

## A SHARP DECAY ESTIMATE FOR POSITIVE NONLINEAR WAVES\*

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**Abstract.** We consider a strictly hyperbolic, genuinely nonlinear system of conservation laws in one space dimension. A sharp decay estimate is proved for the positive waves in an entropy weak solution. The result is stated in terms of a partial ordering among positive measures, using symmetric rearrangements and a comparison with a solution of Burgers’s equation with impulsive sources.

**Key words.** hyperbolic conservation laws, positive nonlinear waves, Burgers’s equation

**AMS subject classifications.** 35L65

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**1. Introduction.** Consider a strictly hyperbolic system of  $n$  conservation laws

$$(1.1) \quad u_t + f(u)_x = 0$$

and assume that all characteristic fields are genuinely nonlinear. Call  $\lambda_1(u) < \dots < \lambda_n(u)$  the eigenvalues of the Jacobian matrix  $A(u) \doteq Df(u)$ . We shall use bases of left and right eigenvectors  $l_i(u), r_i(u)$  normalized so that

$$(1.2) \quad \nabla \lambda_i(u) r_i(u) \equiv 1, \quad l_i(u) r_j(u) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Given a function  $u : \mathbb{R} \mapsto \mathbb{R}^n$  with small total variation following [BC], [B], one can define the measures  $\mu^i$  of  $i$ -waves in  $u$  as follows. Since  $u \in BV$ , its distributional derivative  $D_x u$  is a Radon measure. We define  $\mu^i$  as the measure such that

$$(1.3) \quad \mu^i \doteq l_i(u) \cdot D_x u$$

restricted to the set where  $u$  is continuous, while, at each point  $x$  where  $u$  has a jump, we define

$$(1.4) \quad \mu^i(\{x\}) \doteq \sigma_i,$$

where  $\sigma_i$  is the strength of the  $i$ -wave in the solution of the Riemann problem with data  $u^- = u(x-)$ ,  $u^+ = u(x+)$ . In accordance with (1.2), if the solution of the Riemann problem contains the intermediate states  $u^- = \omega_0, \omega_1, \dots, \omega_n = u^+$ , the strength of the  $i$ -wave is defined as

$$(1.5) \quad \sigma_i \doteq \lambda_i(\omega_i) - \lambda_i(\omega_{i-1}).$$

Observing that

$$\sigma_i = l_i(u^+) \cdot (u^+ - u^-) + O(1) \cdot |u^+ - u^-|^2,$$

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we can find a vector  $l_i(x)$  such that

$$(1.6) \quad |l_i(x) - l_i(u(x+))| = \mathcal{O}(1) \cdot |u(x+) - u(x-)|,$$

$$(1.7) \quad \sigma_i = l_i(x) \cdot (u(x+) - u(x-)).$$

We can thus define the measure  $\mu^i$  equivalently as

$$(1.8) \quad \mu^i \doteq l_i \cdot D_x u,$$

where  $l_i(x) = l_i(u(x))$  at points where  $u$  is continuous, while  $l_i(x)$  is some vector which satisfies (1.6)–(1.7) at points of jump. For all  $x \in \mathbb{R}$  there holds

$$(1.9) \quad |l_i(x) - l_i(u(x))| = \mathcal{O}(1) \cdot |u(x+) - u(x-)|.$$

We call  $\mu^{i+}$ ,  $\mu^{i-}$ , respectively, the positive and negative parts of  $\mu^i$ , so that

$$(1.10) \quad \mu^i = \mu^{i+} - \mu^{i-}, \quad |\mu^i| = \mu^{i+} + \mu^{i-}.$$

It is our purpose to prove a sharp estimate on the decay of the density of the measures  $\mu^{i+}$ . This will be achieved by introducing a partial ordering within the family of positive Radon measures. In the following,  $meas(A)$  denotes the Lebesgue measure of a set  $A$ .

DEFINITION 1. *Let  $\mu, \mu'$  be two positive Radon measures. We say that  $\mu \preceq \mu'$  if and only if*

$$(1.11) \quad \sup_{meas(A) \leq s} \mu(A) \leq \sup_{meas(B) \leq s} \mu'(B) \quad \text{for every } s > 0.$$

In some sense, the above relation means that  $\mu'$  is more singular than  $\mu$ . Namely, it has a greater total mass, concentrated on regions with higher density. Notice that the usual order relation

$$(1.12) \quad \mu \leq \mu' \quad \text{if and only if} \quad \mu(A) \leq \mu'(A) \quad \text{for every } A \subset \mathbb{R}$$

is much stronger. Of course  $\mu \leq \mu'$  implies  $\mu \preceq \mu'$ , but the converse does not hold.

Following [BC], [B], together with the measures  $\mu^i$ , we define the Glimm functionals

$$(1.13) \quad V(u) \doteq \sum_i |\mu^i|(\mathbb{R}),$$

$$(1.14) \quad Q(u) \doteq \sum_{i < j} (|\mu^j| \otimes |\mu^i|) \{(x, y); x < y\} + \sum_i (\mu^{i-} \otimes |\mu^i|) \{(x, y); x \neq y\}.$$

Now let  $u = u(t, x)$  be an entropy weak solution of (1.1). If the total variation of  $u$  is small and the constant  $C_0$  is large enough, it is well known that the quantities

$$(1.15) \quad Q(t) \doteq Q(u(t)), \quad \Upsilon(t) \doteq V(u(t)) + C_0 Q(u(t))$$

are nonincreasing in time. The decrease in  $Q$  controls the amount of interaction, while the decrease in  $\Upsilon$  controls both the interaction and the cancellation in the solution.

An accurate estimate on the measure  $\mu_t^{i+}$  of positive  $i$ -waves in  $u(t, \cdot)$  will be obtained by a comparison with a solution of Burgers's equation with source terms.

THEOREM 1. For some constant  $\kappa$  and for every small BV solution  $u = u(t, x)$  of the system (1.1) the following holds. Let  $w = w(t, x)$  be the solution of the scalar Cauchy problem with impulsive source term

$$(1.16) \quad w_t + (w^2/2)_x = -\kappa \operatorname{sgn}(x) \cdot \frac{d}{dt} Q(u(t)),$$

$$(1.17) \quad w(0, x) = \operatorname{sgn}(x) \cdot \sup_{\operatorname{meas}(A) < 2|x|} \frac{\mu_0^{i+}(A)}{2}.$$

Then, for every  $t \geq 0$ ,

$$(1.18) \quad \mu_t^{i+} \preceq D_x w(t).$$

As shown in the next section, the initial data in (1.17) represents the *odd rearrangement* of the function  $v_i(x) \doteq \mu_0^{i+}([\cdot - \infty, x])$ . The above theorem improves the earlier estimate derived in [BC]. For a scalar conservation law with strictly convex flux, a classical decay estimate was proved by Oleinik [O]. In the case of genuinely nonlinear systems, results related to the decay of nonlinear waves were also obtained in [GL], [L1], [L2], [L3], [BG]. An application of the present analysis can be found in [BY], where Theorem 1 plays a key role in the estimate of the rate of convergence of vanishing viscosity approximations.

**2. Lower semicontinuity.** Let  $\mu$  be a positive Radon measure on  $\mathbb{R}$ , so that  $\mu \doteq D_x v$  is the distributional derivative of some bounded, nondecreasing function  $v : \mathbb{R} \mapsto \mathbb{R}$ . We can decompose

$$\mu = \mu^{\operatorname{sing}} + \mu^{ac}$$

as the sum of a singular and an absolutely continuous part, w.r.t. Lebesgue measure. The absolutely continuous part corresponds to the usual derivative  $z \doteq v_x$ , which is a nonnegative  $\mathbf{L}^1$  function defined at a.e. point. We shall denote by  $\hat{z}$  the *symmetric rearrangement* of  $z$ , i.e., the unique even function such that

$$(2.1) \quad \hat{z}(x) = \hat{z}(-x), \quad \hat{z}(x) \geq \hat{z}(x') \quad \text{if } 0 < x < x',$$

$$(2.2) \quad \operatorname{meas}(\{x; \hat{z}(x) > c\}) = \operatorname{meas}(\{x; z(x) > c\}) \quad \text{for every } c > 0.$$

Moreover, we define the *odd rearrangement* of  $v$  as the unique function  $\hat{v}$  such that (Figure 1)

$$(2.3) \quad \hat{v}(-x) = -\hat{v}(x), \quad \hat{v}(0+) = \frac{1}{2} \mu^{\operatorname{sing}}(\mathbb{R}),$$

$$(2.4) \quad \hat{v}(x) = \hat{v}(0+) + \int_0^x z(y) dy \quad \text{for } x > 0.$$

By construction, the function  $\hat{v}$  is convex for  $x < 0$  and concave for  $x > 0$ .

