# On the Convergence Rate of Vanishing Viscosity Approximations

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**Abstract**

Given a strictly hyperbolic, genuinely nonlinear system of conservation laws, we prove the a priori bound $\|u(t,\cdot) - u^\varepsilon(t,\cdot)\|_{\mathbf{L}^1} = \mathcal{O}(1)(1+t)\cdot\sqrt{\varepsilon}\,|\ln\varepsilon|$ on the distance between an exact BV solution $u$ and a viscous approximation $u^\varepsilon$, letting the viscosity coefficient $\varepsilon \to 0$. In the proof, starting from $u$ we construct an approximation of the viscous solution $u^\varepsilon$ by taking a mollification $u * \varphi_{\sqrt{\varepsilon}}$ and inserting viscous shock profiles at the locations of finitely many large shocks for each fixed $\varepsilon$. Error estimates are then obtained by introducing new Lyapunov functionals that control interactions of shock waves in the same family and also interactions of waves in different families. 

## 1 Introduction

Consider a strictly hyperbolic system of conservation laws

$$u_t + f(u)_x = 0 \tag{1.1}$$

together with the viscous approximations

$$u_t^\varepsilon + A(u^\varepsilon)u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon\,. \tag{1.2}$$

Here $A(u) \doteq Df(u)$ is the Jacobian matrix of $f$. Given initial data $u(0,x) = \bar{u}(x)$ having small total variation, the recent analysis in [2] has shown that the corresponding solutions $u^\varepsilon$ of (1.2) exist for all $t \geq 0$, have uniformly small total variation, and converge to a unique solution of (1.1) as $\varepsilon \to 0$. The aim of the present paper is to estimate the distance $\|u^\varepsilon(t) - u(t)\|_{\mathbf{L}^1}$, thus providing a convergence rate for these vanishing viscosity approximations.

We use the Landau notation $\mathcal{O}(1)$ to denote a quantity whose absolute value remains uniformly bounded, while $o(1)$ indicates a quantity that approaches zero as $\varepsilon \to 0$. Our main result is the following:

THEOREM 1.1 *Let system* (1.1) *be strictly hyperbolic and assume that each characteristic field is genuinely nonlinear. Then, given any initial data* $u(0,\cdot) = \bar{u}$ *with*

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*small total variation, for every* $\tau > 0$ *the corresponding solutions* $u$ *and* $u^\varepsilon$ *of* (1.1) *and* (1.2) *satisfy the estimate*

$$\|u^\varepsilon(\tau,\cdot) - u(\tau,\cdot)\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot(1+\tau)\sqrt{\varepsilon}\,|\ln\varepsilon|\,\text{Tot. Var.}\{\bar u\}\,. \tag{1.3}$$

REMARKS 1.2

(i) For a fixed time $\tau > 0$, a similar convergence rate was proved in [10] for approximate solutions generated by the Glimm scheme, namely,

$$\|u^{\text{Glimm}}(\tau,\cdot) - u(\tau,\cdot)\|_{\mathbf{L}^1} = o(1)\cdot\sqrt{\varepsilon}\,|\ln\varepsilon|\,.$$

Here $\varepsilon \approx \Delta x \approx \Delta t$ measures the mesh of the grid.

(ii) For a scalar conservation law, the method of Kuznetsov [13] shows that the convergence rate in (1.3) is $\mathcal{O}(1)\cdot\varepsilon^{1/2}$. As shown in [14], this rate is sharp in the general case.

In the case of hyperbolic systems, Goodman and Xin [12] studied the viscous approximation of piecewise smooth solutions having a finite number of noninteracting shocks. With these regularity assumptions, they obtain the convergence rate $\mathcal{O}(1)\cdot\varepsilon^\gamma$ for any $\gamma < 1$. On the other hand, estimate (1.3) applies to a general BV solution, possibly with a countable-everywhere dense set of shocks.

To appreciate the estimate in (1.3), call $S_t$ and $S_t^\varepsilon$ the semigroups generated by systems (1.1) and (1.2), respectively. As proven in [2, 7, 9], they are Lipschitz continuous with respect to the initial data, namely

$$\left\|S_t\bar u - S_t\bar v\right\|_{\mathbf{L}^1} \le L\|\bar u - \bar v\|_{\mathbf{L}^1}\,, \tag{1.4}$$

$$\left\|S_t^\varepsilon\bar u - S_t^\varepsilon\bar v\right\|_{\mathbf{L}^1} \le L\|\bar u - \bar v\|_{\mathbf{L}^1}\,. \tag{1.5}$$

The Lipschitz constant $L$ here does not depend on $t$ and $\varepsilon$. By (1.4), a trivial error estimate is

$$\begin{aligned}\|u^\varepsilon(\tau) - u(\tau)\|_{\mathbf{L}^1} &= L\cdot\int_0^\tau\left\{\lim_{h\to0+}\frac{\|u^\varepsilon(t+h) - S_hu^\varepsilon(t)\|_{\mathbf{L}^1}}{h}\right\}dt\\ &= L\cdot\int_0^\tau\left\|\varepsilon u_{xx}^\varepsilon(t)\right\|_{\mathbf{L}^1}dt\,.\end{aligned}$$

However, $\|u_{xx}^\varepsilon(t)\|_{\mathbf{L}^1}$ grows like $\varepsilon^{-1}$; hence the right-hand side in the above estimate does not converge to zero as $\varepsilon \to 0$.

We thus need to take a different approach, relying on (1.5). Let $\varepsilon > 0$ be given. It is well-known (see [4]) that one can construct an $\varepsilon'$-approximate front-tracking solution $\tilde u$ of (1.1), with

$$\|\tilde u(0) - \bar u\|_{\mathbf{L}^1} < \varepsilon'\,, \quad \|\tilde u(\tau) - u(\tau)\|_{\mathbf{L}^1} < \varepsilon'\,,$$

and such that the total strength of all nonphysical fronts is $< \varepsilon'$. Here we can take, for example, $\varepsilon' = e^{-1/\varepsilon}$. Since the errors due to the front-tracking approximation are of order $\varepsilon' \ll \varepsilon$, in the following computations we shall neglect terms of order $\mathcal{O}(1)\cdot\varepsilon'$, since they can be made arbitrarily small by a suitable choice of $\varepsilon'$. For sake of definiteness, we shall always work with the right-continuous version of a BV function. Since all characteristic fields are genuinely nonlinear, it is convenient to measure the (signed) strength of an $i$-rarefaction or of an $i$-shock front connecting the states $u^-$ and $u^+$ as

$$\sigma \doteq \lambda_i(u^+) - \lambda_i(u^-),$$

where $\lambda_i$ denotes the $i^{\text{th}}$ eigenvalue of the matrix $A(u)$. We follow here the notations in [4] and call

$$V(u) = \sum_\alpha |\sigma_\alpha|, \quad Q(u) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \sigma_\beta|, \tag{1.6}$$

respectively, the *total strength of waves* and the *interaction potential* in a front-tracking solution $u$. The second summation here ranges over the set $\mathcal{A}$ of all couples of approaching wave fronts.

For notational convenience, we shall simply call $u$ the $\varepsilon'$-approximate front-tracking approximation; also assume that $\bar{u} = u(0)$ is piecewise constant. Since $\varepsilon' \ll \varepsilon$, this will not have any consequence for our estimates. In the sequel, we shall construct a further approximation $v = v(t, x)$ having the following properties:

Let $0 = t_0 < \cdots < t_N = \tau$ be the interaction times in the front-tracking solution $u$. Then $v$ is smooth on each strip $[t_{i-1}, t_i[ \times \mathbb{R}$. Moreover, calling $\delta_0 \doteq$ Tot. Var.$\{\bar{u}\}$, one has

$$\|v(0) - \bar{u}\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \delta_0 \sqrt{\varepsilon}, \quad \|v(\tau) - u(\tau)\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \delta_0 \sqrt{\varepsilon}, \tag{1.7}$$

$$\int_0^\tau \int |v_t + A(v)v_x - \varepsilon v_{xx}| dx\, dt = \mathcal{O}(1) \cdot \delta_0 (1+\tau)\sqrt{\varepsilon}\, |\ln \varepsilon|, \tag{1.8}$$

$$\sum_{1 \le i \le N} \int |v(t_i, x) - v(t_i -, x)| dx = \mathcal{O}(1) \cdot \delta_0 \sqrt{\varepsilon}\, |\ln \varepsilon|. \tag{1.9}$$

Having achieved this step, by the Lipschitz continuity of the semigroup $S_t^\varepsilon$ in (1.5) we can then conclude

$$\begin{aligned}
&\|u^\varepsilon(\tau) - u(\tau)\|_{\mathbf{L}^1} \\
&\quad \le \|S_\tau^\varepsilon \bar{u} - v(\tau)\|_{\mathbf{L}^1} + \|v(\tau) - u(\tau)\|_{\mathbf{L}^1} \\
&\quad \le L\|\bar{u} - v(0)\|_{\mathbf{L}^1} + L\int_0^\tau \int |v_t + A(v)v_x - \varepsilon v_{xx}| dx\, dt \\
&\qquad + L \sum_{1 \le i \le N} \|v(t_i, x) - v(t_i -, x)\|_{\mathbf{L}^1} + \|v(\tau) - u(\tau)\|_{\mathbf{L}^1} \\
&\quad = \mathcal{O}(1) \cdot \delta_0 (1+\tau) \sqrt{\varepsilon}\, |\ln \varepsilon|.
\end{aligned} \tag{1.10}$$

To construct the approximate solution $v$, we first consider a mollification of $u$ with respect to the space variable $x$. Let $\varphi : \mathbb{R} \mapsto [0, 1]$ be a smooth function such that

$$\varphi(s) = 0 \quad \text{if } |s| > \frac{2}{3},\ s\varphi'(s) \le 0\,,$$
$$\varphi(s) = \varphi(-s)\,, \qquad \int \varphi(s)ds = 1\,.$$

For $\delta > 0$ small, define the rescalings $\varphi_\delta(s) \doteq \delta^{-1}\varphi(x/\delta)$ and the mollified solutions $v^\delta(t) \doteq u(t) * \varphi_\delta$ so that

$$v^\delta(t,x) = \int u(t,y)\varphi_\delta(x-y)dy\,.$$

Recalling that $\delta_0 \doteq \text{Tot. Var.}\{\bar{u}\}$, one has

$$\text{Tot. Var.}\{u(t)\}\,, \quad \left\|u_x^\varepsilon(t)\right\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\delta_0 \quad \text{for all } t \ge 0\,. \tag{1.11}$$

We now observe that

$$\begin{aligned} \|v^\delta - u\|_{\mathbf{L}^1} &= \int \left|\int (u(x)-u(y))\varphi_\delta(x-y)dy\right|dx \\ &\le \int \text{Tot. Var.}\{u;\,[x-\delta, x+\delta]\}dx = \mathcal{O}(1)\cdot\delta_0\delta\,. \end{aligned} \tag{1.12}$$

To estimate the distance between $v^\delta$ and $u^\varepsilon$, we first compute

$$\begin{aligned} \int \left|\varepsilon v^\delta_{xx}(x)\right|dx &= \varepsilon\int \left|(u_x * \varphi_{\delta,x})(x)\right|dx \\ &\le \varepsilon\|u_x\|_{\mathbf{L}^1}\cdot\|\varphi_{\delta,x}\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\delta_0\frac{\varepsilon}{\delta}\,, \end{aligned} \tag{1.13}$$

$$\begin{aligned} &\int \left|v^\delta_t + A(v^\delta)v^\delta_x\right|dx \\ &= \int\left|\int \big(A(v^\delta(x))u_x(y) - A(u(y))u_x(y)\big)\varphi(x-y)dy\right|dx \\ &\le \int\left(\int \left|A(v^\delta(x)) - A(u(y))\right|\varphi(x-y)dx\right)|u_x(y)|dy \\ &= \mathcal{O}(1)\cdot\|DA\|_{\mathcal{C}^0}\int \text{Osc}\{u;\,[y-\delta, y+\delta]\}|u_x(y)|dy\,. \end{aligned} \tag{1.14}$$

For simplicity, formulae (1.13) and (1.14) are here written in the case where the function $u$ is absolutely continuous. In the general case, the same estimates hold by replacing $|u_x|dx$ with the measure $|D_x u|$ of total variation of $u \in \text{BV}$.

If $u$ is a Lipschitz-continuous solution of (1.1), the oscillation of $u$ on any interval of length $2\delta$ is $\mathcal{O}(1)\cdot\delta$. Hence, performing the above mollifications, we would obtain

$$\int \left|v^\delta_t + A(v^\delta)v^\delta_x\right|dx = \mathcal{O}(1)\cdot\delta\delta_0\,. \tag{1.15}$$

Choosing $\delta \doteq \sqrt{\varepsilon}$, by (1.12)–(1.15) we thus conclude

$$
\begin{aligned}
\|u^\varepsilon(\tau) - u(\tau)\|_{\mathbf{L}^1} &\le \left\| S^\varepsilon_\tau \bar u - v^\delta(\tau)\right\|_{\mathbf{L}^1} + \left\| v^\delta(\tau) - u(\tau)\right\|_{\mathbf{L}^1} \\
&\le L \left\| \bar u - v^\delta(0)\right\|_{\mathbf{L}^1} \\
&\quad + L \int_0^\tau \int \left| v^\delta_t + A(v^\delta) v^\delta_x - \varepsilon v^\delta_{xx}\right| dx\, dt \\
&\quad + \left\| v^\delta(\tau) - u(\tau)\right\|_{\mathbf{L}^1} \\
&= \mathcal{O}(1) \cdot \delta_0 (1+\tau)\sqrt{\varepsilon}\,.
\end{aligned}
\tag{1.16}
$$

In general, however, the solution $u$ is not Lipschitz-continuous. The most one can say is that $u$ is a function with bounded variation, possibly with countably many shocks. Hence the easy estimate (1.16) does not hold. For genuinely nonlinear systems, the additional error terms due to centered rarefaction waves can be controlled by carefully estimating the decay rate of these waves. Error terms due to small shocks will be estimated by suitable Lyapunov functionals. However, there is one type of wave front that is responsible for large errors in (1.14), namely, the large shocks of strength $\ll \sqrt{\varepsilon}$. In a neighborhood of each one of these shocks, a more careful approximation is needed. Instead of a mollification, we shall insert an approximate viscous shock profile.

Our construction goes as follows. By the same argument as in [5, prop. 2, p. 17] given $\rho > 0$ one can select finitely many shock fronts

$$
t \mapsto x_\alpha(t)\,, \quad t \in T_\alpha \doteq \,]t^-_\alpha, t^+_\alpha[\,, \quad \alpha = 1, \dots, \nu,
$$

with $\nu = \mathcal{O}(1)\cdot\delta_0/\rho$, having the following properties: For every $t \in T_\alpha$ (apart from finitely many interaction points) the left and right states $u^-_\alpha, u^+_\alpha$ are connected by a shock, say of the family $k_\alpha$, with strength $|\sigma_\alpha(t)| \ge \rho/2$, while $|\sigma_\alpha(t^*)| \ge \rho$ for some $t^* \in T_\alpha$. Moreover, every shock in the front-tracking solution $u$ with strength $\ge \rho$ is included in one of the above fronts.

