A Bidding Game with Heterogeneous Players

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Abstract

A one-sided limit order book is modeled as a noncooperative game for several players. Agents offer various quantities of an asset at different prices, competing to fulfill an incoming order, whose size is not known a priori. Players can have different payoff functions, reflecting different beliefs about the fundamental value of the asset and probability distribution of the random incoming order. In [4] the existence of a Nash equilibrium was established by means of a fixed point argument.

The main issue discussed in the present paper is whether this equilibrium can be obtained from the unique solution to a two-point boundary value problem, for a suitable system of discontinuous ODEs. Some additional assumptions are introduced, which yield a positive answer. In particular, this is true when there are exactly two players, or when all players assign the same exponential probability distribution to the incoming order. In both of these cases, we also prove that the Nash equilibrium is unique. A counterexample shows that these assumptions cannot be removed, in general.

Keywords: optimality conditions, discontinuous ODE, optimal pricing strategy, bidding game, Nash equilibrium, limit order book.

1 Introduction

This paper is concerned with a continuum model of the limit order book in a stock market, viewed as a noncooperative game for n players. As in [3] our main goal is to study the existence of a Nash equilibrium, determining the optimal bidding strategies of the various agents who submit limit orders.

We assume that an external buyer asks for a random amount of $X > 0$ of shares of a certain asset. This external agent will buy the amount X at the lowest available price, as long as this price does not exceed a given upper bound $\overline{P}$. One or more sellers offer various quantities of this asset at different prices, competing to fulfill the incoming order, whose size is not known a priori.

Having observed the prices asked by his competitors, each seller must determine an optimal strategy, maximizing his expected payoff. Of course, when other sellers are present, asking a higher price for a stock reduces the probability of selling it.

The model introduced in [3] was extended in [4], assuming that agents differ from each other in various respects.

- Each agent assigns a different probability distribution to the random variable X, based on his own beliefs. An optimistic seller expects a large incoming order, which will fill most of the outstanding bids. A pessimistic seller will expect a small order, filling only the lowest priced bids. In the following, we denote by

$$\psi_i(s) \;=\; Prob.\{X > s\} \tag{1.1}$$

  the probability distribution assigned by the i-th player to the random variable X.

- Each agent assigns a different fundamental value p_i to the assets he is putting on sale. In other words, to the i-th agent it would be indifferent to sell his assets at unit price p_i or to keep them.

Existence of a Nash equilibrium, in this more general setting, was recently proved in [4] by means of a topological technique. However, this technique did not provide information about the uniqueness of the solution, or how to construct it. Aim of the present paper is to show that, in several cases, this Nash equilibrium solution can be found by solving a a Boundary Value Problem for a system of ODEs.

Let κ_i be the total amount of assets offered for sale by the i-th agent. We use the Lagrangian variable $\beta \in [0, \kappa_i]$ to label one particular asset. By a **pricing strategy** for the i-th seller we mean a nondecreasing map $\phi_i : [0, \kappa_i] \mapsto [0, \overline{P}]$.

To compute the expected payoff achieved by this strategy, let ϕ_j, $j = 1 \ldots, n$, $j \neq i$ be the pricing strategies adopted by the other agents, and define

$$F_j(p) \;\doteq\; \text{meas}\Big(\{\beta \in [0, \kappa_j]\,; \quad \phi_j(\beta) < p\}\Big). \tag{1.2}$$

Moreover, let

$$\Phi_i(p) \;\doteq\; \sum_{j \neq i} F_j(p) \tag{1.3}$$

be the total amount of assets put on sale at price $< p$ by all the other agents. The expected payoff for the i-th player is then measured by

$$J_i(\phi_i, \Phi_i) \;\doteq\; \int_0^{\kappa_i} \Big(\phi_i(\beta) - p_i\Big) \cdot \psi_i\Big(\beta + \Phi_i(\phi_i(\beta))\Big)\, d\beta\,. \tag{1.4}$$

The integrand in (1.4) contains two factors. The term $\phi_i(\beta) - p_i$ is the difference between the price at which the asset β is put on sale and its actual value to the i-th player. The term $\psi_i\Big(\beta + \Phi_i(\phi_i(\beta))\Big)$ is the subjective probability (according to the i-th player) that the asset β will be actually sold.

For future reference, we record some basic assumptions on the probability distributions (1.1).

(A1) *All maps $s \mapsto \psi_i(s)$, $i = 1, \ldots, n$ are continuously differentiable and satisfy*

$$\psi_i(0) \;=\; 1\,, \qquad \psi_i(+\infty) \;=\; 0, \qquad \psi_i'(s) \;<\; 0 \qquad \textit{for all } s > 0\,, \tag{1.5}$$

$$(\ln \psi_i(s))'' \;\geq\; 0 \quad \textit{for all } s > 0\,. \tag{1.6}$$

Example 1. The assumptions (A1) are satisfied if $\psi_i(s) = e^{-\alpha s}$ or if $\psi_i(s) = (1+s)^{-\alpha}$, for some $\alpha > 0$. On the other hand, (1.6) fails if $\psi_i(s) = e^{-s^2}$

Definition 1. *Let $(\phi_1^*, \ldots, \phi_n^*)$ be an n-tuple of pricing strategies such that no two players put a positive amount of assets for sale at exactly the same price. We say that these strategies provide a* **Nash equilibrium** *if, calling $\Phi_1^*, \ldots, \Phi_n^*$ the corresponding functions in (1.2)-(1.3), one has*

$$J_i(\phi_i, \Phi_i^*) \;\leq\; J_i(\phi_i^*, \Phi_i^*) \tag{1.7}$$

for every $i = 1, \ldots, n$ and every other pricing strategy $\phi_i : [0, \kappa_i] \mapsto [0, \overline{P}]$ for the i-th player.

Remark 1. As a basic modeling assumption, the asset will always be bought from whoever seller offers the lowest price. However, if two or more sellers put a positive amount of asset for sale exactly at the same price, one needs to specify which of the agents has selling priority. This would require an additional discussion of the model. However, in a Nash equilibrium this situation never happens, because the player that does not have priority can always improve his expected payoff by slightly reducing his price.

In the general case where agents have different payoff functions, the existence of at least one Nash equilibrium was recently established in [4]. The proof relied on a sequence of discrete approximations, combined with a topological fixed point argument. Apart from the case of players with the same payoff functions, studied in [3], the uniqueness of Nash equilibria remains an open problem.

In the present paper we seek a more explicit way to construct the functions $(F_1, \ldots, F_n)$ in (1.2), and hence determine the equilibrium solution. Toward this goal, we introduce a boundary value problem for a a system on n ODEs, determining the functions F_j. These equations are obtained by adding some auxiliary inequalities to the set of necessary conditions for optimality derived in [3]. Compared with classical literature, this problem is far from standard. It consists of a system of ODEs

$$F_i'(p) \;=\; Q_i(p, F_1(p), \ldots, F_n(p)), \qquad\qquad i = 1, \ldots, n, \tag{1.8}$$

where the right hand sides are discontinuous along the hyperplanes where $F_j = \kappa_j$. The boundary data take the form

$$F_i(\overline{P}) \;=\; \kappa_i \qquad \text{for all } i \text{ except at most one,} \tag{1.9}$$

together with

$$F_i(p_A) \;=\; 0 \qquad\qquad i = 1, \ldots, n, \tag{1.10}$$

where $p_A \in [0, \overline{P}]$ is a point to be determined.

In Section 2 we give a precise description of the right hand side of (1.8), and study its properties. In Section 3 we prove the existence and uniqueness of the solution to the boundary value problem (1.8)–(1.10). As shown by a counterexample, this solution may not yield a Nash equilibrium, in general. This is because the ODEs in (1.8) are obtained by imposing some additional inequalities which are not implied by the optimality conditions. Our main result, proved in Section 4, provides various sufficient conditions in order that the solution to (1.8)–(1.10) yield a Nash equilibrium. In particular, this is true if either (i) there are exactly two players, or (ii) in (1.4) the probability distribution functions have the form $\psi_1(s) = \cdots = \psi_n(s) = e^{-\lambda s}$, while $p_1, \ldots, p_n$ can be arbitrary. In these two cases, the analysis in Section 5 shows that the Nash equilibrium is unique. Namely, there are no other equilibria besides the one provided by the two-point boundary value problem (1.8)–(1.10). We recall that, in the special case of n identical players, a uniqueness result was proved in [3].

In recent literature, models of the limit order book have been studied in [1, 5, 8, 9, 11]. For a general introduction to game theory and Nash equilibria we refer to [2, 6, 12].

2 An algebraic problem

The main goal of this section is to study a class of functions Q_i appearing on the right hand side of the ODEs (1.8). As a first step, we consider a set of linear equations with constraints.

Lemma 1. *Given $n \geq 2$ numbers*

$$0 < a_1 \leq a_2 \leq \cdots \leq a_n, \tag{2.1}$$

there exists a unique n-tuple $(x_1, x_2, \ldots, x_n)$ of non-negative numbers with the following properties:

$$\sum_{j \neq i} x_j \leq a_i \qquad i = 1, \ldots, n, \tag{2.2}$$

$$x_i > 0 \qquad \Longrightarrow \qquad \sum_{j \neq i} x_j = a_i, \tag{2.3}$$

$$\sum_{i=1}^{n} x_i > 0. \tag{2.4}$$

Proof. 1. Consider the integer

$$m = \min \left\{ k \in \{2, \ldots, n\}; \quad \frac{1}{k-1} \sum_{j=1}^{k} a_j \leq a_{k+1} \right\}, \tag{2.5}$$

where $a_{n+1} \doteq +\infty$, so that the inequality in (2.5) is always satisfied when $k = n$. It is straightforward to check that all conditions (2.2)–(2.4) are satisfied by setting

$$x_i = \frac{1}{m-1} \left(\sum_{1 \leq j \leq m,\ j \neq i} a_j - (m-2) a_i \right) \qquad i = 1, \ldots, m, \tag{2.6}$$

$$x_{m+1} = x_{m+2} = \cdots = x_n = 0. \tag{2.7}$$

2. It remains to prove that the solution is unique. As a first step, we claim that any solution $(x_1, \ldots, x_n)$ must be of the form (2.6)-(2.7), for some integer $m \in \{2, \ldots, n\}$. Indeed, assume that $x_\ell = 0$ but $x_k > 0$ for some $\ell < k$. Then (2.2) implies

$$\sum_{j \neq k} x_j < \sum_{j=1}^{n} x_j \leq a_\ell \leq a_k .$$

Hence (2.3) is not satisfied when $i = k$.

3. For a given m, the formulas (2.6)-(2.7) uniquely determine the n-tuple $(x_1, \ldots, x_n)$. To complete the proof of uniqueness, it remains to show that there exists at most one integer m such that this n-tuple satisfies the conditions (2.2)–(2.4).

