Stackelberg Solutions of Feedback Type for Differential Games with Random Initial Data

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Abstract

The paper is concerned with Stackelberg solutions for a differential game with deterministic dynamics but random initial data, where the leading player can adopt a strategy in feedback form: $u_1 = u_1(t, x)$. The first main result provides the existence of a Stackelberg equilibrium solution, assuming that the family of feedback controls $u_1(t, \cdot)$ available to the leading player are constrained to a finite dimensional space. A second theorem provides necessary conditions for the optimality of a feedback strategy. Finally, in the case where the feedback u_1 is allowed to be an arbitrary function, an example illustrates a wide class of systems where the minimal cost for the leading player corresponds to an impulsive dynamics. In this case, a Stackelberg equilibrium solution does not exists, but a minimizing sequence of strategies can be described.

1 Introduction

Consider a differential game for two players. Let $x \in \mathbb{R}^n$ describe the state of a system, which evolves according to the differential equation

$$\dot{x}(t) = f\big(t, x(t), u_1(t), u_2(t)\big) \tag{1.1}$$

with initial condition

$$x(0) = \bar{x}\,. \tag{1.2}$$

Here the upper dot denotes a derivative w.r.t. time. The functions $u_1(\cdot)$, $u_2(\cdot)$ are the controls implemented by the two players, taking values inside admissible sets $U_1, U_2 \subseteq \mathbb{R}^m$, respectively. For $i = 1, 2$, the goal of the i-th player is to minimize his own cost, given by

$$J_i \doteq \int_0^T L_i\big(t, x(t), u_1(t), u_2(t)\big)\, dt\,. \tag{1.3}$$

According to the definition of Stackelberg equilibrium [4, 6, 10], we assume that Player 1 (the leader) announces his feedback strategy in advance, say $u_1 = u_1(t, x)$. Given the initial point

$\bar{x}$, Player 2 (the follower) then chooses his strategy $u_2 = u_2(t; u_1, \bar{x})$ in order to minimize his own cost J_2.

We remark that, in a standard control problem, an optimal feedback control yields the minimum cost in connection with *every* initial data $\bar{x}$. However, this important property cannot be achieved for a Stackelberg solution to a differential game. Namely, the optimal feedback strategy for the leading player usually depends heavily on the initial data [12, 14]. To obtain a meaningful mathematical problem, in connection with a large set of initial data, in this paper we shall thus consider a probability distribution μ on the set of initial data $\bar{x} \in I\!R^n$.

Definition 1. Given a feedback control $u_1 = u_1(t, x)$ for the first player and an initial data $\bar{x} \in I\!R^n$, we define $\mathcal{R}_2(u_1, \bar{x})$ as the set of **best replies** for Player 2. These are the controls $u_2 : [0, T] \mapsto I\!R^m$ which yield the minimum cost for the optimization problem

$$\text{minimize:} \qquad \int_0^T L_2\Big(t,\, x(t),\, u_1(t, x(t)),\, u_2(t)\Big)\, dt\,, \tag{1.4}$$

$$\text{subject to:} \qquad \dot{x} \;=\; f(t, x(t), u_1(t, x(t)),\, u_2(t))\,, \qquad\qquad x(0) \;=\; \bar{x}\,. \tag{1.5}$$

In the following, given a probability measure μ on the set of initial data, we denote by E^μ the expected value of a quantity depending on $\bar{x}$.

Definition 2. Let $\mathcal{F}$ be a family of admissible feedback functions. A feedback $u_1^* \in \mathcal{F}$ is an **optimal strategy** for the leading player if for μ-a.e. initial data $\bar{x}$ one can choose a best reply $u_2(\cdot; u_1^*, \bar{x}) \in \mathcal{R}_2(u_1^*, \bar{x})$ in such a way that

$$\begin{aligned} E^\mu[J(u_1;\, \bar{x})] \;&=\; \int \left[\int_0^T L_1\Big(t,\, x(t),\, u_1^*(t, x(t)),\, u_2(t, u_1^*, \bar{x})\Big)\, dt\right] d\mu(\bar{x}) \\ &\leq\; \int \left[\int_0^T L_1\Big(t,\, x(t),\, u_1(t, x(t)),\, u_2(t, u_1, \bar{x})\Big)\, dt\right] d\mu(\bar{x})\,, \end{aligned} \tag{1.6}$$

for every other feedback $u_1 \in \mathcal{F}$ and any selection of best replies $u_2(\cdot\,; u_1, \bar{x}) \in \mathcal{R}_2(u_1, \bar{x})$.

Notice that, if Player 2 has several best replies all yielding the same minimum cost, we are here assuming that he choses the one most favorable to Player 1.

In Section 2 we assume that, for each $t \in [0, T]$, the feedback $u_1(t, \cdot)$ depends on a finite set of parameters. For example, $u_1(t, \cdot)$ is a polynomial in x, or a piecewise affine function. Under natural convexity hypotheses on the cost functions L_1, L_2, our first main result provides the existence of an optimal strategy for the leading player.

Section 3 is devoted to necessary conditions for the optimality of a strategy of the leading player. As it is well known in the literature, one here encounters a fundamental difficulty. Namely, replies of the second player that satisfy the Pontryagin necessary conditions are not necessarily optimal. It thus makes sense to consider a wider set of "weakly optimal replies".

Definition 3. Given a feedback control $u_1 = u_1(t, x)$ for the first player and an initial data $\bar{x} \in I\!R^n$, we call $\mathcal{R}_2^w(u_1, \bar{x})$ the set of **weakly optimal replies** for Player 2. These are the controls $u_2^*:[0,T]\mapsto I\!R^m$ which satisfy the Pontryagin necessary conditions for optimality:

$$u_2^*(t) \;=\; \arg\min_{\omega\in U_2}\Big\{\xi(t)\cdot f\big(t,x^*(t),u_1(t,x^*(t)),\omega\big)+L_2\big(t,x^*(t),u_1(t,x^*(t)),\omega\big)\Big\}\,, \tag{1.7}$$

$$\left\{\begin{array}{rcl}\dot x(t) &=& f\big(t,x^*(t),u_1(t,x^*(t)),u_2^*(t)\big)\,,\\[2mm] \dot\xi(t) &=& -\,\xi(t)\cdot\left[\dfrac{\partial f}{\partial x}\big(t,x^*(t),u_1(t,x^*(t)),u_2^*(t)\big)+\dfrac{\partial f}{\partial u_1}\big(t,x^*(t),u_1(t,x^*(t)),u_2^*(t)\big)\dfrac{\partial u_1}{\partial x}(t,x^*(t))\right]\\[4mm] && -\left[\dfrac{\partial L_2}{\partial x}\big(t,x^*(t),u_1(t,x^*(t)),u_2^*(t)\big)+\dfrac{\partial L_2}{\partial x}\big(t,x^*(t),u_1(t,x^*(t)),u_2^*(t)\big)\dfrac{\partial u_1}{\partial x}(t,x^*(t))\right],\end{array}\right. \tag{1.8}$$

$$x(0)\;=\;\bar x \qquad\qquad \xi(T)\;=\;0\,. \tag{1.9}$$

Throughout the following, we say that a feedback $u^*\in\mathcal{F}$ is a **weakly optimal strategy** for Player 1 if it satisfies Definition 2, with $R_2(u^*,\bar x)$ replaced by $R_2^w(u^*,\bar x)$.

Remark 1. Clearly, the above definitions are meaningful only if the feedback u_1^* is sufficiently regular so that the evolution equations (1.8) are well defined and have at least a solution for μ-a.e. initial data $\bar x$. If this solution is unique, then $R_2(u_1^*,\bar x)=R_2^w(u_1^*,\bar x)$.

