On the Control of Non Holonomic Systems by Active Constraints

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Abstract

The paper is concerned with mechanical systems which are controlled by implementing a number of time-dependent, frictionless holonomic constraints. The main novelty is due to the presence of additional non-holonomic constraints. We develop a general framework to analyze these problems, deriving the equations of motion and studying the continuity properties of the "control-to-trajectory" maps. Various geometric characterizations are provided, in order that the equations be affine w.r.t. the time derivative of the control. In this case the system is *fit for jumps*, and the evolution is well defined also in connection with discontinuous control functions. The classical Roller Racer provides an example where the non-affine dependence of the equations on the derivative of the control is due only to the non-holonomic constraint. This is a case where the presence of quadratic terms in the equations can be used for controllability purposes.

1 Introduction

The control of mechanical systems provides a rich area for mathematical investigation. In a commonly adopted model [2, 3, 12, 22], the controller can modify the time evolution of the system by applying additional external forces. This leads to a control problem in standard form, where vector fields governing the state variables depend continuously on the control function.

In an alternative model, also physically meaningful, the controller acts on the system by directly assigning the values of some of the coordinates, as functions of time. Stated in a more intrinsic fashion, this means that the controller assigns the leaves of a foliation of the configuration manifold as functions of time. Here the basic framework consists of a manifold with coordinates $q = (q^1, \ldots, q^N, q^{N+1}, \ldots, q^{N+M})$. We assume that the values of the last M coordinates can be prescribed: $q^{N+1} = u^1(t), \ldots, q^{N+M} = u^M(t)$. Instead of forces, these controls thus take the form of time-dependent holonomic constraints. The evolution of the remaining free coordinates $q^1, \ldots, q^N$ is then determined by a control system where the right hand side depends not only on the control itself, but also (linearly or quadratically) on the time derivative of the control function.

This alternative point of view was introduced, independently, in [9] and in [19]. A considerable amount of literature is now available on mechanical systems controlled by active constraints. The form of the basic equations of motion, in relation with geometrical properties of the system, was studied in [1, 9, 13, 17, 23, 24]. Differential equations of impulsive nature, where the right hand side contains a measure, such as the distributional derivative of a discontinuous control, were considered in [5, 6, 7, 18, 20, 25, 28]. These works focus, in particular, on the continuity of the "control-to-trajectory" map, in various topologies. In addition, problems of stabilization and of optimal control for this kind of "hyper-impulsive" mechanical systems were studied in [1, 8, 16] and in [6, 7, 10, 11], respectively. See also [4] for a survey.

The goal of the present paper is to extend this theory to the case where the system is subject to some additional non-holonomic constraints. In this case, the velocity vector $\dot q = (\dot q^1, \ldots, \dot q^{N+M})$ satisfies an additional set of ν linear relations

$$\sum_{i=1}^{N+M} \omega_i^k\, \dot q^i \;=\; 0 \qquad\qquad k = 1, \ldots, \nu.$$

A major focus of our analysis is on the form of the resulting equations, In general, the right hand side of the evolution equations turns out to be a *quadratic* polynomial of the time derivative $\dot u(\cdot)$. Therefore, the evolution problem is well posed as soon as the control function satisfies $u(\cdot) \in W^{1,2}$. There are, however, important cases where the right hand side is an *affine* function of $\dot u(\cdot)$. Our main result in this direction, Theorem 6.1, yields a number of equivalent analytic and geometric conditions for this to happen. In the positive case, the "control-to-trajectory" map can be extended by continuity to a larger family of (possibly discontinuous) control functions. Following [9], we then say that the system is "fit for jumps". Section 7 provides an additional geometric characterization of this property. Let

$$\Lambda_u \;\doteq\; \{(q^1, \ldots, q^{N+M});\;\; q^{N+\alpha} = u^\alpha\,,\; 1 \le \alpha \le M\} \tag{1.1}$$

be the leaf of the foliation corresponding to the control value u. Roughly speaking, we show that the system is "fit for jumps" if and only if the "infinitesimal non-holonomic distance" between leaves of the foliation remains constant, along directions compatible with the non-holonomic constraint.

For sake of illustration, in the last section we show how the present framework applies to the control of the Roller Racer. An interesting feature of this example is that, without the non-holonomic constraint, the equations of motion would be affine w.r.t. the time derivative $\dot u$ of the control function. However, the additional non-holonomic constraint renders the system *not* fit for jumps. It is indeed the presence of a quadratic term in the derivative of the control that makes forward motion possible.

2 Non holonomic systems with active constraints as controls

Let N, M, ν be positive integers such that $\nu \le N + M$. Let $\mathcal{Q}$ be an $(N+M)$-dimensional differential manifold, which will be regarded as the space of configurations of a mechanical system.

Let Γ be a distribution on $\mathcal{Q}$, i.e. a vector sub-bundle of the tangent space $T\mathcal{Q}$. Throughout the following, we consider trajectories $t \mapsto \mathbf{q}(t) \in \mathcal{Q}$ of the mechanical system which are continuously differentiable and satisfy the geometric constraint

$$\dot{\mathbf{q}}(t) \;\in\; \Gamma_{\mathbf{q}(t)}\,. \tag{2.1}$$

We do not assume Γ to be integrable, so that in general (2.1) is a *non holonomic constraint.*

In our model, the system will be controlled by means of an active (holonomic, time-dependent) constraint. To describe this constraint, let an M-dimensional differential manifold $\mathcal{U}$ be given, together with a submersion

$$\pi : \mathcal{Q} \mapsto \mathcal{U}. \tag{2.2}$$

The fibers $\pi^{-1}(\mathbf{u}) \subset \mathcal{Q}$ will be regarded as the states of the active constraint. The set of all these fibers can be identified with the *control manifold* $\mathcal{U}$.

Let $\mathcal{I}$ be a time interval and let $\mathbf{u} : \mathcal{I} \mapsto \mathcal{U}$ be a continuously differentiable map. We say that *a trajectory* $\mathbf{q} : \mathcal{I} \to \mathcal{Q}$ *agrees with the control* $\mathbf{u}(\cdot)$ *if*

$$\pi \circ \mathbf{q}(t) \;=\; \mathbf{u}(t) \qquad\qquad \textit{for all} \;\; t \in \mathcal{I}\,. \tag{2.3}$$

For each $\mathbf{q} \in \mathcal{Q}$, consider the subspace of the tangent space at $\mathbf{q}$ given by

$$\Delta_{\mathbf{q}} \;\doteq\; \textit{ker}\; T_{\mathbf{q}}\pi\,.$$

Here $T_{\mathbf{q}}\pi$ denotes the linear tangent map between the tangent spaces $T_{\mathbf{q}}\mathcal{Q}$ and $T_{\pi(\mathbf{q})}\mathcal{U}$. Clearly, Δ is the (holonomic) distribution whose integral manifolds are precisely the fibers $\pi^{-1}(\mathbf{u})$.

2.1 General setting

1) The manifold $\mathcal{Q}$ is endowed with a Riemannian metric $\mathbf{g} = \mathbf{g}_{\mathbf{q}}[\cdot,\cdot]$, the so-called *kinetic metric*, which defines the kinetic energy $\mathcal{T}$. More precisely, for each $\mathbf{q} \in \mathcal{Q}$ and $\mathbf{v} \in T_{\mathbf{q}}\mathcal{Q}$ one has

$$\mathcal{T}(\mathbf{q},\mathbf{v}) \;\doteq\; \frac{1}{2}\mathbf{g}_{\mathbf{q}}[\mathbf{v},\mathbf{v}]\,. \tag{2.4}$$

We shall use the notation $\mathbf{v} \mapsto \mathbf{g}_{\mathbf{q}}(\mathbf{v})$ to denote the isomorphism from $T_{\mathbf{q}}\mathcal{Q}$ to $T^*_{\mathbf{q}}\mathcal{Q}$ induced by the scalar product $\mathbf{g}_{\mathbf{q}}[\cdot,\cdot]$. Namely, for every $\mathbf{v} \in T_{\mathbf{q}}\mathcal{Q}$, the 1-form $\mathbf{g}_{\mathbf{q}}(\mathbf{v})$ is defined by

$$\langle \mathbf{g}_{\mathbf{q}}(\mathbf{v}), \mathbf{w}\rangle \;\doteq\; \mathbf{g}_{\mathbf{q}}[\mathbf{v},\mathbf{w}] \qquad \text{for all} \;\; \mathbf{w} \in T_{\mathbf{q}}\mathcal{Q}\,, \tag{2.5}$$

where $\langle\cdot,\cdot\rangle$ is the natural duality between the tangent space $T_{\mathbf{q}}\mathcal{Q}$ and the cotangent space $T^*_{\mathbf{q}}\mathcal{Q}$.

If $\mathbf{q} \in \mathcal{Q}$ and $W \subset T_{\mathbf{q}}$, $W^{\perp}$ denotes the subspace of $T_{\mathbf{q}}$ consisting of all vectors that are orthogonal to every vector in W:

$$W^{\perp} \;\doteq\; \{\mathbf{v} \in T_{\mathbf{q}} \quad|\quad \mathbf{g}_{\mathbf{q}}[\mathbf{v},\mathbf{w}] = 0 \quad \text{for all} \;\; \mathbf{w} \in W\}\,.$$

For a given distribution $E \subset T\mathcal{Q}$, the *orthogonal distribution* $E^{\perp} \subset T\mathcal{Q}$ is defined by setting $E^{\perp}_{\mathbf{q}} \doteq (E_{\mathbf{q}})^{\perp}$, for every $\mathbf{q} \in \mathcal{Q}$.

2) Throughout the following, we shall assume that the holonomic distribution Δ and the non-holonomic distribution Γ satisfy the *tranversality condition*

$$\Delta_{\mathbf{q}} + \Gamma_{\mathbf{q}} \;=\; T_{\mathbf{q}}\mathcal{Q} \qquad\qquad \text{for all} \;\; \mathbf{q} \in \mathcal{Q}\,. \tag{2.6}$$

Notice that this is equivalent to

$$\Delta^{\perp} \cap \Gamma^{\perp} \;=\; \{0\}\,, \tag{2.7}$$

and implies $\nu \leq N$.

3) The mechanical system is subject to *forces*. In the Hamiltonian formalism, these are represented by vertical vector fields on the cotangent bundle $T^*\mathcal{Q}$. We recall that, in a natural system of coordinates (q,p), the fact that $\mathbf{F}$ is *vertical* means that its q-component is zero, namely $\mathbf{F} = \sum_{i=1}^{N+M} F_i \frac{\partial}{\partial p_i}$.

4) The constraints (2.1) and (2.3) are dynamically implemented by reaction forces obeying

D'ALEMBERT CONDITION: *If $t \mapsto \mathbf{q}(t)$ is a trajectory which satisfies both (2.1) and (2.3), and $\mathbf{R}(t)$ is the constraint reaction at a time t, then*

$$\mathbf{R}(t) \;\in\; \ker\Big(\Delta_{\mathbf{q}(t)} \cap \Gamma_{\mathbf{q}(t)}\Big). \tag{2.8}$$

In other words, regarding the reaction force $R(t)$ as an element of the cotangent space $T^*_{\mathbf{q}(t)}\mathcal{Q}$, one has

$$\langle \mathbf{R}(t)\,,\; \mathbf{v}\rangle \;=\; 0 \qquad\qquad \text{for all } \;\mathbf{v} \;\in\; \Delta_{\mathbf{q}(t)} \cap \Gamma_{\mathbf{q}(t)} \;\subseteq\; T_{\mathbf{q}(t)}\mathcal{Q}\,. \tag{2.9}$$

2.2 Equations of motion

For each $\mathbf{q} \in \mathcal{Q}$, we shall use $\mathbf{g}_\mathbf{q}^{-1}$ to denote the inverse of the isomorphism $\mathbf{g}_\mathbf{q}$ at (2.5). Moreover we define the scalar product on the cotangent space $\mathbf{g}_\mathbf{q}^{-1}[\cdot,\cdot] : \; T^*_\mathbf{q}\mathcal{Q} \times T^*_\mathbf{q}\mathcal{Q} \;\mapsto\; I\!R$ by setting

$$\mathbf{g}_\mathbf{q}^{-1}[\mathbf{p},\tilde{\mathbf{p}}] \;\doteq\; \mathbf{g}_\mathbf{q}\,[\mathbf{g}_\mathbf{q}^{-1}(\mathbf{p}),\, \mathbf{g}_\mathbf{q}^{-1}(\tilde{\mathbf{p}})] \qquad\qquad \text{for all} \quad (\mathbf{p},\tilde{\mathbf{p}}) \;\in\; T^*_\mathbf{q}\mathcal{Q} \times T^*_\mathbf{q}\mathcal{Q}\,.$$

For every $\mathbf{q} \in \mathcal{Q}$, we shall use $\mathcal{H}(\mathbf{q},\cdot)$ to denote the Legendre transform of the map $\mathbf{v} \to \mathcal{T}(\mathbf{q},\mathbf{v})$, so that

$$\mathcal{H}(\mathbf{q},\mathbf{p}) \;=\; \frac{1}{2}\mathbf{g}_\mathbf{q}^{-1}[\mathbf{p},\mathbf{p}] \qquad \Big(= \mathcal{T}(\mathbf{q}, \mathbf{g}^{-1}(\mathbf{p}))\Big) \qquad \text{for all} \quad \mathbf{p} \in T^*_\mathbf{q}\mathcal{Q}\,. \tag{2.10}$$

The map $\mathcal{H} : T^*\mathcal{Q} \to I\!R$ will be called the *Hamiltonian corresponding to the kinetic energy* $\mathcal{T}$.