Assume, on the contrary, that $(x_1, \ldots, x_m, 0, \ldots, 0)$ and $(x'_1, \ldots, x'_{m'}, 0, \ldots, 0)$ are two distinct solutions, with

$$2 \leq m < m' \leq n, \qquad x'_{m'} > 0 .$$

Since $(x_1, \ldots, x_m, 0, \ldots, 0)$ is a solution, by (2.2) it follows

$$\frac{1}{m-1} \sum_{j=1}^{m} a_j = \sum_{j=1}^{m} x_j \leq a_{m+1} . \tag{2.8}$$

This implies

$$\begin{aligned}
x'_{m'} &= \frac{1}{m'-1} \left(\sum_{j=1}^{m'-1} a_j - (m'-2) a_{m'} \right) \\
&= \frac{1}{m'-1} \left(\sum_{j=1}^{m} a_j + a_{m+1} + \sum_{j=m+2}^{m'-1} a_j - (m'-2) a_{m'} \right) \\
&\leq \frac{1}{m'-1} \Big((m-1) a_{m+1} + a_{m+1} + (m'-m-2) a_{m'-1} - (m'-2) a_{m'} \Big) \\
&\leq \frac{1}{m'-1} \Big((m'-2) a_{m'-1} - (m'-2) a_{m'} \Big) \leq 0 .
\end{aligned}$$

This contradicts the assumption $x'_{m'} > 0$, proving that the solution is unique. □

Corollary 1. *Given any n-tuple $a = (a_1, a_2, \ldots, a_n)$ of strictly positive numbers (not necessarily ordered as in (2.1)), there exists a unique vector $x = (x_1, \ldots, x_n)$ satisfying the conditions (2.2)-(2.4).*

Indeed, we can always find a permutation π of the set of indices $\{1, \ldots, n\}$ such that

$$0 < a_{\pi(1)} \leq a_{\pi(2)} \leq \cdots \leq a_{\pi(n)} , \tag{2.9}$$

and apply Lemma 1. □

Given $a = (a_1, a_2, \ldots, a_n)$, we shall denote by

$$G(a) \;=\; (G_1(a), \ldots, G_n(a)) \;=\; (x_1, \ldots, x_n) \tag{2.10}$$

the unique solution of (2.2)-(2.4). Observe that, if k, ℓ are two indices such that $a_k = a_\ell$, then by uniqueness it follows $G_k(a) = G_\ell(a)$. The next lemma collects some properties of the map G.

Lemma 2. *The map $G = (G_1, \ldots, G_n)$ is Lipschitz continuous and quasi-monotone. Namely, given two n-tuples $a = (a_1, \ldots, a_n)$ and $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n)$, if*

$$a_i = \tilde{a}_i, \qquad a_j \leq \tilde{a}_j \qquad \text{for all} \quad j \neq i\,, \tag{2.11}$$

then $G_i(a) \leq G_i(\tilde{a})$.

Proof. 1. By Corollary 1, the map G is well defined.

As shown by the proof of Lemma 1, for any given a there exists a unique subset of indices $I(a) \subseteq \{1, \ldots, n\}$ with cardinality $m \doteq \#I(a)$ such that

$$x_i \;=\; G_i(a) \;=\; \begin{cases} \dfrac{1}{m-1}\left(\displaystyle\sum_{j\in I(a),\; j\neq i} a_j - (m-2)a_i\right) \;>\; 0 & \text{if} \quad i \in I(a)\,, \\[2ex] 0 & \text{if} \quad i \notin I(a)\,. \end{cases} \tag{2.12}$$

This implies the a priori bounds

$$0 \;\leq\; x_i \;=\; G_i(a) \leq \;\sum_j a_j\,. \tag{2.13}$$

2. From the equations (2.2)-(2.4) it follows that the map G has closed graph. Namely, given sequences $a^\nu = (a_1^\nu, \ldots, a_n^\nu)$, $x^\nu = (x_1^\nu, \ldots, x_n^\nu)$, $\nu \geq 1$ such that

$$a^\nu \to \bar{a}\,, \qquad x^\nu \to \bar{x}\,, \qquad a^\nu = G(x^\nu) \;\; \text{for every} \;\; \nu \geq 1\,,$$

it follows that $\bar{x} = G(\bar{a})$. Being a locally bounded function with closed graph, G is continuous.

3. The conclusion of the lemma is clearly a consequence of the following claim:

(C) Fix any $i \in \{1\,, \ldots\,, n\}$ and let any $(n-1)$-tuple of numbers $(a_1, \cdots, a_{i-1}, a_{i+1}, \ldots, a_n)$ be given. Then the maps

$$s \;\mapsto\; G_j(a_1, \cdots,\, a_{i-1},\, s,\, a_{i+1},\, \cdots\,,\, a_n) \tag{2.14}$$

are Lipschitz continuous and monotone. Namely, G_i is decreasing, while all other G_j for $j \neq i$ are increasing w.r.t. s.

A proof of (C) will be worked out in the remaining steps.

4. Up to a permutation of indices, it is not restrictive to assume that $i = n$ and $0 < a_1 \leq a_2 \leq \cdots \leq a_{n-1}$.

Call $a(s) = (a_1, \cdots, a_{n-1}, s)$. By the continuity of G it follows that the maps $s \mapsto G_j(a(s))$ are all continuous.
If $]s^-, s^+[$ is an open interval where the set of indices $I(a(s))$ remains constant, from the formula (2.12) it follows that the maps G_j are all uniformly Lipschitz continuous, with $s \mapsto G_i(a(s))$ decreasing while $s \mapsto G_j(a(s))$ increasing for all $j \neq i$.

5. To complete the proof of (C), it remains to show that, as s increases, there are finitely many values $0 < s_1 < s_2 < \cdots < s_N$ such that the set $I(a(s))$ is constant on each open interval $]s_{k-1}, s_k[$. In turn, for any $i \in \{1, \ldots, n-1\}$, it suffices to show that the set $I(a(s))$ changes only finitely many times as s ranges in the subinterval $]a_i, a_{i+1}[$. This clearly follows as a consequence of (2.5). □

Remark 2. According to (2.5), if

$$a_k \;\geq\; \sum_{j \neq k} a_j \,,$$

then $x_k = 0$. Hence, for $i \neq k$, the values $x_i = G_i(a_1, \ldots, a_n)$ provide the unique solution to the system of $(n-1)$ equations

$$\sum_{j \neq i,k} x_j \;\leq\; a_i \qquad\qquad i \in \{1, \ldots, n\} \setminus \{k\}, , \tag{2.15}$$

$$x_i \;>\; 0 \qquad \Longrightarrow \qquad \sum_{j \neq i,k} x_j \;=\; a_i \,, \tag{2.16}$$

$$\sum_{i \neq k} x_i \;>\; 0 \,. \tag{2.17}$$

obtained from the (2.2)–(2.4) by removing a_k.

Lemma 3. *Consider any n-tuple $a = (a_1, \ldots, a_n)$ of strictly positive numbers, and let $a' = (a_1, \ldots a_{k-1}, a_{k+1}, \ldots a_n)$ be the $(n-1)$-tuple obtained by removing the entry a_k. For $i = 1, \ldots, n$, let $x_i = G_i(a)$ be the solution of (2.2)–(2.4). Moreover, for $i \neq k$ let $x'_i = G'_i(a')$ be the solution of the corresponding system of $(n-1)$ equations obtained by removing a_k. Then*

$$x'_j \geq x_j \qquad \textit{for all} \quad j \neq k \,. \tag{2.18}$$

Moreover, if $x_k > 0$, then there are at least two distinct indices $k_1, k_2 \in \{1, \ldots, n\} \setminus \{k\}$ such that

$$x'_{k_1} > x_{k_1} \,, \qquad x'_{k_2} > x_{k_2} \,. \tag{2.19}$$

Proof. 1. For each $i \neq k$, as a consequence of Remark 2 we have

$$x'_i - x_i \;=\; \int_{a_k}^{\infty} \frac{\partial}{\partial s} G_i(a_1, \ldots, a_{k-1}, s, a_{k+1}, \ldots, a_n)\, ds \,. \tag{2.20}$$

Clearly the integrand is non-negative. Recalling (2.6) we see that

$$\frac{\partial}{\partial s} G_i(a_1,\ldots,a_{k-1},s,a_{k+1},\ldots,a_n) \;=\; \begin{cases} \frac{1}{m(s)-1} & \text{if } \; x_i(s)>0 \text{ and } x_k(s)>0, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

Here we use the notation $x_j(s) \doteq G_j(a_1,\ldots,a_{k-1},s,a_{k+1},\ldots,a_n)$, while $m(s)$ denotes the number of non-zero components in this solution: $m(s) \doteq \#\{j\,; \quad x_j(s)\neq 0\}$. From the representation (2.20) it is clear that $x_j' \geq x_j$ for all $j\neq k$, so that (2.18) holds.

2. To prove (2.19), we observe that it is not restrictive to assume that the a_i are arranged in increasing order, as in (2.1). We consider two cases.

Case 1: $k\geq 3$, so that $m(a_k)\geq k\geq 3$. In this case we have $x_1(a_k)>0$ $x_2(a_k)>0$. By continuity, there exists $\varepsilon>0$ such that $x_1(s),x_2(s)>0$ for all $s\in[a_k,a_k+\varepsilon]$. By (2.20)-(2.21), this implies

$$x_1'-x_1 \;\geq\; \int_{a_k}^{a_k+\varepsilon} \frac{\partial}{\partial s} G_1(a_1,\ldots,a_{k-1},s,a_{k+1},\ldots,a_n)\,ds \;\geq\; \frac{\varepsilon}{n-1} \;>\; 0\,.$$

The same estimate holds for x_2'. The conclusion thus holds with $k_1=1$, $k_2=2$.

Case 2: $k\in\{1,2\}$. To fix the ideas, assume $k=2$, the case $k=1$ being entirely similar. We claim that, when $s=a_3$,

$$G_i(a_1,s,a_3,a_4,\ldots,a_n) \;>\; 0 \qquad \text{for } \; i\in\{1,2,3\}\,.$$

Indeed, the inequality introduced in (2.5)

$$\frac{1}{\ell-1}\sum_{j=1}^{\ell} a_j \;\leq\; a_{\ell+1}$$

here cannot be satisfied when $\ell=2$. Hence $m(s)\geq 3$. By continuity, we can find $\varepsilon>0$ such that $m(s)\geq 3$ for $s\in[a_3-\varepsilon,\,a_3]$. For $i\in\{1,3\}$ we now have

$$\begin{aligned} |x_i'-x_i| \;&=\; \int_{a_2}^{\infty} \frac{\partial}{\partial s} G_i(a_1,s,a_3,\ldots,a_n)\,ds \\ &\geq\; \int_{a_3-\varepsilon}^{a_3} \frac{\partial}{\partial s} G_i(a_1,s,a_3,\ldots,a_n)\,ds \;\geq\; \frac{\varepsilon}{n-1} \;>\; 0\,. \end{aligned} \tag{2.22}$$

Choosing $k_1=1$, $k_2=3$, we reach the desired conclusion. □

Remark 3. The previous analysis shows that the maps $G_i(a_1,\ldots,a_n)$ can be defined also in the case where some of the a_i take the value $+\infty$, provided that there exist at least two distinct indices $j\neq k$ such that $a_j,a_k<\infty$. Indeed, one can simply define $G_i(a)=0$ if $a_i=+\infty$.

For future use, we recall here a standard comparison lemma for solutions to systems of ODEs, originally proved in [7]. We recall that a map $G=(G_1,\ldots,G_n):\mathbb{R}\times R^n\mapsto\mathbb{R}^n$ is called **quasimonotone** if for every $i\in\{1,\ldots,n\}$ the following holds.