Remark 2. In general, a weakly optimal strategy need not be optimal, and an optimal strategy need not be weakly optimal. The two definitions clearly coincide under the assumption that $R_2(u_1,\bar x) = R_2^w(u_1,\bar x)$ for every admissible control u_1 and μ-a.e. initial data $\bar x$. In particular this is true if, for every $u_1,\bar x$, the Pontryagin equations (1.7)–(1.9) have a unique solution.

Toward the derivation of necessary optimality conditions for the Player 1, the usefulness of this concept of weakly optimal strategy becomes clear. We sketch here the main approach.

Motivated by (1.7), for every t,x,u_1,ξ, define the control value

$$u_2^\sharp(t,x,\,u_1,\,\xi)\;\doteq\;\arg\min_{\omega\in U_2}\Big\{\xi\cdot f(t,x,u_1,\omega)-L_2(t,x,u_1,\omega)\Big\}. \tag{1.10}$$

In the following, we assume that the above minimum is attained at a unique point $u_2^\sharp$, contained in the interior of U_2. This is certainly the case under the assumptions

(A) $U_2=I\!R^m$. The function $f(t,x,u_1,u_2)$ is affine in the variable u_2. Moreover, the function $u_2\mapsto L_2(t,x,u_1,u_2)$ is strictly convex and has superlinear growth:

$$\lim_{|u_2|\to\infty}\frac{L_2(t,x,u_1,u_2)}{|u_2|}\;=\;+\infty\,. \tag{1.11}$$

The optimization problem for the leading player can now be written as follows.

$$\text{Minimize:}\qquad E^\mu[J_1]\;=\;\int\left[\int_0^T L_1\big(t,\,x,\,u_1,\,u_2^\sharp(t,x,u_1,\xi)\big)\,dt\right]\,d\mu(\bar x)\,, \tag{1.12}$$

for a solution to the boundary value problem

$$\begin{cases} \dot x & = \; f\,, \\ \dot \xi & = \; -\,\xi\cdot\left(\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial u_1}\,u_{1,x}\right)-\left(\dfrac{\partial L_2}{\partial x}+\dfrac{\partial L_2}{\partial u_1}\,u_{1,x}\right), \end{cases} \qquad \begin{cases} x(0) & = \; \bar x\,, \\ \xi(T) & = \; 0\,. \end{cases} \tag{1.13}$$

In (1.13) it is understood that the right hand sides of the differential equations are computed at the point $\Big(t, x,\, u_1(t,x),\; u_2^\sharp(t,x,u_1(t,x),\xi)\Big)$.

Due to the presence of $u_{1,x}$ on the right hand side of the evolution equation (1.13), the problem of minimizing the expected cost (1.12) is non-standard. In the case where both x and ξ are assigned at the initial time $t = 0$ has been recently studied in [7]. See also [8] for specific examples. As anticipated in [14], if the values $u_1(t,x)$ of the feedback can be freely assigned, in most cases this leads to an ill posed system of equations.

As observed in [8], the term $u_{1,x}$ can often be considered as an additional component of the control, which can be chosen arbitrarily large and comes with no cost. In such cases, the optimal strategy for the leading player would correspond to an impulsive dynamics and can never be exactly attained.

To illustrate this point, Section 4 exhibits a class of nonlinear systems where a minimizing sequence of feedback controls $u_{1,\nu}$ is explicitly constructed, but the infimum cost for the leading player cannot be attained. Roughly speaking, in these cases there is an optimal state $x^\dagger$ and an optimal control value $u_1^\dagger$ which minimize the running cost $L_1 = L_1(x, u_1)$ for Player 1. For any $\varepsilon > 0$ and any $\bar x$ within a bounded set of initial data, the leading player can force Player 2 to keep the system at the state $x^\dagger$ for all times $t \in [\varepsilon,\, T-\varepsilon]$. This is achieved by introducing suitable penalties whenever $x(t) \neq x^\dagger$. As it is often the case for Stackelberg equilibria, these penalties (which would be very costly also to Player 1) are never implemented, because it is not optimal for Player 2 to deviate from the path imposed by the leader.

An introduction to the basic concepts of differential games can be found in [4, 6, 10]. See [12, 13, 14, 15] for related result on Stackelberg equilibria. For the basic theory of optimal control problems and we refer to [3, 5, 9, 11]. Patchy feedbacks, used in Section 4, were introduced in [1]. See [2, 5] for a survey.

2 Existence of an optimal feedback control

In this section we study the existence of an optimal strategy for the leading player, within a family of feedback strategies depending on finitely many parameters. We thus consider the system

$$\dot x \;=\; f\big(t, x, u_1(t,x), u_2\big)\,, \tag{2.1}$$

assuming that the feedback control $u_1(t,x)$ for the leading player has the form

$$u_1(t,x) \;=\; \Psi(x)\cdot v(t) \;\doteq\; \sum_{i=1}^N \psi_i(x)\, v_i(t)\,. \tag{2.2}$$

Here $\psi_1, \ldots, \psi_N$ are smooth functions of x, while $v_1, \ldots, v_N$ are measurable functions of time. In practice, this assumption can be justified by observing that any continuous map $u = u(t,x)$ can be approximated by functions of the form (2.2) uniformly on compact sets.

In the following, we assume that

$$v(t) = (v_1,\, v_2,\, ,\ldots\ldots v_N)(t) \;\in\; U_1\,, \tag{2.3}$$

where U_1 is a compact convex set in $I\!R^N$. After the leading player has announced his feedback control $u_1(t,x)$ as in (2.2), and given an initial data

$$x(0) \;=\; \bar x\,, \tag{2.4}$$

the second player seeks to minimize his expected cost

$$J_2(u_2) \;=\; \int_0^T L_2\big(t,\, x,\, u_1(t,x),\, u_2(t)\big)\, dt\,. \tag{2.5}$$

The minimization takes place over all measurable controls

$$t \;\mapsto\; u_2(t) \;\in\; U_2\,, \tag{2.6}$$

taking values in a compact convex set $U_2 \subset I\!R^{N'}$.

For a given control $t \mapsto v(t)$ for the leading player, we call $R_2(v,\bar x)$ the (possibly empty) family of best replies $t \mapsto u_2(t) \in U_2$ for the second player. Our present goal is to provide conditions on f and on the cost functionals L_1, L_2 which guarantee the existence of a feedback u_1 for the leading player which is optimal w.r.t. a family of initial data. More precisely, given a probability distribution μ on the set of initial data $\bar x$ in (2.4), we seek an admissible control $t \mapsto v^*(t) \in U_1$ and a selection of optimal replies $\bar x \mapsto u_2^*(\cdot,\bar x) \in R_2(v^*,\bar x)$ such that

$$J_1(v^*) \;=\; \int \left[\int_0^T L_1\Big(t, x, u_1(t,x), u_2^*(t,\bar x)\Big)\, dt\right] d\mu(\bar x) \tag{2.7}$$

achieves the global minimum. Here the minimum is sought among all measurable functions $v : [0,T] \mapsto U_1$, and among all selections $\bar x \mapsto u_2(\cdot,\bar x) \in R_2(v,\bar x)$ of optimal replies.

Consider the following assumptions.

(H1) The cost functions L_1, L_2 are non-negative and continuous w.r.t. all arguments. The function f is continuous in all variables, continuously differentiable w.r.t. x, and satisfies the bound

$$\big|f(t,x,u_1(t,x),u_2)\big| \;\le C\, \big(1+|x|\big) \qquad \text{for all} \quad x,\, t,\, u_2 \in U_2 \tag{2.8}$$

and for all u_1 as in (2.2), with $v \in U_1$.