Let (q) and (u) be local coordinates on $\mathcal{Q}$ and $\mathcal{U}$, respectively, such that the domain of the chart (q) is mapped by π into the domain of the chart (u). Let (q,p) the natural local coordinates on $T^*\mathcal{Q}$ corresponding to the coordinates (q). From Nonholonomic Mechanics it follows that, given a smooth control $t \mapsto u(t)$ (here regarded as a time-dependent holonomic constraint), the corresponding motion $t \mapsto (q,p)(t)$ on $T^*\mathcal{Q}$ verifies the relations

$$\left\{\begin{aligned} \dot{q}(t) \;&=\; \frac{\partial H}{\partial p}(q(t),p(t))\,, \\ \dot{p}(t) \;&=\; -\,\frac{\partial H}{\partial q}(q(t),p(t)) + F(t,q(t),p(t)) + R(t)\,, \\ \pi \circ q(t) \;&=\; u(t), \\ p(t) \;&\in\; g_{q(t)}(\Gamma_{q(t)})\,, \\ R(t) \;&\in\; \ker \Delta_{q(t)} + \ker \Gamma_{q(t)}\,. \end{aligned}\right. \tag{2.11}$$

We have used H, F, R, and g to denote the local expressions of $\mathcal{H}$, $\mathbf{F}$, $\mathbf{R}$, and $\mathbf{g}$, respectively. For simplicity, we use the same notation for the distributions Γ, Δ on $\mathcal{Q}$ and their local expressions in coordinates. Moreover, we use the notation $v \mapsto g_q(v)$ to denote the local expression of the isomorphism $\mathbf{v} \mapsto \mathbf{g_q}(\mathbf{v})$.

The first two equations in (2.11) are the dynamical equations written in Hamiltonian form. The first one yields the inverse of the Legendre transform. The third equation represents the (holonomic) control-constraint. Relying on the fact that π is a submersion, we shall always choose local coordinates (q^r) and (u^α) such that $q^{N+\alpha} = u^\alpha$, for all $\alpha = 1, \ldots, M$. We thus regard this equation as prescribing a priori the evolution of the last M coordinates: $q^{N+1}, \ldots, q^{N+M}$. The fourth relation in (2.11) is the Hamiltonian version of the non holonomic constraint (2.1). The fifth relation is clearly equivalent to (2.8), i.e. it represents d'Alembert's condition.

Remark 2.1 Although a global, intrinsic formulation of these equations can be given (see [19]) we are here mainly interested in their local coordinate-wise expression. Indeed, a major goal of our analysis is to understand the functional dependence on the time derivative $\dot{u}$ of the control.

The special case where no active constraints are present can be obtained by taking $\Delta \equiv T^*\mathcal{Q}$, i.e. $\ker(\Delta) = \{0\}$. In this case, (2.11) reduces to the standard Hamiltonian version of the dynamical equations with non-holonomic constraints, namely

$$\begin{cases} \dot{q}(t) &= \dfrac{\partial H}{\partial p}(q(t), p(t))\,, \\[2ex] \dot{p}(t) &= -\dfrac{\partial H}{\partial q}(q(t), p(t)) + F(t, q(t), p(t)) + R(t)\,, \\[2ex] p(t) &\in g_{q(t)}(\Gamma_{q(t)})\,, \\[2ex] R(t) &\in \ker \Gamma_{q(t)}\,, \end{cases} \tag{2.12}$$

3 Orthogonal decompositions of the tangent and the cotangent bundles

To derive a set of equations describing the constrained motion, it will be convenient to decompose both the tangent bundle $T\mathcal{Q}$ and the cotangent bundle $T^*\mathcal{Q}$ as direct sums of three suitable vector sub-bundles. We recall that $\mathcal{Q}$ is a manifold of dimension $N+M$, while Δ and Γ are distributions on $\mathcal{Q}$, having dimensions N and $N+M-\nu$, respectively. (In view of the transversality condition (2.6) one has $\nu \leq N$.)

3.1 Tangent bundle

Definition 3.1 *For every $\mathbf{q} \in \mathcal{Q}$, we define the following three subspaces of $T_\mathbf{q}\mathcal{Q}$.*

$$(T_\mathbf{q}\mathcal{Q})_I \doteq \Delta_\mathbf{q} \cap \Gamma_\mathbf{q}\,, \qquad (T_\mathbf{q}\mathcal{Q})_{II} \doteq \Gamma_\mathbf{q}^\perp, \qquad (T_\mathbf{q}\mathcal{Q})_{III} \doteq (\Delta_\mathbf{q} \cap \Gamma_\mathbf{q})^\perp \cap \Gamma_\mathbf{q}\,. \tag{3.1}$$

Proposition 3.1 *For each $\mathbf{q} \in \mathcal{Q}$, the three subspaces in (3.1) are mutually orthogonal and span the entire tangent space, namely*

$$T_\mathbf{q}\mathcal{Q} = (T_\mathbf{q}\mathcal{Q})_I \oplus (T_\mathbf{q}\mathcal{Q})_{II} \oplus (T_\mathbf{q}\mathcal{Q})_{III}\,, \tag{3.2}$$

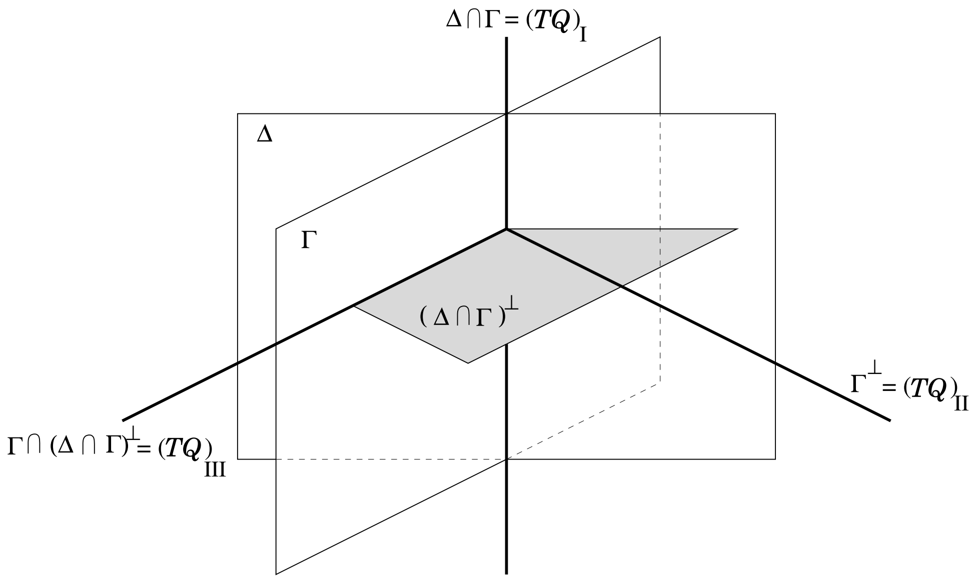

Figure 1: The orthogonal decomposition of $T\mathcal{Q}$.

Moreover,

$$(T_{\mathbf{q}}\mathcal{Q})_I \oplus (T_{\mathbf{q}}\mathcal{Q})_{III} \;=\; \Gamma\,. \tag{3.3}$$

If the transversality condition (2.6) holds, then the above subspaces have dimensions

$$\dim(T_{\mathbf{q}}\mathcal{Q})_I \;=\; N-\nu\,, \qquad \dim(T_{\mathbf{q}}\mathcal{Q})_{II} \;=\; \nu\,, \qquad \dim(T_{\mathbf{q}}\mathcal{Q})_{III} \;=\; M\,. \tag{3.4}$$

Proof. The orthogonality of the subspaces $(T_{\mathbf{q}}\mathcal{Q})_I$ and $(T_{\mathbf{q}}\mathcal{Q})_{II}$ is immediately clear form the definitions. Observing that

$$\Big((T_{\mathbf{q}}\mathcal{Q})_I \oplus (T_{\mathbf{q}}\mathcal{Q})_{II}\Big)^{\perp} \;=\; \Big((T_{\mathbf{q}}\mathcal{Q})_I\Big)^{\perp} \cap \Big((T_{\mathbf{q}}\mathcal{Q})_{II}\Big)^{\perp} \;=\; (\Delta_{\mathbf{q}} \cap \Gamma_{\mathbf{q}})^{\perp} \cap \Gamma_{\mathbf{q}} \;=\; (T_{\mathbf{q}}\mathcal{Q})_{III}\,,$$

we obtain the orthogonal decomposition (3.2). In particular, this implies

$$(T_{\mathbf{q}}\mathcal{Q})_I \oplus (T_{\mathbf{q}}\mathcal{Q})_{III} \;=\; \Big((T_{\mathbf{q}}\mathcal{Q})_{II}\Big)^{\perp} \;=\; \Gamma_{\mathbf{q}}\,.$$

Finally, if (2.6) holds, then

$$\dim(\Delta_{\mathbf{q}} \cap \Gamma_{\mathbf{q}}) \;=\; \dim(\Delta_{\mathbf{q}}) + \dim(\Gamma_{\mathbf{q}}) - \dim(T_{\mathbf{q}}\mathcal{Q}) \;=\; N + (N+M-\nu) - (N+M) \;=\; N-\nu\,.$$

Moreover, $\dim\Big((T_{\mathbf{q}}\mathcal{Q})_{II}\Big) = \dim(T_{\mathbf{q}}\mathcal{Q})$ - $\dim(\Gamma_{\mathbf{q}}) = \nu$. The last equality in (3.4) now follows from (3.2). □

For $J \in \{I, II, III\}$, the orthogonal projection onto the subspace $(T\mathcal{Q})_J$ will be denoted by

$$\mathcal{P}_J : T_{\mathbf{q}}\mathcal{Q} \;\mapsto\; (T_{\mathbf{q}}\mathcal{Q})_J\,. \tag{3.5}$$

Remark 3.1 Under the assumption (2.6), the third subspace in the decomposition (3.2) can be equivalently written as

$$(T_{\mathbf{q}}\mathcal{Q})_{III} \;=\; \mathcal{P}^{\Gamma}(\Delta_{\mathbf{q}}^{\perp}),$$

where $\mathcal{P}^\Gamma$ denotes the orthogonal projection on the space Γ. Indeed, for any vector $\mathbf{v} \in T_\mathbf{q}\mathcal{Q}$ one has $\mathbf{v} \in \mathcal{P}^\Gamma(\Delta_\mathbf{q}^\perp)$ iff

$$\mathbf{v} \in \Gamma_\mathbf{q} \cap (\Delta_\mathbf{q}^\perp + \Gamma_\mathbf{q}^\perp) = \Gamma_\mathbf{q} \cap (\Delta_\mathbf{q} \cap \Gamma_\mathbf{q})^\perp.$$