If $y_i = \tilde{y}_i$ and $y_j \le \tilde{y}_j$ for all $j \ne i$, then $G_i(t, y_1, \ldots, y_n) \le G_i(t, \tilde{y}_1, \ldots, \tilde{y}_n)$.

Lemma 4. *Assume that the map $G : I\!R \times I\!R^n \mapsto I\!R^n$ is Lipschitz continuous and quasimonotone. Let $t \mapsto y(t) = (y_1, \ldots, y_n)(t)$ and $t \mapsto \tilde{y}(t) = (\tilde{y}_1, \ldots, \tilde{y}_n)(t)$ be two solutions of the same systems of ODEs*

$$\dot{y} = G(t, y) \tag{2.23}$$

Then the following comparison properties holds.

(i) If

$$y_i(t_0) \le \tilde{y}_i(t_0) \qquad \text{for all} \quad i = 1, \ldots, n, \tag{2.24}$$

then

$$y_i(t) \le \tilde{y}_i(t) \qquad \text{for all} \quad i = 1, \ldots, n, \quad t \ge t_0. \tag{2.25}$$

(ii) If in addition to (2.24) one has the strict inequality $y_h(t_0) < \tilde{y}_h(t_0)$ for some index $h \in \{1, \ldots, n\}$, then $y_h(t) < \tilde{y}_h(t)$ for all $t \ge t_0$.

3 The two-point boundary value problem

Given an n-tuple of pricing strategies $(\phi_1, \ldots, \phi_n)$, consider the functions

$$F_i(p) = meas\Big(\{\beta \in [0, \kappa_i]; \quad \phi_i(\beta) < p\}\Big) \qquad i \in \{1, \ldots, n\}, \tag{3.1}$$

$$F(p) \doteq \sum_{j=1}^{n} F_j(p). \tag{3.2}$$

If $(\phi_1, \ldots, \phi_n)$ provide a Nash equilibrium in the sense of Definition 1, then the analysis in [3] has shown that the following optimality conditions must hold:

For every $i \in \{1, \ldots, n\}$, if $F_i'(p) > 0$ then

$$\sum_{j \ne i} F_j'(p) = \frac{-\psi_i(F(p))}{(p - p_i)\, \psi_i'(F(p))}. \tag{3.3}$$

For reader's convenience, we briefly recall how (3.3) is obtained. For a given $i \in \{1, \ldots, n\}$, if $F_i'(p) > 0$ this means that the i-th player is putting something on sale at price p, hence $\phi_i(\beta) = p$ for some $\beta \in [0, \kappa_i]$. The optimality of the strategy ϕ_i implies that, by making a small perturbation $\phi_i(\beta) = p + h$, the expected payoff does not increase. Therefore

$$\begin{aligned} 0 &= \frac{d}{dh} \left[(p + h - p_i) \cdot \psi_i\Big(\beta + \sum_{j \ne i} F_j(p + h)\Big) \right]_{h=0} \\ &= \psi_i\Big(\beta + \sum_{j \ne i} F_j(p)\Big) + (p - p_i) \cdot \psi_i'\Big(\beta + \sum_{j \ne i} F_j(p)\Big) \cdot \sum_{j \ne i} F_j'(p) \\ &= \psi_i(F(p)) + (p - p_i) \cdot \psi_i'(F(p)) \cdot \sum_{j \ne i} F_j'(p). \end{aligned}$$

This yields (3.3). For a rigorous derivation under general assumptions we refer to [3].

In the remainder of this section we shall construct functions $F_1, \ldots, F_n$ which satisfy these optimality conditions. In the next section, under some additional assumptions, we will prove that these functions provide a Nash equilibrium solution to the bidding game.

Without loss of generality, we assume that

$$0 \;<\; p_1 \;\leq\; p_2 \;\leq\; \cdots \;\leq\; p_n \;<\; \overline{P}\,. \tag{3.4}$$

Given $F(p)$ as in (3.2), define

$$a_i(p, F(p)) \;\doteq\; \begin{cases} \dfrac{-\psi_i(F(p))}{(p-p_i)\,\psi_i'(F(p))} & \text{if} \quad p > p_i \quad \text{and} \quad F_i(p) < \kappa_i\,, \\[2ex] +\infty & \text{if} \quad p \leq p_i \quad \text{or} \quad F_i(p) = \kappa_i\,. \end{cases} \tag{3.5}$$

Our main goal is to prove the existence of a unique solution to the following free boundary value problem.

$$F_i'(p) \;=\; G_i\Big(a_1(p, F(p)), \ldots, a_n(p, F(p))\Big) \qquad\qquad p \in [p_A, \overline{P}]\,, \quad i = 1, \ldots, n, \tag{3.6}$$

$$F_1(p_A) \;=\; \cdots \;=\; F_n(p_A) \;=\; 0\,, \tag{3.7}$$

$$F_i(\overline{P}) \;=\; \kappa_i \qquad \text{for all } i \text{ with the exception of at most one index } \; i^*. \tag{3.8}$$

Here $G_1, \ldots, G_n$ are the functions introduced at (2.10). The lowest asking price $p_A \in \,]0, \overline{P}[\,$ is regarded as a free boundary. The index i^* in (3.8), corresponding to the unique player that puts a positive amount of assets for sale at the top price $\overline{P}$, needs also to be determined as part of the solution.

Remark 4. The necessary conditions for optimality proved in [3] yield the implication

$$F_i'(p) > 0 \qquad \Longrightarrow \qquad \sum_{j \neq i} F_j'(p) \;=\; \frac{-\psi_i(F(p))}{(p-p_i)\psi_i'(F(p))}\,.$$

These conditions alone do not uniquely determine a system of ODEs for the functions F_i. For example, at each p one could choose any two indices j, k and set

$$F_j'(p) \;=\; \frac{-\psi_k(F(p))}{(p-p_k)\psi_k'(F(p))}\,, \qquad F_k'(p) \;=\; \frac{-\psi_j(F(p))}{(p-p_j)\psi_j'(F(p))}\,, \qquad F_i'(p) \;=\; 0 \quad \text{for} \;\; i \notin \{j, k\}.$$

Applying Lemma 1 with a_i defined at (3.5), we can uniquely determine the values $x_i \doteq F_i'(p)$, provided that we impose the additional inequalities

$$\sum_{j \neq i} F_j'(p) \;\leq\; \frac{-\psi_i(F(p))}{(p-p_i)\psi_i'(F(p))}\,, \qquad\qquad i = 1, \ldots, n, \tag{3.9}$$

corresponding to (2.2). However, one should keep in mind that these additional inequalities do not follow from the optimality conditions. There may be Nash equilibria that do not satisfy

(3.9), while the unique solution of the boundary value problem (3.6)–(3.8) may not yield a Nash equilibrium. This issue will be discussed in detail in Section 4.

Theorem 1. *For $i = 1, \ldots, n$, let the quantities $\kappa_i > 0$ and the prices p_i as in (3.4) be given, together with functions ψ_i satisfying in (1.5)-(1.6). Then the boundary value problem (3.6)–(3.8) has a unique solution.*

Proof. 1. Because of (3.5), the right hand sides of the ODEs in (3.6) are piecewise smooth, with discontinuities occurring when $F_i = \kappa_i$. Our construction will thus be achieved by an inductive algorithm, which restarts every time where the solution reaches a discontinuity.

INITIAL STEP. Consider an initial point $q_0 \in]p_2, \overline{P}[$, whose precise value will be determined later. We begin by solving the system of ODEs (3.6) with initial data

$$F_1(q_0) = \cdots = F_n(q_0) = 0. \tag{3.10}$$

Since $q_0 > p_2$, we have $a_1(q_0, 0) < \infty$, $a_2(q_0, 0) < \infty$. Hence the right hand sides of the ODEs in (3.6) are well defined and locally Lipschitz continuous. By the analysis in Section 2, this Cauchy problem has a unique local solution defined for $p \geq q_0$. This solution can be continued up to the point

$$q_1 \doteq \overline{P} \wedge \min\Big\{p > q_0\,; \quad F_i(p) = \kappa_i \ \text{ for some index } \ i\Big\}. \tag{3.11}$$

Here and in the sequel we use the notation $a \wedge b \doteq \min\{a, b\}$.

INDUCTIVE STEP. Now assume that $q_0 < q_1 < \cdots < q_\nu$ have been determined, and the solution has been constructed on the interval $[q_0, q_\nu]$. If either (i) $q_\nu = \overline{P}$ or (ii) the set of indices $\{i\,; \quad F_i(q_\nu) < \kappa_i\,, \quad q_\nu > p_i\}$ contains less than two elements, then the construction stops.

In the opposite case, we consider the set of indices

$$I_\nu \doteq \Big\{i\,; \quad F_i(q_\nu) < \kappa_i\Big\}. \tag{3.12}$$

The equations (3.5)-(3.6) now yield a system of $N_\nu \doteq \#I_\nu$ (i.e., the cardinality of the set I_ν) differential equations for the components F_i, $i \in I_\nu$. In addition,

$$F_i'(p) = 0\,, \qquad F_i(p) = \kappa_i \qquad \text{for all } \ i \notin I_\nu\,. \tag{3.13}$$

This Cauchy problem, with initial data provided by the inductive step at $p = q_\nu$, has a unique local solution, defined for $p \geq q_\nu$. This solution can be continued up to the point

$$q_{\nu+1} \doteq \overline{P} \wedge \min\Big\{p > q_0\,; \quad F_i(p) = \kappa_i \ \text{ for some index } \ i \in I_\nu\Big\}. \tag{3.14}$$

This achieves the inductive step.

Clearly the algorithm must terminate after at most $n - 2$ steps, yielding a unique solution to (3.6), (3.10), defined on some maximal interval $[q_0, q^\sharp]$. In the remainder of the proof we will show that there exists a unique value for the minimum ask price p_A such that, setting $q_0 = p_A$, one has $q^\sharp = \overline{P}$ and (3.8) holds. More precisely, writing $q^\sharp = q^\sharp(q_0)$ to stress the dependence of $q^\sharp$ on the initial point q_0, one has

$$p_A \doteq \inf \{q_0 \in [0, \overline{P}] ; \quad q^\sharp(q_0) = \overline{P}\}. \tag{3.15}$$

2. In this step we prove that, for any solution of the boundary value problem (3.6)–(3.8) which is defined on the entire interval $[p_A, \overline{P}]$, all functions F_i are Lipschitz continuous with a uniform Lipschitz constant (independent of p_A).

Toward this goal, choose constants m_0, m_1 such that

$$0 < m_0 \leq \frac{-\psi_i(s)}{\psi_i'(s)} \leq m_1 \qquad \text{for all} \quad i = 1, \ldots, n, \qquad 0 \leq s \leq K \doteq \sum_{j=1}^{n} \kappa_j . \tag{3.16}$$

Introducing the constant

$$\delta \doteq (\overline{P} - p_n) \cdot \exp \left\{ -\frac{2Km_1}{m_0^2} \right\} , \tag{3.17}$$

we claim that

$$F_i'(p) = 0 \qquad \text{for all} \quad i = 1, \ldots, n, \quad p < p_i + \delta. \tag{3.18}$$

Notice that, if (3.18) holds, then (3.3) implies

$$F_j'(p) \leq \max_i \max_{p \geq p_i + \delta} \frac{-\psi_i(F(p))}{(p - p_i)\psi'(F(p))} \leq \frac{m_1}{\delta} . \tag{3.19}$$

This provides the uniform upper bound on the Lipschitz constant of all functions F_j, $j = 1 \ldots, n$.