(H2) For every t, x, the function $f(t,x,u_1,u_2)$ is affine w.r.t. the variables u_1, u_2. The cost function L_1 is convex w.r.t. the variables u_1, u_2. Finally, the cost function L_2 has the form

$$L_2(t,x,u_1,u_2) \;=\; L_{21}(t,x)u_1 + L_{22}(t,x,u_2)$$

with L_{22} convex in the variable u_2.

(H3) The set of admissible control values U_1, U_2 are compact, convex.

Under the above assumptions, our main theorem provides the existence of a Stackelberg equilibrium solution.

Theorem 1. *Let the assumptions (H1)–(H3) hold, and let μ be a probability measure on the set of initial data. Assume that the leading player has at least one strategy $v(\cdot)$ with finite expected cost. Then the differential game with dynamics (1.1) and cost functionals (1.3) admits a Stackelberg equilibrium solution, among all feedback controls for the leading player having of the form (2.2).*

Proof. 1. Let $t \mapsto v(t) \in U_1$ be any measurable control for the leading player, yielding the feedback $u_1 = u_1(t,x)$ in (2.2). Under the assumption (H1)–(H3) it is well known that, for every initial data $\bar{x}$, the set of optimal replies $u_2(\cdot,\bar{x})$ is non-empty and weakly closed. See for example [9], or Chapter 5 in [5] for details. More precisely, if $u_{2,\nu} : [0,T] \mapsto U_2$ is a minimizing sequence of control functions for Player 2, and $u_{2,\nu} \rightharpoonup u_2^*$ weakly in $\mathbf{L}^1([0,T])$ as $\nu \to \infty$, then the corresponding trajectories satisfy $x_\nu(t) \to x^*(t)$ uniformly for $t \in [0,T]$. Moreover, for $i = 1\,,2$

$$\int_0^T L_i\Big(t, x^*(t), u_1(t,x^*(t)), u_2^*(t)\Big)\,dt \;\leq\; \liminf_{\nu\to\infty} \int_0^T L_i\Big(t, x_\nu(t), u_1(t,x_\nu(t)), u_{2,\nu}(t)\Big)\,dt\,. \tag{2.9}$$

This proves that, for every fixed initial data $\bar{x}$, the set of best replies $R_2(v,\bar{x})$ is non-empty. A similar argument shows that, within this family of best replies, we can choose one that minimizes the cost to Player 1. Indeed, assume that $u_{2,\nu} \in R_2(v,\bar{x})$ for every $\nu \geq 1$ and

$$\begin{aligned}\lim_{\nu\to\infty} \int_0^T L_1\Big(t, x_\nu(t), u_1(t,x_\nu(t)), u_{2,\nu}(t)\Big)\,dt \;&=\; \inf_{u_2\in R_2(v,\bar{x})} \int_0^T L_1\Big(t, x(t), u_1(t,x(t)), u_2(t)\Big)\,dt\\ &\doteq\; m_1(v,\bar{x})\,.\end{aligned} \tag{2.10}$$

Extracting a subsequence, we can achieve the weak convergence $u_{2,\nu} \rightharpoonup u_2^*$ and the uniform convergence $x_\nu \to x^*$. The convexity of L_1 w.r.t. u_2 now yields

$$\int_0^T L_1\Big(t, x^*(t), u_1(t,x^*(t)), u_2^*(t)\Big)\,dt \;\leq\; \liminf_{\nu\to\infty} \int_0^T L_1\Big(t, x_\nu(t), u_1(t,x_\nu(t)), u_{2,\nu}(t)\Big)\,dt\,. \tag{2.11}$$

We conclude that, given a control v for the first player, for each $\bar{x}$ there exists a best reply $u_2^*(\cdot) \in R_2(v,\bar{x})$ which is optimal for Player 1 (restricted to the set of best replies).

2. In this step we establish a lower semicontinuity property of the map $(v,\bar{x}) \mapsto m_1(v,\bar{x})$ defined at (2.10). Namely, consider a converging sequence of initial points $\bar{x}_k \to \bar{x}$, and a weakly convergent sequence of controls for the leading player: $v_k \rightharpoonup v$. Let $u_{2,k} = u_2(\cdot, v_k, \bar{x}_k) \in R_2(v_k,\bar{x}_k)$ be best replies of Player 2 which are optimal for Player 1 (restricted to the set of best replies). By extracting a subsequence and relabeling, we can assume the weak convergence $u_{2,k} \rightharpoonup u_2^*$ and the uniform convergence of the corresponding trajectories $x_k \to x^*$ on $[0,T]$. We claim that $u_2^* \in R_2(v,\bar{x})$.

Indeed, consider any control $t \mapsto u_2^\sharp(t) \in U_2$ for the second player, and call $x^\sharp, x_k^\sharp$ respectively the solutions of

$$\dot{x}^\sharp(t) \;=\; f(t, x^\sharp(t),\, \Psi(x^\sharp(t))\cdot v(t),\, u_2^\sharp(t))\,, \qquad\qquad x^\sharp(0) \;=\; \bar{x}\,,$$

$$\dot{x}_k^\sharp(t) \;=\; f(t, x_k^\sharp(t),\, \Psi(x_k^\sharp(t))\cdot v_k(t),\, u_2^\sharp(t))\,, \qquad\qquad x_k^\sharp(0) \;=\; \bar{x}_k\,.$$

By the weak convergence $v_k \to v$ and by the linearity of f w.r.t. u_1, u_2, we have the convergence $x_k^\sharp(t) \to x^\sharp(t)$ uniformly on $[0,T]$. Moreover,

$$\begin{aligned}
&\int_0^T \Big[L_{21}\big(t, x^\sharp(t)\big)\Psi(x^\sharp(t))\cdot v(t) + L_{22}\big(t, x^\sharp(t), u_2^\sharp(t)\big)\Big]\,dt \\
&= \lim_{k\to\infty}\int_0^T \Big[L_{21}\big(t, x_k^\sharp(t)\big)\Psi(x_k^\sharp(t))\cdot v_k(t) + L_{22}\big(t, x_k^\sharp(t), u_2^\sharp(t)\big)\Big]\,dt \\
&\geq \liminf_{k\to\infty}\int_0^T \Big[L_{21}\big(t, x_k(t)\big)\Psi(x_k(t))\cdot v_k(t) + L_{22}\big(t, x_k(t), u_{2,k}(t)\big)\Big]\,dt \\
&\geq \int_0^T \Big[L_{21}\big(t, x^*(t)\big)\Psi(x^*(t))\cdot v(t) + L_{22}\big(t, x^*(t), u_2^*(t)\big)\Big]\,dt\,.
\end{aligned}$$

This proves our claim.

As a consequence, given a sequence $(v_k, \bar{x}_k)$ such that $v_k \rightharpoonup v$ and $\bar{x}_k \to \bar{x}$, we have

$$m_1(v, \bar{x}) \;\leq\; \liminf_{k\to\infty} m_1(v_k, \bar{x}_k)\,. \tag{2.12}$$

In particular, for a given control $v(\cdot)$ the map $\bar{x} \mapsto m_1(v, \bar{x})$ is lower semicontinuous, hence measurable. Therefore, for any probability measure μ on the set of initial data, the integral

$$J_1(v) \;\doteq\; \int m_1(v, \bar{x})\,d\mu(\bar{x}) \tag{2.13}$$

is well defined (possibly taking the value $+\infty$).