3.2 Cotangent bundle

Thanks to the isomorphism $\mathbf{g} : T\mathcal{Q} \mapsto T^*\mathcal{Q}$ defined at (2.5), one can use (3.2) to obtain a similar decomposition of the cotangent bundle as the direct sum of three vector bundles:

$$T^*\mathcal{Q} = (T^*\mathcal{Q})_I \oplus (T^*\mathcal{Q})_{II} \oplus (T^*\mathcal{Q})_{III}, \tag{3.6}$$

where, for $J \in \{I, II, III\}$,

$$(T^*_\mathbf{q}\mathcal{Q})_J \doteq \mathbf{g}_\mathbf{q}\Big((T_\mathbf{q}\mathcal{Q})_J\Big). \tag{3.7}$$

We denote by $\mathcal{P}^*_J : T^*_\mathbf{q}\mathcal{Q} \to (T^*_\mathbf{q}\mathcal{Q})_J$ the orthogonal projection w.r.t. the metric $\mathbf{g}^{-1}$. The above construction yields

$$\mathcal{P}^*_J = \mathbf{g} \circ \mathcal{P}_J \circ \mathbf{g}^{-1}. \tag{3.8}$$

Remark 3.2 By (3.8), the orthogonal decompositions of the tangent and cotangent bundles at (3.2) and (3.6) have the following property. Recalling (2.5), assume that $\mathbf{p} = \mathbf{g}(\dot{\mathbf{q}})$, so that $\dot{\mathbf{q}} = \mathbf{g}^{-1}(\mathbf{p})$. Then, for $J \in \{I, II, III\}$,

$$\mathbf{p}_J \doteq \mathcal{P}^*_J(\mathbf{p}) = \mathbf{g}(\mathcal{P}_J(\dot{\mathbf{q}})), \qquad \dot{\mathbf{q}}_J \doteq \mathcal{P}_J(\dot{\mathbf{q}}) = \mathbf{g}^{-1}(\mathcal{P}^*_J(\mathbf{p})). \tag{3.9}$$

In other words, the J-th component of $\mathbf{p}$ depends only on the J-th component of $\dot{\mathbf{q}}$, and viceversa.

In view of Proposition 3.1 one obtains:

Proposition 3.2 *If the transversality condition (2.6) holds, then*

$$\dim\Big((T^*_\mathbf{q}\mathcal{Q})_I\Big) = N - \nu, \qquad \dim\Big((T^*_\mathbf{q}\mathcal{Q})_{II}\Big) = \nu, \qquad \dim\Big((T^*_\mathbf{q}\mathcal{Q})_{III}\Big) = M. \tag{3.10}$$

Moreover, for every $\mathbf{q} \in \mathcal{Q}$ the three subspaces $(T^*_\mathbf{q}\mathcal{Q})_I, (T^*_\mathbf{q}\mathcal{Q})_{II}, (T^*_\mathbf{q}\mathcal{Q})_{III}$ are pairwise orthogonal (w.r.t. the metric $\mathbf{g}^{-1}$). In particular, $(T^*_\mathbf{q}\mathcal{Q})_{II} = \ker(\Gamma_\mathbf{q})$.

Next, we observe that by (2.6) the differential of the submersion $\pi : \mathcal{Q} \mapsto U$ is one-to-one when restricted to $(T_q\mathcal{Q})_{III}$. More precisely, $D\pi : (T_\mathbf{q}\mathcal{Q})_{III} \mapsto T_{\pi(\mathbf{q})}\mathcal{U}$ is a bijective linear map. Its inverse will be denoted as

$$\mathbf{h} : T_{\pi(\mathbf{q})}\mathcal{U} \mapsto (T_\mathbf{q}\mathcal{Q})_{III}. \tag{3.11}$$

For any $\mathbf{v} \in T_{\pi(\mathbf{q})}\mathcal{U}$, by $\mathbf{h}(\mathbf{v})$ we thus denote the unique vector in $(T_\mathbf{q}\mathcal{Q})_{III}$ such that

$$D\pi \cdot (\mathbf{h}(\mathbf{v})) = \mathbf{v}. \tag{3.12}$$

Moreover, we write

$$\mathbf{k}(\mathbf{v}) \doteq \mathbf{g}_\mathbf{q}(\mathbf{h}(\mathbf{v})) \in (T^*_\mathbf{q}\mathcal{Q})_{III}. \tag{3.13}$$

We claim that the vector $\mathbf{h}(\mathbf{v})$ defined above can be characterized as the unique vector $\mathbf{w} \in \Gamma_{\mathbf{q}}$ where the following constrained minimum is attained:

$$\mathbf{h}(\mathbf{v}) \;=\; \underset{\mathbf{z}\in\Gamma_{\mathbf{q}}\,,\; D\pi\cdot\mathbf{z}=\mathbf{v}}{\operatorname{argmin}} \quad \mathbf{g_q}[\mathbf{z},\mathbf{z}]\,. \tag{3.14}$$

Indeed, since the Riemann metric $\mathbf{g_q}$ is positive definite, the right hand side of (3.14) is well defined. Consider any vector $\mathbf{w} \in T_{\mathbf{q}}\mathcal{Q}$ such that

$$\mathbf{w} \in \Gamma_{\mathbf{q}}\,, \qquad D\pi\cdot\mathbf{w} \;=\; \mathbf{v}\,. \tag{3.15}$$

The above equalities imply

$$\mathbf{w} \;=\; \mathbf{h}(\mathbf{v}) + \mathbf{w}_I$$

for some $\mathbf{w}_I \in (T_{\mathbf{q}}\mathcal{Q})_I$. Since the vectors $\mathbf{h}(\mathbf{v}) \in (T_{\mathbf{q}}\mathcal{Q})_{III}$ and $\mathbf{w}_I$ are orthogonal, we have

$$\mathbf{g_q}[\mathbf{w},\mathbf{w}] \;=\; \mathbf{g_q}[\mathbf{h}(\mathbf{v}),\,\mathbf{h}(\mathbf{v})] + \mathbf{g_q}[\mathbf{w}_I,\mathbf{w}_I] \;\geq\; \mathbf{g_q}[\mathbf{h}(\mathbf{v}),\,\mathbf{h}(\mathbf{v})].$$

This proves our claim.

Remark 3.3 Let $\mathbf{u} \in \mathcal{U}$ and choose $\mathbf{q} \in \mathcal{Q}$ such that $\mathbf{u} = \pi(\mathbf{q})$. Identifying the tangent spaces $T_{\mathbf{u}}\mathcal{U} \approx T_{\mathbf{q}}\mathcal{Q}/\Delta_{\mathbf{q}}$, the inner product

$$\langle \mathbf{v},\mathbf{v}'\rangle_{\mathbf{u}} \;\doteq\; \mathbf{g_q}[\mathbf{h}(\mathbf{v}),\,\mathbf{h}(\mathbf{v}')]$$

can be seen as a Riemann metric on the space of leaves. In general, this metric is not canonically defined, because it depends on the choice of a particular point $\mathbf{q}$ on each leaf. From a more precise result proved in Section 7 it will follow that, if this metric is independent of the choice of $\mathbf{q}$, then the system is “fit for jumps”. Namely, the right hand sides of the dynamical equations (4.8) are affine functions of the time derivative $\dot{\mathbf{u}}$ of the control. As a consequence, trajectories of the system can be meaningfully defined also in connection with discontinuous control functions [5, 6].

4 A closed system of control equations

In the original formulation (2.11), the motion is characterized in terms of a family of ODEs coupled with a set of constraints. Relying on the decompositions at (3.2) and (3.6), we will show that the motion can be described by a system of differential equations.

The following result, providing different ways to express the constraint (2.1), is straightforward.

Lemma 4.1 *Let $t \mapsto \mathbf{q}(t) \in \mathcal{Q}$ be a $\mathcal{C}^1$ map. Using the same notation as in (3.9), the nonholonomic constraint $\dot{\mathbf{q}}(t) \in \Gamma_{\mathbf{q}(t)}$ can be expressed in any of the following equivalent forms:*

$$\mathbf{p}(t) \in \mathbf{g}(\Gamma_{\mathbf{q}(t)}) \quad\Longleftrightarrow\quad \dot{\mathbf{q}}_{II}(t) = 0 \quad\Longleftrightarrow\quad \mathbf{p}_{II}(t) = 0. \tag{4.1}$$

On a given, natural chart (q,p) on $T^*\mathcal{Q}$, and for $J \in \{I,\,II,\,III\}$, we denote by P^*_J the (q-dependent) matrix representing the projection $\mathcal{P}^*_J$. Let $g = (\mathbf{g}_{r,s})$ be the matrix representing the Riemannian metric g in the q-coordinates. In turn, the inverse matrix $g^{-1} = (g^{r,s})$ represents the metric g^{-1} on the cotangent space. In this coordinate system, the components of the force and of the constraint reaction will be denoted as $F_J \doteq P^*_J F$ and $R_J \doteq P^*_J R$. The coordinate representations of the linear maps **h** and **k** introduced at (3.11)-(3.13) will be denoted by h and k, respectively.

Throughout the following, we use a local system of *adapted* coordinates, so that $q^{N+\alpha} = u^\alpha$ for $\alpha = 1, \dots, M$. If the holonomic active constraint is satisfied, by (3.9) and (3.12)-(3.13) this implies the identity

$$P_{III}(\dot q(t)) \;=\; h(\dot u(t))\,, \qquad\qquad P^*_{III}(p(t)) \;=\; k(\dot u(t))\,. \tag{4.2}$$

In terms of the orthogonal decompositions (3.2) and (3.6), the non-holonomic and the active holonomic constraints yield the system of equations

$$\begin{cases} \dot q_{II}(t) = \;0 \\ \\ \dot q_{III}(t) = \; h(\dot u(t)) \end{cases} \qquad \left(\text{which holds if and only if} \quad \begin{cases} p_{II}(t) = \;0 \\ \\ p_{III}(t) = \; k(\dot u(t)) \end{cases}\right). \tag{4.3}$$

To complete the description of the motion, it remains to derive the equations for $\dot q_I$ and $\dot p_I$. By (3.9) it follows

$$\dot q_I \;=\; P_I(\dot q) \;=\; P_I(g^{-1}(p)) \;=\; g^{-1}(P^*_I(p))\,. \tag{4.4}$$

Moreover, differentiating $p_I = P^*_I(p)$ w.r.t. time and using (2.11), we obtain

$$\begin{aligned} \dot p_I \;&=\; \left(\frac{\partial P^*_I}{\partial q}\cdot \dot q\right)(p) + P^*_I(\dot p) \\ &=\; \left(\frac{\partial P^*_I}{\partial q}\cdot \Big(g^{-1}(p_I) + h(\dot u)\Big)\right)(p_I + p_{III}) + P^*_I\left(-\frac{\partial H}{\partial q}(q(t),p(t)) + F(t,q(t),p(t)) + R(t)\right) \\ &=\; \left(\frac{\partial P^*_I}{\partial q}\cdot g^{-1}\Big(p_I + k(\dot u)\Big)\right)(p_I + k(\dot u)) - \frac{1}{2}P^*_I\left(\frac{\partial g^{-1}}{\partial q}[p_I + k(\dot u)\,,\; p_I + k(\dot u)]\right) + F_I\,. \end{aligned} \tag{4.5}$$

Indeed, the fifth relation in (2.11), i.e. d'Alembert's condition, implies $P^*_I(R(t)) = 0$.

The first two terms on the right hand side of (4.5) can be written in a simpler form, introducing the bilinear (q-dependent, possibly not symmetric), $\mathbb{R}^{N+M}$-valued map

$$(p,\tilde p) \;\mapsto\; \theta_I[p,\tilde p] \;\doteq\; \left(\frac{\partial P^*_I}{\partial q}\cdot g^{-1}(p)\right)(\tilde p) - \frac{1}{2}P^*_I\left(\frac{\partial g^{-1}}{\partial q}[p,\,\tilde p]\right). \tag{4.6}$$

Setting

$$\Upsilon[p_I\,,\,\dot u] \;\doteq\; \theta_I[p_I\,, k(\dot u)] + \theta_I[k(\dot u)\,,\; p_I], \qquad\qquad \Psi[\dot u,\,\dot u] \;\doteq\; \theta_I[k(\dot u)\,, k(\dot u)], \tag{4.7}$$

the equations (4.4) and (4.5) can be written in the form [1]

$$\begin{cases} \dot q_I(t) \;=\; g^{-1}(p_I)\,, \\ \\ \dot p_I(t) \;=\; \theta_I[p_I, p_I] + \Upsilon[p_I,\dot u] + \Psi[\dot u,\dot u] + F_I\,. \end{cases} \tag{4.8}$$

We can now state the main result of this section.