For each $\alpha$ and each $t \in T_\alpha$ (apart from finitely many interaction points), let $\omega_\alpha$ be the viscous shock profile connecting the states $u^-_\alpha$ and $u^+_\alpha$. Calling $\lambda_\alpha$ the shock speed, we thus have

$$
\omega''_\alpha = (A(\omega_\alpha) - \lambda_\alpha)\omega'_\alpha\,, \qquad \lim_{s\to\pm\infty} \omega_\alpha(s) = u^\pm_\alpha\,.
$$

We choose the parameter $s$ so that the value $s = 0$ corresponds roughly to the center of the traveling profile. This can be achieved by requiring

$$
\int_{-\infty}^0 \left|\omega_\alpha(s) - u^-_\alpha\right| ds = \int_0^\infty \left|\omega_\alpha(s) - u^+_\alpha\right| ds\,.
\tag{1.17}
$$

For system (1.2) with $\varepsilon$-viscosity, the corresponding rescaled shock profile is $s \mapsto \omega^\varepsilon_\alpha(s) \doteq \omega_\alpha(s/\varepsilon)$. On the open interval

$$
J_\alpha(t) \doteq \,]x_\alpha(t) - \delta, x_\alpha(t) + \delta[\,,
$$

we now replace the mollified solution by a shock profile. Define the functions $\varrho_\alpha$ and $\tilde\omega_\alpha$ by setting

$$\varrho_\alpha(x_\alpha+\xi) \doteq u_\alpha^+ \int_{-\infty}^{\xi} \varphi_\delta(y)dy + u_\alpha^- \int_{\xi}^{\infty} \varphi_\delta(y)dy\,, \tag{1.18}$$

$$\tilde\omega_\alpha(x_\alpha+\xi) \doteq \begin{cases} \omega_\alpha^\varepsilon(\phi(\xi)) & \text{if } \xi \in \,]-\delta,\delta[ \\ u_\alpha^+ & \text{if } \xi \geq \delta \\ u_\alpha^- & \text{if } \xi \leq -\delta, \end{cases} \tag{1.19}$$

where

$$\phi(\xi) = \begin{cases} \xi & \text{if } |\xi| \leq \frac{\sqrt{\varepsilon}}{2} \\ \frac{\varepsilon}{4(\sqrt{\varepsilon}-\xi)} & \text{if } \frac{\sqrt{\varepsilon}}{2} \leq \xi < \sqrt{\varepsilon} \\ -\frac{\varepsilon}{4(\sqrt{\varepsilon}+\xi)} & \text{if } -\sqrt{\varepsilon} < \xi < -\frac{\sqrt{\varepsilon}}{2}. \end{cases} \tag{1.20}$$

Notice that $\tilde\omega_\alpha$ is essentially an $\varepsilon$-viscous shock profile, up to a $\mathcal{C}^1$ transformation that squeezes the whole real line onto the interval $J_\alpha(t)$. Moreover, $\varrho_\alpha$ is the mollification of the piecewise constant function taking values $u_\alpha^-$ and $u_\alpha^+$ with a single jump at $x_\alpha$. The above definitions imply that $\tilde\omega_\alpha = \varrho_\alpha$ outside the interval $J_\alpha(t)$. Finally, for every $t \geq 0$ we define

$$v = u * \varphi_\delta + \sum_{\alpha\in\mathcal{BS}} \left(\tilde\omega_\alpha - \varrho_\alpha\right), \tag{1.21}$$

where the summation ranges over all big shock fronts. In the remainder of the paper we will show that, by choosing

$$\delta \doteq \sqrt{\varepsilon}\,, \qquad \rho \doteq 4\sqrt{\varepsilon}\,|\ln\varepsilon|\,, \tag{1.22}$$

all the estimates in (1.7)–(1.9) hold. By (1.10), this will achieve a proof of Theorem 1.1.

## 2 Estimates on Rarefaction Waves

Throughout the following we denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the $A(u) \doteq Df(u)$. Moreover, we shall use bases of left and right eigenvectors $l_i(u)$ and $r_i(u)$ normalized so that

$$\nabla\lambda_i(u)\cdot r_i(u) \equiv 1\,, \quad l_i(u)\cdot r_j(u) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases} \tag{2.1}$$

According to (1.14), outside the large shocks we have to estimate the quantity

$$E(\tau) \doteq \int_0^\tau \int \mathrm{Osc}\{u;[y-\delta,y+\delta]\}|u_x(y)|dy\,dt\,. \tag{2.2}$$

Centered rarefaction waves can have large gradients, and hence give a large contribution to the above integral. However, for genuinely nonlinear families, the density

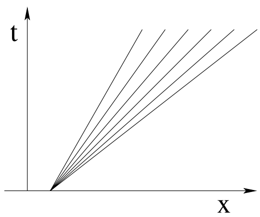


FIGURE 2.1

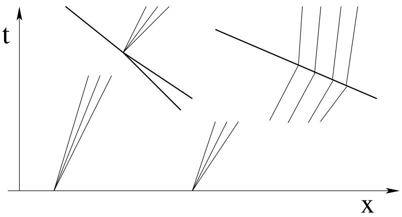


FIGURE 2.2

of these waves decays rapidly, as $t^{-1}$. We now give an example where the integral (2.2) can be easily estimated.

EXAMPLE 2.1 Assume that the solution $u$ consists of a single centered rarefaction wave of the $i^{\text{th}}$ family (Figure 2.1), connecting the states $u^-$ and $u^+$. Call $s \mapsto \omega(s)$ the parametrized $i$-rarefaction curve, so that

$$\dot{\omega} = r_i(\omega)\,, \quad \omega(0) = u^-\,, \quad \omega(\sigma) = u^+$$

for some wave strength $\sigma > 0$. We then have

$$u(t,x) = \begin{cases} u^- & \text{if } \frac{x}{t} < \lambda_i(u^-) \\ \omega(s) & \text{if } \frac{x}{t} = \lambda_i(\omega(s)),\ s \in [0,\sigma], \\ u^+ & \text{if } \frac{x}{t} > \lambda_i(u^+). \end{cases}$$

If $K$ is an upper bound for the length of all eigenvectors $r_i(u)$, we have

$$\mathrm{Osc}\{u(t); [y-\delta, y+\delta]\} \le K \cdot \min\left\{\sigma, \frac{2\delta}{t}\right\}, \qquad \int |u_x(x)|dx \le K\sigma\,.$$

Hence the quantity in (2.2) satisfies

$$E(\tau) \le \int_0^{2\delta/\sigma} K^2\sigma^2\,dt + \int_{2\delta/\sigma}^{\tau} K^2\sigma\frac{2\delta}{t}\,dt = 2K^2\delta\sigma\left(1 + \ln\frac{\sigma\tau}{2\delta}\right). \tag{2.3}$$

The choice $\delta \doteq \sqrt{\varepsilon}$ would thus give the correct order of magnitude $\mathcal{O}(1)\cdot\sqrt{\varepsilon}\,|\ln\varepsilon|\cdot$ Tot. Var.$\{u\}$.

Of course, a general BV solution of the system of conservation laws (1.1) is far more complex than a single rarefaction. It can contain several centered rarefactions originating at $t = 0$ and also at later times as a result of shock interactions (Figure 2.2). Moreover, the crossing of wave fronts of other families may slow down the decay of positive waves. Nevertheless, the forthcoming analysis will show that, in some sense, Example 2.1 represents the worst possible case. Using the sharp decay estimate for positive waves in [11] and a comparison argument, we shall prove that the total error due to steep rarefaction waves for an arbitrary weak solution is no greater than the error computed at (2.3) for a solution containing only one centered rarefaction. In the present section, all the analysis refers to an exact solution. A similar result can then be easily derived for a sufficiently accurate front-tracking approximation.

We begin by recalling the main results in [11]. Given a function $u : \mathbb{R} \mapsto \mathbb{R}^n$ with small total variation, following [4, 5, 6], one can define the measures $\mu^i$ of $i$-waves in $u$ as follows. Since $u \in \mathrm{BV}$, its distributional derivative $D_x u$ is a Radon measure. We define $\mu^i$ as the measure such that

$$\mu^i \doteq l_i(u) \cdot D_x u \tag{2.4}$$

restricted to the set where $u$ is continuous, while, at each point $x$ where $u$ has a jump, we define

$$\mu^i(\{x\}) \doteq \sigma_i \,, \tag{2.5}$$

where $\sigma_i$ is the strength of the $i$-wave in the solution of the Riemann problem with data $u^- = u(x-)$ and $u^+ = u(x+)$. In accordance with (2.1), if the solution of the Riemann problem contains the intermediate states $u^- = \omega_0, \ldots, \omega_n = u^+$, the strength of the $i$-wave is defined as

$$\sigma_i \doteq \lambda_i(\omega_i) - \lambda_i(\omega_{i-1}) \,. \tag{2.6}$$

Together with the measures $\mu^i$ we also define the Glimm functionals

$$V(u) \doteq \sum_i |\mu^i|(\mathbb{R}) \,, \tag{2.7}$$

$$\begin{aligned} Q(u) \doteq & \sum_{i<j} (|\mu^j| \otimes |\mu^i|)\{(x,y) : x < y\} \\ & + \sum_i (\mu^{i-} \otimes |\mu^i|)\{(x,y) : x \neq y\} \,, \end{aligned} \tag{2.8}$$

measuring, respectively, the total strength of waves and the interaction potential.

We call $\mu^{i+}$ and $\mu^{i-}$, respectively, the positive and negative parts of $\mu^i$, so that

$$\mu^i = \mu^{i+} - \mu^{i-} \,, \quad |\mu^i| = \mu^{i+} + \mu^{i-} \,. \tag{2.9}$$

In [11] the authors introduced a partial ordering within the family of positive Radon measures:

DEFINITION 2.2 Let $\mu$ and $\mu'$ be two positive Radon measures. We say that $\mu \preceq \mu'$ if and only if

$$\sup_{\mathrm{meas}(A) \le s} \mu(A) \le \sup_{\mathrm{meas}(B) \le s} \mu'(B) \quad \text{for every } s > 0 \,. \tag{2.10}$$

Here meas$(A)$ denotes the Lebesgue measure of a set $A$. In some sense, the above relation means that $\mu'$ is more singular than $\mu$. Namely, it has a greater total mass, concentrated on regions with higher density. Notice that the usual order relation

$$\mu \le \mu' \quad \text{if and only if} \quad \mu(A) \le \mu'(A) \ \text{ for every } A \subset \mathbb{R}$$

is much stronger. Of course, $\mu \le \mu'$ implies $\mu \preceq \mu'$, but the converse does not hold.

Given a solution $u$ of (1.1), we denote by $\mu_t^{i+}$ the measure of positive $i$-waves in $u(t,\cdot)$. In particular, $\mu_0^{i+}$ refers to the positive $i$-waves in $u$ at the initial time $t=0$. An accurate estimate of these measures is obtained by a comparison with a solution of Burgers' equation with source terms.

PROPOSITION 2.3 *For some constant* $\kappa > 0$ *and for every small BV solution* $u = u(t,x)$ *of the system* (1.1) *the following holds*: *Let* $w = w(t,x)$ *be the solution of the Cauchy problem for Burgers' equation with impulsive source term*

$$w_t + \left(\frac{w^2}{2}\right)_x = -\kappa\,\mathrm{sgn}(x)\cdot\frac{d}{dt}Q(u(t))\,, \tag{2.11}$$

$$w(0,x) = \mathrm{sgn}(x)\cdot\sup_{\mathrm{meas}(A)<2|x|}\frac{\mu_0^{i+}(A)}{2}\,. \tag{2.12}$$

*Then, for every* $t \geq 0$,

$$\mu_t^{i+} \preceq D_x w(t)\,. \tag{2.13}$$

PROOF: For a proof, see [11]. □

The ordering relation (2.10) can be better appreciated in terms of rearrangements. More precisely, let $\mu$ be a positive Radon measure on $\mathbb{R}$, so that $\mu \doteq D_x v$ is the distributional derivative of some bounded, nondecreasing function $v : \mathbb{R} \mapsto \mathbb{R}$. We can decompose

$$\mu = \mu^{\mathrm{sing}} + \mu^{\mathrm{ac}}$$

as the sum of a singular and an absolutely continuous part with respect to Lebesgue measure. The absolutely continuous part corresponds to the usual derivative $z \doteq v_x$, which is a nonnegative $\mathbf{L}^1$ function defined at a.e. point. We shall denote by $\hat{z}$ the *symmetric rearrangement* of $z$, i.e., the unique even function such that

$$\hat{z}(x) = \hat{z}(-x)\,,\quad \hat{z}(x) \geq \hat{z}(x')\,,\quad \text{if } 0 < x < x'\,,$$
$$\mathrm{meas}(\{x : \hat{z}(x) > c\}) = \mathrm{meas}(\{x : z(x) > c\}) \quad \text{for every } c > 0\,.$$

Moreover, we define the *odd rearrangement* of $v$ as the unique function $\hat{v}$ such that

$$\hat{v}(-x) = -\hat{v}(x)\,,\quad \hat{v}(0+) = \frac{1}{2}\mu^{\mathrm{sing}}(\mathbb{R})\,,$$
$$\hat{v}(x) = \hat{v}(0+) + \int_0^x z(y)dy \quad \text{for } x > 0\,.$$

By construction, the function $\hat{v}$ is convex for $x < 0$ and concave for $x > 0$. We now have the following:

PROPOSITION 2.4 *Let* $\mu = D_x v$ *and* $\mu' = D_x v'$ *be positive Radon measures. Call* $\hat{v}$ *and* $\hat{v}'$ *the odd rearrangements of* $v$ *and* $v'$, *respectively. Then* $\mu \preceq D_x\hat{v} \preceq \mu$. *Moreover,*

$$\hat{v}(x) = \mathrm{sgn}(x)\cdot\sup_{\mathrm{meas}(A)\leq 2|x|}\frac{\mu(A)}{2} \tag{2.14}$$

*and*

$$\mu \preceq \mu' \quad \text{if and only if} \quad \hat{v}(x) \le \hat{v}'(x) \ \text{for all } x > 0\,. \tag{2.15}$$

The relevance of the above concepts toward an estimate of the quantity in (2.2) is due to the next three comparison lemmas.

LEMMA 2.5 *Let $u : \mathbb{R} \mapsto \mathbb{R}$ be a nondecreasing BV function and let $\hat{u}$ be its odd rearrangement. Then*

$$\int_{-\infty}^{\infty} \text{Tot. Var.}\{u;\, [x-\rho, x+\rho]\} du(x) \le 3 \int_{-\infty}^{\infty} [\hat{u}(x+\rho) - \hat{u}(x-\rho)] d\hat{u}(x)\,. \tag{2.16}$$

PROOF: We begin by defining a measurable map $x \mapsto \varphi(x)$ from $\mathbb{R}$ onto $\mathbb{R}_+$ with the following properties:

(i) $\varphi(x) = 0$ for all points $x$ in the support of singular part of the measure $u_x$.
(ii) $u_x(x) = \hat{u}_x(\varphi(x))$ for every $x$ where $u$ is differentiable.
(iii) $\text{meas}(\varphi^{-1}(A)) = 2\,\text{meas}(A)$ for every $A \subset \mathbb{R}_+$.

