf{U}, \quad \theta_i : [0, \tau] \mapsto [0, 1], \quad \sum_{i=0}^n \theta_i(t) \equiv 1. \quad (3.8)$$

For any other initial data y' , we can consider the same chattering control $(\tilde{u}, \tilde{\theta})$, always stopping at the same terminal time $t = \tau$. This yields the cost

$$V^{\tilde{u}, \tilde{\theta}, \tau}(y') = \tau + \psi(x(\tau; y', \tilde{u}, \tilde{\theta})). \quad (3.9)$$

The regularity assumptions on f, ψ w.r.t. the variable x imply that, as y' varies in the ball $B_{\bar{\rho}}$, the map $y' \mapsto V^{\tilde{u}, \tilde{\theta}, \tau}(y')$ is twice continuously differentiable. Moreover, its C^2 norm remains bounded:

$$\|V^{\tilde{u}, \tilde{\theta}, \tau}\|_{C^2(B_{\bar{\rho}})} \leq \kappa. \quad (3.10)$$

Since $\tau \in [0, T_{max}]$ while both \tilde{u} and $\tilde{\theta}$ in (3.8) range over compact sets, this bound is uniform, i.e. in (3.10) we can take a constant $\kappa > 1$ which does not depend on the particular chattering control, or on the time τ . Observing that

$$V(y) = V^{\tilde{u}, \tilde{\theta}, \tau}(y), \quad V(y') \leq V^{\tilde{u}, \tilde{\theta}, \tau}(y') \quad \forall y' \in B_{\bar{\rho}},$$

the inequality (3.6) follows from (3.10), choosing $\mathbf{w} = \nabla V^{\tilde{u}, \tilde{\theta}, \tau}(y)$.

4. As shown in the previous step, the value function

$$V(y) = \min_{\tilde{u}, \tilde{\theta}, \tau} V^{\tilde{u}, \tilde{\theta}, \tau}(y)$$

is Lipschitz continuous on the ball $B_{\bar{\rho}}$. In fact, the constant $\kappa > 1$ in (3.10) also provides a Lipschitz constant for V , namely

$$V(x) - V(y) \leq \kappa |x - y| \quad \forall x, y \in B_{\bar{\rho}}. \quad (3.11)$$

By Rademacher's theorem, V is differentiable almost everywhere. At each point $x \in B_{\bar{\rho}}$ where the gradient $\nabla V(x)$ exists, if $V(x) < \psi(x)$ then one has the well known relation [5, 10, 16]

$$\min_{u \in \mathbf{U}} \{\nabla V(x) \cdot f(x, u)\} + 1 = 0. \quad (3.12)$$

Consider the open set

$$\mathcal{D} \doteq \{x; V(x) < \psi(x)\}.$$

Given $\delta > 0$, we can choose finitely many points $y_1, \dots, y_m \in B_{\bar{\rho}} \cap \mathcal{D}$ such that $\nabla V(y_i)$ is well defined for each $i = 1, \dots, m$, and moreover

$$B_{\bar{\rho}} \cap \mathcal{D} \subseteq \bigcup_{i=1}^m B(y_i, \delta). \quad (3.13)$$

Define the approximate value function

$$W(x) \doteq \min \{\psi(x), W_1(x), \dots, W_m(x)\}, \quad (3.14)$$

where

$$W_i(x) \doteq V(y_i) + \nabla V(y_i) \cdot (x - y_i) + \kappa |x - y_i|^2. \quad (3.15)$$

We claim that, by choosing $\delta > 0$ sufficiently small, for all $x \in B_{\bar{\rho}}$ the following relations hold.

$$V(x) \leq W(x) \leq V(x) + \varepsilon, \quad (3.16)$$

$$\left| \min_{u \in \mathbf{U}} \{ \nabla W_i(x) \cdot f(x, u) \} + 1 \right| \leq \varepsilon \quad \text{whenever } W_i(x) = W(x), \quad (3.17)$$