[1] We recall that $\dot q_I(t) \equiv P_I(\frac{dq}{dt})$ *while* $\dot p_I(t) \equiv \frac{d}{dt}(P^*_I(p))$

Theorem 4.1 *Let $\mathcal{I}$ be a time interval and let $u : \mathcal{I} \to I\!R^M$ be a $\mathcal{C}^1$ control function. Let $q : \mathcal{I} \mapsto I\!R^N$ be a $\mathcal{C}^1$ path and set $p(t) \doteq g(\dot{q}(t))$. Then the path $(q,p) : \mathcal{I} \to T^*\mathcal{Q}$ satisfies the non holonomic equations (2.11) (with constraints as controls) if and only if the following two conditions are satisfied.*

(i) For $J \in \{I, II, III\}$, the components $\dot{q}_J \doteq P_J(\dot{q}(t))$ and $p_J(t) \doteq P_J^*(p(t))$ satisfy the equations in (4.3) and (4.8).

(ii) At some time $t_0 \in \mathcal{I}$ one has $q^{N+\alpha}(t_0) = u^\alpha(t_0)$ for all $\alpha = 1, \ldots, M$.

Proof. 1. By the previous analysis, if all relations in (2.11) are satisfied, then the equations (4.3) and (4.8) hold. Moreover, the condition $\pi \circ q(t) = u(t)$ implies (ii), for all times $t \in \mathcal{I}$.

2. Next, assume that all the equations in (4.3) and (4.8) hold, and $\pi(q(t_0)) = u(t_0)$ at some time $t_0 \in \mathcal{I}$. By (4.1), the equation $\dot{q}_{II} = 0$ implies the fourth relation in (2.11).

For any $t \in \mathcal{I}$, if $\pi(q(t)) - u(t) = 0$ then

$$\frac{d}{dt}\Big(\pi(q(t)) - u(t)\Big) \;=\; D\pi \cdot \dot{q}(t) - \dot{u}(t) \;=\; D\pi \cdot \dot{q}_{III}(t) - \dot{u}(t) \;=\; 0\,.$$

From the assumption $\pi(q(t_0)) - u(t_0) = 0$ and the regularity of the coefficients of the equation, we conclude that $\pi(q(t)) - u(t) = 0$ for all $t \in \mathcal{I}$. Hence the third relation in (2.11) holds.

The equations for $\dot{q}$ in (4.3) and (4.8) imply

$$\dot{q} \;=\; \dot{q}_I + \dot{q}_{II} + \dot{q}_{III} \;=\; g^{-1}(p_I) + h(\dot{u}) \;=\; g^{-1}(p_I + p_{II} + p_{III}) \;=\; \frac{\partial H}{\partial p}(q,p)\,,$$

This yields the first equation in (2.11).

Finally, by defining

$$R(t) \;\doteq\; \dot{p} + \frac{\partial H}{\partial q}(q,p) - F(t, q(t), p(t)), \tag{4.9}$$

the second equation in (2.11) is clearly satisfied. It remains to check the fifth equation in (2.11), namely $R(t) \in \ker \Delta_q(t) + \ker \Gamma_q(t)$. Since by construction $\ker \Delta + \ker \Gamma = (T^*\mathcal{Q})_{II} + (T^*\mathcal{Q})_{III}$, this is equivalent to proving that $R_I = 0$. Using the second equation in (4.8) we find

$$\begin{aligned}
R_I \;&=\; P_I^*(R) \;=\; P_I^*(\dot{p}) + P_I^*\left(\frac{\partial H}{\partial q}(q,p)\right) - F_I \\
&=\; \dot{p}_I - \left(\frac{\partial P_I^*}{\partial q} \cdot \dot{q}\right)(p) + P_I^*\left(\frac{\partial H}{\partial q}(q,p)\right) - F_I \\
&=\; \dot{p}_I - \left(\frac{\partial P_I^*}{\partial q} \cdot g^{-1}\Big(p_I + k(\dot{u})\Big)\right)(p_I + k(\dot{u})) + \frac{1}{2} P_I^*\left(\frac{\partial g^{-1}}{\partial q}[p_I + k(\dot{u})\,,\; p_I + k(\dot{u})]\right) - F_I \\
&=\; 0\,.
\end{aligned}$$

□

5 The equations in Δ-adapted coordinates

Let $q = (q^1, \ldots, q^{N+M})$ be Δ-adapted coordinates on a open set $\mathcal{O} \subseteq \mathcal{Q}$, so that

$$\Delta \;=\; span\Big\{\frac{\partial}{\partial q^1}\,,\,\ldots\,,\,\frac{\partial}{\partial q^N}\Big\}.$$

The main goal of this section is to write the equations of motion explicitly in terms of these state coordinates, together with adjoint coordinates ξ_j, $1 \le j \le N+M$ corresponding to suitable bases of the cotangent bundle $T^*\mathcal{Q}$, decomposed as in (3.6). It will turn out that the relevant equations will involve only the $2N-\nu$ variables $q^1, \ldots, q^N, \xi_1, \ldots, \xi_{N-\nu}$.

Consider a family $\{\mathbf{V}_1,\,\ldots\,,\mathbf{V}_{N+M}\}$ of smooth, linearly independent vector fields on $\mathcal{Q}$, such that, at least on the open set $\mathcal{O}$,

$$\begin{aligned}
(T_{\mathbf{q}}\mathcal{Q})_I \;&=\; span\Big\{\mathbf{V}_1(\mathbf{q}), \ldots, \mathbf{V}_{N-\nu}(\mathbf{q})\Big\}\,,\\
(T_{\mathbf{q}}\mathcal{Q})_{II} \;&=\; span\Big\{\mathbf{V}_{N-\nu+1}(\mathbf{q}), \ldots, \mathbf{V}_N(\mathbf{q})\Big\}\,,\\
(T_{\mathbf{q}}\mathcal{Q})_{III} \;&=\; span\Big\{\mathbf{V}_{N+1}(\mathbf{q}), \ldots, \mathbf{V}_{N+M}(\mathbf{q})\Big\}.
\end{aligned} \tag{5.1}$$

Throughout the following, we assume that the vectors $\{\mathbf{V}_1,\,\ldots\,,\mathbf{V}_{N-\nu}\}$, which generate $(T_{\mathbf{q}}\mathcal{Q})_I$, are mutually orthogonal, i.e.

$$\mathbf{g}[\mathbf{V}_r, \mathbf{V}_s] \;=\; 0 \qquad \text{for all } \; r,s \in \{1, \ldots, N-\nu\}, \quad r \neq s\,. \tag{5.2}$$

In addition, for $i = 1, \ldots, N+M$, we define the basis of cotangent vectors

$$\mathbf{\Omega}_i \;\doteq\; \frac{\mathbf{g}(\mathbf{V}_i)}{\mathbf{g}[\mathbf{V}_i,\,\mathbf{V}_i]}, \tag{5.3}$$

By (3.7), this yields

$$\begin{aligned}
(T^*_{\mathbf{q}}\mathcal{Q})_I \;&=\; span\Big\{\mathbf{\Omega}_1(\mathbf{q}), \ldots, \mathbf{\Omega}_{N-\nu}(\mathbf{q})\Big\}\,,\\
(T^*_{\mathbf{q}}\mathcal{Q})_{II} \;&=\; span\Big\{\mathbf{\Omega}_{N-\nu+1}(\mathbf{q}), \ldots, \mathbf{\Omega}_N(\mathbf{q})\Big\}\,,\\
(T^*_{\mathbf{q}}\mathcal{Q})_{III} \;&=\; span\Big\{\mathbf{\Omega}_{N+1}(\mathbf{q}), \ldots, \mathbf{\Omega}_{N+M}(\mathbf{q})\Big\}\,.
\end{aligned}$$

By (5.2) and (5.3), the differential forms $\{\mathbf{\Omega}_1,\,\ldots\,,\mathbf{\Omega}_{N-\nu}\}$ are mutually orthogonal with respect to the metric $\mathbf{g}^{-1}$, namely

$$\mathbf{g}^{-1}[\mathbf{\Omega}_r,\,\mathbf{\Omega}_s] \;=\; 0 \qquad \text{for all } \; r,s \in \{1, \ldots, N-\nu\}, \quad r \neq s\,. \tag{5.4}$$

Moreover, the basis $\Big\{\mathbf{\Omega}_1(\mathbf{q}), \ldots, \mathbf{\Omega}_{N-\nu}(\mathbf{q})\Big\}$ is dual to the basis $\Big\{\mathbf{V}_1(\mathbf{q}), \ldots, \mathbf{V}_{N-\nu}(\mathbf{q})\Big\}$, i.e.

$$\mathbf{\Omega}_r(\mathbf{q})\Big(\mathbf{V}_s(\mathbf{q})\Big) \;=\; \delta_{r,s} \qquad \text{for all } \; r,s = 1, \ldots, N-\nu.$$

This choice of orthogonal bases makes it easy to compute the projections $\mathcal{P}_I$ and $\mathcal{P}_I^*$. Indeed, for any tangent vector $\mathbf{w}$ and any cotangent vector $\mathbf{p}$ one has

$$\mathcal{P}_I(\mathbf{w}) \;=\; \sum_{\ell=1}^{N-\nu} \frac{\mathbf{g}[\mathbf{w}, \mathbf{V}_\ell]}{\mathbf{g}[\mathbf{V}_\ell, \mathbf{V}_\ell]}\mathbf{V}_\ell, \tag{5.5}$$

$$\mathcal{P}_I^*(\mathbf{p}) \;=\; \sum_{\ell=1}^{N-\nu} \frac{\mathbf{g}^{-1}[\mathbf{p}, \mathbf{\Omega}_\ell]}{\mathbf{g}^{-1}[\mathbf{\Omega}_\ell,\, \mathbf{\Omega}_\ell]}\, \mathbf{\Omega}_\ell\,. \tag{5.6}$$

In the following, to simplify notation, whenever repeated indices taking values from 1 to $N+M$ are summed, the summation symbol will be omitted. On the other hand, summations ranging over a smaller set of indices will be explicitly written. As before, we let $g = (\mathbf{g}_{r,s})$ be the matrix representing the Riemannian metric $\mathbf{g}$ in the q-coordinates. In turn, the inverse matrix $g^{-1} = (g^{r,s})$ represents the metric $\mathbf{g}^{-1}$ on the cotangent space.

For $\ell = 1, \ldots, N+M$, let ${V_\ell}^1, \ldots, {V_\ell}^{N+M}$ be the q-components of $\mathbf{V}_\ell$, so that $\mathbf{V}_\ell = {V_\ell}^i \frac{\partial}{\partial q_i}$. If the (column) vector $w = (w^s) \in I\!R^{N+M}$ yields the coordinate representation of $\mathbf{w}$, then by (5.5) the projected vector $\mathcal{P}_I(\mathbf{w})$ is a column vector with coordinates $(P_I\, w)^s \;=\; (P_I)^s_r w^r$, where the $(N+M)\times(N+M)$ matrix P_I is defined by

$$(P_I)^s_r \;\doteq\; \sum_{\ell=1}^{N-\nu} \frac{\mathbf{g}_{r,k}{V_\ell}^k {V_\ell}^s}{\mathbf{g}_{i,j}{V_\ell}^i {V_\ell}^j} \qquad\qquad r, s = 1, \ldots, N+M\,.$$

Similarly, let $\Omega_{r,1}, \ldots, \Omega_{r,N+M}$ be the components of $\mathbf{\Omega}_r$, so that $\mathbf{\Omega}_r = \Omega_{r,s} dq^s$. If the (row) vector $p \in I\!R^{M+N}$ yields the coordinate representation of the covector $\mathbf{p}$, then by (5.6) the projected the vector $\mathcal{P}_I^*(\mathbf{p})$ has coordinates given by $(p\, P_I^*)_s = p_r (P_I^*)^r_s$, where the $(N+M)\times(N+M)$ matrix P_I^* is defined by

$$(P_I^*)^r_s \;\doteq\; \sum_{\ell=1}^{N-\nu} \frac{g^{r,k}\, \Omega_{\ell,k}\, \Omega_{\ell,s}}{g^{i,j}\, \Omega_{\ell,i}\Omega_{\ell,j}}\,. \tag{5.7}$$

In order to write the second set of equations in (4.3), we need an explicit expression of the (q-dependent) matrices h and k. Let us define the $(N+M)\times M$ matrix V_{III} and the $M\times M$ matrix ${V_{III}}^{III}$ by setting

$$V_{III} \doteq \Big({V_{N+\alpha}}^r\Big), \qquad {V_{III}}^{III} \doteq \Big({V_{N+\alpha}}^{N+\beta}\Big)$$

Here and in the sequel, Greek indices such as α, β range from 1 to M, while Latin indices such as r, s range from 1 to $N+M$. Notice that, by (5.1), the columns of V_{III} span $(T_q\mathcal{Q})_{III}$.