We now have

$$\int_{-\infty}^{\infty} \text{Tot. Var.}\{u;\, [x - \rho, x + \rho]\} du(x) = \left( \int_{\varphi(x)\le\rho} + \int_{\varphi(x)>\rho} \right) [u(x+\rho) - u(x-\rho)] du(x) \doteq I_1 + I_2\,.$$

We now estimate $I_1$ and $I_2$ separately as follows:

$$\begin{aligned} I_1 &= \int_{\varphi(x)\le\rho} [u(x+\rho) - u(x-\rho)] du(x) \\ &\le \int_{\varphi(x)\le\rho} 2\hat{u}(\rho) du(x) \\ &\le 4(\hat{u}(\rho))^2 \le 2\int_{-\rho}^{\rho} [\hat{u}(x+\rho) - \hat{u}(x-\rho)] d\hat{u}(x)\,, \end{aligned} \tag{2.17}$$

$$\begin{aligned} I_2 &\le \int_{\varphi(x)>\rho} \int_{-\rho}^{\rho} [u_x(x)\, D_x u(x+s)] ds\, dx \\ &\le 4\rho \int_{\rho}^{\infty} [\hat{u}_x(x)\, D_x \hat{u}(x-\rho)] dx \\ &= 4\rho \int_{0}^{\infty} \hat{u}_x(x+\rho) d\hat{u}(x) \end{aligned} \tag{2.18}$$

$$\leq 2\int_0^{\infty} [\hat{u}(x+\rho) - \hat{u}(x-\rho)]d\hat{u}(x)$$
$$= \int_{-\infty}^{\infty} [\hat{u}(x+\rho) - \hat{u}(x-\rho)]d\hat{u}(x)\,.$$

For $x > \rho$, we are here using the inequality

$$2\rho\hat{u}_x(x) \leq \hat{u}(x) - \hat{u}(x-2\rho)\,.$$

Moreover, calling $\tilde{f}$ and $\tilde{g}$ the nonincreasing even rearrangements of two positive, integrable functions $f$ and $g$, one always has

$$\int_{-\infty}^{\infty} f(x)g(x)dx \leq \int_{-\infty}^{\infty} \tilde{f}(x)\tilde{g}(x)dx\,. \tag{2.19}$$

Together, (2.17) and (2.18) yield (2.16). □

LEMMA 2.6 *Let $v$ and $w$ be two nondecreasing BV functions. If $D_x v \preceq D_x w$, then the odd rearrangements $\hat{v}$ and $\hat{w}$ satisfy*

$$\int_{-\infty}^{\infty} [\hat{v}(x+\rho) - \hat{v}(x-\rho)]d\hat{v}(x) \leq \int_{-\infty}^{\infty} [\hat{w}(x+\rho) - \hat{w}(x-\rho)]d\hat{w}(x)\,. \tag{2.20}$$

PROOF: By using an approximation argument, we can assume that $\hat{v}$ and $\hat{w}$ are smooth. Without loss of generality, we can assume $\hat{v}(\pm\infty) = \hat{w}(\pm\infty)$. By assumptions, $\hat{v}(x) \leq \hat{w}(x)$ for all $x > 0$. We consider a parabolic equation with smooth coefficients

$$z_t = a(t,x)z_{xx}\,, \tag{2.21}$$

with $a(t,x) = a(t,-x) \geq 0$, having a solution such that

$$z(0,x) = \hat{w}(x)\,, \qquad \lim_{t\to\infty} z(t,x) = \hat{v}(x)\,,$$

where the limit holds uniformly for $x$ in bounded sets. To construct $a(t,x)$, one can first define a smooth function $\tilde{a} = \tilde{a}(t,x,z)$ such that

$$\tilde{a}(t,\,-x,z) = \tilde{a}(t,x,z) = \begin{cases} 1 & \text{if } |z - \hat{v}(x)| \geq \frac{2}{t} \\ 0 & \text{if } |z - \hat{v}(x)| \leq \frac{1}{t}. \end{cases}$$

Then we solve the quasilinear Cauchy problem

$$z_t = \tilde{a}(t,x,z)z_{xx}\,, \qquad z(0,x) = \hat{w}(x)\,,$$

and set $a(t,x) \doteq \tilde{a}(t,x,z(t,x))$. We now claim that

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty}\int_{x-\rho}^{x+\rho} z_x(x)z_x(y)dy\,dx\right) \leq 0\,. \tag{2.22}$$

Indeed, calling $\phi \doteq z_x \geq 0$ and using (2.21) we compute

$$\phi_t = (a(t,x)\phi_x)_x\,,$$

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{-\infty}^{\infty}\int_{x-\rho}^{x+\rho}\phi(x)\phi(y)dy\,dx\right)\\
&\quad=\int_{-\infty}^{\infty}\int_{x-\rho}^{x+\rho}\big[(a\phi_x(x))_x\phi(y)+\phi(x)(a\phi_x(y))_x\big]dy\,dx\\
&\quad=\int_{-\infty}^{\infty}[a\phi_x(y+\rho)-a\phi_x(y-\rho)]\phi(y)dy\\
&\qquad+\int_{-\infty}^{\infty}[a\phi_x(x+\rho)-a\phi_x(x-\rho)]\phi(x)dx\\
&\quad=2\int_{-\infty}^{\infty}\phi(x)[a\phi_x(x+\rho)-a\phi_x(x-\rho)]dx\\
&\quad=\int_{-\infty}^{\infty}a\phi_x(x)[\phi(x-\rho)-\phi(x+\rho)]dx\\
&\quad\le 0\,,
\end{aligned}$$

because $\phi(t,\cdot)$ is an even function, nonincreasing for $x \ge 0$. From (2.22) it follows

$$\int_{-\infty}^{\infty}\int_{x-\rho}^{x+\rho}\hat{v}_x(x)\hat{v}_x(y)dy\,dx \le \int_{-\infty}^{\infty}\int_{x-\rho}^{x+\rho}\hat{w}_x(x)\hat{w}_x(y)dy\,dx\,.$$

□

LEMMA 2.7 *Let u be a solution of* (1.1) *defined for* $t \in [0,\tau]$ *and let* $w = w(t,x)$ *as in* (2.11)–(2.12). *Set*

$$\bar{\sigma} \doteq \frac{1}{2}\mu_0^{i+}(\mathbb{R}) + \kappa\big[Q(u(0)) - Q(u(\tau))\big]\,, \tag{2.23}$$

*and let*

$$v(t,x) = \begin{cases} \frac{x}{t} & \text{if } \frac{|x|}{t} \le \bar{\sigma} \\ \operatorname{sgn}(x)\cdot\bar{\sigma} & \text{if } \frac{|x|}{t} > \bar{\sigma} \end{cases} \tag{2.24}$$

*be a solution of Burgers' equation consisting of one single centered rarefaction wave of strength* $2\bar{\sigma}$. *Then*

$$\begin{aligned}
&\int_0^{\tau}\int_{-\infty}^{\infty}[w(t,x+\rho)-w(t,x-\rho)]w_x(t,x)dx\,dt \le\\
&\qquad 2\int_0^{\tau}\int_{-\infty}^{\infty}[v(t,x+\rho)-v(t,x-\rho)]v_x(t,x)dx\,dt\,.
\end{aligned} \tag{2.25}$$

PROOF: To compare the integrals in (2.25) a change of variables will be useful. We define (Figure 2.3)

$$x(t,\xi) \doteq t\xi\,,\quad \xi\in[0,\bar{\sigma}],\ t\in[0,\tau]\,.$$

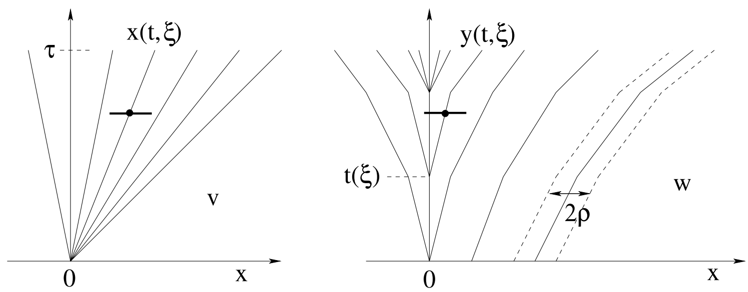


FIGURE 2.3

For $t \in [0, \tau]$ and $Q(t) - Q(\tau) < \xi \le \bar\sigma$, we also consider the point $y(t, \xi) > 0$ implicitly defined by

$$w(t, \infty) - w(t, y(t, \xi)) = \bar\sigma - \xi\,.$$

Notice that $y(t, \xi)$ is defined only for $t \in [t(\xi), \tau]$, or equivalently $\xi \in [\xi(t), \bar\sigma]$, where

$$\xi(t) \doteq \kappa[Q(t) - Q(\tau)]\,, \quad t(\xi) \doteq \inf\{t \ge 0 : [Q(t) - Q(\tau)] \le \xi\}\,.$$

For $0 < \xi_1 < \xi_2 < \bar\sigma$ and $s > 0$ we have

$$\begin{aligned} &y(t(\xi_1) + s, \xi_2) - y(t(\xi_1) + s, \xi_1) \\ &\quad = y(t(\xi_1), \xi_2) - y(t(\xi_1), \xi_1) + (\xi_2 - \xi_1)s \\ &\quad \ge (\xi_2 - \xi_1)s = x(s, \xi_2) - x(s, \xi_1)\,. \end{aligned} \tag{2.26}$$

Observe that since $w$ is odd and nondecreasing,

$$w^+(t, y - \rho) \doteq \max\{w(t, y - \rho), 0\} = w(t, \max\{y - \rho, 0\})\,.$$

Of course, the same is also true for $v$. Calling $I_w$ and $I_v$ the two integrals in (2.25) and using (2.26) at the key step, we obtain

$$\begin{aligned} I_w &= 2\int_0^\tau \int_{\xi(t)}^{\bar\sigma} [w(t, y(t, \xi) + \rho) - w(t, y(t, \xi) - \rho)]d\xi\, dt \\ &\le 4\int_0^\tau \int_{\xi(t)}^{\bar\sigma} [w(t, y(t, \xi) + \rho) - w^+(t, y(t, \xi) - \rho)]d\xi\, dt \\ &= 4\int_0^\tau \iint_{|y(t,\xi_1) - y(t,\xi_2)| < \rho} d\xi_1\, d\xi_2\, dt \\ &= 4\iint \text{meas}\{t \in [0, \tau] : |y(t, \xi_1) - y(t, \xi_2)| < \rho\}d\xi_1\, d\xi_2 \end{aligned}$$

$$\le 4 \iint \text{meas}\{t \in [0,\tau] : |x(t,\xi_1) - x(t,\xi_2)| < \rho\} d\xi_1 \, d\xi_2$$

$$= 4 \int_0^\tau \int_0^{\bar\sigma} [v(t, x(t,\xi) + \rho) - v^+(t, x(t,\xi) - \rho)] dy\xi \, dt$$

$$\le 2 I_\nu \, .$$

□

COROLLARY 2.8 *Assume that all characteristic fields for system* (1.1) *are genuinely nonlinear. Let $u$ be a solution with initial data $u(0,x) = \bar u(x)$ having small total variation. Then, for every $\tau, \delta > 0$, the measures $\mu_t^{i+}$ of positive waves in $u(t,\cdot)$ satisfy the estimate*

$$\sum_{i=1}^n \int_0^\tau \big(\mu_t^{i+} \otimes \mu_t^{i+}\big)\big(\{(x,y) : |x-y| \le \delta\}\big) dt = \mathcal{O}(1) \cdot (\ln(2+\tau) + |\ln \delta|)\delta \cdot \text{Tot. Var.}\{\bar u\} \, . \tag{2.27}$$

PROOF: By Proposition 2.3 and the previous comparison lemmas, for every $i = 1, \dots, n$ the integral on the left-hand side of (2.27) has the same order of magnitude as in the case of a solution with a single centered rarefaction wave of magnitude $\sigma \doteq \text{Tot. Var.}\{\bar u\} < 1$. Looking back at Example 2.1, from (2.3) we thus obtain

$$\begin{aligned} &\int_0^\tau \big(\mu_t^{i+} \otimes \mu_t^{i+}\big)\big(\{(x,y) : |x-y| \le 2\delta\}\big) \\ &\quad = \mathcal{O}(1) \cdot \delta\sigma \left(1 + \ln \frac{\sigma\tau}{2\delta}\right) \\ &\quad = \mathcal{O}(1) \cdot (\ln(2+\tau) + |\ln \delta|)\delta \cdot \text{Tot. Var.}\{\bar u\} \, . \end{aligned} \tag{2.28}$$

□

REMARK 2.9 All of the above estimates refer to an exact solution $u$ of (1.1). If $u_\nu \to u$ is a convergent sequence of front-tracking approximations, the corresponding measures of $i$-waves in $u_\nu(t,\cdot)$ converge weakly: $\mu_{\nu,t}^i \rightharpoonup \mu_t^i$ for all $i = 1, \dots, n$ and $t \ge 0$. Unfortunately, this does not guarantee the weak convergence of the signed measures

$$\mu_{\nu,t}^{i+} \rightharpoonup \mu_t^{i+} \, , \quad \mu_{\nu,t}^{i-} \rightharpoonup \mu_t^{i-} \, . \tag{2.29}$$

For example (Figure 2.4), on a fixed interval $[a,b]$ every $u_\nu$ might contain an alternating sequence of small positive and negative waves that cancel only in the limit as $\nu \to \infty$. However, by a small modification of these front-tracking solutions $u_\nu$, one can achieve the weak convergence (2.29) for each $t$ in a discrete set of times $\{j\tau/N : j = 0, 1, \dots, N\}$, with $N \gg \varepsilon^{-1}$. As a result, we obtain an arbitrarily ac-

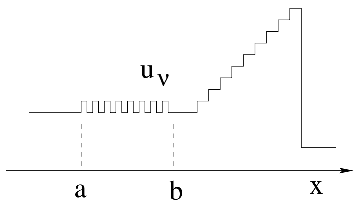


FIGURE 2.4

curate front-tracking approximation (still called $u$) satisfying an estimate entirely analogous to (2.27), namely,

$$\sum_{i=1}^{n}\int_0^{\tau}\Bigg(\sum_{\alpha,\beta\in\mathcal{R}_i,\,|x_\alpha-x_\beta|\le 8\sqrt{\varepsilon}}|\sigma_\alpha\sigma_\beta|\Bigg)dt = \mathcal{O}(1)\cdot(\ln(2+\tau)+|\ln\varepsilon|)\sqrt{\varepsilon}\cdot\text{Tot. Var.}\{\bar{u}\}\,, \tag{2.30}$$

where we replace the $\delta$ in (2.27) by $8\sqrt{\varepsilon}$ for the application in Section 4. Here $\mathcal{R}_i$ denotes the set of rarefaction fronts of the $i^{\text{th}}$ family, and summation is over all possible pairs, including the case where the two indices $\alpha$ and $\beta$ coincide.