In the remainder of this step we thus work toward a proof of (3.18). To fix the ideas, fix an index i and assume

$$p_{j-1} < p_j = p_{j+1} = \cdots = p_i = \cdots = p_k < p_{k+1} .$$

for some indices $j \leq i \leq k$. Assume that

$$F_i'(p_i + \varepsilon) = G_i\Big(a_1(p_i + \varepsilon), \ \ldots \ , a_n(p_i + \varepsilon)\Big) > 0. \tag{3.20}$$

In this case, as shown by the proof of Lemma 1, the sum of the two smallest elements in the n-tuple $(a_1, \ldots, a_n)$ must be greater than a_i. Hence,

$$\min{}_2\Big(a_1(p_i + \varepsilon), \ldots, a_n(p_i + \varepsilon)\Big) > \frac{a_i(p_i + \varepsilon)}{2} = \frac{-\psi_i(F(p))}{2\varepsilon\, \psi_i'(F(p))} \geq \frac{m_0}{2\varepsilon} . \tag{3.21}$$

Here and in the sequel we use the notation $\min_2(a_1, \ldots, a_n)$ to denote the second smallest element of the n-tuple $(a_1, \ldots, a_n)$.

Next, assume that

$$a_j(p_i + \varepsilon) > \frac{m_0}{2\varepsilon} \tag{3.22}$$

for some index j. For $p > p_i + \varepsilon$, if $a_j(p) < \infty$ then

$$\frac{m_1}{p - p_j} \;\geq\; a_j(p) \;=\; \frac{-\psi_j(F(p))}{(p - p_j)\,\psi_j'(F(p))} \;\geq\; \frac{m_0}{p - p_j}\,. \tag{3.23}$$

To estimate the right hand side of (3.23) we consider two cases.

If $p_j \geq p_i$ then

$$a_j(p) \;\geq\; \frac{m_0}{p - p_j} \;\geq\; \frac{m_0}{p - p_i}\,. \tag{3.24}$$

On the other hand, if $p_j < p_i$ then the assumption (3.22) implies

$$\frac{m_1}{p_i + \varepsilon - p_j} \;\geq\; a_j(p_i + \varepsilon) \;>\; \frac{m_0}{2\varepsilon}\,.$$

Therefore, for $p \geq p_i + \varepsilon$ we have

$$\frac{p - p_i}{p - p_j} \;\geq\; \frac{\varepsilon}{p_i + \varepsilon - p_j} \;\geq\; \frac{m_0}{2m_1}\,,$$

$$a_j(p) \;\geq\; \frac{m_0}{p - p_i} \cdot \frac{p - p_i}{p - p_j} \;\geq\; \frac{m_0^2}{2m_1(p - p_i)}\,. \tag{3.25}$$

From (3.21), 3.24), and (3.25) we conclude

$$\min{}_2\Big(a_1(p), \ldots, a_n(p)\Big) \;\geq\; \frac{m_0^2}{2m_1(p - p_i)} \qquad \text{for all} \quad p > p_i + \varepsilon\,. \tag{3.26}$$

Observing that

$$\sum_{i=1}^{n} F_i'(p) \;\geq\; \min{}_2\Big(a_1(p), \ldots, a_n(p)\Big)$$

and setting $\delta_0 \doteq \overline{P} - p_n$, from (3.26) we deduce

$$K \;\geq\; \int_{p_i+\varepsilon}^{\bar{P}} \sum_{i=1}^{n} F_i'(p)\,dp \;\geq\; \int_{p_i+\varepsilon}^{p_i+\delta_0} \sum_{i=1}^{n} F_i'(p)\,dp \;\geq\; \int_{p_i+\varepsilon}^{p_i+\delta_0} \frac{m_0^2}{2m_1(p - p_i)}\,dp \;=\; \frac{m_0^2}{2m_1} \ln\left(\frac{\delta_0}{\varepsilon}\right).$$

Hence

$$\varepsilon \;\geq\; \delta_0 \cdot \exp\left\{-\frac{2Km_1}{m_0^2}\right\}, \tag{3.27}$$

proving our claim (3.18).

3. By choosing $q_0 = \overline{P} - \varepsilon$ with $\varepsilon > 0$ sufficiently small, it is clear that the solution of (3.6), (3.10) is well defined on $[q_0, \overline{P}]$ and satisfies $F_i(\overline{P}) < \kappa_i$ for every i. Hence the set on the right hand side of (3.15) is nonempty and the value p_A is well defined.

We claim that, if the minimum asking price p_A is defined as in (3.15), then solution of the Cauchy problem (3.6)-(3.7) satisfies the terminal condition (3.8) as well.

Indeed, consider a decreasing sequence of initial points $q_\nu \to p_A$, with $q^\sharp(q_\nu) = \overline{P}$ for each $\nu \geq 1$. Let $(F_{1,\nu}, \ldots, F_{n,\nu})$ be the corresponding solution to (3.6) with initial data

$$F_{1,\nu}(q_\nu) \;=\; \cdots \;=\; F_{n,\nu}(q_\nu) \;=\; 0, \tag{3.28}$$

defined on the interval $[p_\nu, \overline{P}]$. By the uniform Lipschitz continuity of the functions $F_{i,\nu}$, proved in step 2, we can extract a subsequence converging to an n-tuple of Lipschitz functions $(F_1, \ldots, F_n)$. It is straightforward to check that these functions provide a solution to the Cauchy problem (3.6)–(3.7) on the interval $[p_A, \overline{P}]$. Hence $q^\sharp(p_A) = \overline{P}$ and the infimum in (3.15) is actually attained as a minimum.

To prove that this solution $(F_1, \ldots, F_n)$ also satisfies the terminal condition (3.8), we assume that, on the contrary, $F_j(\overline{P}) < \kappa_j$ and $F_\ell(\overline{P}) < \kappa_\ell$ for two distinct indices $j \neq \ell$. We can then find a constant $\varepsilon > 0$ such that

$$F_j(\overline{P}) \;<\; \kappa_j - \varepsilon\,, \qquad F_\ell(\overline{P}) \;<\; \kappa_\ell - \varepsilon\,. \tag{3.29}$$

Consider a strictly increasing sequence of initial points $q_{0,\nu} \to p_A$. For each $\nu \geq 1$ let $F_\nu = (F_{1,\nu}, \ldots, F_{n,\nu})$ be the corresponding solution to the Cauchy problem (3.6) with initial data (3.28). By assumption, this solution is defined on some maximal interval $[q_{0,\nu},\, q_\nu^\sharp]$ with $q_\nu^\sharp < \overline{P}$. By quasi-monotonicity, the sequence of solutions is monotone decreasing. More precisely, for any two indices $\mu < \nu$ and any $i \in \{1, \ldots, n\}$ we have

$$F_{i,\nu}(p) \;\leq\; F_{i,\mu}(p) \qquad\qquad \text{for all } \; p \in [q_\mu,\, q^\sharp(q_\mu)] \cap [q_\nu,\, q^\sharp(q_\nu)].$$

We now observe that the pointwise limit $\widetilde{F} = (\widetilde{F}_1, \ldots, \widetilde{F}_n)$, defined as

$$\widetilde{F}_i(p) \;=\; \inf_{\nu \geq 1} F_{i,\nu}(p) \qquad\qquad p_A \leq p < \sup_{\nu \geq 1} q^\sharp(q_\nu) \tag{3.30}$$

provides a solution to the Cauchy problem (3.6)-(3.7). By uniqueness, $\widetilde{F} = F$ and hence $q^\sharp(q_\nu) \to \overline{P}$ as $\nu \to \infty$.

A contradiction is now obtained as follows. By the analysis in step 2, the derivatives $F'_{i,\nu}$ remain uniformly bounded. In particular, we can assume

$$F'_{j,\nu}(p) \;\leq\; M, \qquad\qquad F'_{\ell,\nu}(p) \;\leq\; M \tag{3.31}$$

for some constant M and all $\nu \geq 1$. Choose $\delta > 0$ such that $M\delta < \varepsilon/2$. By (3.30) and (3.29), for ν sufficiently large we have

$$F_{j,\nu}(\overline{P} - \delta) \;<\; F_j(\overline{P} - \delta) + \frac{\varepsilon}{2} \;\leq\; \kappa_j - \frac{\varepsilon}{2}\,, \qquad\qquad F_{\ell,\nu}(\overline{P} - \delta) \;\leq\; F_\ell(\overline{P} - \delta) + \frac{\varepsilon}{2} \;\leq\; \kappa_\ell - \frac{\varepsilon}{2}\,.$$

By (3.31), this implies

$$F_{j,\nu}(p) \;<\; \kappa_j\,, \qquad F_{\ell,\nu}(p) \;<\; \kappa_\ell \qquad\qquad \text{for all } p \in [\overline{P} - \delta,\, \overline{P}].$$

Therefore, $q^\sharp(q_\nu) = \overline{P}$, the minimality of p_A. This contradiction proves our claim, i.e. the terminal condition (3.8) is satisfied.

4. It now remains to prove that the solution to the boundary value problem (3.6)–(3.8) is unique. Clearly, as soon as the initial point p_A is chosen, the solution to the Cauchy problem (3.6)-(3.7) is unique. Consider two starting points $p_A < p_{\widetilde{A}}$ and let $F, \widetilde{F}$ be the solutions to the corresponding problems

$$\left\{ \begin{array}{lll} F'_i(p) \;=\; G_i\Big(a_1(p, F(p)), \ldots, a_n(p, F(p))\Big) & & p \in [p_A, \overline{P}]\,, \quad i = 1, \ldots, n, \\ & & \\ F_1(p) \;=\; \cdots \;=\; F_n(p) \;=\; 0 & & p \in [0, p_A]\,, \end{array} \right. \tag{3.32}$$

$$
\begin{cases}
\widetilde{F}_i'(p) \;=\; G_i\Big(a_1(p,\widetilde{F}(p)),\ldots,a_n(p,\widetilde{F}(p))\Big) & p\in[p_{\widetilde{A}},\overline{P}]\,, \quad i=1,\ldots,n,\\[2ex]
\widetilde{F}_1(p) \;=\; \cdots \;=\; \widetilde{F}_n(p) \;=\; 0 & p\in[0,p_{\widetilde{A}}]\,,
\end{cases}
\tag{3.33}
$$

To prove uniqueness for the boundary value problem it suffices to show:

(C) *For every* $p\in\,]p_A,\overline{P}]$ *one has*

$$
\widetilde{F}_i(p) \;\leq\; F_i(p) \qquad \textit{for all } i\in\{1,\ldots,n\}\,. \tag{3.34}
$$

Moreover, there exist at least two indices j,k *(possibly varying with* p*), such that*

$$
\widetilde{F}_j(p) \;<\; F_j(p), \qquad\qquad \widetilde{F}_k(p) \;<\; F_k(p). \tag{3.35}
$$

Indeed, if $F=(F_1,\ldots,F_n)$ is a solution to the boundary value problem (3.6)–(3.8), then $\widetilde{F}$ cannot be a second solution, because for some indices j,k

$$
\widetilde{F}_j(\overline{P}) \;<\; F_j(\overline{P}) \;\leq\; \kappa_j, \qquad\qquad \widetilde{F}_k(\overline{P}) \;<\; F_k(\overline{P}) \;\leq\; \kappa_k,
$$

hence the terminal condition (3.8) fails. The uniqueness of the solution thus follows from our claim (C).