3. Next, consider a minimizing sequence of controls $v_k(\cdot)$ for Player 1, so that

$$\lim_{k\to\infty} \int m_1(v_k, \bar{x})\,d\mu(\bar{x}) \;=\; \inf_{v(\cdot)} \int m_1(v, \bar{x})\,d\mu(\bar{x})\,. \tag{2.14}$$

By possibly extracting a subsequence and reabeling, we can assume the weak convergence $v_k \rightharpoonup v^*$ in $\mathbf{L}^1([0,T])$. We claim that v^* is an optimal strategy for Player 1. Indeed, all functions $\bar{x} \mapsto m_1(v_k, \bar{x})$ are non-negative and lower semicontinuous. By (2.12) and Fatou's lemma,

$$\begin{aligned}
\int m_1(v^*, \bar{x})\,d\mu(\bar{x}) \;&\leq\; \int \liminf_{k\to\infty} m_1(v_k, \bar{x})\,d\mu(\bar{x}) \\
&\leq\; \liminf_{k\to\infty} \int m_1(v_k, \bar{x})\,d\mu(\bar{x}) \;=\; \inf_{v(\cdot)} \int m_1(v, \bar{x})\,d\mu(\bar{x})\,.
\end{aligned}$$

3 Necessary conditions for optimality

By the analysis at (1.12)-(1.13), a weakly optimal strategy $u_1^*(t,x)$ for the leading player must be optimal for the following problem.

$$\text{Minimize:} \qquad J[u] \;\doteq\; \int \left[\int_0^T L\Big(t, x(t), \xi(t), u(t, x(t))\Big)\,dt\right]\,d\mu(\bar{x})\,, \tag{3.1}$$

$$
\text{subject to:} \qquad \begin{cases} \dot{x} &= f(t,x,\xi,u)\,, \\ \\ \dot{\xi} &= g(t,x,\xi,u,\,u_x)\,, \end{cases} \qquad \begin{cases} x(0) &= \bar{x}\,, \\ \\ \xi(T) &= 0\,. \end{cases} \tag{3.2}
$$

Here $x, \xi \in I\!R^n$. Moreover, with slight abuse of notation, we rename

$$
\begin{aligned} f(t,x,\xi,u) &\doteq f\big(t,x,u,u_2^\sharp(t,x,u,\xi)\big)\,, \\ \\ L(t,x,\xi,u) &\doteq L_1\big(t,x,u,u_2^\sharp(t,x,u,\xi)\big)\,, \end{aligned} \tag{3.3}
$$

with $u^\sharp$ given at (1.10). Finally,

$$
g(t,x,\xi,u,v) \doteq -\xi\cdot\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u_1}\,v\right)-\left(\frac{\partial L_2}{\partial x}+\frac{\partial L_2}{\partial u_1}\,v\right), \tag{3.4}
$$

where f and L_2 are evaluated at the point $(t,x,u,u_2^\sharp(t,x,u,\xi))$. We regard (3.1)-(3.2) as a problem of optimal control on the infinite dimensional space whose elements are couples of functions $(x,\xi): I\!R^n \mapsto I\!R^n \times I\!R^n$, depending on the initial point $\bar{x} \in I\!R^n$. The infimum is sought over all measurable control functions $u : [0,T] \mapsto \mathcal{U}$, where $\mathcal{U}$ is a family of admissible functions $\omega : I\!R^n \mapsto I\!R^m$, sufficiently regular so that the corresponding evolution of the variables x, ξ in (3.2) is well defined. For example, we can impose that for each fixed t the function $u(t,\cdot)$ be affine. In this case, $\mathcal{U}$ would be the family of all polynomial functions of degree one in the variables $x_1,\ldots,x_n$. Another natural choice is to take $\mathcal{U}$ as the family of all $\mathcal{C}^2$ functions of the variable x.

In the following, we seek necessary conditions for optimality of an admissible control $u^* : [0,T] \mapsto \mathcal{U}$, for the problem (3.1)-(3.2). After a renaming of variables, these immediately yield necessary conditions for the weak optimality of a feedback control $u_1 = u_1^*(t,x)$ for the leading player in a Stackelberg game.

Given a control $u = u(t,x)$, consider a family of perturbed solutions of (3.2) having the form

$$
\begin{cases} x^\varepsilon(t,\bar{x}) &= x(t,\bar{x}) + \varepsilon X(t,\bar{x}) + o(\varepsilon)\,, \\ \\ \xi^\varepsilon(t,\bar{x}) &= \xi(t,\bar{x}) + \varepsilon Z(t,\bar{x}) + o(\varepsilon)\,. \end{cases} \tag{3.5}
$$

Linearizing (3.2) around the reference trajectory $t \to (x(t,\bar{x}), \xi(t,\bar{x}))$, we obtain a linear equation for the first order perturbations X, Z, namely

$$
\begin{pmatrix} \dot{X} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} f_x + f_u u_x & f_\xi \\ g_x + g_u u_x + g_v u_{xx} & g_\xi \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix}, \tag{3.6}
$$

with boundary conditions

$$
X(0,\bar{x}) = 0\,, \qquad\qquad Z(T,\bar{x}) = 0\,. \tag{3.7}
$$

Next, for each fixed $\bar{x} \in I\!R^n$, let the couple of functions $(P,Q) : [0,T] \mapsto I\!R^n \times I\!R^n$ provide a solution to the dual system

$$
\begin{pmatrix} \dot{P} \\ \dot{Q} \end{pmatrix} = -\begin{pmatrix} f_x + f_u u_x & g_x + g_u u_x + g_v u_{xx} \\ f_\xi & g_\xi \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} - \begin{pmatrix} L_x + L_u u_x \\ L_\xi \end{pmatrix}, \tag{3.8}
$$

with boundary conditions

$$Q(0,\bar{x}) \;=\; 0\,, \qquad\qquad P(T,\bar{x}) \;=\; 0\,. \tag{3.9}$$

By construction, for any solution $\begin{pmatrix} X \\ Z \end{pmatrix}$ of (3.6) and any solution $\begin{pmatrix} P \\ Q \end{pmatrix}$ of (3.9), integrating w.r.t. the measure μ one obtains

$$\frac{d}{dt}\int \Big[X(t,\bar{x})\,P(t,\bar{x}) + Z(t,\bar{x})\,Q(t,\bar{x})\Big]\,d\mu(\bar{x}) \;=\; -\int \Big[(L_x + L_u u_x)X + L_\xi Z\Big]\,d\mu(\bar{x})\,. \tag{3.10}$$

In the following discussion, we shall always assume that the functions f, g, L satisfy the following assumptions.

(A1) The functions f, g, L are continuous in all variables, and continuously differentiable w.r.t. x, ξ, u, v.

(A2) Every admissible feedback control $\omega(\cdot) \in \mathcal{U}$ is twice continuously differentiable.

(A3) The probability measure μ has compact support.

In the following, given initial data $(\bar{x}, \bar{\xi})$, we denote by $t \mapsto x(t;\, \bar{x}, \bar{\xi})$ the solution to the evolution equations in (3.2) with initial data

$$x(0) \;=\; \bar{x}\,, \qquad\qquad \xi(0) \;=\; \bar{\xi}\,.$$

Moreover, we call $\xi^\sharp(\bar{x}) = \xi(0)$ the initial value for ξ in the solution to the boundary value problem (3.2).