## 3 Estimates on Shock Fronts

We begin by estimating the sum in (1.9). The approximation $v$ is discontinuous precisely at those times $t_i$ when an interaction occurs involving a large shock. Indeed, at such times the left and right states $u_\alpha^-$ and $u_\alpha^+$ across a large shock located at $x = x_\alpha$ suddenly change. As a consequence, the viscous shock profile connecting these two states is modified. The two smooth functions $v(t_i-)$ and $v(t_i)$ will thus be different over the interval $[x_\alpha - \sqrt{\varepsilon}\,, x_\alpha + \sqrt{\varepsilon}]$. To estimate the $\mathbf{L}^1$ norm of this difference, the following elementary observation is useful: Given a smooth function $\phi = \phi(\sigma, \sigma')$, its size satisfies the bounds

$$\text{if}\quad \phi(\sigma,0)=0\ \text{ for all }\sigma,\quad \text{then}\quad \phi(\sigma,\sigma')=\mathcal{O}(1)\cdot|\sigma'|\,,$$
$$\text{if}\quad \phi(\sigma,0)=\phi(0,\sigma')=0\ \text{ for all }\sigma,\sigma',\quad \text{then}\quad \phi(\sigma,\sigma')=\mathcal{O}(1)\cdot|\sigma\sigma'|\,.$$

We now distinguish various cases.

*Case* 1. At time $t_i$ a new large shock is created, say of strength $|\sigma_\alpha| \ge \rho/2$. In this case, since the new viscous shock profile is inserted on an interval of length $2\sqrt{\varepsilon}$, we have

$$\|v(t_i) - v(t_i-)\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\sqrt{\varepsilon}\,|\sigma_\alpha|\,.$$

According to our construction, every large shock not present at time $t = 0$ must grow from a strength $< \rho/2$ up to a strength $\ge \rho$ at some later time $\tau$. Therefore, the sum of the strengths of all large shocks, at the time $t_i$ when they are created, is $\mathcal{O}(1)\cdot\delta_0$, where $\delta_0 \doteq \text{Tot. Var.}\{\bar{u}\}$. The total contribution due to these terms is thus $\mathcal{O}(1)\cdot\sqrt{\varepsilon}\,\delta_0$.

*Case* 2. At time $t_i$ a large shock is terminated. Since every large shock must have strength $\geq \rho$ at some time and is terminated when its strength becomes $< \rho/2$, every such case involves an amount of interaction and cancellation $\geq \rho/2$. Therefore, the total contribution of these terms to the sum in (1.9) is again $\mathcal{O}(1) \cdot \sqrt{\varepsilon}\,\delta_0$.

*Case* 3. A front $\sigma_\beta$ of a different family crosses one large shock $\sigma_\alpha$. In this case we have

$$\|v(t_i) - v(t_i-)\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \sqrt{\varepsilon}\,|\sigma_\alpha|\,|\sigma_\beta|\,.$$

These terms are thus controlled by the decrease in the interaction potential $Q(u)$. Their total sum is $\mathcal{O}(1) \cdot \sqrt{\varepsilon}\,\delta_0^2$.

*Case* 4. A small front $\sigma_\beta$ of the same family impinges on the large shock $\sigma_\alpha$. In this case we have

$$\|v(t_i) - v(t_i-)\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \sqrt{\varepsilon}\,|\sigma_\beta|\,.$$

Since any small front can join at most one large shock of the same family, the total contribution of these terms is $\mathcal{O}(1) \cdot \sqrt{\varepsilon}\,\delta_0$.

*Case* 5. Two large $k$-shocks of the same family, say of strengths $\sigma_\alpha$ and $\sigma_\beta$, merge together. In this case

$$\|v(t_i) - v(t_i-)\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \sqrt{\varepsilon}\,\min\{|\sigma_\alpha|, |\sigma_\beta|\}\,.$$

As will be shown in (3.23), all these interactions are controlled by the decrease in a suitable functional $Q^\sharp(u)$ by noticing that $|\sigma_\alpha|, |\sigma_\beta| > 2\sqrt{\varepsilon}\,|\ln\varepsilon|$. The sum of all these terms is thus found to be $\mathcal{O}(1) \cdot \delta_0\sqrt{\varepsilon}\,|\ln\varepsilon|$.

Putting together all these five cases, one obtains the bound (1.9).

Next, we need to estimate the running error in (1.8) related to the big shocks, namely

$$E_{\mathcal{BS}} \doteq \int_0^\tau \sum_{\alpha\in\mathcal{BS}(t)} \int_{x_\alpha-\sqrt{\varepsilon}}^{x_\alpha+\sqrt{\varepsilon}} |v_t + A(v)v_x - \varepsilon v_{xx}|\,dx\,dt\,. \tag{3.1}$$

Here the summation ranges over all big shocks in $v(t,\cdot)$.

We first consider the simplest case, where the interval

$$I_\alpha(t) \doteq \big[x_\alpha(t) - 2\sqrt{\varepsilon},\, x_\alpha(t) + 2\sqrt{\varepsilon}\,\big] \tag{3.2}$$

does not contain any other wave front. In this case, observing that

$$\big[A(\omega_\alpha^\varepsilon(s)) - \lambda_\alpha\big]\frac{\partial}{\partial s}\omega_\alpha^\varepsilon(s) - \varepsilon\frac{\partial^2}{\partial s^2}\omega_\alpha^\varepsilon(s) = 0\,,$$

and recalling (1.19)–(1.20), the error relative to the shock at $x_\alpha$ can be written as

$$E_\alpha(t) = \left( \int_{-\sqrt{\varepsilon}}^{-\sqrt{\varepsilon}/2} + \int_{\sqrt{\varepsilon}/2}^{\sqrt{\varepsilon}} \right) \Big\{ \big[A\big(\omega_\alpha^\varepsilon(\phi(\xi))\big) - \lambda_\alpha\big] \frac{\partial}{\partial s}\omega_\alpha^\varepsilon(\phi(\xi))\phi'(\xi) - \varepsilon \frac{\partial}{\partial s}\omega_\alpha^\varepsilon(\phi(\xi))\phi''(\xi) - \varepsilon \frac{\partial^2}{\partial s^2}\omega_\alpha^\varepsilon(\phi(\xi))(\phi'(\xi))^2 \Big\}\, d\xi\,. \tag{3.3}$$

Using the bounds

$$\left|\frac{\partial}{\partial s}\omega_\alpha^\varepsilon(s)\right| = \mathcal{O}(1)\cdot \frac{|\sigma_\alpha|^2}{\varepsilon} e^{-|s\sigma_\alpha|/\varepsilon}\,, \qquad \left|\frac{\partial^2}{\partial s^2}\omega_\alpha^\varepsilon(s)\right| = \mathcal{O}(1)\cdot \frac{|\sigma_\alpha|^3}{\varepsilon^2} e^{-|s\sigma_\alpha|/\varepsilon}\,, \tag{3.4}$$

from (1.20) we deduce

$$\begin{aligned} E_\alpha(t) &= \mathcal{O}(1)\cdot \int_{\sqrt{\varepsilon}/2}^{\sqrt{\varepsilon}} \exp\left\{-\frac{|\sigma_\alpha|}{\varepsilon}\phi(\xi)\right\}\cdot \left(\frac{|\sigma_\alpha|^2}{\varepsilon}\phi'(\xi) + \frac{|\sigma_\alpha|^3}{\varepsilon}\phi''(\xi)\right) d\xi \\ &= \mathcal{O}(1)\cdot \int_{\sqrt{\varepsilon}/2}^{\sqrt{\varepsilon}} \exp\left\{-\frac{|\sigma_\alpha|}{4(\sqrt{\varepsilon}-\xi)}\right\} \frac{|\sigma_\alpha|^3}{(\sqrt{\varepsilon}-\xi)^3}\, d\xi \\ &= \mathcal{O}(1)\cdot \int_{2/\sqrt{\varepsilon}}^{\infty} \exp\left\{-\frac{|\sigma_\alpha| s}{4}\right\} |\sigma_\alpha|^3 s^3 \frac{ds}{s^2} \\ &= \mathcal{O}(1)\cdot |\sigma_\alpha| \exp\left\{-\frac{|\sigma_\alpha|}{2\sqrt{\varepsilon}}\right\}\left(1 + \frac{2|\sigma_\alpha|}{\sqrt{\varepsilon}}\right). \end{aligned}$$

Since by assumption $|\sigma_\alpha| \geq \rho/2 = 2\sqrt{\varepsilon}\,|\ln\varepsilon|$, the above estimate implies

$$E_\alpha(t) = \mathcal{O}(1)\cdot \varepsilon(1 + |\ln\varepsilon|)|\sigma_\alpha|\,. \tag{3.5}$$

In the general case, our error estimate must also take into account the presence of other wave fronts within the intervals $I_\alpha(t)$. Indeed, for every point $x_\alpha$ where a large shock is located, we have

$$\begin{aligned} E_\alpha(t) &\doteq \int_{x_\alpha-\sqrt{\varepsilon}}^{x_\alpha+\sqrt{\varepsilon}} |v_t + A(v)v_x - \varepsilon v_{xx}|\,dx\,dt \\ &= \mathcal{O}(1)\cdot \varepsilon(1 + |\ln\varepsilon|)|\sigma_\alpha| \\ &\quad + \mathcal{O}(1)\Bigg(\sum_{x_\beta, x_\gamma \in I_\alpha(t), |x_\beta - x_\gamma| \leq 2\sqrt{\varepsilon}} |\sigma_\beta\sigma_\gamma| - \sum_{x_\theta \in I_\alpha(t), \theta\in\mathcal{BS}} |\sigma_\theta|^2\Bigg). \end{aligned} \tag{3.6}$$

In the following, we introduce three different functionals, which account for

- products $|\sigma_\alpha\sigma_\beta|$ of fronts of different families,
- products $|\sigma_\alpha\sigma_\beta|$ where $\sigma_\alpha$ is a large shock and $\sigma_\beta$ is a rarefaction of the same family, and
- products $|\sigma_\alpha\sigma_\beta|$ of shocks of the same family.

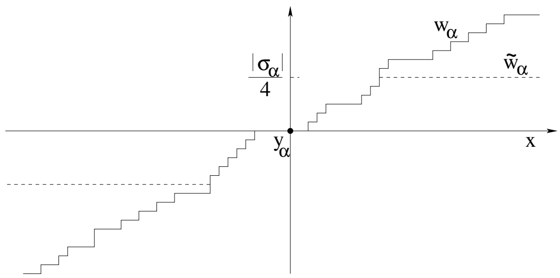


FIGURE 3.1

By combining these three, we form a functional $\widehat{Q}(u)$ such that the map $t \mapsto \widehat{Q}(u(t))$ is nonincreasing except at times where a new large shock is introduced. Moreover, the total increase in this functional at times where large shocks are created will be shown to be $\mathcal{O}(1) \cdot \sqrt{\varepsilon}\,|\ln \varepsilon|$ Tot. Var.$\{\bar{u}\}$. We begin by defining

$$Q^\flat(u) \doteq \sum_{k_\beta \neq k_\alpha} W^\flat_{\alpha\beta} |\sigma_\alpha \sigma_\beta| \,. \tag{3.7}$$

where the sum extends over all couples of fronts of different families (small shocks, big shocks, rarefactions). The weights $W^\flat_{\alpha\beta} \in [0, 1]$ are defined as follows: If $k_\beta < k_\alpha$, then

$$W^\flat_{\alpha\beta} \doteq \begin{cases} 0 & \text{if } x_\beta < x_\alpha - 2\sqrt{\varepsilon} \\ \frac{1}{2} + \frac{x_\beta - x_\alpha}{4\sqrt{\varepsilon}} & \text{if } x_\beta \in [x_\alpha - 2\sqrt{\varepsilon}\,, x_\alpha + 2\sqrt{\varepsilon}\,] \\ 1 & \text{if } x_\beta > x_\alpha + 2\sqrt{\varepsilon}\,. \end{cases}$$

If instead $k_\beta > k_\alpha$, we set

$$W^\flat_{\alpha\beta} \doteq \begin{cases} 1 & \text{if } x_\beta < x_\alpha - 2\sqrt{\varepsilon} \\ \frac{1}{2} - \frac{x_\beta - x_\alpha}{4\sqrt{\varepsilon}} & \text{if } x_\beta \in [x_\alpha - 2\sqrt{\varepsilon}\,, x_\alpha + 2\sqrt{\varepsilon}\,] \\ 0 & \text{if } x_\beta > x_\alpha + 2\sqrt{\varepsilon}\,. \end{cases}$$

By strict hyperbolicity, we expect that the functional $Q^\flat$ will be decreasing in time. Indeed, its rate of decrease dominates the sum

$$\sum_{k_\alpha \neq k_\beta,\ |x_\alpha - x_\beta| < 2\sqrt{\varepsilon}} |\sigma_\alpha \sigma_\beta| \,,$$

containing products of nearby waves of different families.

Next, given a big shock $\sigma_\alpha$ of the $k_\alpha^{\text{th}}$ family located at $x_\alpha$, we write

$\mathcal{R}_\alpha$ to denote the set of all rarefaction fronts of the same family $k_\alpha$,

$\mathcal{S}_\alpha$ to denote the set of all shock fronts of the same family $k_\alpha$.

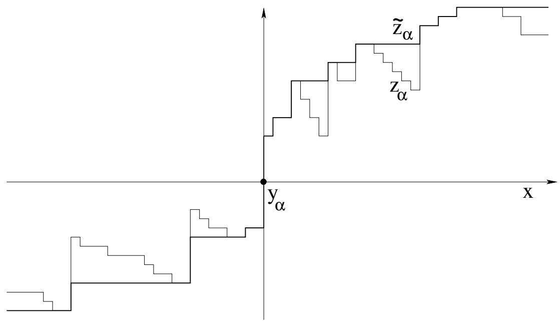


FIGURE 3.2

To control the interaction between large shocks and rarefactions of the same family, we define the weight

$$W_\alpha^\natural(x) \doteq \min\left\{\frac{1}{2} + \frac{|x - x_\alpha|}{4\sqrt{\varepsilon}}\,,1\right\} \tag{3.8}$$

and the function (Figure 3.1)

$$w_\alpha(x) \doteq \begin{cases} \sum_{\beta\in\mathcal{R}_\alpha,\, x_\beta\in[x,\,x_\alpha]}(-\sigma_\beta) & \text{if } x < x_\alpha \\ \sum_{\beta\in\mathcal{R}_\alpha,\, x_\beta\in[x_\alpha,\,x]}\sigma_\beta & \text{if } x > x_\alpha. \end{cases}$$

Calling

$$\tilde{w}_\alpha(x) \doteq \begin{cases} -\frac{|\sigma_\alpha|}{4} & \text{if } w_\alpha(x) < -\frac{|\sigma_\alpha|}{4} \\ w_\alpha(x) & \text{if } |w_\alpha(x)| \le \frac{|\sigma_\alpha|}{4} \\ \frac{|\sigma_\alpha|}{4} & \text{if } w_\alpha(x) > \frac{|\sigma_\alpha|}{4}, \end{cases}$$

we then define

$$Q^\natural(u) \doteq \sum_{\alpha\in\mathcal{BS}} \int W_\alpha^\natural(x) D_x \tilde{w}_\alpha\,. \tag{3.9}$$

By using the function with cutoff $\tilde{w}_\alpha$ instead of $w_\alpha$, in (3.9) we are taking into account only the rarefaction fronts $\sigma_\beta$ of the same family $k_\alpha$ such that the total amount of rarefactions inside the interval $[x_\alpha, x_\beta]$ is $\le |\sigma_\alpha|/4$. If no other fronts of different families are present, this guarantees that all these rarefactions $\sigma_\beta$ are strictly approaching the big shock $\sigma_\alpha$. Indeed, the difference in speed is $|\dot{x}_\beta - \dot{x}_\alpha| \ge |\sigma_\alpha|/4$. As a result, the functional $Q^\natural(u)$ will be strictly decreasing. On the other hand, if the interval $[x_\alpha, x_\beta]$ also contains waves of different families, the above estimate may fail. In this case, however, the decrease in the functional $Q^\flat(u)$ compensates the possible increase in $Q^\natural(u)$.