To prove (C), we proceed by induction. For $i=1,\ldots,n$, define

$$
\begin{aligned}
P_i \;&\doteq\; \min\Big\{p\geq p_{\widetilde{A}}\,;\;\; F_i(p)=\kappa_i\Big\}\,,\\[1ex]
\widetilde{P}_i \;&\doteq\; \min\Big\{p\geq p_{\widetilde{A}}\,;\;\; \widetilde{F}_i(p)=\kappa_i\Big\}\,.
\end{aligned}
\tag{3.36}
$$

Rearranging these values in increasing order, we can write

$$
\{p_{\widetilde{A}},\; P_1,\ldots,P_n,\, \widetilde{P}_1,\ldots,\widetilde{P}_n\} \;=\; \{\tau_0,\tau_1,\ldots,\tau_N\},
$$

with

$$
p_{\widetilde{A}} \;=\; \tau_0 \;<\; \tau_1 \;<\; \cdots \;<\tau_N \;=\; \overline{P}.
$$

We compare the solutions of the two Cauchy problems (3.32), (3.33). The inequalities in (3.34) can be proved by induction on $\ell=1,\ldots,N$. Indeed, they trivially hold when $p=p_{\widetilde{A}}$. Assuming that (3.34) holds for $p=\tau_\ell$, by (i) in Lemma 4 and the quasi-monotonicity of the right hand sides of (3.32)-(3.33) we conclude that the same inequalities are true for $p\in[\tau_\ell,\,\tau_{\ell+1}]$.

The strict inequalities in (3.35) will also be proved by induction on the intervals $[\tau_\ell,\tau_{\ell+1}[$.

INITIAL STEP. For every $\tau\in[p_A,\,p_{\widetilde{A}}]$, by Lemma 1 there are two indices $j\neq k$ (possibly depending on τ) such that

$$
F_j'(\tau)>0\,, \qquad F_k'(\tau)>0\,.
$$

For every $p\in\,]p_A,\,p_{\widetilde{A}}]$, integrating over the interval $[p_A,\,p]$ we conclude that there are at least two indices $j\neq k$ such that

$$
F_j(p) \;>\; 0 \;=\; \widetilde{F}_j(p)\,, \qquad F_k(p) \;>\; 0 \;=\; \widetilde{F}_k(p)\,. \tag{3.37}
$$

INDUCTIVE STEP. Assume that the inequality in (3.35) has been proved for all $p \in]p_A, \tau_\ell[$, for some $0 \le \ell < N$. We show that it remains valid on the interval $[\tau_\ell, \tau_{\ell+1}[$ as well.

For any p, define the sets of indices

$$L(p) \doteq \{i; \quad F_i(p) < \kappa_i\} = \{i; \quad p < P_i\}.$$

To achieve the inductive step, consider any $\bar p \in]\tau_{\ell-1}, \tau_\ell[$. By the inductive hypothesis, there exists two indices $j \neq k$ such that

$$\widetilde F_j(\bar p) < F_j(\bar p), \qquad \widetilde F_k(\bar p) < F_k(\bar p). \tag{3.38}$$

Two cases will be considered.

CASE 1: $\tau_\ell \notin \{\widetilde P_j, \widetilde P_k\}$. Observe that this implies

$$\widetilde F_j(p) < \kappa_j, \qquad \widetilde F_k(p) < \kappa_k \qquad \text{for all } p < \tau_{\ell+1}.$$

In this case, using part (ii) of Lemma 4 we first conclude that the inequalities (3.38) hold for all $p \in [\bar p, \tau_\ell]$. A second application of Lemma 4 shows that the same strict inequalities hold also for $p \in [\tau_\ell, \tau_{\ell+1}[$.

CASE 2: $\tau_\ell \in \{\widetilde P_j, \widetilde P_k\}$. To fix the ideas, assume $\tau_\ell = \widetilde P_j \doteq \min\{p; \quad \widetilde F_j(p) = \kappa_j\}$. Observe that in this case we must have $F_j(p) = \kappa_j$ for all $p \ge \tau_{\ell-1}$. Otherwise the relations

$$\widetilde F_j(\bar p) < F_j(\bar p), \qquad \widetilde F_j(\tau_\ell) = F_j(\tau_\ell)$$

would provide a contradiction with part (ii) of Lemma 4. We are thus in the situation shown in Fig. 1.

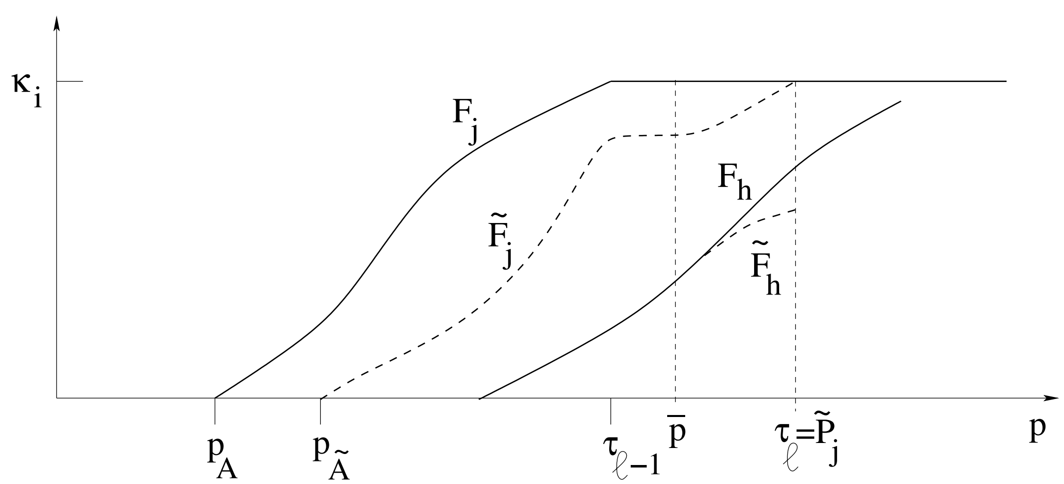

Figure 1: At $p = \tau_\ell = \widetilde P_j$ the functions F_j and $\widetilde F_j$ become equal. However, during the previous interval $[\tau_{\ell-1}, \tau_\ell]$ there are at least two other indices h, h' such that $F_h' > \widetilde F_h'$ and $F_{h'}' > \widetilde F_{h'}'$ on a set with positive measure.

We claim that there exists at least two distinct indices $h, h' \in \{1, \dots, n\}$ such that

$$\widetilde F_h(\tau_\ell) < F_h(\tau_\ell), \qquad \widetilde F_{h'}(\tau_\ell) < F_{h'}(\tau_\ell). \tag{3.39}$$

Suppose on the contrary that (3.39) fails. Then there exists an index k^* such that

$$\widetilde{F}_i(\tau_\ell) \;=\; F_i(\tau_\ell) \qquad\qquad \text{for all } i \neq k^*. \tag{3.40}$$

To achieve a contradiction, observe that (3.40) implies

$$\widetilde{F}_i(p) \;=\; F_i(p) \qquad\qquad \text{for all } p \in [\bar{p},\, \tau_\ell[\,, \quad i \in L(p),\; i \neq k^*. \tag{3.41}$$

Since $\widetilde{F}_j(\bar{p}) < \widetilde{F}_j(\tau_\ell) = \kappa_j$, there is a subset $S \subset [\bar{p}, \tau_\ell]$ of positive measure such that $\widetilde{F}_j'(p) > 0$ for every $p \in S$. By (2.19) in Lemma 3, for every $p \in S$ we can find at least one index $h \in L(p)$, $h \neq k^*$ (possibly depending on p), such that

$$F_h'(p) \;>\; \widetilde{F}_h'(p). \tag{3.42}$$

This is clearly in contradiction with (3.41). We thus conclude that (3.39) holds.

If now $\tau_\ell = \overline{P}$, we are done. Otherwise, using again part (ii) of Lemma 4 we conclude that

$$F_h(p) \;>\; \widetilde{F}_h(p), \qquad\qquad F_{h'}(p) \;>\; \widetilde{F}_{h'}(p) \qquad\qquad \text{for all } p \in [\tau_\ell,\, \tau_{\ell+1}[\,.$$

This completes the inductive step in the proof of our claim (C). □

4 Computing the Nash equilibrium

Let $(F_1, \ldots, F_n)$ be a solution of the boundary value problem (3.6)–(3.8). These functions determine a unique n-tuple of bidding strategies. Namely, for every $p \leq \overline{P}$, the value $F_i(p)$ determines the total amount of assets put on sale by the i-th player at price $< p$. By (3.8) there can be at most one player, say the agent i^*, who puts a positive amount of assets for sale exactly at the price $\overline{P}$. However, as explained in Remark 4, this solution to the boundary value problem does not necessarily yield a Nash equilibrium. We further illustrate this point by an example.

Example 2. Consider a bidding game for three sellers. We assume

$$p_1 = 1, \quad p_2 = p_3 = 4, \qquad\qquad \psi_1(s) \;=\; e^{-s}\,, \quad \psi_2(s) \;=\; \psi_3(s) \;=\; e^{-4s}\,, \tag{4.1}$$

The values $0 < \kappa_1 < \kappa_2 = \kappa_3$ and $\overline{P}$ will be chosen later, so that the minimum ask price will turn out to be

$$p_A \;=\; 5\,. \tag{4.2}$$

Let (F_1, F_2, F_3) be the solution to the BVP (3.6)–(3.8) constructed in Theorem 1. Since Players 2 and 3 have the same payoff function, by uniqueness we have $F_2(p) = F_3(p)$ for all p. In a small interval of the form $[p_A, p_A + \delta]$, by (3.3) it follows

$$\begin{cases} 2F_2'(p) \;=\; \dfrac{1}{p - p_1} \;=\; G_1(p)\,, \\[2ex] F_1'(p) + F_2'(p) \;=\; \dfrac{1}{4(p - p_2)} \;=\; G_2(p) \;=\; G_3(p)\,. \end{cases} \tag{4.3}$$

Therefore

$$F_1'(p) \;=\; \frac{1}{4(p-p_2)} - \frac{1}{2(p-p_1)}\,, \qquad F_2'(p) \;=\; F_3'(p) \;=\; \frac{1}{2(p-p_1)}\,. \tag{4.4}$$

We choose $\kappa_1 = \varepsilon > 0$ sufficiently small. Observe that, when $p \approx p_A = 5$, by (4.1) the right hand sides of (4.4) take the values $F_i'(p) \approx 1/8$. Therefore, we can uniquely determine the value

$$q(\varepsilon) \;\doteq\; \min\;\{p > p_A\,;\;\; F_1(p) = \varepsilon\} \;=\; 5 + 8\varepsilon + o(\varepsilon). \tag{4.5}$$

For $p > q(\varepsilon)$ one has

$$F_1(p) \equiv \varepsilon, \qquad\qquad F_2'(p) \;=\; F_3'(p) \;=\; \frac{1}{4(p-4)}\,. \tag{4.6}$$

Given $\varepsilon > 0$ sufficiently small and $\overline{P} >> 5$ we set

$$\kappa_1 \;=\; \varepsilon\,, \qquad\qquad \kappa_2 \;=\; \kappa_3 \;=\; \int_5^{q(\varepsilon)} \frac{dp}{2(p-1)} + \int_{q(\varepsilon)}^{\overline{P}} \frac{dp}{4(p-4)}\,. \tag{4.7}$$

Notice that, (4.5) and (4.7) together imply

$$\kappa_2 \;=\; \kappa_3 \;=\; \frac{1}{4}\ln(\overline{P}-4) + \mathcal{O}(\varepsilon). \tag{4.8}$$

With the above choice of κ_i, the triple (F_1, F_2, F_3) provides the unique solution to the boundary value problem.