The relation between the odd rearrangement  $\hat{v}$  and the partial ordering (1.10) is clarified by the following result, which is an easy consequence of the definitions.

PROPOSITION 1. Let  $\mu = D_x v$  and  $\mu' = D_x v'$  be positive Radon measures. Call  $\hat{v}, \hat{v}'$  the odd rearrangements of  $v, v'$ , respectively. Then  $\mu \preceq D_x \hat{v} \preceq \mu$  and moreover

$$(2.5) \quad \hat{v}(x) = \operatorname{sgn}(x) \cdot \sup_{\operatorname{meas}(A) \leq 2|x|} \frac{\mu(A)}{2},$$

$$(2.6) \quad \mu \preceq \mu' \quad \text{if and only if} \quad \hat{v}(x) \leq \hat{v}'(x) \quad \text{for all } x > 0.$$

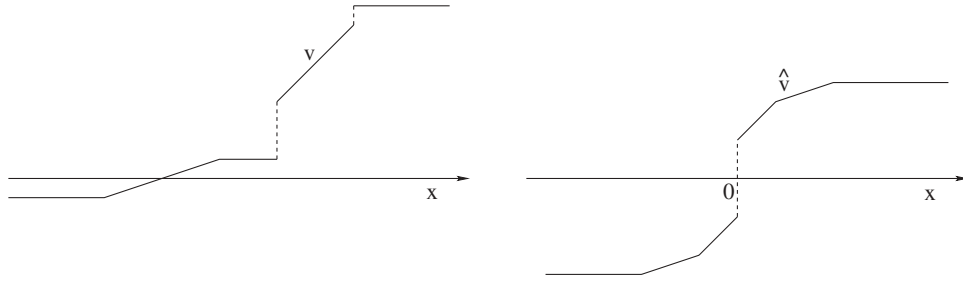


FIG. 1.

Two more results will be used in what follows. By the restriction of a measure  $\mu$  to a set  $J$ , we mean the measure

$$(\mu \lfloor J)(A) \doteq \mu(A \cap J).$$

**PROPOSITION 2.** *Let  $\mu, \mu'$  be positive measures. Consider any finite partition  $\mathbb{R} = J_1 \cup \dots \cup J_N$ . If the restrictions of  $\mu, \mu'$  to each set  $J_\ell$  satisfy  $\mu \lfloor J_\ell \preceq \mu' \lfloor J_\ell$ , then  $\mu \preceq \mu'$ .*

**PROPOSITION 3.** *Assume that  $\mu \preceq D_s w$  for some nondecreasing odd function  $w$ . If  $|\mu^\# - \mu|(\mathbb{R}) \leq \varepsilon$ , then*

$$\mu^\# \preceq D_s \left[ w + \operatorname{sgn}(s) \cdot \frac{\varepsilon}{2} \right].$$

The next result is concerned with the lower semicontinuity of the partial ordering  $\preceq$  w.r.t. weak convergence of measures.

**PROPOSITION 4.** *Consider a sequence of measures  $\mu_\nu$  converging weakly to a measure  $\mu$ . Assume that the positive parts satisfy  $\mu_\nu^+ \preceq D w_\nu$  for some odd, nondecreasing functions  $s \mapsto w_\nu(s)$ , concave for  $s > 0$ . Let  $w$  be the odd function such that*

$$w(s) \doteq \liminf_{\nu \rightarrow \infty} w_\nu(s) \quad \text{for } s > 0.$$

*Then the positive part of  $\mu$  satisfies*

$$(2.7) \quad \mu^+ \preceq D_s w.$$

*Proof.* By possibly taking a subsequence, we can assume that  $w_\nu(s) \rightarrow w(s)$  for all  $s \neq 0$ . Moreover, we can assume the weak convergence

$$\mu_\nu^+ \rightharpoonup \tilde{\mu}^+, \quad \mu_\nu^- \rightharpoonup \tilde{\mu}^-$$

for some positive measures  $\tilde{\mu}^+, \tilde{\mu}^-$ . We thus have

$$(2.8) \quad \mu = \tilde{\mu}^+ - \tilde{\mu}^-, \quad \mu^+ \leq \tilde{\mu}^+, \quad \mu^- \leq \tilde{\mu}^-.$$

By (2.8) it suffices to prove that  $\tilde{\mu}^+ \preceq D_s w$ , i.e.,

$$(2.9) \quad \operatorname{meas}(A) \leq 2s \quad \implies \quad \tilde{\mu}^+(A) \leq 2w(s)$$

for every  $s > 0$  and every Borel measurable set  $A \subset \mathbb{R}$ . If (2.9) fails, there exists  $s > 0$  and a set  $A$  such that

$$\operatorname{meas}(A) = 2s, \quad \tilde{\mu}^+(A) > 2w(s) = 2 \lim_{\nu \rightarrow \infty} w_\nu(s).$$

Since  $w$  is continuous for  $s > 0$ , we can choose an open set  $A' \supseteq A$  such that, setting  $s' \doteq \text{meas}(A')/2$ , one has  $2w(s') < \tilde{\mu}^+(A)$ . By the weak convergence  $\mu_\nu^+ \rightharpoonup \tilde{\mu}^+$  one obtains

$$\tilde{\mu}^+(A') \leq \liminf_{\nu \rightarrow \infty} \mu_\nu^+(A') \leq 2w(s') < \tilde{\mu}^+(A),$$

reaching a contradiction. Hence (2.9) must hold.  $\square$

Toward the proof of Theorem 1 we shall need a lower semicontinuity property for wave measures, similar to what was proved in [BaB]. In the following,  $C_0$  is the same constant as in (1.15).

LEMMA 1. *Consider a sequence of functions  $u_\nu$  with uniformly small total variation and call  $\mu_\nu^{i+}$  the corresponding measures of positive  $i$ -waves. Let  $s \mapsto w_\nu(s)$ ,  $\nu \geq 1$ , be a sequence of odd, nondecreasing functions, concave for  $s > 0$ , such that*

$$(2.10) \quad \mu_\nu^{i+} \preceq D_s \left[ w_\nu + C_0 \text{sgn}(s)(Q_0 - Q(u_\nu)) \right]$$

for some  $Q_0$ . Assume that  $u_\nu \rightarrow u$  and  $w_\nu \rightarrow w$  in  $\mathbf{L}_{\text{loc}}^1$ . Then the measure of positive  $i$ -waves in  $u$  satisfies

$$(2.11) \quad \mu^{i+} \preceq D_s \left[ w + C_0 \text{sgn}(s)(Q_0 - Q(u)) \right].$$

*Proof.* The main steps follow the proof of Theorem 10.1 in [B].

1. By possibly taking a subsequence we can assume that  $u_\nu(x) \rightarrow u(x)$  for every  $x$  and that the measures of total variation converge weakly, say,

$$(2.12) \quad |\mu_\nu| \doteq |D_x u_\nu| \rightharpoonup \mu^\sharp$$

for some positive Radon measure  $\mu^\sharp$ . In this case one has  $\mu^\sharp \geq |\mu|$ , in the sense of (1.12).