Finally, to control the interactions among shocks of the same family, for each shock front $\sigma_\alpha$ (of any size, big or small) located at $x_\alpha$, we begin by defining (Figure 3.2)

$$z_\alpha(x) \doteq \begin{cases} -\frac{|\sigma_\alpha|}{2} - \sum_{\beta\in\mathcal{S}_\alpha, x<x_\beta<x_\alpha} |\sigma_\beta| + \sum_{\beta\in\mathcal{R}_\alpha, x<x_\beta<x_\alpha} 3\sigma_\beta & \text{if } x < x_\alpha \\ \frac{|\sigma_\alpha|}{2} + \sum_{\beta\in\mathcal{S}_\alpha, x_\alpha<x_\beta<x} |\sigma_\beta| - \sum_{\beta\in\mathcal{R}_\alpha, x_\alpha<x_\beta<x} 3\sigma_\beta & \text{if } x > x_\alpha. \end{cases}$$

Then we set

$$\tilde{z}_\alpha(x) = \begin{cases} \min\{z_\alpha(x') : x < x' < x_\alpha\} & \text{if } x < x_\alpha \\ \max\{z_\alpha(x') : x_\alpha < x' < x\} & \text{if } x > x_\alpha. \end{cases}$$

Notice that $\tilde{z}_\alpha$ is a nondecreasing, piecewise constant function, with $(x - x_\alpha)\tilde{z}_\alpha(x) > 0$ for $x \neq x_\alpha$.

Using the weights

$$W_\alpha^\sharp(x) \doteq \begin{cases} [\varepsilon - \tilde{z}_\alpha(x-)]^{-1} & \text{if } x < x_\alpha \\ [\varepsilon + \tilde{z}_\alpha(x+)]^{-1} & \text{if } x > x_\alpha, \end{cases}$$

we now define

$$Q^\sharp(u) \doteq \sum_{\alpha\in\mathcal{S}} |\sigma_\alpha| \int W_\alpha^\natural(x)\, W_\alpha^\sharp(x) D_x\tilde{z}_\alpha\,. \tag{3.10}$$

Notice that in this case the summation runs over all shock fronts. If $\sigma_\beta$ is a shock located at $x_\beta$, then $[W_\alpha^\sharp(x_\beta)]^{-1}$ roughly describes the amount of shock waves inside the interval $[x_\alpha, x_\beta]$ in excess of three times the amount of rarefactions. If the interval $[x_\alpha, x_\beta]$ does not contain waves of other families and the function $x \mapsto \tilde{z}_\alpha(x)$ has a jump at $x = x_\beta$, then the two shocks $\sigma_\alpha$ and $\sigma_\beta$ are strictly approaching; hence the functional $Q^\sharp(u)$ will decrease. On the other hand, if waves of different families are present, the above estimate may fail. In this case, however, the decrease in the functional $Q^\flat(u)$ compensates the possible increase in $Q^\sharp(u)$.

In the definition of $z_\alpha$, notice that the strength of rarefactions is multiplied by 3 to make sure that couples of shocks $\sigma_\alpha$ and $\sigma_\beta$ entering the definition of $Q^\sharp(u)$ are always approaching each other (except for the presence of fronts of different families in between). An example is shown in Figure 3.3, where two nearby shocks move apart from each other because there are sufficiently many rarefaction waves in the middle. Because of the factor 3, the function $x \mapsto \tilde{z}_\alpha(x)$ will be constant at the point $x_\beta$. Hence the product $|\sigma_\alpha\sigma_\beta|$ will not appear within the definition of $Q^\sharp(u)$.

We now consider the composite functional

$$\widehat{Q}(u) \doteq \sqrt{\varepsilon}\,|\ln\varepsilon| \cdot \big(C_1\Upsilon(u) + C_2Q^\flat(u) + C_3Q^\natural(u)\big) + \sqrt{\varepsilon}\, Q^\sharp(u)\,. \tag{3.11}$$

Here

$$\Upsilon(u) \doteq V(u) + C_0Q(u) \tag{3.12}$$

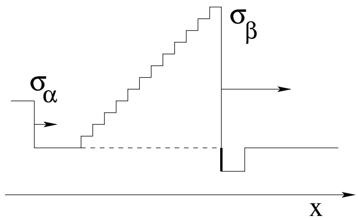


FIGURE 3.3

is a quantity that is decreasing at every interaction time. Its decrease dominates both the amount of interaction and of cancellation in the front-tracking solution $u$. Observe that

$$\widehat{Q}(u) = \mathcal{O}(1) \cdot \sqrt{\varepsilon}\,|\ln \varepsilon|\ \text{Tot. Var.}\{u\}\,. \tag{3.13}$$

Indeed, by the definition of $W_\alpha^\sharp(x)$, we have

$$\int W_\alpha^\sharp(x) D_x \tilde{z}_\alpha = \mathcal{O}(1) \cdot \int_0^{\text{Tot.Var.}\{u\}} \frac{1}{s+\varepsilon}\,ds = \mathcal{O}(1) \cdot |\ln \varepsilon|\,. \tag{3.14}$$

Using (3.14), it is now clear that

$$\begin{aligned} Q^\sharp(u) &= \mathcal{O}(1) \cdot |\ln \varepsilon|\,\text{Tot. Var.}\{u\}\,,\\ \Upsilon(u),\ Q^\flat(u),\ Q^\natural(u) &= \mathcal{O}(1) \cdot \text{Tot. Var.}\{u\}\,. \end{aligned} \tag{3.15}$$

The bound on (3.11) now follows from (3.15).

LEMMA 3.1 *For a suitable choice of the constants* $C_1 \gg C_2 \gg C_3 \gg 1$, *if* Tot. Var.$\{u\}$ *remains small, then at each time* $t^*$ *where an interaction occurs, the following holds*: *If a new large shock of strength* $|\sigma_\alpha| > 2\sqrt{\varepsilon}\,|\ln \varepsilon|$ *is created, then*

$$\Delta\widehat{Q} \doteq \hat{Q}(\tau+) - \hat{Q}(\tau-) = \mathcal{O}(1) \cdot \sqrt{\varepsilon}\,|\ln \varepsilon|\,|\sigma_\alpha|\,. \tag{3.16}$$

*If no large shock is created, then*

$$\Delta\widehat{Q} \le 0\,. \tag{3.17}$$

PROOF: Notice that the weight $W_{\alpha,\beta}^\flat$ is always $\le 1$. For a newly created large shock $\sigma_\alpha$, the increase in the functional $Q^\flat(u)$ can be estimated as

$$\Delta Q^\flat(u) = \mathcal{O}(1) \cdot |\sigma_\alpha|\ \text{Tot. Var.}\{u\}\,. \tag{3.18}$$

Similarly, since $W_\alpha^\natural \le 1$, it is clear that the increase of $Q^\natural(u)$ due to a new large shock $\sigma_\alpha$ is

$$\Delta Q^\natural(u) = \mathcal{O}(1) \cdot |\sigma_\alpha|\,. \tag{3.19}$$

The estimate on the increase of the functional $Q^\sharp(u)$ is different. In this case, the integral

$$\int W_\alpha^\natural(x) W_\alpha^\sharp(x) D_x \tilde{z}_\alpha$$

is bounded by

$$\mathcal{O}(1) \cdot \int_0^{\text{Tot.Var.}\{u\}} \frac{1}{\varepsilon + x}\, dx = \mathcal{O}(1)|\ln \varepsilon| .$$

Hence,

$$\Delta Q^\sharp(u) = \mathcal{O}(1) \cdot |\sigma_\alpha|\, |\ln \varepsilon| . \tag{3.20}$$

Together, (3.18)–(3.20) imply (3.16).

Next, we prove (3.17). Assume that at time $t^*$ an interaction occurs without the introduction of any new large shock. We will show that the functional $\widehat{Q}(u(t))$ decreases.

First we look at the change in $Q^\flat(u)$ and $Q^\natural(u)$. Since the weights $W^\flat$ and $W^\natural$ are uniformly bounded, it is straightforward to check that the change in these two functionals at time $t^*$ is bounded by a constant times the decrease in the Glimm functional $\Upsilon(u(t))$ in (3.12). Hence, by choosing $C_1 \ll C_2 \ll C_3$, the quantity

$$C_1 \Upsilon(u) + C_2 Q^\flat(u) + C_3 Q^\natural(u)$$

is not increasing in time.

The analysis of $Q^\sharp(u)$ is a bit harder. We will show that the change of $Q^\sharp(u)$ at the interaction time $t^*$ is of the same order of magnitude as $|\ln \varepsilon||\Delta\Upsilon(u)|$. Here and in the following, $\Delta\Upsilon$ denotes the change in $\Upsilon(u(t))$ across the interaction time. As a preliminary, we notice a basic property of the weight function $W^\sharp_\alpha(x)$. For any fixed location $x = x_0$, we have

$$\sum_{\alpha\in S} |\sigma_\alpha| W^\sharp_\alpha(x_0) = \mathcal{O}(1) \cdot |\ln \varepsilon| , \tag{3.21}$$

$$\sum_{\alpha\in S,\, x(\alpha)<x_0} |\sigma_\alpha| \int_{x_0}^{\infty} \big(W^\sharp_\alpha(x)\big)^2 D_x \tilde{z}_\alpha + \sum_{\alpha\in S,\, x(\alpha)>x_0} |\sigma_\alpha| \int_{-\infty}^{x_0} \big(W^\sharp_\alpha(x)\big)^2 D_x \tilde{z}_\alpha = \mathcal{O}(1) \cdot |\ln \varepsilon| . \tag{3.22}$$

The proof of the estimates in (3.21) and (3.22) is straightforward by noticing that the functions $f(x) = 1/(x+\varepsilon)$ and $g(x) = 1/(x+\varepsilon)^2$ are convex and bounded away from zero for $x \geq 0$; the left-hand sides of (3.21) and (3.22) are bounded by the following single and double integrals, respectively:

$$\int_0^{\text{Tot.Var.}\{u\}} f(x)dx = \mathcal{O}(1)|\ln \varepsilon| ,$$

$$\int_0^{\text{Tot.Var.}\{u\}} \int_x^{\text{Tot.Var.}\{u\}} g(y)dy\, dx = \mathcal{O}(1) \cdot |\ln \varepsilon| .$$

Now we are ready to estimate the change in $Q^\sharp(u(t))$ at time $t^*$. Note that, in some cases, it is possible that the interaction does not change the functional. In the following, we will consider the case where $Q^\sharp(u)$ does change across the interaction. Depending on the types and families of the waves involved in the interaction, we have the following four cases:

*Case* 1: *Two shocks of the same family interact.* Let $\beta_1$ and $\beta_2$ be the two interacting shocks, say of the $i^{\text{th}}$ family, and call $\beta$ the outgoing $i$-shock. We also let $\alpha_1$ and $\alpha_2$ be any two shock fronts on the left and right of the interaction point, respectively, so that $x_{\alpha_1} < x_\beta < x_{\alpha_2}$ at time $t = t^*$. For any shock front $\sigma_\alpha$ at time $t^*$, set

$$Q^\sharp_\alpha = |\sigma_\alpha| \int W^\natural_\alpha(x) W^\sharp_\alpha(x) D_x \tilde{z}_\alpha \,.$$

Observe that

$$Q^\sharp_\beta - \big(Q^\sharp_{\beta_1} + Q^\sharp_{\beta_2}\big) \leq -\frac{|\sigma_{\beta_1}\sigma_{\beta_2}|}{|\sigma_{\beta_1} + |\sigma_{\beta_2}| + \varepsilon} + \mathcal{O}(1) \cdot |\ln \varepsilon|\, |\Delta\Upsilon| \,. \tag{3.23}$$

Indeed,

$$\sigma_\beta = \sigma_{\beta_1} + \sigma_{\beta_2} + \mathcal{O}(1) \cdot |\Delta\Upsilon| \,.$$

Moreover, recalling (3.8), we see that after the interaction we lose the term

$$W^\natural_{\beta_1}(x_{\beta_2})\big(W^\sharp_{\beta_1}(x_{\beta_2}) + W^\sharp_{\beta_2}(x_{\beta_1})\big)|\sigma_{\beta_1}\sigma_{\beta_2}| \geq \frac{|\sigma_{\beta_1}\sigma_{\beta_2}|}{|\sigma_{\beta_1}| + |\sigma_{\beta_2}| + \varepsilon} \,.$$

Notice that $W^\sharp_{\alpha_1}(x)$ (respectively, $W^\sharp_{\alpha_2}(x)$) does not change across the interaction for $x < x_\beta$ ($x > x_\beta$). The change in the $Q^\sharp_{\alpha_i}$, $i = 1, 2$, can be estimated as follows: when $\alpha_i$ is of the same family of $\beta_j$, $i$, $j = 1, 2$, by (3.22) we have

$$\begin{aligned}
\Delta Q^\sharp_{\alpha_1} = &\bigg(\frac{|\sigma_{\alpha_1}\sigma_\beta|}{\varepsilon + |\sigma_{\alpha_1}| + |\sigma_\beta| + I} - \frac{|\sigma_{\alpha_1}\sigma_{\beta_1}|}{\varepsilon + |\sigma_{\alpha_1}| + |\sigma_{\beta_1}| + I} \\
&\quad - \frac{|\sigma_{\alpha_1}\sigma_{\beta_2}|}{\varepsilon + |\sigma_{\alpha_1}| + |\sigma_{\beta_1}| + |\sigma_{\beta_2}| + I}\bigg) W^\natural_\alpha(x(\beta)) \\
&\quad + \mathcal{O}(1) \cdot |\Delta\Upsilon|\, |\sigma_{\alpha_1}| \int_{x_\beta}^\infty \big(W^\sharp_{\alpha_1}\big)^2(x) D_x \tilde{z}_{\alpha_1} \\
\leq\ & \mathcal{O}(1) \cdot |\sigma_{\alpha_1}|\, |\Delta\Upsilon| W^\sharp_{\alpha_1}(x_\beta) + \mathcal{O}(1) \cdot |\Delta\Upsilon|\, |\sigma_{\alpha_1}| \int_{x_\beta}^\infty \big(W^\sharp_{\alpha_1}\big)^2(x) D_x \tilde{z}_{\alpha_1} \,.
\end{aligned}$$

Here and in the following, we assume that the whole strength of $\beta_i$, $i = 1, 2$, and of $\beta$ appear in the functional $Q^\sharp_{\alpha_1}$. Moreover, $I$ represents the sum of the strengths of the $i$-shocks between $\beta_1$ and $\alpha_1$ that appear in $Q^\sharp_{\alpha_1}$. The other cases when part or none of the above wave strengths appears in $Q^\sharp_{\alpha_1}$ can be treated similarly.