Theorem 2 (necessary conditions for optimality). *Let the above assumptions (A1)–(A3) hold. Let $u = u(t,x)$ be an optimal feedback control for the problem (3.1)–(3.2), piecewise continuous w.r.t. time. Call $x(t,\bar{x}), \xi(t,\bar{x})$ the corresponding trajectories, which we assume remain uniformly bounded as $\bar{x}$ ranges in the support of μ. Let the couple of dual functions $(P,Q) : [0,T]\times \mathbb{R} \mapsto \mathbb{R}^n \times \mathbb{R}^n$ provide a solution to (3.8)-(3.9) and assume that the $n\times n$ Jacobian matrix*

$$D_{\bar{\xi}}\,\xi(T;\,\bar{x},\,\bar{\xi}) \;=\; \left(\frac{\partial \xi_i(T;\,\bar{x},\,\bar{\xi})}{\partial \bar{\xi}_j}\right) \tag{3.11}$$

is invertible, for all $\bar{x}$ in the support of μ and $\bar{\xi} = \xi^\sharp(\bar{x})$.

Then, for a.e. $t \in [0,T]$, the feedback control $u(t,\cdot) \in \mathcal{U}$ provides a global minimizer to the functional

$$\begin{aligned} J\Big(t,\omega(\cdot)\Big) \;\doteq\; & \int P(t,\bar{x})\cdot f\Big(t,\, x(t,\bar{x}),\, \xi(t,\bar{x}),\, \omega(t,x(t,\bar{x}))\Big)\,d\mu(\bar{x}) \\ & + \int Q(t,\bar{x})\cdot g\Big(t,\, x(t,\bar{x}),\, \xi(t,\bar{x}),\, \omega(t,x(t,\bar{x})),\, \omega_x(t,x(t,\bar{x}))\Big)\,d\mu(\bar{x}) \\ & + \int L\Big(t, x(t,\bar{x}), \xi(t,\bar{x}), \omega(t,x(t,\bar{x}))\Big)\,d\mu(\bar{x})\,. \end{aligned} \tag{3.12}$$

within the family of all admissible control functions $\omega(\cdot) \in \mathcal{U}$.

Proof. 1. Assume that, for some $\tau \in]0,T]$ where u is continuous, the feedback $u(\tau,\cdot)$ does not satisfy the above minimality condition. Then there exists an admissible control function $\omega : I\!R^n \mapsto I\!R^m$, $\omega(\cdot) \in \mathcal{U}$ such that

$$J\big(\tau, \omega(\cdot)\big) \ < \ J\big(\tau, u(\tau,\cdot)\big). \tag{3.13}$$

Consider the family of "needle variations" of u, defined as

$$u_\varepsilon(t,x) \ = \ \begin{cases} \omega(x) & \text{if } \ t \in [\tau-\varepsilon,\,\tau]\,, \\ u(t,x) & \text{if } \ t \notin [\tau-\varepsilon,\,\tau]\,. \end{cases} \tag{3.14}$$

We claim that, for $\varepsilon > 0$ sufficiently small, $J[u_\varepsilon] < J[u]$, contradicting the optimality of u.

2. We claim that, for all $\varepsilon \in]0, \varepsilon_0]$ sufficiently small and every initial data $\bar x$ in the support of μ, a solution $\big(x_\varepsilon(t,\bar x),\, \xi_\varepsilon(t,\bar x)\big)$ of (3.2) corresponding to the control u_ε does exist.

Indeed, consider the map

$$\Phi^{\bar x,\varepsilon} : \bar\xi \ \mapsto \ \xi(T),$$

where $\xi(T)$ is the terminal value of the solution to the Cauchy problem

$$\begin{cases} \dot x \ = \ f(t,x,\xi,u_\varepsilon)\,, \\[2mm] \dot \xi \ = \ g(t,x,\xi,u_\varepsilon,\, u_{\varepsilon,x})\,, \end{cases} \tag{3.15}$$

with initial data

$$x(0) \ = \ \bar x\,, \qquad\qquad \xi(0) \ = \ \bar\xi\,. \tag{3.16}$$

For any $\bar x$ in the support of the probability measure μ, when $\varepsilon = 0$ and $\bar\xi = \xi^\sharp(\bar x)$, by assumption we have $\xi(T) = 0$. Using the transversality assumption (3.11) and the implicit function theorem, we obtain a neighborhood $\mathcal{N}_{\bar x}$ of $\bar x$ and $\varepsilon_{\bar x} > 0$ with the following property. For all $\varepsilon \in [0, \varepsilon_{\bar x}]$ and every initial point $\bar y \in \mathcal{N}_{\bar x}$ there exists an initial value $\bar\xi = \bar\xi_\varepsilon(\bar y)$ such that the corresponding solution of (3.15) with initial data

$$x(0) \ = \ \bar y\,, \qquad\qquad \xi(0) \ = \ \bar\xi_\varepsilon(\bar y)$$

satisfies $\xi(T) = 0$. Since the support of μ is bounded, hence compact, it can be covered with finitely many neighborhoods $\mathcal{N}_{\bar x_i}$, $i = 1, \ldots, \kappa$. Choosing

$$\varepsilon_0 \ = \ \min\{\varepsilon_{\bar x_i}\,; \ \ i = 1, \ldots, \kappa\}$$

our claim is proved.

3. We now estimate the difference in the costs: $J[u_\varepsilon] - J[u]$. For $t < \tau - \varepsilon$, the first order perturbations $X(t,\bar x)$, $Z(t,\bar x)$ in (3.5) satisfy (3.6), (3.10). Moreover, at the time $t = \tau$ where the needle variation in the control takes place, we have

$$\begin{aligned} X(\tau+,\,\bar x) \ &= \ \lim_{\varepsilon\to 0+} \frac{x^\varepsilon(\tau,\bar x) - x(\tau,\bar x)}{\varepsilon} \\ &= \ X(\tau-,\,\bar x) + f\big(\tau,\, x(\tau,\bar x),\, \xi(\tau,\bar x),\, \omega(x(\tau,\bar x))\big) - f\big(\tau,\, x(\tau,\bar x),\, \xi(\tau,\bar x),\, u(\tau, x(\tau,\bar x))\big)\,, \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
Z(\tau+,\bar{x}) &= \lim_{\varepsilon\to 0+} \frac{\xi^\varepsilon(\tau,\bar{x}) - \xi(\tau,\bar{x})}{\varepsilon} \\
&= Z(\tau-,\,\bar{x}) + g\Big(\tau,\, x(\tau,\bar{x}),\, \xi(\tau,\bar{x}),\, \omega(x(\tau,\bar{x})),\, \omega_x(x(\tau,\bar{x}))\Big) \\
&\qquad - g\Big(\tau,\, x(\tau,\bar{x}),\, \xi(\tau,\bar{x}),\, u(\tau,x(\tau,\bar{x})),\, u_x(\tau,x(\tau,\bar{x}))\Big)\,.
\end{aligned}
\tag{3.18}
$$