Recalling the definitions (3.12)-(3.13), and the identities $q^{N+\alpha} = u^\alpha$, it is easy to check that the injective linear map $\mathbf{h} : T\mathcal{U}_{\pi(\mathbf{q})} \mapsto (T_\mathbf{q}\mathcal{Q})_{III} \subset T_\mathbf{q}\mathcal{Q}$ is represented by the $(N+M)\times M$ matrix

$$h \;=\; V_{III} \cdot \Big({V_{III}}^{III}\Big)^{-1}.$$

In turn, the linear map $\mathbf{k}(\cdot) \;=\; \mathbf{g}(\mathbf{h}(\cdot))$ is represented by the $(N+M)\times M$ matrix

$$k \doteq g \cdot h \;=\; g \cdot V_{III} \cdot \Big({V_{III}}^{III}\Big)^{-1}.$$

Using this particular system of coordinates, the equations of motion (4.3), (4.8) for the non-holonomic system with active constraints can be written in the following form.

Proposition 5.1 *Let $(q, p, u)(\cdot)$ be as in Theorem 4.1. Moreover, let $(\xi,\, \eta,\, \lambda)$ be the components of $\mathbf{p} = p_i dq^i$ w.r.t. the frame $\{\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_{N+M}\}$, so that*

$$\mathbf{p}(t) \;=\; \sum_{\ell=1}^{N-\nu} \xi_\ell(t)\mathbf{\Omega}_\ell + \sum_{\ell=N-\nu+1}^{N} \eta_\ell(t)\mathbf{\Omega}_\ell + \sum_{\ell=N+1}^{N+M} \lambda_\ell(t)\mathbf{\Omega}_\ell\,.$$

Then the curve $t \mapsto (q(t), p(t), u(t))$ provides a solution to the system (2.11) if and only if the curve $t \mapsto (q(t), \xi(t), \eta(t), \lambda(t), u(t))$ is a solution of

$$\begin{cases} \dot{q}^r &= \displaystyle\sum_{\ell=1}^{N-\nu} \frac{V_\ell{}^r}{\mathbf{g}_{i,j} V_\ell{}^i V_\ell{}^j}\, \xi_\ell + \sum_{\alpha=1}^{M} h_\alpha{}^r\, \dot{u}^\alpha & r = 1, \ldots, N\,, \\ q^{N+\alpha} &= u^\alpha & \alpha = 1, \ldots, M\,, \\ \dot{\xi}_m &= \tilde{\theta}_m[\xi, \xi] + \widetilde{\Upsilon}_m\,[\xi\,, \dot{u}] + \widetilde{\Psi}_m[\dot{u}, \dot{u}] + \widetilde{F}_m & m = 1, \ldots, N-\nu\,, \\ \eta &= 0\,, \\ \lambda &= (V_{III})^T \cdot k \cdot \dot{u}\,, \end{cases} \tag{5.8}$$

where the superscript ^{T} denotes transposition and, recalling (4.6)-(4.7), for every $m = 1, \ldots, N-\nu$ we set

$$\tilde{\theta}_m[\xi, \hat{\xi}] \doteq \textstyle\sum_{a,b=1}^{N-\nu} \tilde{\theta}_m^{a,b}\, \xi_a \hat{\xi}_b, \quad \textit{with} \quad \tilde{\theta}_m^{a,b} \doteq \left([\theta_I]_i^{r,s} \Omega_{r,a} \Omega_{s,b} - \displaystyle\sum_{j=1}^{N} \frac{\partial \Omega_{i,a}}{\partial q^j} g^{j,s} \Omega_{s,b} \right) V_m{}^i,$$

$$\widetilde{\Upsilon}_m\,[\xi\,, \dot{u}] \doteq \textstyle\sum_{a=1}^{N-\nu} \widetilde{\Upsilon}_{\alpha,m}^a\, \xi_a\, \dot{u}^\alpha, \quad \textit{with} \quad \widetilde{\Upsilon}_{\alpha,m}^a \doteq \left(\Upsilon_{\alpha,i}^r\, \Omega_{r,a} - \dfrac{\partial \Omega_{i,a}}{\partial q^{N+\alpha}} - \displaystyle\sum_{j=1}^{N} \frac{\partial \Omega_{i,a}}{\partial q^j} h_\alpha{}^j \right) V_m{}^i,$$

$$\widetilde{\Psi}_m\,[w, \tilde{w}] \doteq \widetilde{\Psi}_{\alpha,\beta,m}\, w^\alpha \tilde{w}^\beta, \quad \textit{with} \quad \widetilde{\Psi}_{\alpha,\beta,m} \doteq \Psi_{\alpha,\beta,i} V_m{}^i\,, \qquad \widetilde{F}_m \doteq F_i V_m{}^i\,. \tag{5.9}$$

Indeed, (5.9) provides the representation of the maps θ_I, Υ, and Ψ in (4.6)-(4.7), with the present choice of coordinates.

6 Systems which are "fit for jumps"

We now examine the connections between the following properties:

i) The continuity of the control-to-trajectory map $u(\cdot) \mapsto (q(\cdot), p(\cdot))$.

ii) The vanishing of the "centrifugal" term $\Psi[\dot{u}, \dot{u}]$ in (4.7).

iii) The invariance of the distribution $\Gamma \cap (\Gamma \cap \Delta)^\perp$ in Γ-constrained inertial motions.

We recall that, if $\Psi[\dot{u}, \dot{u}] \equiv 0$, then the equations of motion (5.8) are affine w.r.t. the time derivative $\dot{u}$ of the control function. In this case, the control-to-trajectory map $u(\cdot) \mapsto (q(\cdot), p(\cdot))$ can be continuously extended to a family of possibly discontinuous control functions $u(\cdot)$ with bounded variation [5]. As in [9], we then say that the system is *fit for jumps.* On the other hand, if the quadratic term $\Psi[\dot{u}, \dot{u}]$ does not vanish, this means that small vibrations of the control produce a centrifugal force. As shown in [8], this feature can be exploited as an additional way to control or stabilize the system.

As in Section 2, $\mathcal{T}$ will denote the kinetic energy associated with the metric $\mathbf{g}$, defined at (2.4), while Γ denotes a non-holonomic distribution on the manifold $\mathcal{Q}$.

Definition 6.1 *Let $\mathcal{I}$ be a time interval. A $\mathcal{C}^2$ map $\mathbf{q} : \mathcal{I} \to \mathcal{Q}$ will be called a* free inertial motion *if, in any set of coordinates, its local expression $q(\cdot)$ provides a solution to the Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}} - \frac{\partial \mathcal{T}}{\partial q} \;=\; 0. \tag{6.1}$$

Definition 6.2 *A $\mathcal{C}^2$ map $q : \mathcal{I} \to \mathcal{Q}$ will be called a* Γ-constrained inertial motion *if, in any set of coordinates, its local expression $q(\cdot)$ is a solution of*

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}} - \frac{\partial \mathcal{T}}{\partial q} \;\in\; \ker \Gamma \tag{6.2}$$

and satisfies the non holonomic constraints

$$\dot{q} \;\in\; \Gamma. \tag{6.3}$$

Definition 6.3 *Let $S \subseteq T\mathcal{Q}$ be any set. We say that S is* invariant for the free inertial flow, (*or equivalently:* inertially invariant), *if, for every free inertial motion $\mathbf{q} : \mathcal{I} \to \mathcal{Q}$ such that $(\mathbf{q}, \dot{\mathbf{q}})(t_0) \in S$ for some $t_0 \in \mathcal{I}$, one has*

$$(\mathbf{q}, \dot{\mathbf{q}})(t) \in S \qquad \textit{for all} \;\; t \in \mathcal{I}. \tag{6.4}$$

Definition 6.4 *Let $S \subseteq \mathcal{T}\mathcal{Q}$ be any set. We say that S is* invariant for the Γ-contrained inertial flow, (*or equivalently:* Γ-inertially invariant), *if, for every Γ-constrained inertial motion (see Def. 6.2) $\mathbf{q} : \mathcal{I} \to \mathcal{Q}$ such that $(\mathbf{q}, \dot{\mathbf{q}})(t_0) \in S$ for some $t_0 \in \mathcal{I}$, one has*

$$(\mathbf{q}, \dot{\mathbf{q}})(t) \in S \qquad \textit{for all} \;\; t \in \mathcal{I}. \tag{6.5}$$

In the next theorem we state a characterization for the Γ-inertially invariance of set $(T\mathcal{Q})_{III} = \Gamma \cap (\Gamma \cap \Delta)^{\perp}$.

Theorem 6.1 *The following conditions are equivalent:*

(1) *If $F \equiv 0$, for every $\mathcal{C}^1$ control $u(\cdot)$ and every solution $q(\cdot)$ of the control differential equation (on $\mathcal{Q}$)*

$$\dot{q} \;=\; h(\dot{u}),$$

the map

$$t \;\mapsto\; (q(t), p(t)) \;\doteq\; \Big(q(t)\,,\, k(\dot{u}(t))\Big)$$

is a solution of (2.11). *In particular, $p_I(t) = 0$ and hence $\dot{q}_I(t) = 0$ for all $t \in \mathcal{I}$.*

(2) *For every local chart (q) the (vector-valued) quadratic form Ψ defined at (4.7) vanishes identically.*

(3) *For every local chart (q), the (vector-valued) quadratic form $v \mapsto \theta_I[v, v]$ in* (4.6) *vanishes on the subspace $P^*_{III}(I\!R^{N+M})$.*

(4) *The sub-bundle $(T\mathcal{Q})_{III} = \Gamma \cap (\Gamma \cap \Delta)^{\perp}$ is Γ-inertially invariant.*

Proof. We first prove the implication (1)$\Longrightarrow$(4). Let (1) hold, and let $q : \mathcal{I} \mapsto \mathcal{Q}$ be a Γ-constrained inertial motion such that $\dot{q}_I(t_0) = 0$. Choosing $u(t) = \pi(q(t))$, it is clear that the equations (6.2)-(6.3) imply (2.11), with $F \equiv 0$. Hence, if (1) holds, then $\dot{q}_I(t) \equiv 0$ for all $t \in \mathcal{I}$. By construction, the non-holonomic constraint yields $\dot{q}_{II}(t) \equiv 0$. Therefore $\dot{q}(t) = \dot{q}_{III}(t) \in (T_q\mathcal{Q})_{III}$ for all $t \in \mathcal{I}$.

Next, we prove the implication (4)$\Longrightarrow$(2). Fix any point $q(0) = q_0$ and any vector $\omega_0 \in I\!R^M$. Let $t \mapsto q(t)$ be the Γ-constrained inertial motion starting at q_0, with $\dot{q}(0) = h(\omega_0)$. Then $q(\cdot)$ provides a solution to the system (2.11) in connection with the control $u(t) \doteq \pi(q(t))$. If (4) holds, then $\dot{q}_I(t) \equiv 0$ for all t, and hence $p_I(t) \equiv 0$ as well. On the other hand, our construction implies $\dot{u}(0) = \omega_0$. Therefore, at time $t = 0$, the second equation in (4.8) yields

$$0 = \dot{p}_I(0) \; = \; \Psi[\dot{u}(0), \dot{u}(0)] \; = \; \Psi[\omega_0, \omega_0]\,.$$

Since q_0 and ω_0 can be chosen arbitrarily, we conclude that $\Psi \equiv 0$, proving (2).