2. Let any  $\varepsilon > 0$  be given. Since the total mass of  $\mu^\sharp$  is finite, one can select finitely many points  $y_1, \dots, y_N$  such that

$$(2.13) \quad \mu^\sharp(\{x\}) < \varepsilon \quad \text{for all } x \notin \{y_1, \dots, y_N\}.$$

We now choose disjoint open intervals  $I_k \doteq ]y_k - \rho, y_k + \rho[$  such that

$$(2.14) \quad \mu^\sharp(I_k \setminus \{y_k\}) < \frac{\varepsilon}{N}, \quad k = 1, \dots, N.$$

Moreover, we choose  $R > 0$  such that

$$(2.15) \quad \bigcup_{k=1}^N I_k \subset [-R, R], \quad \mu^\sharp(]-\infty, -R] \cup [R, \infty[) < \varepsilon.$$

Because of (2.13), we can now choose points  $p_0 < -R < p_1 < \dots < R < p_r$  which are continuity points for  $u$  and for every  $u_\nu$ , such that

$$(2.16) \quad \mu^\sharp(\{p_h\}) = 0, \quad u_\nu(p_h) \rightarrow u(p_h) \quad \text{for all } h = 0, \dots, r$$

and such that either

$$(2.17) \quad p_h - p_{h-1} < \frac{\varepsilon}{N}, \quad p_{h-1} < y_k < p_h, \quad [p_{h-1}, p_h] \subset I_k$$

for some  $k \in \{1, \dots, N\}$ , or else

$$(2.18) \quad |\mu|([p_{h-1}, p_h]) \leq \mu^\sharp([p_{h-1}, p_h]) < \varepsilon.$$

Call  $J_h \doteq [p_{h-1}, p_h]$ . If (2.18) holds, by weak convergence for some  $\nu_0$  sufficiently large one has

$$(2.19) \quad |\mu_\nu|(J_h) < \varepsilon \quad \text{for all } \nu \geq \nu_0.$$

On the other hand, if (2.17) holds, from (2.14) it follows that

$$(2.20) \quad |\mu|(J_h \setminus \{y_k\}) \leq \mu^\sharp(J_h \setminus \{y_k\}) < \frac{\varepsilon}{N}.$$

In the remainder of the proof, the main strategy is as follows.

- On the intervals  $J_{h(k)}$  containing a point  $y_k$  of large oscillation, we first replace each  $u_\nu$  by a piecewise constant function  $\bar{u}_\nu$  having a single jump at  $y_k$ . The relations between the corresponding measures  $\mu_\nu^i$  and  $\bar{\mu}_\nu^i$  are given by Lemma 10.2 in [B]. Then we take the limit as  $\nu \rightarrow \infty$ .
- On the remaining intervals  $J_h$  with small oscillation, we replace the left eigenvectors  $l_i(u_\nu)$  by a constant vector  $l_i(u_h^*)$ . Then we use Proposition 4 to estimate the limit as  $\nu \rightarrow \infty$ .

3. We first take care of the intervals  $J_h$  containing a point  $y_k$  of large oscillation, so that (2.17) holds. For each  $k = 1, \dots, N$ , let  $h = h(k) \in \{1, \dots, r\}$  be the index such that  $y_k \in J_h \doteq [p_{h-1}, p_h]$ . For every  $\nu \geq 1$  consider the function

$$\bar{u}_\nu(x) \doteq \begin{cases} u_\nu(x) & \text{if } x \notin \cup_k J_{h(k)}, \\ u_\nu(p_{h(k)-1}) & \text{if } x \in ]p_{h(k)-1}, y_k[, \\ u_\nu(p_h) & \text{if } x \in [y_k, p_{h(k)}]. \end{cases}$$

Observe that all functions  $u, \bar{u}_\nu$  are continuous at every point  $p_0, \dots, p_r$  and have jumps at  $y_1, \dots, y_N$ . Call  $\bar{\mu}_\nu^i, i = 1, \dots, n$ , the corresponding measures, defined as in (1.8) with  $u$  replaced by  $\bar{u}_\nu$ . Clearly  $\bar{\mu}_\nu^i = \mu_\nu^i$  outside the intervals  $J_{h(k)}$  of large oscillation. By Lemma 10.2 at page 203 in [B], there holds

$$Q(\bar{u}_\nu) \leq Q(u_\nu), \quad V(\bar{u}_\nu) + C_0 Q(\bar{u}_\nu) \leq V(u_\nu) + C_0 \cdot Q(u_\nu),$$

$$\bar{\mu}_\nu^{i+}(\mathbb{R}) - \mu_\nu^{i+}(\mathbb{R}) \leq C_0 [Q(u_\nu) - Q(\bar{u}_\nu)].$$

As a consequence, from (2.10) we deduce

$$(2.21) \quad \bar{\mu}_\nu^{i+} \preceq D_s \left[ T^\varepsilon w_\nu + C_0 \operatorname{sgn}(s)(Q_0 - Q(\bar{u}_\nu)) \right],$$

where

$$T^\varepsilon w(s) \doteq \begin{cases} w(s + \varepsilon/2) & \text{if } s > 0, \\ w(s - \varepsilon/2) & \text{if } s < 0. \end{cases}$$

Indeed, all the mass which in  $\mu_\nu^{i+}$  lies on the set

$$\Omega \doteq \bigcup_{k=1}^N J_{h(k)}, \quad J_h \doteq [p_{h-1}, p_h]$$

is replaced in  $\bar{\mu}_\nu^{i+}$  by point masses at  $y_1, \dots, y_N$ . We obtain (2.21) by observing that, by (2.17),  $meas(\Omega) < \varepsilon$ . Moreover, the increase in the total mass is  $\leq C_0 [Q(u_\nu) - Q(\bar{u}_\nu)]$ .

Since  $u_\nu(p_h) \rightarrow u(p_h)$  for every  $h$ , there holds

$$\begin{aligned} \left| \mu^i(\{y_k\}) - \bar{\mu}_\nu^i(\{y_k\}) \right| &= \mathcal{O}(1) \cdot \left\{ |u(y_k-) - u(p_{h(k)-1})| + |u(y_k+) - u(p_{h(k)})| \right. \\ &\quad \left. + |u(p_{h(k)-1}) - u_\nu(p_{h(k)-1})| + |u(p_{h(k)}) - u_\nu(p_{h(k)})| \right\} \\ (2.22) \qquad \qquad \qquad &= \mathcal{O}(1) \cdot \frac{\varepsilon}{N} \end{aligned}$$

for each  $k = 1, \dots, N$  and all  $\nu$  sufficiently large. By construction we also have

$$(2.23) \qquad |\bar{\mu}_\nu^i(J_{h(k)} \setminus \{y_k\})| = 0, \qquad |\mu^i(J_{h(k)} \setminus \{y_k\})| = \mathcal{O}(1) \cdot \frac{\varepsilon}{N}.$$

4. Next, call  $\mathcal{S} \doteq \{h; \mu^\sharp(J_h) < \varepsilon\}$  the family of intervals where the oscillation of every  $u_\nu$  is small, so that (2.18) holds. If  $h \in \mathcal{S}$ , for every  $x, y \in J_h$  and  $\nu$  sufficiently large, one has

$$|u_\nu(x) - u_\nu(y)| \leq |\mu_\nu|(J_h) < \varepsilon,$$

$$|u(x) - u(y)| \leq |\mu|(J_h) \leq \mu^\sharp(J_h) < \varepsilon.$$

Set  $u_h^* \doteq u(p_h)$ . By the pointwise convergence  $u_\nu(p_h) \rightarrow u(p_h)$  and the two above estimates it follows that

$$(2.24) \qquad |u_\nu(x) - u_h^*| < \varepsilon, \qquad |u(x) - u_h^*| < \varepsilon \qquad \text{for all } x \in J_h.$$

5. We now introduce the measures  $\hat{\mu}_\nu^i$  such that

$$\hat{\mu}_\nu^i \doteq l_i(u_h^*) \cdot D_x u_\nu$$

restricted to each interval  $J_h$ ,  $h \in \mathcal{S}$ , where the oscillation is small, while

$$\hat{\mu}_\nu^i = \bar{\mu}_\nu^i$$

on each interval  $J_h = J_{h(k)}$  where the oscillation is large. Observe that the restriction of  $\hat{\mu}_\nu^i$  to  $J_{h(k)}$  consists of a single mass at the point  $y_k$ . Namely,  $\hat{\mu}_\nu^i(\{y_k\})$  is precisely the size of the  $i$ th wave in the solution of the Riemann problem with data  $u^- = u_\nu(p_{h(k)-1})$ ,  $u^+ = u_\nu(p_{h(k)})$ .