Indeed, the first inequality in (3.16) follows from (3.6). Next, since f is continuous and \mathbf{U} is compact, we can find $\delta_1 \in]0, 1]$ such that the following conditions hold. If $x \in B_{\bar{\rho}}$, $\mathbf{w} = \nabla V(y)$ exists and

$$\min_{u \in \mathbf{U}} \{ \mathbf{w} \cdot f(y, u) \} + 1 = 0,$$

$$|\mathbf{w}' - \mathbf{w}| \leq 2\kappa\delta_1, \quad |x - y| \leq \delta_1,$$

then

$$\left| \min_{u \in \mathbf{U}} \{ \mathbf{w}' \cdot f(x, u) \} + 1 \right| \leq \varepsilon. \quad (3.18)$$

We now choose $\delta > 0$ such that

$$2\kappa\delta + \kappa\delta^2 \leq \min \left\{ \varepsilon, \frac{\kappa\delta_1^2}{2} \right\}.$$

Given any $x \in B_{\bar{\rho}}$, if j is an index such that $|x - y_j| \leq \delta$, recalling the Lipschitz condition (3.11) we find

$$W(x) \leq V(y_j) + |\nabla V(y_j)| |x - y_j| + \kappa |x - y_j|^2 \leq V(x) + 2\kappa |x - y_j| + \kappa |x - y_j|^2,$$

$$W(x) - V(x) \leq \min \left\{ \varepsilon, \frac{\kappa\delta_1^2}{2} \right\}. \quad (3.19)$$

This already yields (3.16). Comparing (3.6) with (3.15) we notice that

$$W_i(x) - V(x) \geq \kappa \frac{|x - y_i|^2}{2}.$$

Hence from (3.19) it follows

$$|x - y_i| \leq \delta_1, \quad \text{whenever } W_i(x) = W(x). \quad (3.20)$$

Observing that, if $W(x) = W_i(x)$,

$$|\nabla W_i(x) - \nabla W_i(y_i)| \leq 2\kappa |x - y_i| \leq 2\kappa\delta_1,$$

from (3.18) we deduce the inequality (3.17). This establishes our claim.

5. By the definition of W_i , it is clear that all level sets where W_i is constant are spheres. Indeed, for any given constant c we can write

$$\{x; W_i(x) = c\} = \{x; |x - x_i| = r\},$$

with $x_i = y_i - \nabla V(y_i)/2\kappa$ and a suitable radius r .

For each $i = 1, \dots, m$, consider the set

$$\mathcal{D}_i \doteq \{x \in B_{\bar{\rho}}; W_i(x) = W(x)\}. \quad (3.21)$$

In this step we show that there exists a minimum radius $r_{min} > 0$ and a maximum radius r_{max} such that, fixed $x \in \mathcal{D}_i$, the level set where $W_i = W_i(x)$ is a sphere of center x_i and radius r with

$$0 < r_{min} \leq r \leq r_{max}. \quad (3.22)$$

Indeed, since $\varepsilon < 1/2$, by (3.17) it follows

$$|\nabla W_i(x)| |f(x, u)| > \frac{1}{2}. \quad (3.23)$$

Calling

$$M_f \doteq \max_{|x| \leq \bar{\rho}, u \in \mathbf{U}} |f(x, u)| \leq C_f (1 + \bar{\rho}),$$

from (3.23) we deduce

$$|\nabla W_i(x)| > \frac{1}{2M_f}.$$

Therefore, for any ξ such that $W_i(\xi) = W_i(x)$,

$$|\xi - x_i| = |x - x_i| = \frac{|\nabla W_i(x)|}{2\kappa} > \frac{1}{4\kappa M_f} \doteq r_{min}.$$

On the other hand, by (3.15) and (3.20) we have

$$|\nabla W_i(x)| \leq |\nabla W_i(y_i)| + 2\kappa|x - y_i| \leq \kappa + 2\kappa\delta_1 \leq 3\kappa.$$