By summing over $\alpha_1$ and using (3.21) and (3.22), we find that the total change of $Q^\sharp_{\alpha_1}$ is $\mathcal{O}(1) \cdot |\ln \varepsilon||\Delta\Upsilon|$. A similar estimate holds for $\alpha_2$.

Now consider two shock fronts $\alpha_1$ and $\alpha_2$ of the $j^{\text{th}}$ family, with $j \neq i$. Notice that the change of the weight function $W^\sharp_{\alpha_i}(x)$, $i = 1, 2$, is at most of the order of $(W^\sharp_{\alpha_i}(x))^2|\Delta\Upsilon|$ when $x$ lies on the opposite side of $x_{\alpha_i}$ with respect to $x_\beta$. Together with (3.22) this yields

$$\sum_{\alpha_i,\, i=1,2} |\Delta Q^\sharp_{\alpha_i}| = \mathcal{O}(1)\cdot\Bigg(\sum_{\alpha_1}|\Delta\Upsilon|\,|\sigma_{\alpha_1}|\int_{x_\beta}^{\infty}\big(W^\sharp_{\alpha_1}\big)^2(x)D_x\tilde z_{\alpha_1} + \sum_{\alpha_2}|\Delta\Upsilon|\,|\sigma_{\alpha_2}|\int_{-\infty}^{x_\beta}\big(W^\sharp_{\alpha_2}\big)^2(x)D_x\tilde z_{\alpha_2}\Bigg) = \mathcal{O}(1)\cdot|\ln\varepsilon|\,|\Delta\Upsilon|\,.$$

If $\gamma$ is a newly created shock of the $j^{\text{th}}$ family, then the new term $Q^\sharp_\gamma$ has size $\mathcal{O}(1)\cdot|\sigma_\gamma|\,|\ln\varepsilon|$. Hence, the total sum of these new terms over $\gamma$ is of $\mathcal{O}(1)|\ln\varepsilon||\Delta\Upsilon|$; this completes the discussion on this case.

*Case* 2: *Interaction of a shock with a rarefaction front of the same family*. Let $\beta_1$ and $\beta_2$ be a shock and a rarefaction front of the $i^{\text{th}}$ family, interacting at time $t^*$.

First, consider the case where the shock $\beta_1$ is completely cancelled, and hence the decrease in $\Upsilon(u)$ is of the same order as $\beta_1$. In this case the term $Q^\sharp_{\beta_1}$ disappears after the interaction. Let $\alpha_1$ and $\alpha_2$ be shock waves of the $j^{\text{th}}$ family on the left and right of the location of interaction. For both cases when $i=j$ and $i\neq j$, by (3.22) we have

$$\sum_{\alpha_i,\, i=1,2} |\Delta Q^\sharp_{\alpha_i}| = \mathcal{O}(1)\cdot\Bigg(\sum_{\alpha_1}|\Delta\Upsilon|\,|\sigma_{\alpha_1}|\int_{x_\beta}^{\infty}(W^\sharp_{\alpha_1})^2(x)D_x\tilde z_{\alpha_1} + \sum_{\alpha_2}|\Delta\Upsilon|\,|\sigma_{\alpha_2}|\int_{-\infty}^{x_\beta}\big(W^\sharp_{\alpha_2}\big)^2(x)D_x\tilde z_{\alpha_2}\Bigg) = \mathcal{O}(1)\cdot|\ln\varepsilon|\,|\Delta\Upsilon|\,.$$

The same argument applies to the change in $Q^\sharp_\gamma$, related to the newly created shock $\gamma$ of the $j^{\text{th}}$ family, when $j\neq i$. In this case, the total change in $Q^\sharp$ is again $\mathcal{O}(1)\cdot|\ln\varepsilon|\,|\Delta\Upsilon|$.

In the case where the interaction produces an outgoing $i$-shock $\bar\beta_1$ so that the rarefaction $\beta_2$ is completely cancelled, the analysis is as follows: First, notice that the increase in $Q^\sharp(u)$ due to the newly created waves is $\mathcal{O}(1)\cdot|\ln\varepsilon|\,|\Delta\Upsilon|$, with $|\Delta\Upsilon| = \mathcal{O}(1)\cdot|\sigma_{\beta_2}|$.

Next, the difference between $Q^\sharp_{\bar\beta_1}$ and $Q^\sharp_{\beta_1}$ comes from the changes in $(W^\sharp_{\beta_1})^{-1}$ and $\tilde z_{\beta_1}$, which are at most of the order of $\sigma_{\beta_2}$ at each $x$. Hence

$$\begin{aligned} Q^\sharp_{\bar\beta_1} - Q^\sharp_{\beta_1} &\le |\sigma_{\bar\beta_1}|\Bigg(\int W^\natural_{\bar\beta_1}(x)W^\sharp_{\bar\beta_1}(x)D_x\tilde z_{\bar\beta_1} - \int W^\natural_{\beta_1}(x)W^\sharp_{\beta_1}(x)D_x\tilde z_{\beta_1}\Bigg)\\ &= \mathcal{O}(1)\cdot|\sigma_{\beta_2}|\,|\sigma_{\bar\beta_1}|\int_0^{\text{Tot.Var.}\{u\}}\frac{dy}{(\varepsilon+|\sigma_{\bar\beta_1}|+y)^2}\\ &\quad+\mathcal{O}(1)\cdot|\sigma_{\bar\beta_1}|\int_0^{|\sigma_{\beta_2}|}\frac{dy}{\varepsilon+|\sigma_{\bar\beta_1}|+y}\\ &= \mathcal{O}(1)\cdot|\sigma_{\beta_2}| = \mathcal{O}(1)\cdot|\Delta\Upsilon|\,. \end{aligned}$$

The change in $Q^\sharp_{\alpha_i}$ also comes from the change in $W^\sharp_{\alpha_i}(x)$ and $\tilde z_{\alpha_i}(x)$. Since the weight $W^\sharp_{\alpha_i}(x)$ decreases as $x$ moves away from $x_{\alpha_i}$, we have

$$\begin{aligned}\sum_{\alpha_i,\, i=1,2} \left|\Delta Q^\sharp_{\alpha_i}\right| = \mathcal{O}(1)\cdot \Bigg( &\sum_{\alpha_1} |\sigma_{\beta_2}|\,|\sigma_{\alpha_1}| \int_{x_\beta}^{\infty} \big(W^\sharp_{\alpha_1}\big)^2(x) D_x \tilde z_{\alpha_1} \\ &+ \sum_{\alpha_2} |\sigma_{\beta_2}|\,|\sigma_{\alpha_2}| \int_{-\infty}^{x_\beta} \big(W^\sharp_{\alpha_2}\big)^2(x) D_x \tilde z_{\alpha_2}\Bigg) \\ &+ \mathcal{O}(1)\cdot|\sigma_{\beta_2}|\Bigg(\sum_{\alpha_1}|\sigma_{\alpha_1}| W^\sharp_{\alpha_1}(x_{\beta_1}) + \sum_{\alpha_2}|\sigma_{\alpha_2}| W^\sharp_{\alpha_2}(x_{\beta_1})\Bigg) \\ &= \mathcal{O}(1)\cdot|\ln\varepsilon|\,|\Delta\Upsilon|\,.\end{aligned}$$

*Case* 3: *Interaction of a shock and a rarefaction front of different families*. To fix the ideas, let $\beta_1$ be a shock of the $i^{\text{th}}$ family and $\beta_2$ be a rarefaction wave of the $j^{\text{th}}$ family with $i > j$. Assume $\beta_1$ and $\beta_2$ interact at time $t^*$ and denote the outgoing wave of the $i^{\text{th}}$ family by $\bar\beta_1$, and the $j^{\text{th}}$ family wave $\bar\beta_2$. Moreover, let $\gamma$ be a newly created shock front of the $k^{\text{th}}$ family, $k \neq i, j$. By a standard interaction estimate, we have

$$|\sigma_{\beta_i} - \sigma_{\bar\beta_i}| = \mathcal{O}(1)\cdot|\Delta\Upsilon|\,, \quad |\sigma_\gamma| = \mathcal{O}(1)\cdot|\Delta\Upsilon|\,, \quad i = 1, 2.$$

Thus, if we consider two shock waves $\alpha_i$, $i = 1, 2$, of the $k^{\text{th}}$ family located on the left and right of the interaction point, respectively, as in the analysis of Case 2, we have

$$\begin{aligned}\sum_{\alpha_i,\, i=1,2} \left|\Delta Q^\sharp_{\alpha_i}\right| = \mathcal{O}(1)\cdot \Bigg( &\sum_{\alpha_1} |\Delta\Upsilon|\,|\sigma_{\alpha_1}| \int_{x_\beta}^{\infty} \big(W^\sharp_{\alpha_1}\big)^2(x) D_x \tilde z_{\alpha_1} \\ &+ \sum_{\alpha_2} |\Delta\Upsilon|\,|\sigma_{\alpha_2}| \int_{-\infty}^{x_\beta} \big(W^\sharp_{\alpha_2}\big)^2(x) D_x \tilde z_{\alpha_2}\Bigg) \\ &= \mathcal{O}(1)\cdot|\ln\varepsilon|\,|\Delta\Upsilon|\,.\end{aligned}$$

In addition,

$$\left|Q^\sharp_{\beta_1} - Q^\sharp_{\bar\beta_1}\right| = \mathcal{O}(1)\cdot|\ln\varepsilon|\,|\Delta\Upsilon|\,, \quad \left|Q^\sharp_\gamma\right| = \mathcal{O}(1)\cdot|\ln\varepsilon|\,|\Delta\Upsilon|\,.$$

Here we assume that $\bar\beta_1$ is a shock wave. In the other case, we have $Q^\sharp_{\beta_1} = \mathcal{O}(1)\cdot|\ln\varepsilon||\Delta\Upsilon|$.

*Case* 4: *Interaction of rarefaction fronts of different families*. The change of $Q^\sharp(u)$ in this case only comes from the new fronts created by the interaction. Therefore, as in the analysis of Case 3, the total change in $Q^\sharp$ is bounded by $\mathcal{O}(1)\cdot|\ln\varepsilon||\Delta\Upsilon|$.

Based on the analysis of the above four cases, we see that by choosing $C_1$ to be sufficiently large, the nonlinear functional $\widehat Q(u)$ is nonincreasing at the interaction time when no new large shocks are introduced. This completes the proof of the lemma. □

## 4 Proof of the Main Theorem

Relying on the analysis of the two previous sections, we can now conclude the proof of Theorem 1.1. We briefly recall the main argument. If one defines the mollification $v^\delta \doteq u * \varphi_\delta$ with $\delta = \sqrt{\varepsilon}$, the estimates (1.7) hold, while (1.13)–(1.14) imply

$$\begin{aligned}\int_0^\tau \int \left|v_t^\delta + A(v^\delta)v_x^\delta - \varepsilon v_{xx}^\delta\right| dx\,dt \\ = \mathcal{O}(1) \cdot \int_0^\tau \int \mathrm{Osc}\{u : [y-\delta, y+\delta]\} |du(y)|\,dt \\ = \mathcal{O}(1) \cdot \int_0^\tau \sum_{|x_\alpha(t) - x_\beta(t)| \le \delta} |\sigma_\alpha \sigma_\beta|\,dt\,. \end{aligned} \tag{4.1}$$

In this case, the presence of big shocks gives a large contribution to the right-hand side (4.1), namely,

$$\int_0^\tau \sum_{\alpha \in \mathcal{BS}} |\sigma_\alpha(t)|^2\,dt\,. \tag{4.2}$$

To get a more accurate estimate, in a neighborhood of each big shock we replaced the mollification with a (modified) viscous traveling wave, according to (1.21). By doing this, we picked up more error terms, namely:

- The terms related to the interactions of big shocks with other fronts. The analysis at the beginning of Section 3 has shown that the total contribution of all these terms satisfies the bound (1.9).
- The errors due to the difference between the rescaled profiles $\tilde{\omega}_\alpha$ in (1.19) and the exact traveling wave profiles $\omega_\alpha$. According to (3.4), the total strength of these terms is

$$\int_0^\tau \sum_{\alpha \in \mathcal{BS}} E_\alpha(t)dt = \mathcal{O}(1) \cdot \tau\varepsilon(1 + |\ln \varepsilon|)\,\mathrm{Tot.Var.}\{\bar{u}\}\,. \tag{4.3}$$

On the other hand, we removed the contributions of all terms in (4.2). For the function $v$ defined at (1.21) we thus have

$$\begin{aligned}\int_0^\tau \int |v_t + A(v)v_x - \varepsilon v_{xx}|\,dx\,dt \\ = \mathcal{O}(1) \cdot \tau\varepsilon(1 + |\ln \varepsilon|)\,\mathrm{Tot.Var.}\{\bar{u}\} \\ + \mathcal{O}(1) \cdot \int_0^\tau \left( \sum_{|x_\beta - x_\gamma| \le 2\sqrt{\varepsilon}} |\sigma_\beta \sigma_\gamma| - \sum_{\alpha \in \mathcal{BS}} |\sigma_\alpha|^2 \right) dt\,. \end{aligned} \tag{4.4}$$

The main goal of this section is to show that the last integral in (4.4) can be estimated as

$$\begin{aligned}
&\int_0^\tau \Bigg( \sum_{|x_\beta - x_\gamma| \le 2\sqrt{\varepsilon}} |\sigma_\beta \sigma_\gamma| - \sum_{\alpha \in \mathcal{BS}} |\sigma_\alpha|^2 \Bigg) dt \\
&\quad = \mathcal{O}(1) \cdot \sum_{i=1}^{n} \int_0^\tau \Bigg( \sum_{\beta,\gamma \in \mathcal{R}_i\,,\, |x_\beta - x_\gamma| \le 8\sqrt{\varepsilon}} |\sigma_\beta \sigma_\gamma| \Bigg) dt \\
&\qquad + \mathcal{O}(1) \cdot \int_0^\tau \left| \frac{d}{dt} \widehat{Q}(u(t)) \right| dt + \mathcal{O}(1) \cdot \sqrt{\varepsilon}\, |\ln \varepsilon| \tau \ \text{Tot. Var.}\{\bar{u}\}\,.
\end{aligned} \tag{4.5}$$

Using estimate (2.30) on the spreading of positive wave fronts and the bounds (3.15)–(3.17) concerning $\widehat{Q}(u)$, from (4.5) we obtain

$$\begin{aligned}
&\int_0^\tau \Bigg( \sum_{|x_\beta - x_\gamma| \le 2\sqrt{\varepsilon}} |\sigma_\beta \sigma_\gamma| - \sum_{\alpha \in \mathcal{BS}} |\sigma_\alpha|^2 \Bigg) dt = \\
&\qquad\qquad \mathcal{O}(1) \cdot (1 + \tau) \sqrt{\varepsilon}\, |\ln \varepsilon| \cdot \text{Tot. Var.}\{\bar{u}\}\,.
\end{aligned}$$

This will complete the proof of estimate (1.3).