We define  $\hat{w}_\nu$  as the nondecreasing odd function such that

$$(2.25) \qquad \hat{w}_\nu(s) \doteq \sup_{meas(A) \leq 2s} \frac{\hat{\mu}_\nu^{i+}(A)}{2}, \qquad s > 0.$$

By possibly taking a further subsequence we can assume the convergence

$$Q(\bar{u}_\nu) \rightarrow \bar{Q}, \qquad \hat{\mu}_\nu^i \rightharpoonup \hat{\mu}^i, \qquad \hat{w}_\nu(s) \rightarrow \hat{w}(s).$$

Using (2.16), we can apply Proposition 4 on each interval  $J_h$  and obtain

$$(2.26) \qquad \hat{\mu}^{i+} \preceq D_s \hat{w}.$$

6. Observe that, by (2.24) and (2.19),

$$(2.27) \quad |\hat{\mu}_\nu^i - \mu_\nu^i|(J_h) = \mathcal{O}(1) \cdot \varepsilon \mu^\sharp(J_h), \quad h \in \mathcal{S}.$$

From (2.21) and the definition of  $\hat{w}_\nu$  at (2.25) it thus follows that

$$(2.28) \quad \hat{w}_\nu(s) \leq T^\varepsilon w_\nu(s) + C_0 [Q_0 - Q(\bar{u}_\nu)] + \mathcal{O}(1) \cdot \varepsilon, \quad s > 0.$$

Letting  $\nu \rightarrow \infty$  we obtain

$$(2.29) \quad \hat{w}(s) \leq T^\varepsilon w(s) + C_0 [Q_0 - \bar{Q}] + \mathcal{O}(1) \cdot \varepsilon, \quad s > 0,$$

$$(2.30) \quad \bar{Q} = \lim_{\nu \rightarrow \infty} Q(\bar{u}_\nu) \geq \lim_{\nu \rightarrow \infty} Q(u_\nu) - \mathcal{O}(1) \cdot \varepsilon \geq Q(u) - \mathcal{O}(1) \cdot \varepsilon,$$

because of the lower semicontinuity of the functional  $u \mapsto Q(u)$ . From (2.26), (2.29), and (2.30) we deduce

$$\hat{\mu}^{i+} \preceq D_s \left[ T^\varepsilon w + \operatorname{sgn}(s) (C_0 [Q_0 - Q(u)] + \mathcal{O}(1) \cdot \varepsilon) \right].$$

By (2.22)–(2.24), our construction of the measure  $\hat{\mu}^i$  achieves the property

$$|\mu^{i+} - \hat{\mu}^{i+}|(\mathbb{R}) = \mathcal{O}(1) \cdot \varepsilon.$$

Hence, by Proposition 3,

$$\mu^{i+} \preceq D_s \left[ T^\varepsilon w + \operatorname{sgn}(s) (C_0 [Q_0 - Q(u)] + \mathcal{O}(1) \cdot \varepsilon) \right].$$

Since  $\varepsilon > 0$  was arbitrary, this proves (2.11).  $\square$

**3. A decay estimate.** The second basic ingredient in the proof is the following lemma, which refines the estimate in [BC].

LEMMA 2. *For some constant  $\kappa > 0$  the following holds. Let  $u = u(t, x)$  be any entropy weak solution of (1.1), with initial data  $u(0, x) = \bar{u}(x)$  having small total variation. Then the measure  $\mu_t^{i+}$  of positive  $i$ -waves in  $u(t, \cdot)$  can be estimated as follows.*

Let  $w : [0, \tau[ \times \mathbb{R} \mapsto \mathbb{R}$  be the solution of Burgers’s equation

$$(3.1) \quad w_t + (w^2/2)_x = 0$$

with initial data

$$(3.2) \quad w(0, x) = \operatorname{sgn}(x) \cdot \sup_{\operatorname{meas}(A) \leq 2|x|} \frac{\mu_0^{i+}(A)}{2}.$$

Set

$$(3.3) \quad w(\tau, x) = w(\tau-, x) + \kappa \operatorname{sgn}(x) \cdot [Q(\bar{u}) - Q(u(\tau))].$$

Then

$$(3.4) \quad \mu_\tau^{i+} \preceq D_x w(\tau).$$

*Proof.* The main steps follow the proof of Theorem 10.3 in [B]. We first prove the estimate (3.3) under the following additional hypothesis.



(H) There exist points  $y_1 < \dots < y_m$  such that the initial data  $\bar{u}$  is smooth outside such points, constant for  $x < y_1$  and  $x > y_m$ , and the derivative component  $l_i(u)u_x$  is constant on each interval  $]y_\ell, y_{\ell+1}[$ . Moreover, the Glimm functional  $t \mapsto Q(u(t))$  is continuous at  $t = \tau$ .

1. The solution  $u = u(t, x)$  can be obtained as the limit of front tracking approximations. In particular, we can consider a particular converging sequence  $(u_\nu)_{\nu \geq 1}$  of  $\varepsilon_\nu$ -approximate solutions with the following additional properties:

(i) Each  $i$ -rarefaction front  $x_\alpha$  travels with the characteristic speed of the state on the right:

$$\dot{x}_\alpha = \lambda_i(u(x_\alpha+)).$$

(ii) Each  $i$ -shock front  $x_\alpha$  travels with a speed strictly contained between the right and the left characteristic speeds:

$$(3.5) \quad \lambda_i(u(x_\alpha+)) < \dot{x}_\alpha < \lambda_i(u(x_\alpha-)).$$

(iii) As  $\nu \rightarrow \infty$ , the interaction potentials satisfy

$$(3.6) \quad Q(u_\nu(0, \cdot)) \rightarrow Q(\bar{u}).$$

2. Let  $u_\nu$  be an approximate solution constructed by the front tracking algorithm. By a (*generalized*)  $i$ -characteristic we mean an absolutely continuous curve  $x = x(t)$  such that

$$\dot{x}(t) \in [\lambda_i(u_\nu(t, x-)), \lambda_i(u_\nu(t, x+))]$$

for a.e.  $t$ . If  $u_\nu$  satisfies the above properties (i)–(ii), then the  $i$ -characteristics are precisely the polygonal lines  $x : [0, \tau] \mapsto \mathbb{R}$  for which the following holds. For a suitable partition  $0 = t_0 < t_1 < \dots < t_m = \tau$ , on each subinterval  $[t_{j-1}, t_j]$  either  $\dot{x}(t) = \lambda_i(u_\nu(t, x))$ , or else  $x$  coincides with a wave front of the  $i$ th family. For a given terminal point  $\bar{x}$  we shall consider the *minimal backward  $i$ -characteristic* through  $\bar{x}$ , defined as

$$y(t) = \min \{x(t); \ x \text{ is an } i\text{-characteristic, } x(\tau) = \bar{x}\}.$$

Observe that  $y(\cdot)$  is itself an  $i$ -characteristic. By (3.5), it cannot coincide with an  $i$ -shock front of  $u$  on any nontrivial time interval.

In connection with the exact solution  $u$ , we define an  $i$ -characteristic as a curve

$$t \mapsto x(t) = \lim_{\nu \rightarrow \infty} x_\nu(t)$$

which is the limit of  $i$ -characteristics in a sequence of front tracking solutions  $u_\nu \rightarrow u$ .

3. Let  $\varepsilon > 0$  be given. If the assumption (H) holds, the measure  $\mu_\tau^{i+}$  of  $i$ -waves in  $u(\tau)$  is supported on a bounded interval and is absolutely continuous w.r.t. Lebesgue measure. We can thus find a piecewise constant function  $\psi^\tau$  with jumps at points  $x_1(\tau) < \bar{x}_2(\tau) < \dots < \bar{x}_N(\tau)$  such that

$$(3.7) \quad \int \left| \frac{d\mu_\tau^{i+}}{dx} - \psi^\tau \right| dx < \varepsilon, \quad \int_{x_j(\tau)}^{x_{j+1}(\tau)} \left( \frac{d\mu_\tau^{i+}}{dx} - \psi^\tau \right) dx = 0, \quad j = 1, \dots, N-1.$$

To prove the lemma in this special case, relying on Proposition 2, it thus suffices to find  $i$ -characteristics  $t \mapsto x_j(t)$  such that the following hold (Figure 2):

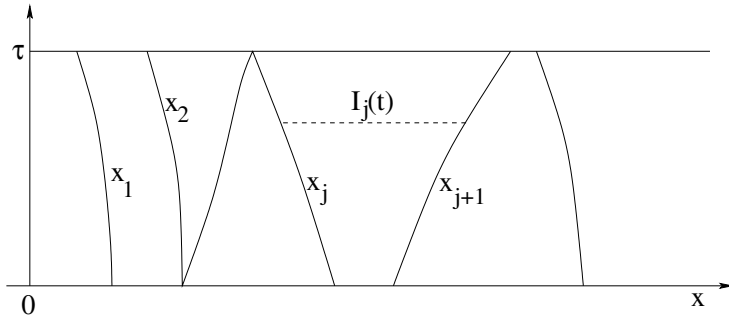


FIG. 2.