Differentiating the total cost w.r.t. ε at $\varepsilon = 0+$, and using (3.10), the boundary conditions (3.7) and (3.9) to eliminate boundary terms, and finally (3.17)-(3.18), we obtain

$$
\begin{aligned}
&\frac{d}{d\varepsilon} \int \left[\int_0^T L\Big(t, x^\varepsilon(t,\bar{x}), \xi^\varepsilon(t,\bar{x}), u^\varepsilon(t, x^\varepsilon(t,\bar{x}))\Big)\,dt\right] d\mu(\bar{x}) \Bigg|_{\varepsilon=0+} \\
&= \int \left[\int_0^T \Big((L_x + L_u u_x)X + L_\xi Z\Big)\,dt\right] d\mu(\bar{x}) \\
&\quad + \int \Big[L\Big(\tau, x(\tau,\bar{x}), \xi(\tau,\bar{x}), \omega(x(\tau,\bar{x}))\Big) - L\Big(\tau, x(\tau,\bar{x}), \xi(\tau,\bar{x}), u(\tau,x)\Big)\Big]\,d\mu(\bar{x}) \\
&= -\int_0^T \frac{d}{dt}\left[\int (XP + ZQ)\,d\mu(\bar{x})\right] dt + \int \Big[L(\tau,\omega) - L\big(\tau, u(\tau)\big)\Big]\,d\mu(\bar{x}) \\
&= \int \Big[-P(T,\bar{x})\,X(T,\bar{x}) - Z(T,\bar{x})\,Q(T,\bar{x}) + P(0,\bar{x})\,X(0,\bar{x}) + Z(0,\bar{x})\,Q(0,\bar{x})\Big]\,d\mu(\bar{x}) \\
&\quad + \int \Big[P(\tau,\bar{x})\big(X(\tau+,\bar{x}) - X(\tau-,\bar{x})\big) + Q(\tau,\bar{x})\big(Z(\tau+,\bar{x}) - Z(\tau-,\bar{x})\big)\Big]\,d\mu(\bar{x}) \\
&\quad + \int \Big[L(\tau,\omega) - L\big(\tau,u(\tau)\big)\Big]\,d\mu(\bar{x}) \\
&= \int \Big[P(\tau)\,\big(f(\tau,\omega) - f(\tau,u(\tau))\big) + Q(\tau)\,\big(g(\tau,\omega,\omega_x) - g(\tau,u(\tau),u_x(\tau))\big)\Big]\,d\mu(\bar{x}) \\
&\quad + \int \Big[L(\tau,\omega) - L\big(\tau,u(\tau)\big)\Big]\,d\mu(\bar{x}) \\
&= J\Big(\tau,\omega(\cdot)\Big) - J\Big(\tau,u(\tau,\cdot)\Big) \;<\; 0\,.
\end{aligned}
\tag{3.19}
$$

because of the assumption (3.13). This shows that, for $\varepsilon > 0$ small, the feedback control u_ε in (3.14) achieves a lower cost, contradicting the assumption that u is optimal. □

In the case where g does not depend on u_x, the minimizer of the functional $J(t,\,\omega(\cdot))$ can be constructed pointwise, for each given $\bar{x} \in \mathbb{R}^n$. The dependence on the first derivative u_x makes the problem "non-classical".

As stated in Theorem 2, the necessary conditions are hard to implement, because at each time τ the optimization has to be carried out w.r.t. a probability measure μ given on the set of initial data, rather than on the probability distribution of the state at time τ. Following [1, 2], in the one-dimensional case one can write these equations in a more appealing form.

We begin by representing the probability measure μ as the push-forward of Lebesgue measure on $[0,1]$ by a nondecreasing map $y \mapsto \bar{x}(y)$. This map is defined by the property

$$\mu\big(]-\infty,\alpha]\big) \;=\; \sup\{y\in[0,1]\,;\;\bar{x}(y)\le\alpha\} \qquad\qquad \text{for every } \alpha\in I\!R\,.$$

Denoting by $x(t,y),\xi(t,y)$ the solution of (3.2) with boundary data

$$x(0) \;=\; \bar{x}(y)\,, \qquad\qquad \xi(T) \;=\; 0\,,$$

the expected cost in (3.1) can be rewritten as

$$J[u] \;=\; \int_0^1 \left[\int_0^T L\Big(t,x(t,y),\xi(t,y),u(t,x(t,y))\Big)\,dt\right]dy\,. \tag{3.20}$$

Assume that the map $y\mapsto x(t,y)$ is strictly increasing, for every $t\in[0,T]$. Observe that the map $\phi(t,y)\doteq[x_y(t,y)]^{-1}$ can then be obtained as the solution to

$$\phi_t \;=\; -\,(f_x x_y + f_\xi \xi_y + f_u u_y)\,\phi^2\,, \qquad\qquad \phi(0,y) \;=\; \frac{1}{x_y(0,y)}\,.$$

If $\mathcal{U}$ is the family of all $\mathcal{C}^2$ control functions and u is an optimal feedback control satisfying the assumptions of Theorem 2, then at a.e. time $\tau\in[0,T]$ the map $u(\tau,\cdot)$ should be a global minimizer for (3.12). Replacing an integration w.r.t. the probability measure μ with an integration w.r.t. $y\in[0,1]$, we find that the function $u(t,y)=u(t,x(t,y))$ must satisfy

$$\begin{aligned} 0 \;&=\; \int_0^1 \Big(P\,f_u\,w + Qg_u w + Qg_v w_x + L_u w\Big)\,dy \\ &=\; \int_0^1 \Big(P\,f_u + Qg_u - (Qg_v\phi)_y + L_u\Big)w\,dy\,, \end{aligned} \tag{3.21}$$

for every function $w\in\mathcal{C}^2([0,1])$ which vanishes at $y=0$ and at $y=1$. Notice that in (3.21) one of the terms was integrated by parts, using the identity $w_x=\phi(t,y)w_y$. Since w is arbitrary, the above necessary conditions yield

$$Pf_u + Qg_u - \big(Qg_v\phi\big)_y + L_u \;=\; 0\,. \tag{3.22}$$

At a given time t, it is understood that the left hand side of (3.22) should be computed at the point $\big(t,x(t,y),\xi(t,y),u(t,x(t,y))\big)$, for any $y\in[0,1]$. Moreover, by (3.2), (3.9), and choosing a perturbation w which does not vanish on the boundary, we obtain the following boundary conditions:

$$\begin{aligned} (Qg_v)\Big|_{y=0} \;=\; (Qg_v)\Big|_{y=1} \;&=\; 0\,, \\ x(0,y) \;=\; \bar{x}(y)\,, \qquad \Big(Qg_u - \big(Qg_v\phi\big)_y + L_u\Big)\Big|_{t=T} \;&=\; 0\,. \end{aligned} \tag{3.23}$$

4 Nonexistence of optimal feedback strategies

Aim of this section is to exhibit a class of differential games where no optimal feedback control for the leading player exists, if this feedback is allowed to range over the family of all piecewise constant functions of t, x.

Consider a scalar system which is linear w.r.t. the control variables

$$\dot{x} \;=\; f_0(x) + f_1(x)u_1 + f_2(x)u_2\,, \tag{4.1}$$

where f_0, f_1, f_2 are $\mathcal{C}^1$ functions such that

$$|f_0(x)| \;\leq\; C\,, \qquad\qquad \frac{1}{C} \;\leq\; f_1(x),\; f_2(x) \;\leq\; C \tag{4.2}$$

for some constant $C > 0$ and every $x \in I\!R$. Let the cost functions for the two players have the form

$$J_i \;=\; \int_0^T L_i\big(x(t),\, u_i(t)\big)\, dt \qquad\qquad i = 1, 2\,, \tag{4.3}$$

with

$$L_2(x, u_2) \;\geq\; \alpha\big(|x| + |u_2|\big) - \beta\,, \qquad \text{for all} \quad \big(x, u_2\big) \in I\!R^2, \tag{4.4}$$

for some constants $\alpha, \beta > 0$. Let $\big(\bar{x}_1, \bar{u}_1\big)$ be the global minimizer for $L_1(x, u_1)$. Without loss of generality we can assume that $\bar{x}_1 = 0$, $\bar{u}_1 = 0$, and

$$L_1(0, 0) \;=\; 0 \;\leq\; L_1(x, u_1) \qquad\qquad \text{for all } (x, u_1) \in I\!R^2. \tag{4.5}$$

Under these assumptions we claim that, for any $\bar{\eta}$ and $\varepsilon > 0$, the leading player can implement a piecewise constant "patchy" feedback $u_1 = u_1(t, x)$ such that, for every initial data $\bar{x} \in [-\bar{\eta}, \bar{\eta}]$ and any best reply $u_2(\cdot;\, u_1, \bar{x})$ of Player 2, its cost is $\leq \varepsilon$. As a consequence, if μ is any probability measure with bounded support, for any $\varepsilon > 0$ there exists a patchy feedback u_1 for the leading player which yields an expected cost $\leq \varepsilon$.