We claim that, if $\overline{P}$ is sufficiently large, the above solution is not a Nash equilibrium, because the strategy of Player 1 is not optimal. Indeed, when Player 1 puts on sale a total amount $\kappa_1 = \varepsilon$ of assets at price $p \approx p_A = 5$, for $\varepsilon > 0$ small his expected payoff is

$$J_1(\varepsilon) \;=\; 4\varepsilon + o(\varepsilon). \tag{4.9}$$

On the other hand, if he puts all his assets for sale at the top price $\overline{P}$, his expected payoff is

$$\begin{aligned} J_1^\dagger(\varepsilon) \;&=\; (\overline{P}-1)\int_0^\varepsilon \exp\{-\kappa_1-\kappa_2-s\}\,ds \\ &=\; (\overline{P}-1)\,e^{-2\kappa_1}\cdot\varepsilon + o(\varepsilon) \;=\; \frac{\overline{P}-1}{\sqrt{\overline{P}-4}}\cdot\varepsilon + o(\varepsilon). \end{aligned} \tag{4.10}$$

When $\overline{P}$ sufficiently large we have $J_1^\dagger(\varepsilon) > J_1(\varepsilon)$, hence the first strategy is not optimal.

In the remainder of this section we seek additional conditions, which guarantee that the n-tuple of strategies $(F_1, \ldots, F_n)$ obtained by solving the boundary value problem (3.6)–(3.8) provides a Nash equilibrium to the bidding game. Toward this goal, we shall use

Lemma 5. *A sufficient condition in order that the strategy ϕ_i for the i-th player be optimal is*

$$(\phi_i(\beta)-p_i)\psi_i\Big(\beta+\Phi_i(\phi_i(\beta))\Big) \;=\; \max_{p\in[0,\overline{P}]}\Big\{(p-p_i)\psi_i\Big(\beta+\Phi_i(p)\Big)\Big\} \qquad \textit{for a.e.}\;\; \beta \in [0,\kappa_i]\,. \tag{4.11}$$

Proof. Intuitively, the above statement should be clear. If ϕ_i yields the maximum expected payoff from the sale of each single asset $\beta \in [0, \kappa_i]$, then $\phi(\cdot)$ is optimal.

To prove the lemma, for any admissible strategy $\varphi : [0, \kappa_i] \mapsto [0, \overline{P}]$ we simply observe that

$$\begin{aligned} J(\varphi, \Phi_i) &= \int_0^{\kappa_i} (\varphi(\beta) - p_i)\psi_i\Big(\beta + \Phi_i(\varphi(\beta))\Big)\, d\beta \\ &= \leq \int_0^{\kappa_i} \max_{p\in[0,\overline{P}]} \Big\{(p - p_i)\psi_i\Big(\beta + \Phi_i(p)\Big)\Big\}\, d\beta \\ &= \int_0^{\kappa_i} (\phi_i(\beta) - p_i)\psi_i\Big(\beta + \Phi_i(\phi_i(\beta))\Big)\, d\beta \;=\; J(\phi_i, \Phi_i)\,. \end{aligned}$$

□

To proceed further, we need to introduce an additional condition:

(H) For every $i \in \{1, \ldots, n\}$ and every $\tilde{p} < \overline{P}$, the following implication holds.

If $G_i(a_1(\tilde{p}), \ldots, a_i(\tilde{p}), \ldots a_n(\tilde{p})) \; > \; 0$ *for some* $\tilde{p} \in]p_i, \overline{P}]$*, then for every* $p \in]\tilde{p}, \overline{P}[$ *one has*

$$G_i\left(a_1(p), \ldots, a_{i-1}(p), \frac{-\psi_i(F(p))}{(p-p_i)\,\psi_i'(F(p))}, a_{i+1}(p), \ldots, a_n(p)\right) \; > \; 0\,. \tag{4.12}$$

Roughly speaking, the assumption (H) means that, if i-th player puts some asset for sale at price $\tilde{p}$, then he continues to put assets for sale at every price $p \in [\tilde{p}, q_i]$, for some q_i such that $F_i(q_i) = \kappa_i$. The only reason for which he does not put assets for sale at prices $p > q_i$ is that he simply does not have anything more to sell. In this case, recalling (3.5), we have

$$a_i(p) \;\doteq\; \begin{cases} \dfrac{-\psi_i(F(p))}{(p-p_i)\,\psi_i'(F(p))} & \text{if} \quad p < q_i\,, \\[2ex] +\infty & \text{if} \quad p \in [q_i,\, \overline{P}]\,. \end{cases} \tag{4.13}$$

Lemma 6. *Under the same assumptions as in Theorem 1, let* $p \mapsto (F_1, \ldots F_n)(p)$ *be a solution to the boundary value problem (3.6)– (3.8). If the condition (H) holds, then the corresponding pricing strategies yield a Nash equilibrium to the bidding game with payoffs (1.3)-(1.4).*

Proof. 1. For every $i \in \{1, \ldots, n\}$ we need to show that the strategy ϕ_i of the i-th player is a best reply to the strategies adopted by all other players. By Lemma 5, this is the case if (4.11) holds. Calling

$$E(\beta) \;\doteq\; \frac{d}{dp}\Big[(p-p_i)\cdot\psi_i(\beta+\Phi_i(p))\Big] \;=\; \psi_i(\beta+\Phi_i(p)) + (p-p_i)\cdot\psi_i'(\beta+\Phi_i(p))\cdot\Phi_i'(p), \tag{4.14}$$

our conclusion will be reached by proving that

$$\begin{cases} E(p) \;\geq\; 0 & \text{if} \quad p < \phi(\beta), \\ E(p) \;\leq\; 0 & \text{if} \quad p > \phi(\beta). \end{cases} \tag{4.15}$$

2. When $p < \phi_i(\beta)$, since $F_i(p) \le \beta < \kappa_i$ and the equations (3.6) are satisfied, recalling (2.2) we have

$$\Phi_i'(p) \;=\; \sum_{j\neq i} F_j'(p) \;\le\; a_i(p) \;\doteq\; \frac{-\psi_i(F(p))}{(p-p_i)\psi_i'(F(p))}\,. \tag{4.16}$$

Inserting (4.16) in (4.14) and recalling that $\psi_i' < 0$, for $p < \phi_i(\beta)$ we thus obtain

$$\begin{aligned}\frac{d}{dp}\Big[(p-p_i)\cdot\psi_i(\beta+\Phi_i(p))\Big] \;&\ge\; \psi_i(\beta+\Phi_i(p)) + (p-p_i)\cdot\psi_i'(\beta+\Phi_i(p))\cdot\frac{-\psi_i(F(p))}{(p-p_i)\psi_i'(F(p))}\\[2mm] &=\; \psi_i'(\beta+\Phi_i(p))\cdot\left[\frac{\psi_i(\beta+\Phi_i(p))}{\psi_i'(\beta+\Phi_i(p))} - \frac{\psi_i(F(p))}{\psi_i'(F(p))}\right] \;\ge\; 0.\end{aligned} \tag{4.17}$$

Indeed, $\beta+\Phi_i(p) \ge F(p)$. Moreover, by the assumption (1.6) the map $s \mapsto \psi_i(s)/\psi_i'(s)$ is non-increasing.

3. Next, assume $p > \phi_i(\beta)$. Using the assumption (H) with $\tilde p = \phi_i(\beta)$, we obtain

$$\Phi_i'(p) \;\ge\; \frac{-\psi_i(F(p))}{(p-p_i)\,\psi_i'(F(p))}\,, \tag{4.18}$$

with equality holding if $F_i(p) < \kappa_i$. A similar computation as in (4.17) now yields

$$\begin{aligned}E(p) \;&\le\; \psi_i(\beta+\Phi_i(p)) + (p-p_i)\cdot\psi_i'(\beta+\Phi_i(p))\cdot\frac{-\psi_i(F(p))}{(p-p_i)\psi_i'(F(p))}\\[2mm] &=\; \psi_i'(\beta+\Phi_i(p))\cdot\left[\frac{\psi_i(\beta+\Phi_i(p))}{\psi_i'(\beta+\Phi_i(p))} - \frac{\psi_i(F(p))}{\psi_i'(F(p))}\right] \;\le\; 0.\end{aligned} \tag{4.19}$$

Indeed, in this second case we have $\beta+\Phi_i(p) \le F(p)$.

This establishes the second inequality in (4.15), completing the proof. □

Using Lemma 6, we can give a number of sufficient conditions in order that the pricing strategies constructed in Theorem 1 provide a Nash equilibrium.

Theorem 2. *Let the quantities $\kappa_i > 0$ and the prices p_i as in (3.4) be given, together with functions ψ_i satisfying (1.5)-(1.6). Let $p \mapsto (F_1, \ldots F_n)(p)$ be the unique solution to the boundary value problem (3.6)– (3.8). Then these strategies provide a Nash equilibrium to the bidding game with payoffs (1.4) provided that any one of the following assumptions holds:*

(i) $n = 2$.

(ii) $\psi_1(s) = \cdots = \psi_n(s) = e^{-\lambda s}$ *for some* $\lambda > 0$.

(iii) For every $i, j \in \{1, \ldots, n\}$ *and* $\max\{p_i, p_j\} < p < \overline{P}$, *one has*

$$\frac{d}{dp}\left|\ln\frac{-\psi_i(F(p))}{(p-p_i)\,\psi_i'(F(p))} - \ln\frac{-\psi_j(F(p))}{(p-p_j)\,\psi_j'(F(p))}\right| \;\le\; 0. \tag{4.20}$$

Proof. **1.** In the case of two players, let (F_1, F_2) be the solution to the boundary value problem (3.6)-(3.8). Then

$$F_1'(p) = \frac{-\psi_2(F(p))}{(p-p_2)\psi_2'(F(p))} = a_2(p) > 0, \qquad F_2'(p) = \frac{-\psi_1(F(p))}{(p-p_1)\psi_1'(F(p))} = a_1(p) > 0, \tag{4.21}$$

for all $p \in]p_A, \overline{P}[$. Hence the assumption (H) in Lemma 6 trivially holds. Part (i) of the theorem is thus a consequence of Lemma 6.