- (i) For each  $j = 1, \dots, N$ , the function  $\psi^\tau$  is constant on the interval  $]x_j(\tau), x_{j+1}(\tau)[$  and (3.7) holds. Moreover, either  $x_j(0) = x_{j+1}(0)$ , or else the derivative component  $\psi^0 \doteq l^i(u)u_x(0, \cdot)$  is constant on the interval  $]x_j(0), x_{j+1}(0)[$ .
- (ii) An estimate corresponding to (3.3)–(3.4) holds restricted to each subinterval  $]x_j(\tau), x_{j+1}(\tau)[$ .

We need to explain in more detail this last statement. Define

$$I_j(t) \doteq [x_j(t), x_{j+1}(t)[, \quad \Delta_j \doteq \{(t, x) ; t \in [0, \tau], x \in I_j(t)\}.$$

For each  $j$ , we denote by  $\Gamma_j$  the total amount of wave interaction within the domain  $\Delta_j$ . This is defined as in [B], first for a sequence of front tracking approximations  $u_\nu$ , then taking a limit as  $\nu \rightarrow \infty$ . Furthermore, we define the constant values

$$\begin{aligned} \psi_j^\tau &\doteq \psi^\tau(x), & x \in I_j(\tau), \\ \psi_j^0 &\doteq \psi^0(x), & x \in I_j(0). \end{aligned}$$

Call

$$\sigma_j^0 \doteq \lim_{t \rightarrow 0^+} \mu^{i+}(I_j(t))$$

the initial amount of positive  $i$ -waves inside the interval  $I_j$ .

For each interval  $I_j$ , we consider on one hand the function  $w_j^\tau$  corresponding to (3.2)–(3.3), namely,

$$w_j^\tau(s) \doteq \min \left\{ \sigma_j^0, \frac{s}{\tau + (\psi_j^0)^{-1}} \right\} + \kappa \Gamma_j \cdot \text{sgn}(s).$$

Here  $(\psi_j^0)^{-1} \doteq 0$  in the case where  $x_j(0) = x_{j+1}(0)$ . This may happen when the initial data has a jump at  $x_j(0)$ , and the corresponding measure  $\mu^{i+}$  has a Dirac mass (with infinite density) at that point.

On the other hand, we look at the nondecreasing, odd function  $\eta_j$  such that

$$\eta_j(s) \doteq \min \left\{ \psi_j^\tau s, \psi_j^\tau [x_{j+1}(\tau) - x_j(\tau)] \right\}, \quad s > 0.$$

Our basic goal is to prove that (Figure 3)

$$(3.8) \quad \eta_j(s) \leq w_j^\tau(s) \quad \text{for all } s > 0.$$

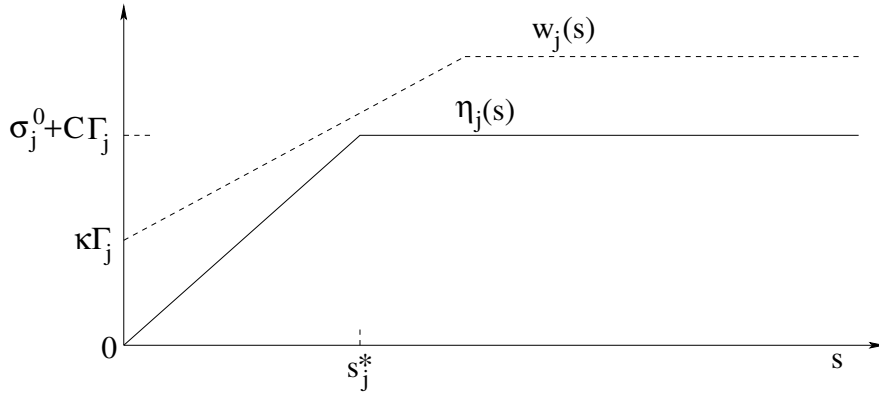


FIG. 3.

Indeed, by (3.7), for  $s > 0$  one has

$$\sup_{\text{meas}(A) \leq 2s} \frac{\mu_\tau^{i+}(A \cap I_j(\tau))}{2} \leq \eta_j(s) + \varepsilon_j$$

with

$$\sum_j \varepsilon_j < \varepsilon.$$

Proving (3.8) for each  $j$  will thus imply

$$\mu_\tau^{i+} \leq w(\tau, x) = w(\tau-, x) + \kappa \operatorname{sgn}(x) \cdot [Q(\bar{u}) - Q(u(\tau)) + \mathcal{O}(1) \cdot \varepsilon].$$

Since  $\varepsilon > 0$  was arbitrary, this establishes the lemma under the additional assumptions (H).

4. We now work toward a proof of (3.8), in three cases.

Case 1:  $\sigma_j^0 = 0$ .

Case 2:  $x_j(0) = x_{j+1}(0)$  and  $\sigma_j^0 > 0$ .

Case 3:  $x_j(0) < x_{j+1}(0)$  and  $\sigma_j^0 = (x_{j+1}(0) - x_j(0)) \psi_j^0 > 0$ .

In Case 1 the proof is easy. Indeed, the total amount of positive  $i$ -waves in  $I_j(\tau)$  is here bounded by a constant times the total amount of interaction taking place inside the domain  $\Delta_j$ , i.e.,

$$\mu_\tau^{i+}(I_j(\tau)) \leq C_0 \cdot \Gamma_j$$

for some constant  $C_0$ . On the other hand

$$w_j^\tau(s) = \kappa \Gamma_j \cdot \operatorname{sgn}(s).$$

Choosing  $\kappa > C_0$  we achieve (3.8).

5. Since Case 2 can be obtained from Case 3 in the limit as  $x_{j+1} - x_j \rightarrow 0$ , we shall only give a proof for Case 3.

We can again distinguish two cases. If the amount of interaction  $\Gamma_j$  is large compared with the initial amount of  $i$ -waves, say

$$\Gamma_j \geq \frac{1}{6C_0} \sigma_j^0,$$

then the bound (3.8) is readily achieved choosing  $\kappa > 8C_0$ . Indeed, for  $s > 0$  we have

$$\eta_j(s) \leq \frac{1}{2} \mu_\tau^{i+}(I_j(\tau)) \leq C_0 \Gamma_j + \sigma_j^0 \leq 7C_0 \Gamma_j.$$

The more difficult case to analyze is when  $\Gamma_j$  is small, say

$$(3.9) \quad \Gamma_j < \sigma_j^0 / 6C_0.$$

Looking at Figure 3, it clearly suffices to prove (3.8) for the single value

$$s = s_j^* \doteq \frac{x_{j+1}(\tau) - x_j(\tau)}{2}.$$

Equivalently, calling

$$z_j(t) \doteq x_{j+1}(t) - x_j(t)$$

the length of the interval  $I_j(t)$  and

$$\sigma_j^\tau \doteq \mu_\tau^{i+}(I_j(\tau)) = z_j(\tau) \psi_j^\tau$$

the total amount of positive  $i$ -waves inside  $I_j(\tau)$ , we need to show that

$$(3.10) \quad \sigma_j^\tau \leq 2\kappa \Gamma_j + \min \left\{ \sigma_j^0, \frac{2s_j^*}{\tau + (\psi_j^0)^{-1}} \right\}.$$

By the approximate conservation of  $i$ -waves over the region  $\Delta_j$ , we can write

$$(3.11) \quad \sigma_j^\tau \leq \sigma_j^0 + C_0 \Gamma_j.$$

Using (3.11) in (3.10), our task is reduced to showing that

$$(3.12) \quad \sigma_j^\tau \leq 2\kappa \Gamma_j + \frac{2s_j^*}{\tau + (\psi_j^0)^{-1}}$$

for a suitably large constant  $\kappa$ . Because of (3.11), it suffices to show that

$$(3.13) \quad \begin{aligned} z_j(\tau) &\geq (\sigma_j^0 - C' \Gamma_j)(\tau + (\psi_j^0)^{-1}) \\ &= [z_j(0) + \tau \sigma_j^0] - C'(\tau + (\psi_j^0)^{-1}) \Gamma_j \end{aligned}$$

for a suitable constant  $C'$ .