The remaining part of this section is devoted to a proof of (4.5), which is a consequence of the following lemma:

LEMMA 4.1 *Outside interaction times, one has* $\frac{d}{dt}\widehat{Q}(u(t)) \le 0$ *and*

$$\begin{aligned}
&\sum_{|x_\beta - x_\gamma| \le 2\sqrt{\varepsilon}} |\sigma_\beta \sigma_\gamma| - \sum_{\alpha \in \mathcal{BS}} |\sigma_\alpha|^2 \\
&\quad = \mathcal{O}(1) \cdot \sum_{i=1}^{n} \Bigg( \sum_{\beta,\gamma \in \mathcal{R}_i,\, |x_\beta - x_\gamma| \le 8\sqrt{\varepsilon}} |\sigma_\beta \sigma_\gamma| \Bigg) \\
&\qquad + \mathcal{O}(1) \cdot \left| \frac{d}{dt} \widehat{Q}(u(t)) \right| + \mathcal{O}(1) \cdot \sqrt{\varepsilon}\, |\ln \varepsilon|\, \text{Tot. Var.}\{\bar{u}\}\,.
\end{aligned} \tag{4.6}$$

To help the reader work his way through the technicalities of the proof, we first describe the heart of the matter in plain words.

After removing the terms in (4.2) related to large shocks, the left-hand side of (4.6) still contains the sum

$$\sum_{\alpha \in \mathcal{SS}} |\sigma_\alpha|^2\,,$$

where $\mathcal{SS}$ denotes the set of all small shocks. According to (1.22), the maximum-strength small shock is $\le 4\sqrt{\varepsilon}\, |\ln \varepsilon|$. Hence the above sum is estimated by $\mathcal{O}(1) \cdot \sqrt{\varepsilon}\, |\ln \varepsilon|\, \text{Tot. Var.}\{\bar{u}\}$.

Next, consider any interval $J$ of length $2\sqrt{\varepsilon}$. We first estimate the restriction of (4.6) to fronts inside $J$, i.e.,

$$\Theta \doteq \sum_{x_\alpha\in J, |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon}} |\sigma_\alpha \sigma_\beta| - \sum_{x_\alpha \in J,\, \alpha\in\mathcal{BS}} |\sigma_\alpha|^2 .$$

It is convenient to split $\Theta$ into various sums:

$$\Theta^\flat \doteq \sum_{x_\alpha\in J, |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon},\ k_\alpha\neq k_\beta} |\sigma_\alpha\sigma_\beta| ,$$

$$\Theta_i^{\text{raref}} \doteq \sum_{x_\alpha\in J, |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon},\, \alpha,\beta\in\mathcal{R}_i} |\sigma_\alpha\sigma_\beta| ,$$

$$\Theta_i^\natural \doteq \sum_{x_\alpha\in J, |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon},\ \alpha\in\mathcal{S}_i, \beta\in\mathcal{R}_i} |\sigma_\alpha\sigma_\beta| ,$$

$$\Theta_i^\sharp \doteq \sum_{x_\alpha\in J, |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon},\, \alpha,\beta\in\mathcal{S}_i} |\sigma_\alpha\sigma_\beta| .$$

If $\Theta^\flat$ dominates all other terms, then the whole sum $\Theta$ can be controlled by the rate of decrease in the functional $Q^\flat$, related to products of fronts of different families. The alternative case is when $J$ contains almost only waves of one single family, say of the $i^{\text{th}}$ family. If $\Theta_i^\natural$ is the dominant term, then $\Theta$ is controlled by the decrease of the functional $Q^\natural$ and $Q^\sharp$. If $\Theta_i^\sharp$ dominates, then $J$ contains mainly $i$-shocks, and $\Theta$ is controlled by the decrease in $Q^\sharp$. Finally, if $\Theta_i^{\text{raref}}$ dominates, then there is nothing to prove, because the sum over all couples of nearby rarefactions appears explicitly also on the right-hand side of (4.6).

Covering the real line with countably many intervals $J_\ell$ of fixed length, we eventually obtain the desired result.

PROOF OF LEMMA 4.1: Since the system is strictly hyperbolic, the definition of the functional $Q^\flat(u)$ implies

$$\frac{d}{dt}Q^\flat(u) = -\sum_{|x_\beta - x_\gamma|\le 2\sqrt{\varepsilon},\, k_\beta\neq k_\gamma} |\sigma_\beta\sigma_\gamma| \frac{|\dot{x}_\beta - \dot{x}_\gamma|}{4\sqrt{\varepsilon}} = -\frac{c_2}{\sqrt{\varepsilon}} \sum_{|x_\beta - x_\gamma|\le 2\sqrt{\varepsilon},\, k_\beta\neq k_\gamma} |\sigma_\beta\sigma_\gamma| \tag{4.7}$$

for some constant $c_2 > 0$ related to the minimum gap between different characteristic speeds. Hence the terms containing a product of two waves of different families on the left-hand side of (4.6) are controlled by the decreasing rate of $Q^\flat(u)$.

In the following, we only need to show that the products involving one or two shock waves of the same family can be controlled by the decreasing rate of the nonlinear functional $\widehat{Q}(u)$, plus the quantity in (2.30) and $\sqrt{\varepsilon}\,|\ln\varepsilon|$ Tot. Var.$\{\bar{u}\}$.

By the definition of $Q^\natural(u)$, we know that the rarefaction waves located in $I_\alpha(t)$ involved in $Q_\alpha^\natural(u)$ approach to the large shock wave $\alpha$ unless there are waves of the other families in between. Hence, if we use $\mathcal{BS}'$ to denote the set of big shocks $\alpha$ such that the total strength of small wave fronts within the interval $I_\alpha(t)$ is $\leq |\sigma_\alpha|/4$, then for $\alpha \in \mathcal{BS}'$, we have

$$
\begin{aligned}
\frac{d}{dt} Q_\alpha^\natural(u) &= -\sum_{x_\beta \in I_\alpha(t),\, \beta \in \mathcal{R}_\alpha} |\sigma_\beta| \frac{|\dot{x}_\alpha - \dot{x}_\beta|}{4\sqrt{\varepsilon}} \\
&\leq -\frac{c_3}{\sqrt{\varepsilon}} \sum_{x_\beta \in I_\alpha(t),\, \beta \in \mathcal{R}_\alpha} |\sigma_\alpha \sigma_\beta| + \frac{\mathcal{O}(1)}{\sqrt{\varepsilon}} \sum_{\beta, x_\beta \in I_\alpha(t),\, k_\beta \neq k_\alpha} |\sigma_\alpha \sigma_\beta|,
\end{aligned}
\tag{4.8}
$$

where

$$
Q_\alpha^\natural(u) = \int W_\alpha^\natural(x) D_x \tilde{w}_\alpha \,.
$$

On the other hand, the functional $Q^\sharp(u)$ is defined for all shock waves no matter how small or large. In this way, its time derivative yields mainly the product of two shock waves of the same family with distance $\leq 2\sqrt{\varepsilon}$. Let

$$
Q_\alpha^\sharp(u) \doteq |\sigma_\alpha| \int W_\alpha^\natural(x)\, W_\alpha^\sharp(x) D_x \tilde{z}_\alpha \,.
$$

Since there is a factor 3 in front of the summation of rarefaction waves in the definition of $Q^\sharp(u)$, this guarantees that all the shock waves appearing in $Q_\alpha^\sharp$ approach the shock wave $\alpha$ if there are no waves of other families in between. Notice that there is a constant $\varepsilon$ in the denominator of the weight function $W_\alpha^\sharp(x)$. Thus, for any shock $\alpha \in \mathcal{S}$,

$$
\begin{aligned}
\frac{d}{dt} Q_\alpha^\sharp(u) &\leq -\frac{c_4}{\sqrt{\varepsilon}} \widehat{\sum_{\beta \in \mathcal{S}_\alpha,\, x_\beta \in \hat{I}_\alpha(t)}} |\sigma_\alpha \sigma_\beta| \\
&\quad + \frac{c_5}{\sqrt{\varepsilon}} \sum_{\beta \in \mathcal{R}_\alpha,\, x_\beta \in I_\alpha(t)} |\sigma_\alpha \sigma_\beta| + \frac{\mathcal{O}(1)}{\sqrt{\varepsilon}} \sum_{\beta, x_\beta \in I_\alpha(t),\, k_\beta \neq k_\alpha} |\sigma_\alpha \sigma_\beta| \\
&\leq -\frac{c_4}{\sqrt{\varepsilon}} \sum_{\beta \in \mathcal{S}_\alpha,\, x_\beta \in \hat{I}_\alpha(t)} |\sigma_\alpha \sigma_\beta| + \frac{c_5}{\sqrt{\varepsilon}} \sum_{\beta \in \mathcal{R}_\alpha,\, x_\beta \in I_\alpha(t)} |\sigma_\alpha \sigma_\beta| \\
&\quad + \frac{\mathcal{O}(1)}{\sqrt{\varepsilon}} \sum_{\beta, x_\beta \in I_\alpha(t),\, k_\beta \neq k_\alpha} |\sigma_\alpha \sigma_\beta| + \mathcal{O}(1)\sqrt{\varepsilon}\, |\sigma_\alpha| \,,
\end{aligned}
\tag{4.9}
$$

where $c_4, c_5 > 0$ are constants independent of $\varepsilon$, $\hat{I}_\alpha(t) = [x_\alpha - 2\sqrt{\varepsilon}\,, x_\alpha[\cup]x_\alpha, x_\alpha + 2\sqrt{\varepsilon}\,]$, and $\widehat{\sum}$ means that the summation is over all shocks $\beta$ with the property that the total strength of all shock fronts between $\alpha$ and $\beta$ with $x_\beta \in I_\alpha(t)$ is $\geq \varepsilon$.

By noticing that the time derivative of $\Upsilon(u)$ is zero outside interaction times and by choosing $C_2 \gg C_3 \gg 1$, based on estimates (4.7)–(4.9), the increase of (4.6) can be given as follows, by considering separately the products involving large shocks and those involving only small wave fronts:

For a large shock front $\alpha$, consider the summation

$$\sum_{x_\beta \in \hat{I}_\alpha(t)} |\sigma_\alpha \sigma_\beta| \,. \tag{4.10}$$

Since the sum of all products $|\sigma_\alpha\sigma_\beta|$ when $k_\alpha \neq k_\beta$ is controlled by (4.7), we have

$$\sum_{x_\beta \in \hat{I}_\alpha(t)} |\sigma_\alpha \sigma_\beta| \leq \mathcal{O}(1) \cdot \left| \frac{d}{dt} Q_\alpha^\flat(u) \right| , \tag{4.11}$$

provided that waves of different families dominate, say

$$\sum_{\beta, x_\beta \in I_\alpha(t),\, k_\beta \neq k_\alpha} |\sigma_\beta| \geq \frac{1}{4} \sum_{\beta, x_\beta \in \hat{I}_\alpha(t)} |\sigma_\beta| \,. \tag{4.12}$$

It thus remains to consider the case when (4.12) does not hold. We then have

$$\sum_{x_\beta \in \hat{I}_\alpha(t),\, k_\beta = k_\alpha} |\sigma_\beta| \geq \frac{3}{4} \sum_{\beta, x_\beta \in \hat{I}_\alpha(t)} |\sigma_\beta| \,. \tag{4.13}$$

In this case, if $\alpha \in \mathcal{BS}'$, then the summation of $|\sigma_\alpha\sigma_\beta|$ for $\beta \in \mathcal{R}_\alpha \cup \mathcal{S}_\alpha$ is controlled by (4.8) and (4.9) together with (4.7). Therefore

$$\begin{aligned} \sum_{\alpha \in \mathcal{BS}'} \sum_{x_\beta \in \hat{I}_\alpha(t)} |\sigma_\alpha \sigma_\beta| &\leq \mathcal{O}(1) \cdot \left( \sum_{i=1}^{n} \sum_{\beta, \gamma \in \mathcal{R}_i,\, |x_\beta - x_\gamma| \leq 2\sqrt{\varepsilon}} |\sigma_\beta \sigma_\gamma| \right) \\ &\quad + \mathcal{O}(1) \cdot \left| \frac{d}{dt} \widehat{Q}(u(t)) \right| + \mathcal{O}(1) \cdot \varepsilon \ \text{Tot. Var.}\{\bar{u}\} \,. \end{aligned} \tag{4.14}$$

Moreover, by (4.9), for $\alpha \in \mathcal{S}$, if

$$\sum_{\beta \in \mathcal{S}_\alpha,\, x_\beta \in \hat{I}_\alpha(t)} |\sigma_\beta| \geq \frac{2c_5}{c_4} \sum_{\beta \in \mathcal{R}_\alpha,\, x_\beta \in \hat{I}_\alpha(t)} |\sigma_\beta| \,, \tag{4.15}$$

then

$$\sum_{x_\beta \in \hat{I}_\alpha(t)} |\sigma_\alpha \sigma_\beta| \leq \mathcal{O}(1) \cdot \left( -\frac{d}{dt} \big( C_2 \sqrt{\varepsilon}\, |\ln \varepsilon| Q_\alpha^\natural + \sqrt{\varepsilon}\, Q_\alpha^\sharp \big) + \varepsilon |\sigma_\alpha| \right) . \tag{4.16}$$

For a large shock wave, it now remains to consider the case when $\alpha \in \mathcal{BS}'' = \mathcal{BS} - \mathcal{BS}'$ satisfies (4.13) and

$$\sum_{\beta \in \mathcal{S}_\alpha, x_\beta \in \hat{I}_\alpha(t)} |\sigma_\beta| \leq \frac{2c_5}{c_4} \sum_{\beta \in \mathcal{R}_\alpha,\, x_\beta \in \hat{I}_\alpha(t)} |\sigma_\beta| \,.$$

We denote the set consisting of all these large shock waves by $\mathcal{BS}'''$. Notice that this is a subset of $\mathcal{BS}''$. Roughly speaking, for $\alpha \in \mathcal{BS}'''$, the small wave fronts is not small compared to $\alpha$, and rarefaction waves of the $k_\alpha^{\text{th}}$ family dominate in $I_\alpha(t)$. Hence, for $\alpha \in \mathcal{BS}'''$, one has

$$\sum_{\theta\in\mathcal{BS}'''_\alpha, x_\theta\in I_\alpha(t)} \sum_{\beta, x_\beta\in\hat{I}_\theta(t)} |\sigma_\theta\sigma_\beta| \leq \mathcal{O}(1)\cdot \sum_{\beta,\gamma\in\mathcal{R}_\alpha,\, x_\beta, x_\gamma\in[x_\alpha-4\sqrt{\varepsilon}, x_\alpha+4\sqrt{\varepsilon}]} |\sigma_\beta\sigma_\gamma|\,, \tag{4.17}$$

which is controlled by the corresponding part of (2.30) in the interval $[x_\alpha - 4\sqrt{\varepsilon}\,, x_\alpha + 4\sqrt{\varepsilon}\,]$. Hence

$$\sum_{\alpha\in\mathcal{BS}'''} \sum_{\beta, x_\beta\in\hat{I}_\alpha(t)} |\sigma_\alpha\sigma_\beta| \leq \mathcal{O}(1)\cdot \sum_{\alpha,\beta\in\mathcal{R},\, |x_\alpha-x_\beta|\leq 8\sqrt{\varepsilon}, k_\alpha=k_\beta} |\sigma_\alpha\sigma_\beta|\,, \tag{4.18}$$

which is estimated by (2.30).