We now prove the implication (2)$\Longrightarrow$(1). Let a continuously differentiable control $u(\cdot)$ be given. If condition (2) holds true and $F \equiv 0$, then in the ODE (4.8) for $\dot{p}_I$ each term on the right hand side is a linear or quadratic w.r.t. p_I. Therefore, if $p_I(t_0) = 0$ at some time t_0 then $p_I(t) \equiv 0$ for all times $t \in I$, proving (1).

Finally, we show that (2)$\Longleftrightarrow$(3). Since the linear map $\mathbf{k} : T_{\pi(\mathbf{q})}\mathcal{U} \mapsto (T_{\mathbf{q}}\mathcal{Q})_{III}$ is a bijection, the equivalence of (2) and (3) follows directly by the definition of θ_I and Ψ. □

Remark 6.1 In the holonomic case the the quadratic term in $\dot{u}$, which prevents the system to be "fit for jumps", is completely determined by the relation between the kinetic metric and the distribution Δ. In particular this quadratic term accounts for the curvature of the orthogonal distribution $\Delta^\perp$ (see [8] for a precise result in this direction). On the other hand, when an additional non holonomic constraint $\dot{q} \in \Gamma$ is present, the orthogonal distribution $\Delta^\perp$ can have zero curvature[2] while the dynamical equations still contain a term which is quadratic in $\dot{u}$. This is the case of the Roller Racer, our first example in Section 8.

Given a holonomic system which is fit for jumps, Theorem 6.2 below provides a sufficient condition in order that the system remains fit for jumps also after the addition of non-holonomic constraints. This happens in the case of the rolling ball, our second example in Section 8.

Theorem 6.2 *Assume that the external force F does not depend on velocity*[3]*. Suppose that the following conditions are satisfied:*

(i) The orthogonal distribution $\Delta^\perp$ has zero curvature, i.e.: without the nonholonomic constraint $\dot{q} \in \Gamma$ the system would be fit for jumps.

(ii) There exists a Δ-adapted system of coordinates $(q) = (q_\sharp, q_\flat)$, with $q_\sharp \doteq (q^1, \ldots, q^N)$, $q_\flat \doteq (q^{N+1}, \ldots, q^{N+M})$, $\Delta = span\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^N}\}$, such that the Hamiltonian H is independent of the coordinates $q_\flat$.

[2]We recall that $\Delta^\perp$ has zero curvature if the following holds. Let $s \mapsto q(s)$ be any geodesic of the manifold $\mathcal{Q}$ which intersects perpendicularly one of the leaves of the foliation generated by the integrable distribution Δ. Then every other leaf touched by this geodesic is also crossed perpendicularly [8, 24, 26, 27].

[3]We can relax this hypothesis by allowing F to be affine in the velocity.

(iii) There exists a basis $\{\mathbf{V}_1, \ldots, \mathbf{V}_{N+M}\}$ *for* $T\mathcal{Q}$ *such that, in this basis, the projection* $\mathcal{P}_I^*$ *is represented by a constant (i.e., independent of* $\mathbf{q}$*) matrix* P_I^**.*

Then, even after the addition of nonholonomic constraints $\dot{q} \in \Gamma$*, the system remains fit for jumps .*

Proof. We need to show that the time derivative $\dot{p}_I$ is an affine function of $\dot{u}$. Recalling (4.5) we have

$$\dot{p}_I \;=\; \left(\frac{\partial P_I^*}{\partial q} \cdot \dot{q}\right)(p) + P_I^*(\dot{p})\,. \tag{6.6}$$

By the assumption (iii), $\frac{\partial P_I^*}{\partial q} = 0$. Therefore

$$\dot{p}_I \;=\; P_I^*(\dot{p}),$$

where $p(\cdot)$ solves the second equation in (2.12), namely

$$\dot{p} \;=\; -\frac{\partial H}{\partial q} + F + R.$$

By the hypotheses on F and since $R_I = 0$ it is then sufficient to show that $-\dfrac{\partial H}{\partial q}$ can be expressed as a function of $(q, p_I, \dot{u})$ which is affine in $\dot{u}$. The assumption (ii) yields

$$-\frac{\partial H}{\partial q} \;=\; \left(-\frac{\partial H}{\partial q_\sharp}, -\frac{\partial H}{\partial q_\flat}\right) \;=\; \left(-\frac{\partial H}{\partial q_\sharp}, 0\right).$$

On the other hand, by considering the cotangent natural basis $(dq_\sharp, dq_\flat)$ and the corresponding adjoint coordinates $(p_\sharp, p_\flat)$, we can express $-\dfrac{\partial H}{\partial q_\sharp}$ as a function of $(q, p_\sharp, \dot{q}_\flat)$,

$$-\frac{\partial H}{\partial q_\sharp} \;=\; -\frac{\partial H}{\partial q_\sharp}\Big(q,\, p_\sharp,\, p_\flat(q, p_\sharp, \dot{q}_\flat)\Big),$$

where $p_\flat = p_\flat(q, p_\sharp, \dot{q}_\flat)$ is the function obtained by partially inverting the linear isomorphism $\dot{q} = \dfrac{\partial H}{\partial p}$. By the assumption (i), the function $-\dfrac{\partial H}{\partial q_\sharp}$ (considered as a function of the variables $(q, p_\sharp, \dot{q}_\flat)$) is in fact affine w.r.t. $\dot{q}_\flat$ (see e.g. [4] or [24]). This is equivalent to saying that it is affine w.r.t. $\dot{u}$, because the map $\dot{u} \mapsto \dot{q}_\flat(\dot{u})$ is a linear isomorphism[4]. Let $\pi_\sharp$ denote the projection $p \mapsto \pi_\sharp(p) = (p_\sharp, 0)$. By (4.3) we have

$$(p_\sharp, 0) \;=\; \pi_\sharp(p) \;=\; \pi_\sharp(p_I + p_{III}) \;=\; \pi_\sharp(p_I + k(\dot{u})).$$

Therefore $-\frac{\partial H}{\partial q_\sharp}$ is a function of $(q, p_I, \dot{u})$ which is affine in $\dot{u}$. □

[4]For a suitable choice of local coordinates on $\mathcal{U}$, this isomorphism coincides with the identity.

7 A geometric interpretation of the quadratic term

We begin this section by observing that it is not restrictive to assume that the manifold $\mathcal{Q}$ is an open subset of an Euclidean space.

Indeed, by the isometric embedding theorem for Riemann manifolds [14, 21], there exists a positive integer M' and an embedding $\iota : \mathcal{Q} \mapsto I\!R^{N+M+M'}$ (with $I\!R^{N+M+M'}$ endowed with the usual Euclidean metric) which preserves the Riemann metric on $\mathcal{Q}$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_{N+M+M'}\}$ be the standard orthonormal basis of $I\!R^{N+M+M'}$. Fix any point $\bar{\mathbf{q}} \in \mathcal{Q}$. By the implicit function theorem, we can assume that in a neighborhood of $\iota(\bar{\mathbf{q}})$ the image of $\mathcal{Q}$ can be written as

$$\Big\{(x_1, \ldots, x_{N+M+M'})\,; \quad x_i = f_i(x_1, \ldots, x_{N+M})\,, \quad N+M+1 \leq i \leq N+M+M'\Big\}.$$

For $\mathbf{x} = (x_1, \ldots, x_{M+N+M'})$, we denote by $\mathbf{q}(\mathbf{x})$ the unique point $\mathbf{q}$ in a neighborhood of $\bar{\mathbf{q}} \in \mathcal{Q}$ such that the first $N+M$ coordinates of $\iota(\mathbf{q})$ coincide with $(x_1, \ldots, x_{N+M})$.

Next, consider the enlarged control manifold $\mathcal{U}' \doteq \mathcal{U} \times I\!R^{M'}$. Together with the submersion $\pi : \mathcal{Q} \mapsto \mathcal{U}$, consider the submersion $\pi' : I\!R^{N+M+M'} \mapsto \mathcal{U}'$ by setting

$$\pi'(x_1, \ldots, x_{N+M+M'}) \;\doteq\; \Big(\mathbf{u},\, (u_{M+1}, \ldots, u_{M+M'})\Big) \;\in\; \mathcal{U} \times I\!R^{M'}, \tag{7.1}$$

where

$$\mathbf{u} = \pi(\mathbf{q}(\mathbf{x})), \qquad\qquad u_i \;=\; x_i - f_i(x_1, \ldots, x_{N+M}) \quad \text{for} \quad N+M+1 \leq i \leq N+M+M'.$$

Calling $D\iota$ the differential of the embedding ι at the point $\mathbf{q}(\mathbf{x})$, we use $\Delta'_{\mathbf{x}} \doteq D\iota \circ \Delta_{\mathbf{q}(\mathbf{x})}$ to denote the holonomic distribution on $I\!R^{N+M+M'}$ generated by the submersion (7.1). Given the non-holonomic distribution Γ on $\mathcal{Q}$, the corresponding non-holonomic distribution Γ' on $I\!R^{N+M+M'}$ is defined as

$$\Gamma'_{\mathbf{x}} \;\doteq\; \text{span}\Big\{D\iota \circ \Gamma_{\mathbf{q}(\mathbf{x})}\,,\; \mathbf{e}_{N+M+1}, \;\ldots\;, \mathbf{e}_{N+M+M'}\Big\}. \tag{7.2}$$

Observe that, if (2.6) holds, then we also have

$$\Delta'_{\mathbf{x}} + \Gamma'_{\mathbf{x}} \;=\; I\!R^{N+M+M'}.$$

Trajectories of the non-holonomic system with active constraints defined at (2.1)-(2.2) can now be recovered as trajectories of the non-holonomic system with active constraints (7.1)-(7.2) corresponding to control functions

$$t \;\mapsto\; \Big(\mathbf{u}(t),\, u_{M+1}(t), \ldots, u_{M+M'}(t)\Big) \;\in\; \mathcal{U} \times I\!R^{M'}$$

with $u_{M+1}(t) = \cdots = u_{M+M'}(t) = 0$ for every time t.