Hence, for any ξ such that $W_i(\xi) = W_i(x)$,

$$|\xi - x_i| = |x - x_i| = \frac{|\nabla W_i(x)|}{2\kappa} \leq \frac{3}{2} \doteq r_{max}. \quad (3.24)$$

6. We are now ready to construct the near-optimal patchy feedback. We will define $U(x)$ on the open set

$$\Omega \doteq \{x \in B_{\bar{\rho}}; W(x) < \psi(x)\}, \quad (3.25)$$

and the required terminal set S will be defined as $S \doteq \mathbf{R}^n \setminus \Omega$. Given $\eta > 0$ small, for each point $x \in \mathcal{D}_i$ consider the point (see Figure 1)

$$p_i^x \doteq \frac{2}{3}x + \frac{1}{3}x_i + \eta \frac{x - x_i}{|x - x_i|}$$

and the ball $B_i^x = B(p_i^x, |x - x_i|/3)$ centered at p_i^x with radius $r = |x - x_i|/3$. By (3.17), there exists a nearly-optimal control value $u = u_i^x \in \mathbf{U}$ such that

$$\nabla W_i(x) \cdot f(x, u_i^x) \leq -1 + \varepsilon. \quad (3.26)$$

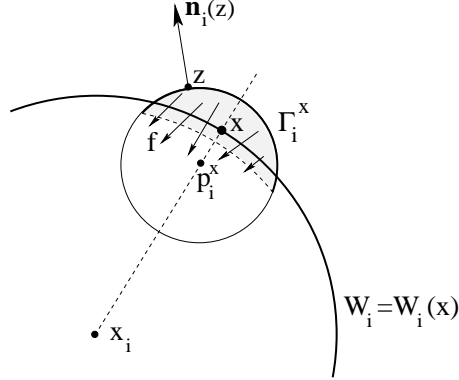


Figure 1. Construction of a lens-shaped patch.

Consider the lens-shaped region

$$\Gamma_i^x \doteq B\left(p_i^x, \frac{|x - x_i|}{3}\right) \setminus \overline{B}(x_i, |x - x_i| - \eta). \quad (3.27)$$

Its upper boundary will be denoted as

$$\partial^+ \Gamma_i^x \doteq \partial \Gamma_i^x \setminus \overline{B}(x_i, |x - x_i| - \eta). \quad (3.28)$$

Moreover, for $z \in \partial^+ \Gamma_i^x$, we write $\mathbf{n}_i(z)$ for the outer unit normal at the point z .

We claim that, by choosing $\eta > 0$ sufficiently small, the following holds:

$$\nabla W_i(z) \cdot f(z, u_i^x) \leq -1 + 2\varepsilon \quad \forall z \in \Gamma_i^x, \quad (3.29)$$

$$\mathbf{n}_i(z) \cdot f(z, u_i^x) \leq -\eta \quad \forall z \in \partial^+ \Gamma_i^x. \quad (3.30)$$

Moreover, the constant $\eta > 0$ can be chosen uniformly valid for all $i = 1, \dots, m$ and all $x \in \mathcal{D}_i$.

For fixed i, x this is clear because, as $\eta \rightarrow 0$, the diameter of the set Γ_i^x approaches zero. Moreover, as z varies on the upper boundary $\partial^+ \Gamma_i^x$, all the unit normals $\mathbf{n}_i(z)$ approach the vector $\nabla W_i(x)/|\nabla W_i(x)|$. Therefore, both inequalities (3.29)-(3.30) follow from (3.26).

We now observe that $f = f(x, u)$ is uniformly continuous on the compact domain $B_{\bar{\rho}} \times \mathbf{U}$. Moreover, on each set \mathcal{D}_i , the gradient $\nabla W_i(x)$ is uniformly Lipschitz continuous and bounded away from zero. Finally, the radius of each level set, where W_i is constant, by (3.22) is uniformly bounded above and below. This allows us to choose a constant $\eta > 0$ uniformly valid for all i, x .