6. We now prove (3.13). Notice that, by genuine nonlinearity and the normalization (1.2), if no other waves were present in the region  $\Delta_j$  we would have  $\Gamma_j = 0$  and

$$\frac{d}{dt} z_j(t) \equiv \sigma_j^0.$$

In this case, the equality would hold in (3.13).

To handle the general case, we represent the solution  $u$  as a limit of front tracking approximations  $u_\nu$ , where for each  $\nu \geq 1$  the function  $u_\nu(0, \cdot)$  contains exactly  $\nu$  rarefaction fronts equally spaced along the interval  $I_j(0)$  (Figure 4). Each of these fronts has initial strength  $\sigma_\alpha(0) = \sigma_j^0 / \nu$ . For  $\alpha = 1, \dots, \nu$ , let  $y_\alpha(t) \in I_j(t)$  be the

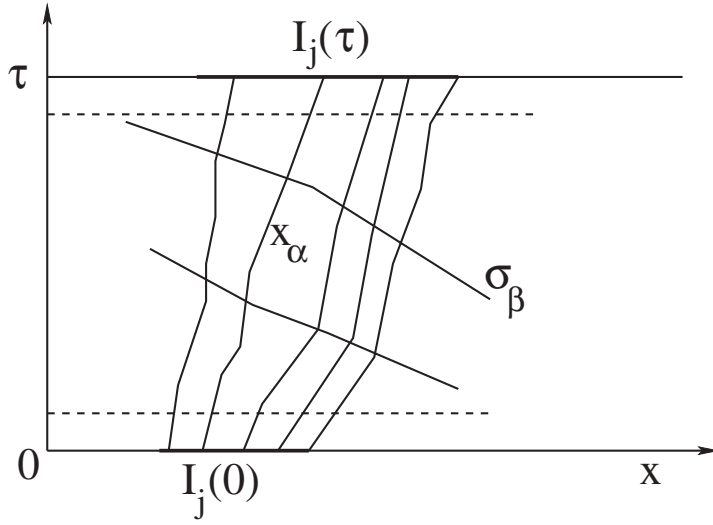


FIG. 4.

location of one of these fronts at time  $t \in [0, \tau]$ , and let  $\sigma_\alpha(t) > 0$  be its strength. Moreover, call

$$J_\alpha(t) \doteq [y_\alpha(t), y_{\alpha+1}(t)], \quad \Delta_\alpha \doteq \{(t, x); t \in [0, \tau], x \in J_\alpha(t)\},$$

and let  $\Gamma_\alpha$  be the total amount of interaction in  $u_\nu$  taking place inside the domain  $\Delta_\alpha$ .

We define a subset of indices  $\mathcal{I} \subseteq \{1, \dots, \nu\}$  by setting  $\alpha \in \mathcal{I}$  if

$$(3.14) \quad 5C_0\Gamma_\alpha > \sigma_\alpha(0) = \sigma_j^0/\nu.$$

Observe that, if  $\alpha \notin \mathcal{I}$ , then

$$\left| \frac{\sigma_\alpha(t)}{\sigma_\alpha(0)} - 1 \right| < \frac{1}{2} \quad \text{for all } t \in [0, \tau].$$

In particular, if  $\alpha, \alpha + 1 \notin \mathcal{I}$ , then the interval  $J_\alpha(t)$  is well defined for all  $t \in [0, \tau]$ . Its length

$$z_\alpha(t) \doteq y_{\alpha+1}(t) - y_\alpha(t)$$

satisfies the differential inequality

$$(3.15) \quad \frac{d}{dt} z_\alpha(t) \geq W_\alpha(t) - C_1 \cdot \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta|$$

for some constant  $C_1$ . Here

$$(3.16) \quad \begin{aligned} W_\alpha(t) &\doteq [\text{amount of } i\text{-waves inside the interval } J_\alpha(t)] \\ &\geq \sigma_\alpha(0) - C_0\Gamma_\alpha, \end{aligned}$$

while  $\mathcal{C}_\alpha(t)$  refers to the set of all wave fronts of different families which are crossing the interval  $J_\alpha$  at time  $t$ . Calling  $W'_\alpha$  the total amount of waves of families  $\neq i$  which lie inside  $J_\alpha(0)$ , we can now write

$$(3.17) \quad \int_0^\tau \left( \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| \right) dt \leq \left( \max_{t \in [0, \tau]} z_\alpha(t) \right) \cdot \frac{2\nu}{\sigma_j^0} \cdot \Gamma_\alpha + \mathcal{O}(1) \cdot \tau \Gamma_\alpha + \mathcal{O}(1) \cdot \left( \frac{z_j(0) + 1}{\nu} \right) W'_\alpha.$$

Indeed, by strict hyperbolicity, every front  $\sigma_\beta$  of a different family can spend at most a time  $\mathcal{O}(1) \cdot z_\alpha$  inside  $J_\alpha$ . Either it is located inside  $J_\alpha$  already at time  $t = 0$ , or else, when it enters, it crosses  $y_\alpha$  or  $y_{\alpha+1}$ . In this case, since  $\alpha, \alpha + 1 \notin \mathcal{I}$ , by (3.14) it will produce an interaction of magnitude  $|\sigma_\beta \sigma_\alpha| \geq |\sigma_\beta \cdot \sigma_j^0|/2\nu$ . The second term on the right-hand side of (3.17) takes care of the new wave fronts which are generated through interactions inside  $J_\alpha$ . The last term takes into account wave fronts of different families that initially lie already inside  $J_\alpha$  at time  $t = 0$ . Integrating (3.15) over the time interval  $[0, \tau]$  and using (3.16)–(3.17) one obtains

$$(3.18) \quad z_\alpha(\tau) \geq z_\alpha(0) + \tau \frac{\sigma_j^0}{\nu} - \mathcal{O}(1) \cdot \tau \Gamma_\alpha - \mathcal{O}(1) \cdot \left( \max_{t \in [0, \tau]} z_\alpha(t) \right) \cdot \frac{2\nu}{\sigma_j^0} \cdot \Gamma_\alpha - \mathcal{O}(1) \cdot \left( \frac{z_j(0) + 1}{\nu} \right) W'_\alpha.$$

7. To proceed in our analysis, we now show that

$$(3.19) \quad \max_{t \in [0, \tau]} z_\alpha(t) \leq 2 z_\alpha(\tau).$$

Indeed, let  $\tau' \in [0, \tau]$  be the time where the maximum is attained. If our claim (3.19) does not hold, there would exist a first time  $\tau'' \in [\tau', \tau]$  such that  $z_\alpha(\tau'') = z_\alpha(\tau')/2$ . From (3.15) and the assumption  $W_\alpha(t) \geq 0$  it follows that

$$(3.20) \quad \int_{\tau'}^{\tau''} C_1 \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| dt \geq \frac{z_\alpha(\tau')}{2}.$$