2. To prove part (iii), let the inequality (4.20) hold.

To understand the basic case, for $i = 1, \ldots, n$ define

$$\alpha_i(p) \doteq \begin{cases} \dfrac{-\psi_i(F(p))}{(p-p_i)\,\psi_i'(F(p))} & \text{if } p > p_i\,, \\ \\ +\infty & \text{if } p \leq p_i\,. \end{cases} \tag{4.22}$$

After a permutation of indices, we can assume that

$$\alpha_1(p) \leq \cdots \leq \alpha_n(p) \qquad \text{for all } p\,. \tag{4.23}$$

Indeed, by (4.20), if $\alpha_i(\tilde{p}) = \alpha_j(\tilde{p}) < \infty$ then $\alpha_i(p) = \alpha_j(p) < \infty$ for all $p \geq \tilde{p}$. Hence the difference $\alpha_i(p) - \alpha_j(p)$ can never change sign. Note that (4.23) implies

$$p_1 \leq \cdots \leq p_n\,. \tag{4.24}$$

Otherwise, if $i < j$ but $p_i > p_j$, then

$$\lim_{p \to p_i+} a_j(p) = \frac{-\psi_j(F(p_i))}{(p_i - p_j)\psi_j'(F(p_i))} < \lim_{p \to p_i+} a_i(p) = +\infty\,,$$

in contradiction with (4.23).

Next, assuming that

$$G_i(\alpha_1(\tilde{p}), \ldots, \alpha_n(\tilde{p})) > 0 \tag{4.25}$$

for some $\tilde{p}$, we claim that

$$G_i(\alpha_1(p), \ldots, \alpha_n(p)) > 0 \qquad \text{for all } p \in [\tilde{p}, \overline{P}[\,. \tag{4.26}$$

Indeed, as shown in the proof of Lemma 1, one has $G_i(\alpha_1(p), \ldots \alpha_n(p)) > 0$ if and only if

$$\frac{1}{k-1}\sum_{j=1}^{k} \alpha_j(p) > \alpha_{k+1}(p) \qquad k = 2, \ldots, i-1.$$

Equivalently, this holds if and only if

$$\frac{1}{k-1}\sum_{j=1}^{k} \frac{\alpha_j(p)}{\alpha_{k+1}(p)} > 1 \qquad k = 2, \ldots, i-1. \tag{4.27}$$

By (4.23) and (4.20) it follows that, for $j \leq k+1$,

$$\ln\left(\frac{\alpha_j(p)}{\alpha_{k+1}(p)}\right) \leq 0\,, \qquad \frac{d}{dp}\left(\frac{\alpha_j(p)}{\alpha_{k+1}(p)}\right) \geq 0.$$

Hence, if (4.27) holds for $p = \tilde p$, the same holds for all $p \geq \tilde p$.

3. The argument in the previous step shows that, if $a_i(p) = \alpha_i(p)$ for all $i = 1, \ldots, n$, then the assumption (H) in Lemma 6 is satisfied. By applying Lemma 6 we conclude that the solution $(F_1, \ldots, F_n)$ of the boundary value problem yields a Nash equilibrium.

To complete the proof of (iii), we need to consider the case where the $a_i(p)$ defined at (4.13) do not necessarily coincide with the $\alpha_i(p)$ in (4.22). This happens precisely when $p > p_i$, $F_i(p) = \kappa_i$ and $a_i(p) = \infty$.

Assume that, at some point $\tilde p$, one has

$$G_i\Big(a_1(\tilde p), \ldots, a_i(\tilde p), \ldots, a_n(\tilde p)\Big) > 0. \tag{4.28}$$

Clearly this implies $a_i(\tilde p) = \alpha_i(\tilde p) < \infty$. Notice that, if $j < i$ and $a_j(\tilde p) = \infty$, then $a_j(p) = \infty$ for all $p \geq \tilde p$. For notational convenience, for any $p \geq \tilde p$ define

$$b_j(p) \doteq \begin{cases} \infty & \text{if } j < i \text{ and } a_j(\tilde p) = \infty\,, \\ \alpha_j(p) & \text{otherwise.} \end{cases}$$

For $p \geq \tilde p$ we then have

$$\begin{aligned} &G_i\Big(a_1(p), \ldots, a_{i-1}(p), \alpha_i(p), a_{i+1}(p), \ldots, a_n(p)\Big) \\ &\quad \geq G_i\Big(b_1(p), \ldots, b_{i-1}(p), \alpha_i(p), b_{i+1}(p), \ldots, b_n(p)\Big) > 0. \end{aligned} \tag{4.29}$$

Indeed, the first inequality is a consequence of the quasi-monotonicity of the maps G_i, proved in Lemma 2. The second inequality is obtained from (4.28), using the arguments in step 2, after discarding the components $j < i$ for which $a_j(\tilde p) = \infty$.

Having proved that the assumption (H) holds, by an application of Lemma 6 we conclude that the solution $(F_1, \ldots, F_n)$ to the boundary value problem yields a Nash equilibrium.

4. To prove part (ii) of the theorem we show that, if $\psi_1(s) = \cdots = \psi_n(s) = e^{-\lambda s}$, then (4.20) holds. Indeed, to fix the ideas assume $p_i \geq p_j$. For $p > p_i$ we then have

$$\frac{d}{dp}\left|\ln\frac{1}{\lambda(p-p_i)} - \ln\frac{1}{\lambda(p-p_j)}\right| = \frac{d}{dp}\Big[\ln(p-p_j) - \ln(p-p_i)\Big] = \frac{1}{p-p_j} - \frac{1}{p-p_i} \leq 0.$$

Therefore (ii) follows as a special case of (iii). □

5 Uniqueness of the Nash equilibrium

Aim of this final section is to show that, if one of the assumptions (i) or (ii) in Theorem 2 holds, then the Nash equilibrium is unique, i.e. there are no other equilibria except the one obtained by solving the boundary value problem (3.6)–(3.8). In the following, a Nash equilibrium will be described in terms of the functions $F_1, \ldots, F_n$ in (3.1)-(3.2).

Lemma 7. *For $i = 1, \ldots, n$, let the quantities $\kappa_i > 0$ and the prices p_i as in (3.4) be given, together with functions ψ_i satisfying (1.5)-(1.6). Let the n-tuple $(F_1, \ldots, F_n)$ provide a Nash equilibrium. Then the following holds.*

(i) There exists a Lipschitz constant C such that

$$F(p') - F(p) \;\le\; C(p' - p) \qquad \text{for all } 0 < p < p' < \overline{P}\,. \tag{5.1}$$

(ii) At most one of the functions F_i can have an upward jump at $p = \overline{P}$, while all the others are Lipschitz continuous on the entire interval $[0, \overline{P}]$.

(iii) There exists a minimum ask price p_A and a constant $\delta_0 > 0$ such that

$$F(p) \;=\; 0 \qquad \text{for all } p \le p_A\,, \qquad\qquad F'(p) \;\ge\; \delta_0 \qquad \text{for a.e. } p \in [p_A, \overline{P}]. \tag{5.2}$$

The proof is identical to the one of Lemma 8.1 in [3] and will not be repeated here.

Theorem 3. *Let the quantities $\kappa_i > 0$ and the prices p_i as in (3.4) be given, together with functions ψ_i satisfying (1.5)-(1.6). Let one of the following assumptions hold:*

(i) $n = 2$.

(ii) $\psi_1(s) = \cdots = \psi_n(s) = e^{-\lambda s}$ *for some* $\lambda > 0$.

Then the bidding game with payoffs (1.4) has a unique Nash equilibrium.

Proof. 1. In the case of two players, the result is an immediate consequence of (iii) in Lemma 7. Indeed, since $F'(p) = F_1'(p) + F_2'(p) \ge \delta_0$ for a.e. $p \in [p_A, \overline{P}]$, the necessary conditions for optimality (3.3) imply

$$F_2'(p) \;=\; \frac{-\psi_1(F(p))}{(p - p_1)\psi_1'(F(p))}\,, \qquad\qquad F_1'(p) \;=\; \frac{-\psi_2(F(p))}{(p - p_2)\psi_2'(F(p))}\,,$$

for a.e. $p \in [p_A, \overline{P}]$. Hence (F_1, F_2) provides a solution to the two-point boundary value problem (3.6)–(3.8).

2. In the remainder of the proof we thus focus on case (ii), where all players assign the same exponential probability distribution to the random incoming order X.

Since the functions F_i are Lipschitz continuous and nondecreasing, almost every point $p \in [0, \overline{P}[$ lies in the set

$$\mathcal{L} \;\doteq\; \{p \in [0, \overline{P}[\,;\;\; p \text{ is a Lebesgue point for every function } F_i',\;\; i = 1, \ldots, n\}.$$

As p ranges in the interval $[p_A, \overline{P}]$ by Lemma 7 the map $p \mapsto F'(p)$ is measurable and bounded above and below. In the next two steps we shall prove the following claim:

(C) *For every* $i = 1, \ldots, n$, *the support of* F_i' *is an interval* $I_i = [a_i, b_i] \subseteq [p_A, \overline{P}]$, *with* $p_A = a_1 \le a_2 \le \cdots \le a_n$.

3. Toward a proof of **(C)**, define $a_i \doteq \inf\{p \in \mathcal{L}\,;\;\; F_i'(p) > 0\}$. Assuming that $a_j < a_i$ for some $i < j$, to achieve a contradiction we consider the following perturbed strategies.

$$F_j^\varepsilon(q) \;=\; \begin{cases} \max\{F_j(q) - \varepsilon,\, 0\} & \text{if } \; q < a_i\,, \\[2mm] F_j(q) & \text{if } \; q \ge a_i\,, \end{cases}$$

$$F_i^\varepsilon(q) \;=\; \begin{cases} 0 & \text{if } \; q < a_j\,, \\[2mm] \max\{F_i(q),\, \varepsilon\} & \text{if } \; q \ge a_j\,. \end{cases}$$

Roughly speaking, the strategy F_j^ε is obtained from F^j by offering an amount ε of assets at price a_i instead of a_j. On the other hand, the strategy F_i^ε is obtained from F^i by offering an amount ε of assets at price a_j instead of a_i. The changes in the expected payoffs achieved by these modified strategies are computed by

$$\begin{aligned} A_j \;&\doteq\; \left\{ \frac{d}{d\varepsilon} \int_0^{\overline{P}} (q - p_j) e^{-\lambda F^\varepsilon(q)} dF_j^\varepsilon(q) \right\}_{\varepsilon = 0} \\ &=\; (a_i - p_j) e^{-\lambda F(a_i)} - (a_j - p_j) e^{-\lambda F(a_j)} + \int_{a_j}^{a_i} (q - p_j) \lambda e^{-\lambda F(q)} F_j'(q)\, dq\,, \end{aligned} \tag{5.3}$$

$$A_i \;\doteq\; \left\{ \frac{d}{d\varepsilon} \int_0^{\overline{P}} (q - p_i) e^{-\lambda F^\varepsilon(q)} dF_i^\varepsilon(q) \right\}_{\varepsilon = 0} \;=\; (a_j - p_i) e^{-\lambda F(a_j)} - (a_i - p_i) e^{-\lambda F(a_i)}\,. \tag{5.4}$$

Summing the right hand sides of (5.3)-(5.4) we obtain

$$A_j + A_i \;\ge\; (p_j - p_i)[e^{-\lambda F(a_j)} - e^{-\lambda F(a_i)}] \;>\; 0.$$

Therefore at least one of the quantities in (5.3)-(5.4) is strictly positive, proving that the corresponding pricing strategy is not optimal.

4. To complete the proof of **(C)** we need to show that the set of points where $F_i' > 0$ is an interval (up to a set of measure zero). This will be proved by induction on $i = 1, \ldots, n$.