Based on the previous remarks, without loss of generality we now analyze the geometric meaning of the term $\Psi[\dot{u}, \dot{u}]$ in (4.7), assuming that $\mathcal{Q} = I\!R^{N+M}$ is a finite dimensional vector space with Euclidean metric. In this case the coefficients of the metric are constant: $\mathbf{g}_{rs} \equiv g^{rs} \equiv \delta_{rs}$. Moreover, with canonical identifications we have

$$p \;=\; \dot{q}\,, \qquad k(\dot{u}) \;=\; h(\dot{u})\,, \qquad \frac{\partial g^{-1}}{\partial q} \;\equiv\; 0 \qquad P_I = P_I^*\,. \tag{7.3}$$

Therefore, (4.6)-(4.7) yield the simpler formula

$$\Psi[\dot{u}, \dot{u}] \;=\; \left(\frac{\partial P_I}{\partial q} \cdot h(\dot{u})\right)(h(\dot{u}))\,. \tag{7.4}$$

In this setting, consider a system of local coordinates $(u_1, \ldots, u_M)$ on $\mathcal{U}$. To fix the ideas, let $v = (1, 0 \ldots, 0) \in I\!R^M$. We seek a geometric characterization of the term

$$\Psi[v, v] \;=\; \left(\frac{\partial P_I}{\partial q} \cdot h(v)\right)(h(v))\,. \tag{7.5}$$

At each point $q \in I\!R^{M+N}$ in a neighborhood of $\bar{q}$, choose an orthogonal basis of unit vectors $\mathbf{V}_1(q), \ldots, \mathbf{V}_{N+M}(q)$ as in (5.1), but with $\mathbf{V}_{N+1}$ being the unit vector parallel to $h(v)$, so that

$$h(v, q) \;=\; \xi(q)\mathbf{V}_{N+1}(q)$$

for some $\xi(q) > 0$. Notice that we now write $h = h(v, q)$, since we regard h as a vector field on $\mathcal{Q} = I\!R^{N+M}$. In the following, we use $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product, and we write $(D\phi)\,\mathbf{V}_i$ for the directional derivative of ϕ in the direction of $\mathbf{V}_i$. Moreover, the map $t \mapsto (\exp t\mathbf{V}_i)(\bar{x})$ denotes the solution of the Cauchy problem

$$\dot{x}(t) \;=\; \mathbf{V}_i(x(t))\,, \qquad\qquad x(0) \;=\; \bar{x}\,.$$

Computing (7.5) in terms of the basis $\{\mathbf{V}_1, \ldots, \mathbf{V}_{N+M}\}$ we obtain

$$P_I(w) \;=\; \sum_{i=1}^{N-\nu} \Big\langle w\,,\; \mathbf{V}_i \Big\rangle\, \mathbf{V}_i \qquad \text{for all} \;\; w \in I\!R^{N+M},$$

$$\begin{aligned} \Psi[v, v] \;&=\; \sum_{i=1}^{N-\nu} \Big\langle \xi\mathbf{V}_{N+1}\,, (D\mathbf{V}_i)\;\xi\mathbf{V}_{N+1} \Big\rangle\, \mathbf{V}_i + \sum_{i=1}^{N-\nu} \Big\langle \xi\mathbf{V}_{N+1}\,,\; \mathbf{V}_i \Big\rangle\, (D\mathbf{V}_i)\xi\mathbf{V}_{N+1} \\ &=\; \sum_{i=1}^{N-\nu} \Big\langle \xi\mathbf{V}_{N+1}\,,\; (D\mathbf{V}_i)\xi\mathbf{V}_{N+1} \Big\rangle\, \mathbf{V}_i\,. \end{aligned} \tag{7.6}$$

We now claim that, for $1 \leq i \leq N - \nu$, the Lie bracket

$$[\xi\mathbf{V}_{N+1}\,,\; \mathbf{V}_i] \;=\; (D\mathbf{V}_i)\xi\mathbf{V}_{N+1} - (D(\xi\mathbf{V}_{N+1}))\mathbf{V}_1$$

lies in the subspace $\Delta_{\mathbf{q}}$ and hence is orthogonal to $\mathbf{V}_{N+1}$. Indeed, for every $\bar{q}$ one has

$$[\xi\mathbf{V}_{N+1}\,,\; \mathbf{V}_i](\bar{q}) \;=\; \lim_{\varepsilon \to 0} \frac{q(\varepsilon) - \bar{q}}{\varepsilon^2}\,,$$

where

$$q(\varepsilon) \;\doteq\; \Big(\exp(-\varepsilon\mathbf{V}_i)\Big) \circ \Big(\exp(-\varepsilon\xi\mathbf{V}_{N+1})\Big) \circ \Big(\exp(\varepsilon\mathbf{V}_i)\Big) \circ \Big(\exp(\varepsilon\xi\mathbf{V}_{N+1})\Big)(\bar{q})\,.$$

However, for any q we have

$$\frac{d}{dt}\pi\Big(\exp(t\mathbf{V}_i)\Big)(q) \;=\; 0\,, \qquad\qquad \frac{d}{dt}\pi\Big(\exp(t\xi\mathbf{V}_{N+1})\Big)(q) \;=\; v\,.$$

Hence, for every ε one has

$$\pi(q(\varepsilon)) \;=\; \Big(\exp(-\varepsilon v)\Big) \circ \Big(\exp(\varepsilon v)\Big)(\pi(\bar{q})) \;=\; \pi(\bar{q})\,.$$

This implies

$$[\xi\mathbf{V}_{N+1}\,,\; \mathbf{V}_i](\bar{q}) \;\in\; ker\, D\pi(\bar{q}) \;=\; \Delta_{\bar{q}}\,,$$

as claimed. Therefore

$$\Big\langle \mathbf{V}_{N+1}\,, (D\mathbf{V}_i)(\xi\mathbf{V}_{N+1})\Big\rangle ~=~ \Big\langle \mathbf{V}_{N+1}\,,~ (D(\xi\mathbf{V}_{N+1}))\mathbf{V}_i\Big\rangle\,. \tag{7.7}$$

Next, we observe that

$$\Big\langle \mathbf{V}_{N+1}\,,~ (D\mathbf{V}_{N+1})\mathbf{V}_i\Big\rangle ~=~ \frac{1}{2}(D|\mathbf{V}_{N+1}|^2)\mathbf{V}_i ~=~ 0\,,$$

because all vectors $\mathbf{V}_{N+1}(q)$ have unit length. From (7.6) and (7.7) we obtain

$$\begin{aligned}\Psi[v,v] ~&=~ \sum_{i=1}^{N-\nu}\Big\langle \xi\mathbf{V}_{N+1}\,,~ (D(\xi\mathbf{V}_{N+1}))\mathbf{V}_i\Big\rangle\,\mathbf{V}_i \\ &=~ \xi\sum_{i=1}^{N-\nu}((D\xi)\mathbf{V}_i)\,\mathbf{V}_i ~=~ \xi\,P_I^*(\nabla\xi)\,.\end{aligned} \tag{7.8}$$

Recalling (3.14), we can think of $\xi(q) = |h(v,q)|$ as the infinitesimal distance between leaves of the foliation generated by the submersion π at the point q, computed along paths that respect the non-holonomic distribution. According to (7.8), if the directional derivative of this distance is zero in all directions of the subspace $(T_q\mathcal{Q})_I = \Delta_q\cap\Gamma_q$, then the term $\Psi[\mathbf{v},\mathbf{v}]$ vanishes for all $\mathbf{v}\in T_{\pi(q)}\mathcal{U}$.

Going back to the general case of a Riemann manifold $\mathcal{Q}$ and a submersion $\pi:\mathcal{Q}\mapsto\mathcal{U}$, we thus obtain the following geometric characterization. For every $\mathbf{u}\in\mathcal{U}$, consider the leaf

$$\Lambda_{\mathbf{u}} ~\doteq~ \{\mathbf{q}\in\mathcal{Q}\,;~~ \pi(\mathbf{q})=\mathbf{u}\}\,. \tag{7.9}$$

Then for every $\mathbf{q}\in\Lambda_{\mathbf{u}}$ and every tangent vector $\mathbf{v}\in T_{\mathbf{u}}\mathcal{U}$ there exists a unique vector $\mathbf{h}(\mathbf{v})\in(T_{\mathbf{q}}\mathcal{Q})_{III}$ such that $D\pi\cdot\mathbf{h}(\mathbf{v})=\mathbf{v}$. Recalling (3.14), one could define an inner product on $\langle\cdot,\cdot\rangle_{\mathbf{u}}$ on $T_{\mathbf{u}}\mathcal{U}$ by setting

$$\langle\mathbf{v},\mathbf{v}\rangle_{\mathbf{u}} ~\doteq~ \mathbf{g}_{\mathbf{q}}[\mathbf{h}(\mathbf{v}),\,\mathbf{h}(\mathbf{v})]\,. \tag{7.10}$$

Of course, in general this inner product depends on the choice of the point $\mathbf{q}$ along the leaf $\Lambda_{\mathbf{u}}$. According to (7.8), the condition $\Psi\equiv 0$ means that the directional derivative of the right hand side of (7.10) vanishes along $\Delta_{\mathbf{q}}\cap\Gamma_{\mathbf{q}}$. More precisely:

Proposition 7.1 *The following conditions are equivalent.*

(i) $\Psi\equiv 0$.

(ii) For every $\mathbf{u}\in\mathcal{U}$, $\mathbf{v}\in T_{\mathbf{u}}\mathcal{U}$, *and every smooth path* $s\mapsto\mathbf{q}(s)\in\Lambda_{\mathbf{u}}$ *such that* $\dot{\mathbf{q}}(s)\in\Delta_{\mathbf{q}(s)}\cap\Gamma_{\mathbf{q}(s)}$, *the inner product* $\mathbf{g}_{\mathbf{q}(s)}[\mathbf{h}(\mathbf{v}),\,\mathbf{h}(\mathbf{v})]$ *is constant.*

Remark 7.1 In general, the condition $\Psi\equiv 0$ does not guarantee the existence of a Riemannian metric on $\mathcal{U}$ defined by (7.10). In order that this metric be well defined, i.e. independent of the point $\mathbf{q}\in\Lambda_{\mathbf{u}}$, one also needs to assume that the distribution $\Delta_{\mathbf{q}}\cap\Gamma_{\mathbf{q}}$ is completely non-integrable restricted to each leaf $\Lambda_{\mathbf{u}}$. More precisely, every two points $\mathbf{q}_1,\mathbf{q}_2\in\Lambda_{\mathbf{u}}$ should be connected by a path $t\mapsto\mathbf{q}(t)\in\Lambda_{\mathbf{u}}$ with $\dot{\mathbf{q}}(t)\in\Delta_{\mathbf{q}(t)}\cap\Gamma_{\mathbf{q}(t)}$ for all t. Notice that this condition is clearly satisfied if the non-holonomic constraint is not present (i.e., $\Gamma_{\mathbf{q}}=T_{\mathbf{q}}\mathcal{Q}$) and the leaves $\Lambda_{\mathbf{u}}$ are connected. From (7.8) we thus recover some earlier characterizations of the property "fit for jumps" given in [9, 24].

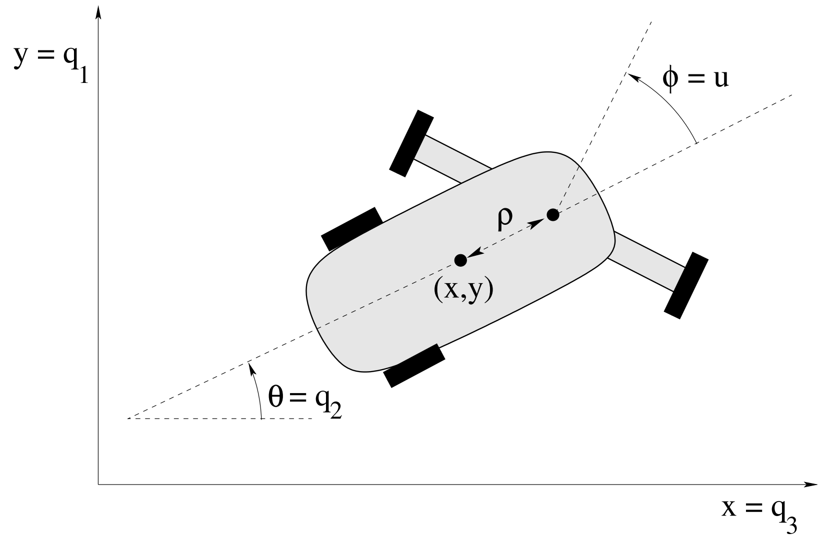

Figure 2: The roller racer.

8 Two examples

8.1 The Roller Racer

As a toy model, consider the *Roller Racer*, which will serve as a simple illustration of the general theory. This is a classical example of a non-holonomic system, widely investigated within the theory of the *momentum map* [2, 12]. It consists of two rigid planar bodies, connected at a point by a rotating joint, as shown in Figure 2. One of the two bodies has a much larger mass than the other. Let ρ be the distance between the joint and the center of mass of the heavier body. To simplify computations we also assume that the center of mass of the lighter body coincides with the joint.

The coordinates (q^1, q^2, q^3, u) are as follows . We let $(q^3, q^1) = (x, y)$ be the Euclidean coordinates of the center of mass of the large body. Moreover, $q^2 = \theta$ is the counter-clockwise angle between the horizontal axis and the major axis of the large body, and $u = \phi$ is the counter-clockwise angle between the major axes of the two bodies.