Let $C_1 > 0$ be a constant such that

$$L_2(x, 0) \;\leq\; C_1 \qquad\qquad \text{for} \quad |x| \leq 1\,. \tag{4.6}$$

Let $\bar{\eta}$ be given. It is clearly not restrictive to assume

$$\frac{\bar{\eta}}{2} \;>\; \frac{C_1 + \beta}{\alpha}\,. \tag{4.7}$$

The feedback u_1 is constructed as follows. Define the continuous function $\eta : [0, T] \mapsto I\!R$ by setting (see Fig. 1)

$$\eta(t) \;\doteq\; \begin{cases} \left(1 - \dfrac{t}{\delta}\right)\bar{\eta} & \text{if } \; t \in [0, \delta]\,, \\[2ex] 0 & \text{if } \; t \in [\delta,\, T - \delta]\,, \\[2ex] \left(1 - \dfrac{T - t}{\delta}\right)\bar{\eta} & \text{if } \; t \in [T - \delta,\, T]\,. \end{cases} \tag{4.8}$$

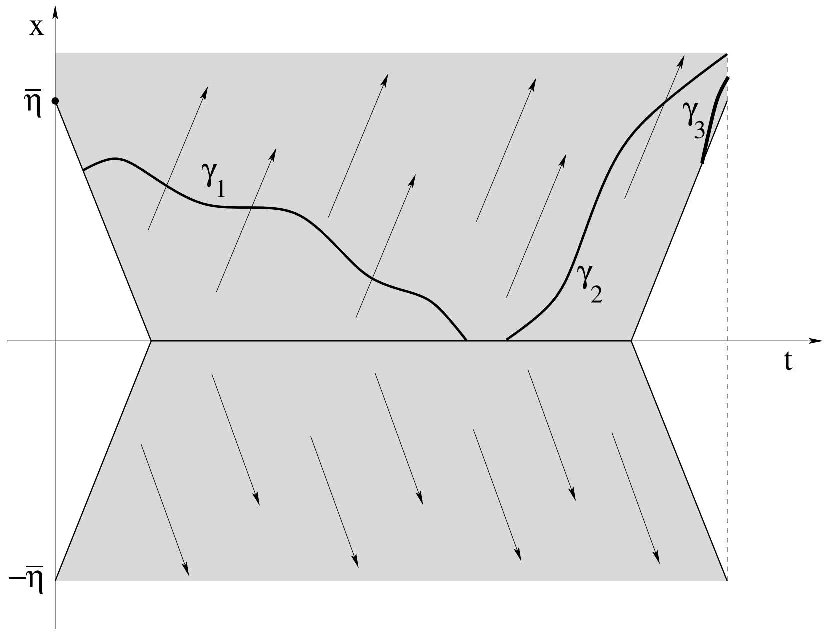

Figure 1: A near-optimal strategy for Player 1. If Player 2 allows the trajectory of the system to enter the shaded area (outside the line $x = 0$), then his strategy is not optimal. The trajectories $\gamma_1, \gamma_2, \gamma_3$ illustrate respectively the cases 1,2,3 in the analysis below.

Since we are assuming that $L_1(0,0) = 0$, we can choose the constant $\delta > 0$ small enough so that

$$\delta C \;<\; 1\,, \qquad\qquad \frac{1}{C}\left(\frac{\bar\eta}{\delta} - C\right) \;\geq\; \bar\eta\,, \tag{4.9}$$

and moreover

$$\int_0^T L_1\big(x(t),0\big)\,dt \;\leq\; \varepsilon \tag{4.10}$$

for every trajectory $x(\cdot)$ such that

$$|x(t)| \leq \eta(t) \qquad\qquad \text{for all} \;\; t \in [0,T]\,. \tag{4.11}$$

Consider a feedback u_1 for the leading player having the form

$$u_1(t,x) \;\doteq\; \begin{cases} -K & \text{if} \quad x < -\eta(t)\,, \\ \\ 0 & \text{if} \quad x \in [-\eta(t)\,,\; \eta(t)]\,, \\ \\ K & \text{if} \quad x > \eta(t)\,, \end{cases} \tag{4.12}$$

We claim that, if the constant K is chosen large enough, then for every initial point $\bar x \in [-\bar\eta,\, \bar\eta]$, the best reply $u_2(\cdot;\, u_1, \bar x)$ of Player 2 yields a trajectory satisfying (4.11). Hence (4.10) holds.

Observe that, by (4.2), Player 2 can steer the system along any absolutely continuous path $x(\cdot)$. Indeed, given $x(\cdot)$, his control function is determined by

$$u_2(t) \;=\; \frac{\dot x(t) - f_0(x(t)) - f_1(x(t))u_1(t,x(t))}{f_2(x(t))}\,. \tag{4.13}$$

In particular, if $x(t)=\eta(t)$, then the corresponding control u_2^η satisfies

$$|u_2^\eta(t)| \;=\; \left|\frac{\dot\eta(t)-f_0(\eta(t))}{f_2(\eta(t))}\right| \;\leq\; \frac{\bar\eta\delta^{-1}+C}{C^{-1}}\,. \tag{4.14}$$

We can now find constants C_1, C_2, not depending of K, such that

$$L_2\big(\eta(t),\,u_2^\eta(t)\big) \;\leq\; C_2 \qquad\qquad \text{for all } t\in[0,T]\,. \tag{4.15}$$

Next, assume that the trajectory $x(\cdot)$ does not satisfy (4.11). To fix the ideas, we assume $x(t)>\eta(t)$ at some time t. If $x(t)<-\eta(t)$, the analysis is entirely similar. Three cases will be considered, illustrated by the paths $\gamma_1,\gamma_2,\gamma_3$ in Fig. 4.

Case 1: $x(t)>\eta(t)$ for $t\in\,]t_1,t_2[$ and $x(t_1)=\eta(t_1)$, $x(t_2)=\eta(t_2)$.

We claim that the trajectory

$$\tilde x(t) \;=\; \begin{cases} \eta(t) & \text{if } \ t\in[t_1,t_2]\,,\\ x(t) & \text{if } \ t\notin[t_1,t_2]\,,\end{cases} \tag{4.16}$$

yields a strictly lower cost to Player 2. Indeed, by (4.13) one has

$$\begin{aligned}\int_{t_1}^{t_2} u_2(t)\,dt \;&=\; \int_{t_1}^{t_2}\frac{\dot x-f_0(x)-f_1(x)K}{f_2(x)}\,dt\\ &\leq\; \frac{|\eta(t_2)-\eta(t_1)|}{C^{-1}}+(t_2-t_1)\frac{C}{C^{-1}}-\frac{C^{-1}K}{C}(t_2-t_1)\\ &\leq\; \big(\delta^{-1}C\bar\eta+C^2-C^{-2}K\big)(t_2-t_1)\,.\end{aligned} \tag{4.17}$$

If K satisfies

$$\frac{K}{C^2} \;>\; \frac{C\bar\eta}{\delta}+C^2+\frac{\beta}{\alpha}+\frac{C_2}{\alpha}\,, \tag{4.18}$$

then by (4.15)

$$\begin{aligned}\int_{t_1}^{t_2} L_2(x,u_2)\,dt \;&\geq\; \int_{t_1}^{t_2}\Big(\alpha\big(|x|+|u_2|\big)\;-\beta\Big)dt \;\geq\; \alpha\Big|\int_{t_1}^{t_2}u_2\,dt\Big|-\beta(t_2-t_1)\\ &\geq\; \Big(\alpha\big(C^{-2}K-\delta^{-1}C\bar\eta-C^2\big)-\beta\Big)(t_2-t_1)\\ &>\; C_2(t_2-t_1) \;\geq\; \int_{t_1}^{t_2}L_2(\eta,u_2^\eta)\,dt\,,\end{aligned} \tag{4.19}$$

This shows that the trajectory (4.16) yields a strictly lower cost to Player 2, as claimed.