7. To achieve a nearly optimal feedback, we would need the inequality

$$\nabla W(z) \cdot f(z, u_i^x) \leq -1 + 4\varepsilon \quad \forall z \in \Gamma_i^x. \quad (3.31)$$

If $W(z) = W_i(z)$ for all $z \in \Gamma_i^x$, this is a trivial consequence of (3.29). However, we must also consider the case where some of the points $z \in \Gamma_i^x$ lie in a region where

$W(z) = W_j(z) < W_i(z)$, for some different index j . For this purpose, we observe that the set where $W_i = W_j$ is always a hyperplane, say

$$\mathcal{H}_{ij} \doteq \{x; W_i(x) = W_j(x)\} = \{x; \mathbf{n}_{ij} \cdot x = c_{ij}\}. \quad (3.32)$$

for a suitable constant c_{ij} and a unit normal vector \mathbf{n}_{ij} . The orientation of \mathbf{n}_{ij} will be chosen so that

$$\{x; W_i(x) < W_j(x)\} = \{x; \mathbf{n}_{ij} \cdot x < c_{ij}\}.$$

We claim that, by choosing $\eta > 0$ sufficiently small, uniformly w.r.t. i, x , one of the following two cases occurs (see Figure 2).

CASE 1: At every point $z \in \Gamma_i^x \cap \mathcal{H}_{ij}$ one has

$$\mathbf{n}_{ij} \cdot f(z, u_i^x) < -\eta. \quad (3.33)$$

CASE 2: At every point $z \in \Gamma_i^x$ one has

$$\nabla W_j(z) \cdot f(z, u_i^x) \leq -1 + 4\varepsilon. \quad (3.34)$$

Indeed, assume that (3.33) fails. Then there exists a point $z^* \in \Gamma_i^x \cap \mathcal{H}_{ij}$ such that

$$\mathbf{n}_{ij} \cdot f(z^*, u_i^x) \geq -\eta. \quad (3.35)$$

By (3.32) and the orientation of the unit vector \mathbf{n}_{ij} , we can write

$$\nabla W_j(z^*) = \nabla W_i(z^*) - \beta \mathbf{n}_{ij} \quad (3.36)$$

for some constant $\beta > 0$. Together, (3.29) and (3.35) now imply

$$\begin{aligned} \nabla W_j(z^*) \cdot f(z^*, u_i^x) &= \nabla W_i(z^*) \cdot f(z^*, u_i^x) - \beta \mathbf{n}_{ij} \cdot f(z^*, u_i^x) \\ &\leq -1 + 2\varepsilon + \beta \eta \leq -1 + 3\varepsilon, \end{aligned} \quad (3.37)$$

provided that we choose $\eta > 0$ sufficiently small. Since f and ∇W_j are uniformly Lipschitz continuous, from (3.37) it follows that (3.34) is valid for all z sufficiently close to z^* . By reducing the size of $\eta > 0$, we can make the diameter of the lens-shaped domain Γ_i^x as small as we like. Hence the inequality (3.34) will hold for all $z \in \Gamma_i^x$.

To prove our claim, it remains to observe that the functions f and ∇W_i are uniformly continuous, and that the constant β in (3.36) remains uniformly bounded. Hence the constant $\eta > 0$ can be chosen uniformly valid for all i, j, x .

We now define the smaller domain

$$\Omega_i^x \doteq \Gamma_i^x \setminus \bigcup_{j \in I_i} \{z \in \mathbf{R}^n; W_j(z) \leq W_i(z)\} \quad (3.38)$$

where $I_i \subset \{1, \dots, m\}$ is the set of indices $j \neq i$ for which CASE 1 applies.