When $i = 1$, fix any $\tilde{p} > a_1 = p_A$ such that $F_1'(\tilde{p}) > 0$ and consider the perturbed strategy

$$F_1^\varepsilon(q) \;=\; \begin{cases} 0 & \text{if } \; q < a_1\,, \\[2mm] \min\{F_1(q) + \varepsilon,\, F_1(\tilde{p})\} & \text{if } \; q \in [a_1, \tilde{p}[\,, \\[2mm] F_1(q) & \text{if } \; q \ge \tilde{p}\,. \end{cases}$$

In essence, the strategy F_1^ε is obtained from F_1 by offering an amount ε of assets at price a_1

instead of $\tilde p$. The change in the expected payoff satisfies

$$
\begin{array}{rcl}
A_1 & \doteq & \left\{ \dfrac{d}{d\varepsilon} \displaystyle\int_0^{\overline{P}} (q-p_1) e^{-\lambda F^\varepsilon(q)} dF_1^\varepsilon(q) \right\}_{\varepsilon=0} \\[2ex]
& = & (a_1-p_1)e^{-\lambda F(a_1)} - (\tilde p - p_1)e^{-\lambda F(\tilde p)} - \displaystyle\int_{a_1}^{\tilde p} (q-p_1)\lambda e^{-\lambda F(q)} F_1'(q)\,dq \\[2ex]
& = & \displaystyle\int_{a_1}^{\tilde p} e^{-\lambda F(q)} \Big[-1 + \lambda(q-p_1)(F'(q)-F_1'(q))\Big]\,dq \\[2ex]
& = & \displaystyle\int_{\{q\in[a_1,\tilde p]\,;\ F_1'(q)=0\}} e^{-\lambda F(q)} \Big[-1 + \lambda(q-p_1)F'(q)\Big]\,dq\,.
\end{array}
\tag{5.5}
$$

Notice that the last equality is true because, if $F_1'(q) > 0$, then by (3.3) the integrand vanishes. If the set $\{q \in [a_1,\tilde p]\,;\ F_1'(q) = 0\}$ has positive measure, then the right hand side of (5.5) is strictly positive. Indeed, if i is an index such that $F_i'(q) > 0$, then the necessary conditions (3.3) yield

$$
F'(q) \;=\; F_i'(q) + \sum_{j\neq i} F_j'(q) \;=\; F_i'(q) + \frac{1}{\lambda(q-p_i)}\,, \tag{5.6}
$$

$$
-1+\lambda(q-p_1)F'(q) \;=\; -1+\lambda(q-p_1)F_i'(q) + \frac{q-p_1}{q-p_i} \;>\; 0.
$$

Therefore, the quantity A_1 in (5.5) is positive and the strategy F_1 is not optimal. This contradiction proves that the set where $F_1' > 0$ is an interval.

Next, assume that the set where $F_j' > 0$ is an interval $[a_j, b_j]$, for every $j = 1,\ldots,k-1$. To prove that this property remains valid for F_k', let $\tilde p \in \mathcal{L}$ be a point where $F_k'(\tilde p) > 0$. Consider the perturbed strategy

$$
F_k^\varepsilon(q) \;=\; \left\{ \begin{array}{cl} 0 & \text{if } \ q < a_k\,, \\[2ex] \min\{F_k(q)+\varepsilon,\ F_k(\tilde p)\} & \text{if } \ q \in \ [a_k, \tilde p[\,, \\[2ex] F_k(q) & \text{if } \ q \geq \tilde p\,. \end{array}\right.
$$

Roughly speaking, the strategy F_k^ε is obtained from F_k by offering an amount ε of assets at price a_k instead of $\tilde p$. The change in the expected payoff satisfies

$$
\begin{array}{rcl}
A_k & \doteq & \left\{ \dfrac{d}{d\varepsilon} \displaystyle\int_0^{\overline{P}} (q-p_k) e^{-\lambda F^\varepsilon(q)} dF_k^\varepsilon(q) \right\}_{\varepsilon=0} \\[2ex]
& = & (a_k-p_k)e^{-\lambda F(a_k)} - (\tilde p - p_k)e^{-\lambda F(\tilde p)} - \displaystyle\int_{a_k}^{\tilde p} (q-p_k)\lambda e^{-\lambda F(q)} F_k'(q)\,dq \\[2ex]
& = & \displaystyle\int_{a_k}^{\tilde p} e^{-\lambda F(q)} \Big[-1 + \lambda(q-p_k)(F'(q)-F_k'(q))\Big]\,dq \\[2ex]
& = & \displaystyle\int_{\{q\in[a_k,\tilde p]\,;\ F_k'(q)=0\}} e^{-\lambda F(q)} \Big[-1 + \lambda(q-p_k)F'(q)\Big]\,dq\,.
\end{array}
\tag{5.7}
$$

Consider a point $q \in [a_k, \tilde{p}]$ with $F_k'(q) = 0$. Two cases can arise.

CASE 1: $F_i'(q) > 0$ for some $i > k$. By (5.6) it then follows

$$-1 + \lambda(q - p_k)F'(q) \;=\; -1 + \lambda(q - p_k)F_i'(q) + \frac{q - p_k}{q - p_i} \;>\; 0.$$

CASE 2: $\mathcal{I}(q) \doteq \{i\,;\; F_i'(q) > 0\} \subseteq \{1, 2, \ldots, k-1\}$. By the inductive assumption, we can then find an intermediate point $z \in \mathcal{L}$, with $a_k < z < q$, such that

$$F_k'(z) > 0, \qquad F_i'(z) > 0, \qquad \text{for all } i \in \mathcal{I}(q)\,.$$

Using (2.5)–(2.7) with $a_i = (z - p_i)^{-1}$, we deduce

$$\frac{1}{z - p_k} \;<\; \frac{1}{\#\mathcal{I}(z) - 1} \sum_{j \in \mathcal{I}(z),\, j \neq k} \frac{1}{z - p_j}$$

Since $\mathcal{I}(q) \subseteq \mathcal{I}(z) \setminus \{k\}$, by Lemma 3, we also have

$$\frac{1}{z - p_k} \;<\; \frac{1}{\#\mathcal{I}(q)} \sum_{j \in \mathcal{I}(q)} \frac{1}{z - p_j}\,. \tag{5.8}$$

Since $z < q$ and $p_k \geq p_j$ for all $j \in \mathcal{I}(q)$, from (5.8) we conclude

$$\frac{1}{\lambda(q - p_k)} \;<\; \frac{1}{\#\mathcal{I}(q)} \sum_{j \in \mathcal{I}(q)} \frac{1}{\lambda(q - p_j)}\,. \tag{5.9}$$

and finally

$$\lambda(q - p_k)F'(q) \;=\; \lambda(q - p_k) \sum_{j \in \mathcal{I}(q)} F_j'(q) \;>\; 1. \tag{5.10}$$

If the set $\{q \in [a_k, \tilde{p}]\,;\; F_k'(q) = 0\}$ has positive measure, then the right hand side of (5.7) is strictly positive, hence F_k is not optimal. This contradiction proves that the set where $F_k' > 0$ is an interval.

5. To complete the proof of the theorem, we need to show that the n-tuple $(F_1, \ldots, F_n)$ provides the unique solution to the boundary value problem (3.6)–(3.8). This amounts to proving that the additional inequalities (3.9) hold. Toward this goal, define

$$a_i^* \;\doteq\; \inf \left\{ p \in \mathcal{L}\,;\;\; p > \max\{p_A, p_i\}, \quad \sum_{j \neq i} F_j'(p) \;\geq\; \frac{1}{\lambda(p - p_i)} \right\}. \tag{5.11}$$

By the previous analysis, it suffices to prove that $a_i^* = a_i \doteq \inf\{p \in \mathcal{L}\,;\; F_i'(p) > 0\}$ for every i. We shall do this by induction.

For $i = 1$ we already know that $a_1 = a_1^* = p_A$. Next, assume that the equality $a_i^* = a_i$ holds for every $i = 1, \ldots, k-1$. If $a_k^* < a_k$, to achieve a contradiction we consider the perturbed strategies

$$F_k^\varepsilon(q) \;=\; \begin{cases} 0 & \text{if } q < a_k^*\,, \\[2ex] \max\{F_k(q),\, \varepsilon\} & \text{if } q \geq a_k^*\,. \end{cases}$$

In essence, the strategy F_k^ε is obtained from F^k by offering an amount ε of assets at price a_k^* instead of a_k. The change in the expected payoff achieved by this modified strategy satisfies

$$\begin{aligned} A_k^* &\doteq \left\{\frac{d}{d\varepsilon}\int_0^{\overline{P}}(q-p_k)e^{-\lambda F^\varepsilon(q)}dF_k^\varepsilon(q)\right\}_{\varepsilon=0} = (a_k^*-p_k)e^{-\lambda F(a_k^*)} - (a_k-p_k)e^{-\lambda F(a_k)} \\ &= \int_{a_k^*}^{a_k} e^{-\lambda F(q)}\Big[-1+\lambda(q-p_k)F'(q)\Big]\,dq\,. \end{aligned} \tag{5.12}$$

We claim that the above integrand is a.e. strictly positive. Indeed, consider any $q\in\mathcal{L}\cap\,]a_k^*,a_k[\,$. By definition, there exists an intermediate point $z\in\mathcal{L}\cap]a_k^*,q[$ such that

$$\frac{1}{\lambda(z-p_k)} \;<\; F'(z) \;=\; \frac{1}{\#\mathcal{I}(z)-1}\sum_{j\in\mathcal{I}(z)}\frac{1}{\lambda(z-p_j)} \tag{5.13}$$

Consider the set

$$\mathcal{I}^*(z) \;\doteq\; \{j\,;\;1\leq j<k,\quad z<b_j\}$$

Recalling the definition of G at (2.10), by (5.13) we have

$$G_k(a_1(z),\ldots,a_k(z)) \;>\; 0 \tag{5.14}$$

where

$$a_j(z) \;=\; \begin{cases} \frac{1}{\lambda(z-p_j)} & \text{if}\quad z<b_j\,,\\[2ex] +\infty & \text{if}\quad z\geq b_j\,.\end{cases}$$

Since $p_1\leq p_2\leq\cdots\leq p_k$, this implies

$$G_j(a_1(z),\ldots,a_k(z)) \;>\; 0 \qquad\qquad \text{for all } j\leq k \text{ such that } z<b_j.$$

By the inductive assumption, in (5.13) we have $\mathcal{I}(z)\doteq\{j;\;F_j'(q)>0\}=\{j<k\,;\;z<b_j\}$. Therefore, $\mathcal{I}(q)\subseteq\mathcal{I}(z)$ for all $z<q<a_i$. By (5.13), the same arguments as in (5.8)-(5.9) yield (5.10). This shows that the integrand on the right hand side of (5.12) is strictly positive for a.e. $q\in[a_k^*,a_k]$. If $a_k^*<a_k$ this contradicts the optimality of the strategy F_k. □

6 Conclusions

In this paper we studied a game theoretical model for the limit order book, where various agents can hold different beliefs about the fundamental value of the asset and the size of the random incoming order. While the existence of a Nash equilibrium was proved in the earlier paper [4], here we introduced conditions which ensure that this equilibrium is unique and can be computed by solving a two-point boundary value problem for a system of ODEs. If these conditions fail, a counterexample shows that the solution to the system of ODEs may not yield a Nash equilibrium. We expect that, in general, the equilibrium may not be unique. However, it remains an open problem to construct an example with multiple Nash equilibria.

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