Using the smallness of the total variation, a contradiction is now obtained as follows. Call

$$\Phi(t) \doteq C_0 Q(t) + \sum_{k_\beta \neq i} \phi_{k_\beta}(t, x_\beta(t)) |\sigma_\beta|,$$

where the sum ranges over all fronts of strength  $\sigma_\beta$  located at  $x_\beta$ , of a family  $k_\beta \neq i$ . The weight functions  $\phi_j$  are defined as

$$\phi_j(t, x) \doteq \begin{cases} 0 & \text{if } x > y_{\alpha+1}(t), \\ \frac{y_{\alpha+1}(t) - x}{y_{\alpha+1}(t) - y_\alpha(t)} & \text{if } x \in [y_\alpha(t), y_{\alpha+1}(t)], \\ 1 & \text{if } x < y_\alpha(t) \end{cases}$$

in the case  $j > i$ , while

$$\phi_j(t, x) \doteq \begin{cases} 1 & \text{if } x > y_{\alpha+1}(t), \\ \frac{x - y_\alpha(t)}{y_{\alpha+1}(t) - y_\alpha(t)} & \text{if } x \in [y_\alpha(t), y_{\alpha+1}(t)], \\ 0 & \text{if } x < y_\alpha(t) \end{cases}$$

in the case  $j < i$ . Because of the term  $C_0Q(t)$ , the functional  $\Phi$  is nonincreasing at times of interactions. Moreover, outside interaction times a computation entirely similar to the one on page 213 of [B] now yields

$$(3.21) \quad -\frac{d}{dt}\Phi(t) \geq \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| \cdot \frac{c_0}{z(t)}$$

for some small constant  $c_0 > 0$  related to the gap between different characteristic speeds. From (3.20) and (3.21), respectively, we now deduce

$$\int_{\tau'}^{\tau''} \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| dt \geq \frac{z_\alpha(\tau')}{2C_1},$$

$$\int_{\tau'}^{\tau''} \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| dt \leq \int_{\tau'}^{\tau''} \left| \frac{d\Phi(t)}{dt} \right| \cdot \frac{z_\alpha(\tau')}{c_0} dt \leq \frac{\Phi(\tau')}{c_0} z_\alpha(\tau').$$

Since  $\Phi(t) = \mathcal{O}(1) \cdot \text{Tot.Var.}\{u(t)\}$ , by the smallness of the total variation we can assume  $\Phi(\tau') < 2C_1/c_0$ . In this case, the two above inequalities yield a contradiction.

8. Using (3.19), from (3.18) we obtain

$$(3.22) \quad \begin{aligned} z_j(\tau) &= \sum_{1 \leq \alpha \leq \nu} z_\alpha(\tau) \geq \sum_{\alpha \notin \mathcal{I}} z_\alpha(\tau) \\ &\geq \sum_{\alpha \notin \mathcal{I}} \left\{ \frac{z_\alpha(0) + \tau \sigma_j^0 / \nu}{1 + C_2(\nu / \sigma_j^0) \Gamma_\alpha} - \mathcal{O}(1) \cdot \tau \Gamma_j - \mathcal{O}(1) \cdot \left( \frac{z_j(0) + 1}{\nu} \right) W'_\alpha \right\} \\ &\geq \sum_{\alpha \notin \mathcal{I}} \left( z_\alpha(0) + \tau \frac{\sigma_j^0}{\nu} \right) \left( 1 - C_2 \frac{\nu}{\sigma_j^0} \Gamma_\alpha \right) - \mathcal{O}(1) \cdot \tau \Gamma_j - \mathcal{O}(1) \cdot \frac{z_j(0) + 1}{\nu} \\ &\geq \sum_{\alpha \notin \mathcal{I}} \left( z_\alpha(0) + \tau \frac{\sigma_j^0}{\nu} \right) - C_2 \frac{z_j(0)}{\sigma_j^0} \Gamma_j - \mathcal{O}(1) \cdot \tau \Gamma_j - \mathcal{O}(1) \cdot \frac{z_j(0) + 1}{\nu}. \end{aligned}$$

By (3.14) the cardinality of the set  $\mathcal{I}$  satisfies

$$\#\mathcal{I} \cdot \frac{\sigma_j^0}{5C_0\nu} \leq \sum_{\alpha \in \mathcal{I}} \Gamma_\alpha \leq \Gamma_j;$$

hence

$$\frac{\#\mathcal{I}}{\nu} \leq \frac{5C_0}{\sigma_j^0} \Gamma_j.$$

In turn, this implies

$$(3.23) \quad \begin{aligned} \sum_{\alpha \notin \mathcal{I}} \left( z_\alpha(0) + \tau \frac{\sigma_j^0}{\nu} \right) &\geq (z_j(0) + \tau \sigma_j^0) \left( 1 - \frac{\#\mathcal{I}}{\nu} \right) \\ &\geq (z_j(0) + \tau \sigma_j^0) - 5C_0 \Gamma_j \frac{z_j(0)}{\sigma_j^0} \Gamma_j - 5C_0 \tau \Gamma_j. \end{aligned}$$

Using (3.23) in (3.22), observing that

$$\frac{z_j(0)}{\sigma_j^0} = \frac{x_{j+1}(0) - x_j(0)}{\sigma_j^0} = (\psi_j^0)^{-1},$$

and letting  $\nu \rightarrow \infty$ , we conclude

$$z_j(\tau) \geq (z_j(0) + \tau\sigma_j^0) - \mathcal{O}(1) \cdot (\psi_j^0)^{-1}\Gamma_j - \mathcal{O}(1) \cdot \tau\Gamma_j.$$

This establishes (3.13) for a suitable constant  $C'$ .

9. In the general case, without the assumptions (H), the lemma is proved by an approximation argument. We construct a convergent sequence of initial data  $\bar{u}_\nu \rightarrow \bar{u}$  which satisfy (H) and such that

$$\bar{u}_\nu \rightarrow \bar{u}, \quad Q(\bar{u}_\nu) \rightarrow Q(\bar{u}), \quad |\mu_{\nu,0}^{i+} - \mu_0^{i+}| \rightarrow 0.$$

Calling  $w_\nu$  the solution of (3.1) with initial data

$$w_\nu(0, x) = \operatorname{sgn}(x) \cdot \sup_{\operatorname{meas}(A) \leq 2|x|} \frac{\mu_{\nu,0}^{i+}(A)}{2},$$

by the previous analysis we have

$$\mu_{\nu,\tau_\nu}^{i+} \leq D_x \left[ w_\nu(\tau_\nu-) + \operatorname{sgn}(x) \cdot [Q(\bar{u}_\nu) - Q(u_\nu(\tau_\nu))] \right].$$

Observe that  $w_\nu(\tau_\nu-) \rightarrow w(\tau-)$  in  $L^1_{\text{loc}}$ . Choosing  $\kappa \geq C_0$ , by the lower semicontinuity result stated in Lemma 1 we now conclude

$$\mu_\tau^{i+} \leq D_x \left[ w(\tau-) + \kappa \operatorname{sgn}(x) \cdot [Q(\bar{u}) - Q(u(\tau))] \right]. \quad \square$$

**4. Proof of the main theorem.** Using the previous lemmas, we now give a proof of Theorem 1. For a given interval  $[0, \tau]$ , the solution of the impulsive Cauchy problem (1.17)–(1.18) can be obtained as follows. Consider a partition  $0 = t_0 < t_1 < \dots < t_N = \tau$ . Construct an approximate solution by requiring that  $w(0, x) = \hat{v}_i(x)$ ,

$$(4.1) \quad w_t + (w^2/2)_x = 0$$

on each subinterval  $[t_{k-1}, t_k[$ , while

$$(4.2) \quad w(t_k, x) = w(t_{k-1}, x) + \kappa \operatorname{sgn}(x) \cdot [Q(t_{k-1}) - Q(t_k)].$$

We then consider a sequence of partitions  $0 = t'_0 < t'_1 < \dots < t'_{N_\nu} = \tau$ , and the corresponding solutions  $w_\nu$ . If the mesh of the partitions approaches zero, i.e.,

$$\lim_{\nu \rightarrow \infty} \sup_k |t'_k - t'_{k-1}| = 0,$$

then the approximate solutions  $w_\nu$  converge to a unique limit, which yields the solution of (1.17)–(1.18).