By the previous analysis, for each j such that $W(z) = W_j(z)$ for some $z \in \Gamma_i^x$, two cases can occur. If CASE 1 applies, then the vector field $f(\cdot, u_i^x)$ is strictly

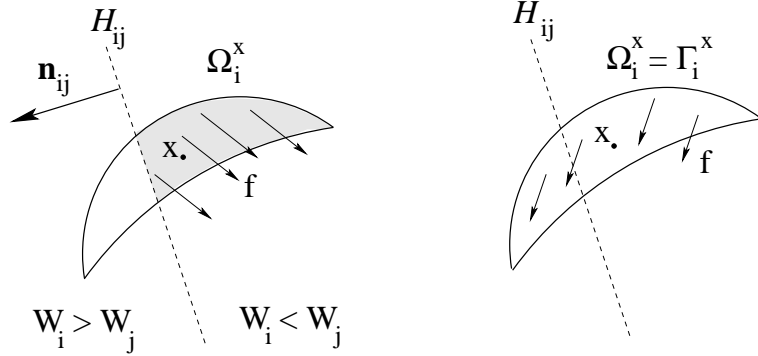


Figure 2. If the domain Γ_i^x intersects the half-space where $W_j < W_i$, two cases must be considered. Left: in Case 1, the vector field $f(\cdot, u_i^x)$ points toward the set where $W_i < W_j$. As a patch we then take the shaded region $\Omega_i^x \subset \Gamma_i^x$. Right: In Case 2, $f(\cdot, u_i^x)$ points toward the set where $W_j < W_i$. We can now take $\Omega_i^x = \Gamma_i^x$, because the control u_i^x is nearly optimal on this whole region.

inward-pointing along the portion of the boundary $\partial\Omega_i^x$ where $W_i = W_j$. On the other hand, if CASE 2 applies, then (3.34) holds on the entire domain Γ_i^x .

8. Consider the family of all domains Ω_i^x , as $i \in \{1, \dots, m\}$ and x ranges over the closure of the set $\Omega \doteq \{x \in B_{\bar{\rho}}; W(x) < \psi(x)\}$. It now remains to select finitely many domains Ω_i^x which cover the compact set $\bar{\Omega}$. This last step, however, must be done with some care because on the lower portion of the boundary

$$\partial^- \Omega_i^x \doteq \partial\Omega_i^x \cap \bar{B}(x_i, |x - x_i| - \eta) \quad (3.39)$$

the vector field $f(\cdot, u_i^x)$ may not be inward-pointing. To cope with this problem, we first observe that there exists a uniform constant $h > 0$ such that

$$W_i(z) \leq W_i(x) - h, \quad (3.40)$$

for every i, x and every $z \in \partial^- \Omega_i^x$.

We now set $M^* \doteq \max \{W(x); x \in B_{\bar{\rho}}\}$, and split the domain Ω in sub-domains of the form

$$\Omega_\ell \doteq \{x \in \Omega; M^* - (\ell + 1)h < W(x) < M^* - \ell h\}. \quad (3.41)$$

For each ℓ , we cover the compact set $\bar{\Omega}_\ell$ with finitely many domains Ω_i^x , constructed as in step **7**, choosing $x \in \bar{\Omega}_\ell$. After a relabelling of both the domains and the correspondent vector fields from (3.26), this yields the patches (see Figure 3)

$$(\Omega_{\ell, \alpha}, f(\cdot, u_{\ell, \alpha})), \quad \alpha = 1, \dots, N_\ell. \quad (3.42)$$

On the collection of all patches (3.42) we define the lexicographic order:

$$(\ell, \alpha) \prec (\ell', \alpha') \quad \text{iff} \quad \text{either } \ell < \ell' \text{ or } \ell = \ell' \text{ and } \alpha < \alpha'.$$

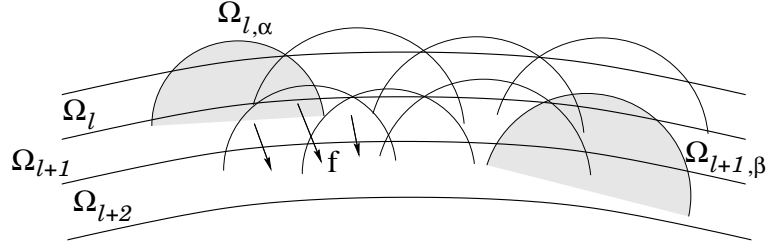


Figure 3. The domain $\Omega = \cup \Omega_\ell$ is covered by a family of patches $\Omega_{\ell, \alpha}$, ordered like tiles on a roof.