Case 2: $x(t)>\eta(t)$ for $t\in\,]t_1,T]$ with $t_1<T-\delta$ and $x(t_1)=\eta(t_1)=0$. Again, we claim that the trajectory (4.16) yields a lower cost to Player 2.

By (4.1) and (4.12) it follows

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^{t} \Big(f_0(x(s)) + f_1(x(s))K + f_2(x(s))u_2(s) \Big)\, ds \\ &\geq (-C + C^{-1}K)(t - t_1) - C \int_{t_1}^{t} |u_2(s)|\, ds\,. \end{aligned} \tag{4.20}$$

Integrating both sides of the above inequality from t_1 to T one obtains

$$\begin{aligned} \int_{t_1}^{T} x(t)\, dt &\geq \int_{t_1}^{T} \Big((-C + C^{-1}K)(t - t_1) - C \int_{t_1}^{t} |u_2(s)|\, ds \Big)\, dt \\ &= \frac{C^{-1}K - C}{2}(T - t_1)^2 - C \int_{t_1}^{T} (T - s)|u_2(s)|\, ds \\ &\geq \frac{C^{-1}K - C}{2}(T - t_1)^2 - C(T - t_1) \int_{t_1}^{T} |u_2(s)|\, ds\,. \end{aligned} \tag{4.21}$$

In turn, this implies

$$\begin{aligned} \int_{t_1}^{T} \big(|x(t)| + |u_2(t)| \big)\, dt &\geq \min_{Z \geq 0} \left\{ \Big| \frac{C^{-1}K - C}{2}(T - t_1)^2 - C(T - t_1)Z \Big|,\ Z \right\} \\ &= \frac{\frac{C^{-1}K - C}{2}(T - t_1)^2}{1 + C(T - t_1)} \ \geq\ \frac{C^{-1}K - C}{2(1 + CT)}\, \delta^2\,. \end{aligned} \tag{4.22}$$

Comparing the second player's costs for the trajectories x and $\tilde{x}$ in (4.16), if the constant K was chosen so that

$$\frac{\alpha\delta^2}{2C(1 + CT)} K \ >\ (C_2 + \beta)T + \frac{C}{2(1 + CT)} \alpha\delta^2, \tag{4.23}$$

we then obtain

$$\begin{aligned} \int_{t_1}^{T} L_2(x(t), u_2(t))\, dt &\geq \int_{t_1}^{T} \Big(\alpha\big(|x| + |u_2|\big) - \beta \Big)\, dt \ \geq\ \frac{C^{-1}K - C}{2(1 + CT)}\, \alpha\delta^2 - \beta T \\ &> C_2 T \ \geq\ \int_{t_1}^{T} L_2(\eta(t), u_2^{\eta}(t))\, dt\,, \end{aligned}$$

proving our claim.

Case 3: $x(t) > \eta(t)$ for $t \in \,]t_1, T]$ with $t_1 \geq T - \delta$ and $x(t_1) = \eta(t_1)$.

Observe that it is not restrictive to assume that $|x(t)| \leq \eta(t)$ for all $t \in [0, t_1]$. Otherwise, as proved in Case 1, the control u_2 for Player 2 would not be optimal. In particular, this assumption implies $x(T - \delta) = 0$. Consider the alternative control function

$$u_2^{\sharp}(t) \ =\ \begin{cases} u_2(t) & \text{if } \ t \in [0,\, T - \delta]\,, \\ 0 & \text{if } \ t \in \,]T - \delta, T]\,. \end{cases} \tag{4.24}$$

and let $x^\sharp(\cdot)$ be the corresponding trajectory. If $\bar\eta/\delta > C$, for $t\in[T-\delta, T]$ this solution is found by solving the Cauchy problem

$$\dot x^\sharp(t) \;=\; f_0(x^\sharp(t))\,, \qquad\qquad x^\sharp(T-\delta) \;=\; 0\,.$$

Since $|f_0(x)|\leq C$ and $C\delta<1$, this implies $|x^\sharp(t)|\leq 1$ for $t\in[T-\delta,\, T]$. By (4.6) we conclude

$$\int_{T-\delta}^T L_2\big(x^\sharp(t), u_2^\sharp(t)\big)\,dt \;=\; \int_{T-\delta}^T L_2\big(x^\sharp(t),\,0\big)\,dt \;\leq\; C_1\delta\,. \tag{4.25}$$

On the other hand, by (4.4) the original trajectory yields a cost

$$\int_{T-\delta}^T L_2\big(x(t), u_2(t)\big)\;dt \;\geq\; \alpha\int_{T-\delta}^{t_1}|u_2(t)|\;dt+\alpha\int_{t_1}^T|x(t)|\;dt-\beta\delta\,. \tag{4.26}$$

Recalling that $x(T-\delta)=0$, we obtain

$$\begin{aligned} x(t_1) \;&=\; \eta(t_1) \;=\; \frac{t_1+\delta-T}{\delta}\,\bar\eta \;=\; \int_{T-\delta}^{t_1}\Big(f_0(x(t))+f_2(x(t))u_2(t)\Big)\,dt \\ &\leq\; C(t_1-T+\delta)+C\int_{T-\delta}^{t_1}|u_2(t)|\,dt\,, \end{aligned}$$

By (4.9), this yields

$$\int_{T-\delta}^{t_1}|u_2(t)|\,dt \;\geq\; \frac{1}{C}\left(\frac{\bar\eta}{\delta}-C\right)(t_1+\delta-T) \;\geq\; (t_1+\delta-T)\,\bar\eta\,. \tag{4.27}$$

Moreover,

$$\int_{t_1}^T|x(t)|\,dt \;\geq\; \int_{t_1}^T\eta(t)\,dt \;=\; \int_{t_1}^T\frac{t+\delta-T}{\delta}\,\bar\eta \;\geq\; (T-t_1)\,\frac{\bar\eta}{2}\,. \tag{4.28}$$

Using (4.27)-(4.28) in (4.26) and recalling the choice of $\bar\eta$ at (4.7) and (4.25), we obtain

$$\begin{aligned} \int_{T-\delta}^T L_2\big(x(t), u_2(t)\big)\;dt \;&\geq\; \alpha\Big((t_1+\delta-T)+(T-t_1)\Big)\frac{\bar\eta}{2}-\beta\delta \;\geq\; \alpha\delta\,\frac{\bar\eta}{2}-\beta\delta \\ &>\; C_1\delta \;\geq\; \int_{T-\delta}^T L_2\big(x^\sharp(t), u^\sharp(t)\big)\,dt\,. \end{aligned}$$

Once again, this shows that the control u_2 does not yield the minimum cost to Player 2.

The previous analysis has shown that, for every $\varepsilon>0$ and every compact set I of initial states, the leading player can design a feedback achieving a cost $\leq\varepsilon$ for every initial point $\bar x\in I$. Clearly, there is no measurable feedback that can yield exactly zero cost. Hence in this setting no optimal feedback can exist.

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