Call  $\mathcal{F}$  the set of nondecreasing odd functions, concave for  $x > 0$ . This set is positively invariant for the flow of Burgers’s equation (4.1). Moreover, this flow is order preserving. Namely, if  $w, w' \in \mathcal{F}$  are solutions of (4.1) with initial data such that  $w(0, x) \leq w'(0, x)$  for all  $x > 0$ , then also

$$w(t, x) \leq w'(t, x) \quad \text{for all } t, x > 0.$$



Equivalently,

$$D_x w(0) \leq D_x w'(0) \implies D_x w(t) \leq D_x w'(t)$$

for every  $t > 0$ . For each fixed  $\nu$ , we can apply Lemma 2 on each subinterval  $[t_{k-1}^\nu, t_k^\nu]$  and obtain

$$\mu_{t_k^\nu}^{i+} \leq D_x w_\nu(t_k^\nu) \implies \mu_{t_{k+1}^\nu}^{i+} \leq D_x w_\nu(t_{k+1}^\nu).$$

By induction on  $k$ , this yields

$$(4.3) \quad \mu_\tau^{i+} \leq D_x w_\nu(\tau),$$

where  $w_\nu$  is the approximate solution constructed according to (4.1)–(4.2). Letting  $\nu \rightarrow \infty$  and using Lemma 1, we achieve a proof of Theorem 1.  $\square$

**5. Examples.**

*Example 1.* Consider first the scalar case. Let  $u = u(t, x)$  be a solution of Burgers’s equation with smooth initial data

$$u_t + (u^2/2)_x = 0, \quad u(0, x) = \bar{u}(x).$$

Define the function  $w$  as the solution to

$$w_t + (w^2/2)_x = 0, \quad w(0, x) = \operatorname{sgn}(x) \cdot \sup_{\operatorname{meas}(A) < 2|x|} \int_A \frac{\bar{u}_x(y)}{2} dy.$$

This corresponds to (1.16)–(1.17) with  $Q \equiv 0$ . According to Oleinik’s estimate we now have  $u_x(t, x) \leq 1/t$  for all  $t > 0$ , and a.e.  $x \in \mathbb{R}$ . Of course, this reflects the fact that, along each characteristic with  $\dot{x} = u(t, x(t))$ , the gradient satisfies

$$(5.1) \quad \frac{d}{dt} u_x(t, x(t)) = -\frac{1}{u_x^2(t, x(t))}.$$

A better estimate on  $u_x(t, x(t))$  when it is positive, based on (5.1), is

$$(5.2) \quad u_x(t, x(t)) \leq \frac{1}{t + [\bar{u}_x(x(0))]^{-1}}.$$

According to (1.18), for every  $t > 0$  we have the relation

$$(5.3) \quad \mu_t^+ \leq D_x w(t),$$

which includes the additional information (5.2). This relation is sharp in the sense that the converse inequality

$$D_x w(t) \leq \mu_t^+$$

also holds, as long as no positive waves are cancelled by interacting with shocks.

Analogous results are valid for scalar equations with more general flux:

$$u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x),$$

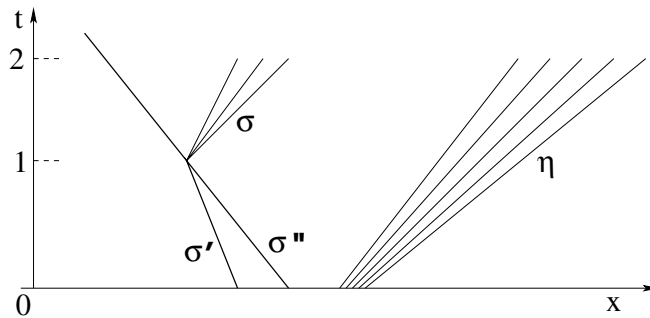


FIG. 5.

under the assumption of genuine nonlinearity  $f'' > 0$ . Notice that in this case the normalization (1.2) yields  $r(u) = 1/f''(u)$ ,  $l(u) = f''(u)$ . As long as the solution  $u(t, \cdot)$  remains smooth, the corresponding wave measure is thus defined as

$$\mu_t(A) = \int_A f''(u(t, x)) \cdot u_x(t, x) dx.$$

*Example 2.* Consider the p-system in Lagrangean coordinates (see [Sm])

$$v_t - u_x = 0, \quad u_t + (K/v^\gamma)_x = 0.$$

Here  $K > 0$  and  $\gamma \geq 1$  are constants. Consider a solution which initially contains two approaching 1-shocks and a 2-rarefaction (Figure 5). Assume that at time  $t = 1$  the two shocks interact. As shown in [Sm], the Riemann problem is then solved in terms of a 1-shock and a 2-rarefaction. Let us look at the measure of positive 2-waves in the solution. Let  $\eta$  be the strength of the centered rarefaction. During the interval  $t \in [0, 1[$  the density of rarefaction waves decays, as in the scalar case. At time  $t = 1$  a new centered rarefaction is created by the interaction. Calling  $\sigma', \sigma''$  the strengths of the incoming shocks, the strength  $\sigma$  of this new rarefaction will satisfy

$$(5.4) \quad \sigma \leq \kappa \cdot |\sigma' \sigma''| \doteq \tilde{\sigma}$$

for a suitable constant  $\kappa > 0$ . Notice that the decrease in the interaction potential at time  $t = 1$  is  $\Delta Q = -|\sigma' \sigma''|$ . The values of the corresponding function  $w = w(t, x)$  in (1.16)–(1.18) at various times are illustrated in Figure 6. For  $t \in [0, 1[$  the estimate (1.18) is sharp, in the sense that

$$(5.5) \quad D_x w(t) \leq \mu_t^{2+} \leq D_x w(t).$$

The first relation in (5.5) will fail for  $t \geq 1$  if  $\tilde{\sigma} < \sigma$ . The accuracy of our estimate in this case depends essentially on the careful choice of the constant  $\kappa$  in (5.4). In particular, if we could choose  $\kappa$  so that  $\sigma = \kappa |\sigma' \sigma''|$ , then both relations in (5.5) would remain valid for all times  $t \geq 0$ .

*Remark.* Concerning compression waves, an estimate of the form (1.18) could be derived also for the negative part of the measures  $\mu_t^i$ . In this case,  $\mu_t^{i-}$  can be compared with the gradient  $D_x w$  of an odd, nonincreasing solution of a perturbed Burgers's equation. However, the result does not appear to be very interesting. Indeed, as time progresses negative waves become ever more singular and a bound such as (1.18)

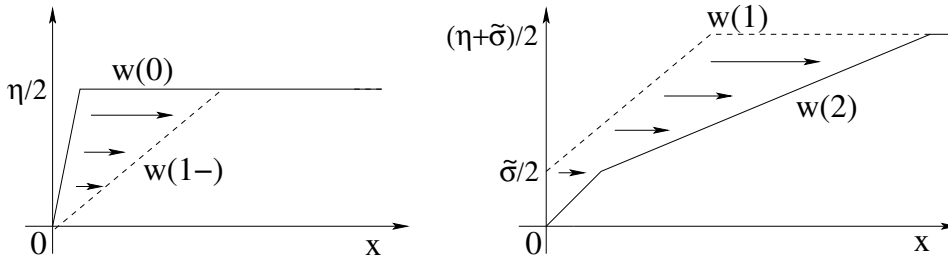


FIG. 6.

retains little content. For large times, a much better way to estimate negative waves is to analyze their cancellation with positive waves of the same family, as in [L1], [L2].

## REFERENCES

- [BaB] P. BAITI AND A. BRESSAN, *Lower semicontinuity of weighted path length in BV*, in Geometrical Optics and Related Topics, F. Colombini and N. Lerner, eds., Birkhäuser, Boston, 1997, pp. 31–58.
- [B] A. BRESSAN, *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy Problem*, Oxford University Press, New York, 2000.
- [BC] A. BRESSAN AND R. M. COLOMBO, *Decay of positive waves in nonlinear systems of conservation laws*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 26 (1998), pp. 133–160.
- [BG] A. BRESSAN AND P. GOATIN, *Oleinik type estimates and uniqueness for  $n \times n$  conservation laws*, J. Differential Equations, 156 (1999), pp. 26–49.
- [BY] A. BRESSAN, AND T. YANG, *On the convergence rate of vanishing viscosity approximations*, Comm. Pure Appl. Math., submitted.
- [GL] J. GLIMM AND P. LAX, *Decay of solutions of systems of nonlinear hyperbolic conservation laws*, Mem. Amer. Math. Soc., 101 (1970).
- [L1] T. P. LIU, *Decay to N-waves of solutions of general systems of nonlinear hyperbolic conservation laws*, Comm. Pure Appl. Math., 30 (1977), pp. 586–611.
- [L2] T. P. LIU, *Linear and nonlinear large-time behavior of solutions of general systems of hyperbolic conservation laws*, Comm. Pure Appl. Math., 30 (1977), pp. 767–796.
- [L3] T. P. LIU, *Admissible solutions of hyperbolic conservation laws*, Mem. Amer. Math. Soc., 30 (1981).
- [O] O. OLEINIK, *Discontinuous solutions of nonlinear differential equations*, Amer. Math. Soc. Transl. (2), 26 (1963), pp. 95–172.
- [Sm] J. SMOLLER, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.