We claim that the above construction yields a patchy vector field:

$$g(x) \doteq f(x, u_{\ell, \alpha}) \quad \text{iff} \quad x \in \Omega_{\ell, \alpha} \setminus \bigcup_{(\ell, \alpha) \prec (\ell', \alpha')} \Omega_{\ell', \alpha'} \quad (3.43)$$

Indeed, according to Remark 1, it suffices to check that, for each patch $\Omega_{\ell, \alpha} = \Omega_i^x$, the vector field $f(\cdot, u_{\ell, \alpha}) = f(\cdot, u_i^x)$ is inward pointing at every point of the set

$$\Omega \cap \partial \Omega_{\ell, \alpha} \setminus \bigcup_{(\ell, \alpha) \prec (\ell', \alpha')} \Omega_{\ell', \alpha'}.$$

In the present case, this is clear, because the only boundary points where $f(\cdot, u_i^x)$ is not inward pointing are those on the lower boundary $\partial^- \Omega_i^x$. Since $x \in \Omega_\ell$ we have $W(x) \leq M^* - \ell h$, and hence by (3.40)

$$W(z) \leq M^* - (\ell + 1)h \quad \forall z \in \partial^- \Omega_i^x.$$

Therefore, given any point $z \in \partial^- \Omega_i^x$, either $W(z) = \psi(z)$ and $z \notin \Omega$, or else z is contained in a patch $\Omega_{\ell', \alpha'}$ with $\ell' > \ell$, as required in Remark 1.

9. To complete the proof, we now check that the patchy feedback that we have constructed is nearly optimal. We recall that, by the analysis in step **7**, for every i, x we have

$$\nabla W_i(z) \cdot f(z, u_i^x) \leq -1 + 4\varepsilon \quad \forall z \in \Omega_i^x. \quad (3.44)$$

Now take any initial point $y \in K$ and let $t \mapsto x(t)$ be any Carathéodory solution of the Cauchy problem

$$\dot{x} = g(x) \quad x(0) = y,$$

with g defined at (3.43). If $y \in K \setminus \Omega$, $\Omega = \{x \in B_{\bar{\rho}}; W(x) < \psi(x)\}$ as in (3.25), we are in the terminal set S and there will be no evolution, since it is more convenient to stay in y than to move along a trajectory. Otherwise, call $\tau \geq 0$ the first time at which $x(t)$ reaches the boundary of the set Ω . By (3.44) we have

$$W(x(\tau)) - W(y) = \int_0^\tau \left[\frac{d}{dt} W(x(t)) \right] dt \leq (-1 + 4\varepsilon)\tau,$$

hence

$$\tau \leq \frac{W(y) - W(x(\tau))}{1 - 4\varepsilon} \leq 2W(y) \leq 2\psi(y) \leq 2M.$$

By (3.4), it follows that $x(\tau)$ cannot be on the boundary of $B_{\bar{\rho}}$. We thus conclude that $W(x(\tau)) = \psi(x(\tau))$. Stopping at time τ , since $W(x) \geq 0$ and $V(y) \leq M$, the total cost can be estimated as

$$\tau + \psi(x(\tau)) \leq \frac{W(y) - W(x(\tau))}{1 - 4\varepsilon} + \psi(x(\tau)) \leq \frac{V(y) + \varepsilon}{1 - 4\varepsilon} \leq V(y) + \frac{\varepsilon(M + 1)}{1 - \varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof.

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