Combining (4.11), (4.14), (4.16), and (4.18) we obtain

$$\begin{aligned}\sum_{\alpha\in\mathcal{BS}} \sum_{x_\beta\in\hat{I}_\alpha(t)} |\sigma_\alpha\sigma_\beta| \leq \mathcal{O}(1)\cdot\left(\sum_{i=1}^n \sum_{\beta,\gamma\in\mathcal{R}_i,\, |x_\beta-x_\gamma|\leq 8\sqrt{\varepsilon}} |\sigma_\beta\sigma_\gamma|\right)\\ + \mathcal{O}(1)\cdot\left|\frac{d}{dt}\widehat{Q}(u(t))\right| + \mathcal{O}(1)\cdot\varepsilon\ \text{Tot. Var.}\{\bar{u}\}\,.\end{aligned} \tag{4.19}$$

Now it remains to show that the sum of products of small wave fronts of the same family satisfies the same bound:

$$\begin{aligned}\sum_{\alpha,\beta\in\mathcal{SS}\cup\mathcal{R}, |x_\alpha-x_\beta|\leq 2\sqrt{\varepsilon},\, k_\alpha=k_\beta} |\sigma_\alpha\sigma_\beta| \leq \mathcal{O}(1)\\ \cdot\left(\sum_{i=1}^n \sum_{\beta,\gamma\in\mathcal{R}_i\,,\, |x_\beta-x_\gamma|\leq 8\sqrt{\varepsilon}} |\sigma_\beta\sigma_\gamma| + \left|\frac{d}{dt}\widehat{Q}(u(t))\right| + \sqrt{\varepsilon}\,|\ln\varepsilon|\ \text{Tot. Var.}\{\bar{u}\}\right).\end{aligned} \tag{4.20}$$

To obtain estimate (4.20), we divide the real line into a union of closed intervals of length $2\sqrt{\varepsilon}$, i.e., $\mathbb{R} = \bigcup_i J_i$ with $J_i \doteq [2i\sqrt{\varepsilon},\ 2(i+1)\sqrt{\varepsilon}\,]$. We denote by $s_i^k$ and $r_i^k$, respectively, the total strengths of small $k$-shock and $k$-rarefaction fronts contained in the interval in $J_i$. We have

$$\sum_{\alpha,\beta\in\mathcal{SS}\cup\mathcal{R}, |x_\alpha-x_\beta|\leq 2\sqrt{\varepsilon},\, k_\alpha=k_\beta} |\sigma_\alpha\sigma_\beta| \leq \sum_i \sum_{\alpha\in\mathcal{SS}\cup\mathcal{R}, x_\alpha\in J_i} \sum_{\beta\in\mathcal{SS}\cup\mathcal{R},\, |x_\alpha-x_\beta|\leq 2\sqrt{\varepsilon},\, k_\alpha=k_\beta} |\sigma_\alpha\sigma_\beta|\,.$$

To estimate the quantity

$$\sum_{\alpha\in\mathcal{SS}\cup\mathcal{R}, x_\alpha\in J_i} \sum_{\beta\in\mathcal{SS}\cup\mathcal{R},\, |x_\alpha-x_\beta|\leq 2\sqrt{\varepsilon},\, k_\alpha=k_\beta} |\sigma_\alpha\sigma_\beta|\,, \tag{4.21}$$

we consider the following two cases:

*Case* 1: *For a given k*, $s_i^k \geq (2c_5/c_4)(r_{i-1}^k + r_i^k + r_{i+1}^k)$. In this case, from (4.9) we deduce

$$\begin{aligned}
&\frac{d}{dt} \sum_{\alpha \in \mathcal{SS}_k,\, x_\alpha \in J_i} Q_\alpha^\sharp(u) \\
&\quad \leq -\frac{c_5}{\sqrt{\varepsilon}} \sum_{\alpha \in \mathcal{SS}_k,\, x_\alpha \in J_i} \sum_{\beta \in \mathcal{S}_\alpha,\, x_\beta \in \hat{I}_\alpha(t)} |\sigma_\alpha \sigma_\beta| \\
&\qquad + \frac{\mathcal{O}(1)}{\sqrt{\varepsilon}} \sum_{\alpha \in \mathcal{SS}_k,\, x_\alpha \in J_i} \sum_{x_\beta \in I_\alpha(t),\, k_\beta \neq k_\alpha} |\sigma_\alpha \sigma_\beta| + \mathcal{O}(1)\sqrt{\varepsilon} \sum_{\alpha \in \mathcal{SS}_k, x_\alpha \in J_i} |\sigma_\alpha| \\
&\quad \leq -\frac{c_5}{\sqrt{\varepsilon}} \sum_{\alpha \in \mathcal{SS}_k,\, x_\alpha \in J_i} \sum_{\beta \in \mathcal{S}_\alpha,\, x_\beta \in I_\alpha(t)} |\sigma_\alpha \sigma_\beta| \\
&\qquad + \frac{\mathcal{O}(1)}{\sqrt{\varepsilon}} \sum_{\alpha \in \mathcal{SS}_k,\, x_\alpha \in J_i} \sum_{x_\beta \in I_\alpha(t),\, k_\beta \neq k_\alpha} |\sigma_\alpha \sigma_\beta| \\
&\qquad + \frac{\mathcal{O}(1)}{\sqrt{\varepsilon}} \sum_{\alpha \in SS_k, x_\alpha \in J_i} |\sigma_\alpha|^2 + \mathcal{O}(1)\sqrt{\varepsilon} \sum_{\alpha \in \mathcal{SS}_k,\, x_\alpha \in J_i} |\sigma_\alpha| \\
&\quad \leq -\frac{c_5}{\sqrt{\varepsilon}} \sum_{\alpha \in \mathcal{SS}_k,\, x_\alpha \in J_i} \sum_{\beta \in \mathcal{S}_\alpha,\, x_\beta \in I_\alpha(t)} |\sigma_\alpha \sigma_\beta| \\
&\qquad + \frac{\mathcal{O}(1)}{\sqrt{\varepsilon}} \sum_{\alpha \in \mathcal{SS}_k,\, x_\alpha \in J_i} \sum_{x_\beta \in I_\alpha(t),\, k_\beta \neq k_\alpha} |\sigma_\alpha \sigma_\beta| \\
&\qquad + \mathcal{O}(1)|\ln \varepsilon| \sum_{\alpha \in SS_k,\, x_\alpha \in J_i} |\sigma_\alpha| + \mathcal{O}(1)\sqrt{\varepsilon} \sum_{\alpha \in \mathcal{SS}_k,\, x_\alpha \in J_i} |\sigma_\alpha| \,.
\end{aligned} \tag{4.22}$$

Here we have used the fact that $\sigma_\alpha \leq 4\sqrt{\varepsilon}\,|\ln \varepsilon|$ for $\alpha \in \mathcal{SS}_k$. By (4.22) we see that those terms containing a product with $\alpha \in \mathcal{SS}_k$ and $\beta \in \mathcal{SS}_k$ in (4.21) can be controlled by $\frac{d}{dt}\widehat{Q}(u)$ up to an error of the order of $\sqrt{\varepsilon}\,|\ln \varepsilon|$ Tot. Var.$\{\bar{u}\}$. Since the total strength of all small $k$-shocks in $J_i$ dominates the total strength of all $k$-rarefactions in $\bigcup_{j=i-1}^{i+1} J_j$, the products of $\alpha \in \mathcal{SS}_k$ and $\beta \in \mathcal{R}_k$ and the products of $\alpha \in \mathcal{R}_k$ with $\beta \in \mathcal{SS}_k$ for $x_\beta \in J_i$ in (4.21) are also controlled by $\frac{d}{dt}\widehat{Q}(u)$ up to an error of the order of $\sqrt{\varepsilon}\,|\ln \varepsilon|$ Tot. Var.$\{\bar{u}\}$. Moreover, those products of $\alpha \in \mathcal{R}_k$ and $\beta \in \mathcal{R}_k$ in (4.21) are controlled by the value of the right-hand side in (2.30) when the locations of the waves lie in the interval $\bigcup_{j=i-1}^{i+1} J_i$.

Hence, it remains to consider the product of $\alpha \in \mathcal{R}_k$ and $\beta \in \mathcal{SS}_k$ with $x_\beta \in J_{i-1} \cup J_{i+1}$. To fix the ideas, we consider the case when $\alpha \in \mathcal{R}_k$ and $\beta \in \mathcal{SS}_k$ with $x_\beta \in J_{i-1}$, i.e.,

$$\sum_{\alpha \in \mathcal{R}_k, \beta \in \mathcal{SS}_k,\, x_\alpha \in J_i,\, x_\beta \in J_{i-1},\, |x_\alpha - x_\beta| \leq 2\sqrt{\varepsilon}} |\sigma_\alpha \sigma_\beta| \,. \tag{4.23}$$

When $s^k_{i-1} \le (2c_5/c_4)(r^k_{i-2} + r^k_{i-1} + r^k_i)$, (4.23) is controlled by the value of the right-hand side in (2.30) when the locations of the waves lie in the interval $\bigcup_{j=i-2}^{i+1} J_j$. Otherwise

$$\sum_{\substack{\alpha\in\mathcal{R}_k,\beta\in\mathcal{SS}_k,\, x_\alpha\in J_i, x_\beta\in J_{i-1},\\ |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon}}} |\sigma_\alpha\sigma_\beta| \le \Bigg(\sum_{\beta, x_\beta\in J_{i-1},\, \beta\in\mathcal{SS}_k} |\sigma_\beta|\Bigg)^2,$$

which can be controlled as in (4.22), using (4.7) to control the $k$-shock fronts in $J_{i-1}$.

*Case* 2: *For a given k, $s^k_i < (2c_5/c_4)(r^k_{i-1} + r^k_i + r^k_{i+1})$.* In this case the total strength of all $k$-rarefaction fronts in $\bigcup_{j=i-1}^{i+1} J_j$ dominates the total strength of $k$-small shocks in $J_i$. As done previously, we only need to consider the case when $\alpha \in \mathcal{SS}_k \cup \mathcal{R}_k$ and $\beta \in \mathcal{SS}_k$ with $x_\beta \in J_{i-1} \cup J_{i+1}$ in (4.21) because all the other terms can be controlled by the value of the right-hand side in (2.30) when the locations of the waves lie in the corresponding interval $\bigcup_{j=i-1}^{i+1} J_j$.

For illustration, we discuss the following two terms,

$$\sum_{\substack{\alpha\in\mathcal{R}_k,\beta\in\mathcal{SS}_k,\, x_\alpha\in J_i,\, x_\beta\in J_{i-1},\\ |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon}}} |\sigma_\alpha\sigma_\beta| \tag{4.24}$$

and

$$\sum_{\substack{\alpha,\beta\in\mathcal{SS}_k,\, x_\alpha\in J_i,\, x_\beta\in J_{i-1},\\ |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon}}} |\sigma_\alpha\sigma_\beta|, \tag{4.25}$$

respectively. The other terms can be handled similarly.

Concerning (4.24), when $s^k_{i-1} \le (2c_5/c_4)(r^k_{i-2} + r^k_{i-1} + r^k_i)$, it can be controlled by the value of the right-hand side in (2.30) when the locations of the waves lie in the interval $\bigcup_{j=i-2}^{i} J_j$. Otherwise, when $s^k_{i-1} > \frac{2c_5}{c_4}(r^k_{i-2} + r^k_{i-1} + r^k_i)$, we have

$$\sum_{\substack{\alpha\in\mathcal{R}_k,\, \beta\in\mathcal{SS}_k,\, x_\alpha\in J_i,\, x_\beta\in J_{i-1},\\ |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon}}} |\sigma_\alpha\sigma_\beta| \le \Bigg(\sum_{\beta\in\mathcal{SS}_k,\, x_\beta\in J_{i-1}} |\sigma_\beta|\Bigg)^2,$$

which can be estimated as in (4.22), using (4.7) to control the $k$-shock fronts in the interval $J_{i-1}$.

Concerning (4.25), when $s^k_{i-1} \ge (2c_5/c_4)(r^k_{i-2} + r^k_{i-1} + r^k_i)$, then a similar argument as in (4.22) can be applied, using (4.7).

Otherwise,

$$\sum_{\substack{\alpha,\beta\in\mathcal{SS}_k,\, x_\alpha\in J_i,\, x_\beta\in J_{i-1},\\ |x_\alpha - x_\beta|\le 2\sqrt{\varepsilon}}} |\sigma_\alpha\sigma_\beta| \le \Bigg(\sum_{j=i-2}^{i+1} r^k_j\Bigg)^2, \tag{4.26}$$

which can be controlled by the value of the right-hand side in (2.30) when the locations of the waves lie in the interval $\bigcup_{j=i-2}^{i+1} J_j$.

Notice that each interval $J_i$ can be counted no more than three times. By combining (4.22)–(4.26), we have the desired estimate on (4.21) for small wave fronts so that (4.20) holds. In summary, (4.19) and (4.20) imply (4.6), completing the proof of the lemma. □

REMARKS 4.2

(1) In the proof of the error estimate (1.3), the three basic ingredients are:

- the existence of uniformly Lipschitz semigroups of approximate (viscous) solutions,
- the decay of positive waves due to genuine nonlinearity, and
- the exponential rate of convergence to steady states in the tails of traveling viscous shocks.

Assuming that all characteristic fields are genuinely nonlinear, we thus conjecture that similar error estimates are valid also for the semidiscrete scheme considered in [1]. In the case of straight-line systems, based on the analysis in [8], it is reasonable to expect that analogous results should also hold for the Godunov scheme.

(2) In the case where all characteristic fields are linearly degenerate, solutions with Lipschitz-continuous initial data having small total variation remain uniformly Lipschitz-continuous for all times, as shown in [3]. Therefore, the easy error estimate (1.16) can be used. For systems having some linearly degenerate and some genuinely nonlinear fields, we still conjecture that the error bound (1.3) is valid. A proof, however, will require some new techniques. Indeed, the contact discontinuities that may be generated by shock interactions at times $t > 0$ can no longer be approximated by viscous traveling profiles.

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