The non-holonomic constraint consists in assuming that each pair of wheels roll without slipping, parallel to the corresponding body. This corresponds to the condition

$$(\dot{q}, \dot{u}) \ \in \ \Gamma \ \doteq \ ker(\omega^1) \cap ker(\omega^2),$$

where

$$\begin{cases} \omega^1 \doteq \cos q^2\, dq_1 - \sin q^2\, dq^3\,, \\ \omega^2 \doteq \cos(q^2 + u)\, dq_1 + \rho\, \cos u\, dq^2 - \sin(q^2 + u)\, dq^3\,. \end{cases} \tag{8.1}$$

In this case the (non integrable) distribution is 2-dimensional. The admissible motions $t \mapsto (q, u)(t)$ are thus subject to

$$\begin{cases} \cos q^2(t)\, \dot{q}^1(t) - \sin q^2(t)\, \dot{q}^3(t) \ = \ 0\,, \\ \\ \cos(q^2(t) + u(t))\, \dot{q}^1(t) + \rho\, \cos u(t)\, \dot{q}^2(t) - \sin(q^2(t) + u(t))\, \dot{q}^3(t) \ = \ 0\,. \end{cases} \tag{8.2}$$

The coordinate $\phi = u$ is regarded as a control. Notice that the transversality condition (2.6) is trivially satisfied, because

$$\Delta^{ker} \;=\; span\{du\}, \qquad \Gamma^{ker} \;=\; span\{\omega^1, \omega^2\},$$

and $span\{\omega^1, \omega^2\} \cap span\{du\} = \{0\}$.

For simplicity, we assume that the large body has unit mass. The moment of inertia of the large body w.r.t. the vertical axis passing through its center of mass is denoted by I. The mass of the small body is regarded as negligible, but its moment of inertia J w.r.t. the vertical axis passing through the center of mass (of the small body) is assumed to be different from zero.[5] The kinetic matrix $\mathbf{G} = (g_{i,j})$ and its inverse $\mathbf{G^{-1}} = (g^{i,j})$ are computed as

$$(g_{i,j}) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I+J & 0 & J \\ 0 & 0 & 1 & 0 \\ 0 & J & 0 & J \end{pmatrix} \qquad (g^{i,j}) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{I} & 0 & -\frac{1}{I} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{I} & 0 & \frac{I+J}{IJ} \end{pmatrix}.$$

Let us start by finding a basis for the decomposition $T\mathcal{Q} = (T\mathcal{Q})_I \oplus (T\mathcal{Q})_{II} \oplus (T\mathcal{Q})_{III}$. Notice that the vectors

$$\mathbf{w}_1 = 2\rho\cos u \sin q^2 \,\frac{\partial}{\partial q^1} + \; 2\sin u \frac{\partial}{\partial q^2} + \; 2\rho\cos u\cos q^2 \frac{\partial}{\partial q^3}\,, \qquad \mathbf{w}_2 = \frac{\partial}{\partial u}\,,$$

form a basis for Γ. Since $\Delta = span\{\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^3}\}$, $\{\mathbf{w}_1\}$ is a basis for $(T_\mathcal{Q})_I = \Gamma \cap \Delta$.

It is also straightforward to verify that

$$(T\mathcal{Q})_{II} \;=\; \Gamma^{\perp} \;=\; \ker\{\mathbf{g}(\mathbf{w}_1), \mathbf{g}(\mathbf{w}_2)\} \;=\; span\,\{\mathbf{v}_2, \mathbf{v_3}\}\;, \tag{8.3}$$

where

$$\mathbf{v}_2 \;=\; \frac{I\csc q^2 \tan u}{\rho}\,\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2} + \frac{\partial}{\partial u}, \qquad \mathbf{v}_3 \;=\; -\cot q^2\,\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^3}\,.$$

Since $(T\mathcal{Q})_{III}$ is orthogonal to $(T\mathcal{Q})_I \oplus (T\mathcal{Q})_{II}$,

$$(T\mathcal{Q})_{III} \;=\; \ker\Big\{\mathbf{g}(\mathbf{w}_1), \mathbf{g}(\mathbf{v}_2), \mathbf{g}(\mathbf{v}_3)\Big\} = \; span\{\mathbf{v}_4\} \tag{8.4}$$

with

$$\mathbf{v}_4 \;\doteq\; \frac{1}{\Delta_0}\left(-\frac{1}{2}J\rho\sin q^2\sin 2u\,\frac{\partial}{\partial q^1} \;- J\sin^2 u\,\frac{\partial}{\partial q^2} \;-\frac{1}{2}J\rho\cos q^2\sin 2u\,\frac{\partial}{\partial q^3}\right) + \frac{\partial}{\partial u}\,,$$

we have $\Delta_0 \doteq \rho^2\cos^2 u + (I+J)\sin^2 u$.

Defining $\mathbf{V}_1 \doteq \mathbf{w}_1, \mathbf{V}_2 \doteq \mathbf{v}_2, \mathbf{V}_3 \doteq \mathbf{v}_3, \mathbf{V}_4 \doteq \mathbf{v}_4$ we obtain a family of vector fields (on an open subset of the configuration manifold) as in (5.1).

To obtain the equations of motion we need to compute the coefficients (5.9). For this purpose, let us begin with observing that the form $\mathbf{\Omega}_1$ generating $(T^*\mathcal{Q})_I$ is given by

$$\begin{aligned} \mathbf{\Omega}_1 \;&=\; \Omega_{1,1}dq^1 + \Omega_{2,1}dq^2 + \Omega_{3,1}dq^3 + \Omega_{4,1}dq^4 \;=\; \mathbf{g}(\mathbf{V}_1) \\ &=\; (2\rho\cos u\sin q^2)dq^1 + (2(I+J)\sin u)dq^2 + (2\rho\cos u\cos q^2)dq^3 + (2J\sin u)dq^4. \end{aligned}$$

[5]This is a standard approximation adopted in the existing literature.

Let us recall that the projection matrix P_I^* is computed as (see (5.7))

$$(P_I^*)^r{}_s = \sum_{k=1}^{4} \frac{g^{r,k}\Omega_{k,1}\Omega_{s,1}}{\sum_{a,b=1}^{4} g_{a,b}V^a{}_1 V^b{}_1}\,.$$

Let us set $\Delta_1 = I + J + \rho^2 + (-I - J + \rho^2)\cos 2u$. Then an elementary computation of (5.8) yields the following system of four differential equations:

$$\begin{cases} \dot q^1 &= 2\rho\cos u\sin q^2\cdot\xi - \frac{J\rho\sin q^2\sin 2u}{2\Delta_0}\cdot\dot u\,, \\ \dot q^2 &= 2\sin u\cdot\xi - \frac{J\sin^2 u}{\Delta_0}\cdot\dot u\,, \\ \dot q^3 &= 2\rho\cos q^2\cos u\cdot\xi - \frac{J\rho\cos q^2\sin 2u}{2\Delta_0}\cdot\dot u\,, \\ \dot\xi &= -\frac{2(I+J-\rho^2)\sin 2u}{\Delta_1}\cdot\xi\,\dot u + \frac{8J\rho^2\cos u}{\Delta_1^2}\cdot\dot u^2. \end{cases} \tag{8.5}$$

Remark 8.1 Assume that we implement a vibrating control, of the form $u(t) = \bar u + \varepsilon K\sin(t/\varepsilon)$ for $K \in I\!R$ some $\varepsilon > 0$ small. By the results from [7, 8], each solution of the system

$$\begin{cases} \dot q^1 &= 2\rho\cos\bar u\sin q^2\cdot\xi \\ \dot q^2 &= 2\sin\bar u\cdot\xi \\ \dot q^3 &= 2\rho\cos\bar u\cos q^2\cdot\xi \\ \dot\xi &= \dfrac{4J\rho^2\cos\bar u\cdot K^2}{\Delta_1^2} \end{cases} \tag{8.6}$$

(formally obtained by neglecting higher order terms and then averaging), can be uniformly approximated by the solutions of the original system 8.5 [6]. It is then easy to verify a kinematical well-known behavior of the Roller Racer: fast oscillations of the handlebar around a angle $\bar u$ produce a motion which, asymptotically, is a superposition of forward motion and a rotation of the angle q^2 (which is the angle between the principal axis of the large body and the x-axis, namely the q^3 axis). In particular if the oscillations are around the central position $u = 0$, then q^2 is constant and the motion approaches asymptotically a forward motion. Of course this agrees with results obtained with methods based on momentum maps, where, in particular, controls are forces or torques. See [2] or the extensive analysis of the Roller Racer in [15].

8.2 A ball rolling on rotating disc

The next example illustrates an application of Theorem 6.2. It represents a modification of the classical case (see [2]) of a mechanical system consisting of a ball that rolls without sliding on a disc which rotates with constant speed (Figure 3). The somehow surprising fact is that the ball does not move away toward infinity, regardless of the angular speed of the disc. In

[6] More generally, one could approximate any solution of a (possibly) impulsive system obtained 8.5 by replacing $(\dot u, \dot u^2)$ with a pair (a, μ), where a is a L^1 control and μ is a positive measure such that $a^2 \le \mu$ (in the sense of measures)

fact, the barycenter of the ball remains confined within a bounded region, depending on the initial conditions. The "surprise" actually comes from the fact that intuition suggests a sort of centrifugal effect, pushing the ball outward in the radial direction.

We slightly modify this example by allowing the rotational speed of the disc to be *not constant*. More precisely, we consider the rotational angle u of the disc as a (state-space) parameter u, regarded as a control. We will show that, as in the case of constant rotational speed, a small rapid oscillation of the disc around the center does not instantly push the ball toward infinity. As a consequence, the system is "fit for jumps". To recast this system in our general framework, consider a material disc D, centered at a point O with inertial moment $I > 0$. If $u(t)$ designates the counter-clockwise angle of the disc with respect to inertial coordinate axes, its kinetic energy is given by $\frac{1}{2}I\dot{u}^2$. Actually, in our setting the value of I is irrelevant for the motion of the system, because the angle u is taken as a control variable. The configuration manifold of the rotating disc together with the rolling ball can be identified with $\mathcal{Q} \doteq I\!R^2 \times SO(3) \times S^1$. The first two coordinates $(x, y) \in I\!R^2$ describe the position of the contact point w.r.t the inertial frame. A unitary matrix $R \in SO(3)$ represents the rotation of the ball, while the angle $u \in S^1$ describes the rotation of the disc. Here the submersion describing the control constraint is simply the projection

$$\pi : \mathcal{Q} \ \rightarrow \ S^1, \qquad \qquad \pi(x, y, R, u) \ = \ u\,.$$

In addition, the non-holonomic constraints are represented by the linear equations

$$\begin{cases} \dot{x} + r\omega_2 + \dot{u}y \ = \ 0\,, \\ \\ \dot{y} - r\omega_1 - \dot{u}x \ = \ 0\,, \end{cases} \tag{8.7}$$

where r is the radius of the ball and $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector of the ball w.r.t. the inertial frame. For simplicity, we assume that the inertial moment of the disc w.r.t. its center is $I = 1$ and that the ball has homogeneous density and total mass equal to 1. The kinetic energy T of the system is then given by

$$T \ = \ \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \kappa^2 \sum_{i=1}^{3} \omega_i^2 + \dot{u}^2\right), \tag{8.8}$$

where κ^2 is the moment of inertia of the ball w.r.t. any of its axes.

Let $(q^1, \ldots, q^6)$ be local coordinates for $\mathcal{Q}$, where $q^4 \doteq x$, $q^5 \doteq y$, $q^6 \doteq u$, while (q^1, q^2, q^3) are local coordinates for $SO(3)$. Recalling the meaning of angular velocity ω and the form of the kinetic energy T, we can find an orthonormal basis $\{\mathbf{V}_1, \ldots, \mathbf{V}_6\}$ of the tangent space $T\mathcal{Q}$ such that the following holds:

$$\mathbf{V}_i \ \doteq \ \frac{\partial}{\partial q^i} \qquad \text{for} \quad i \in \{4, 5, 6\}\,,$$

$$\mathbf{V}_j \ = \ \sum_{h=1}^{3} A_j^h(q^1, q^2, q^3)\frac{\partial}{\partial q^h} \qquad \text{for} \quad j \in \{1, 2, 3\}. \tag{8.9}$$

Moreover, if $\omega = (\omega_1, \omega_2, \omega_3)$ describes the angular velocity of the ball, then

$$(\dot{q}_1, \dot{q}_2, \dot{q}_3) \ = \ \sum_{j=1}^{3} \kappa\, \omega_j \mathbf{V}_j\,.$$

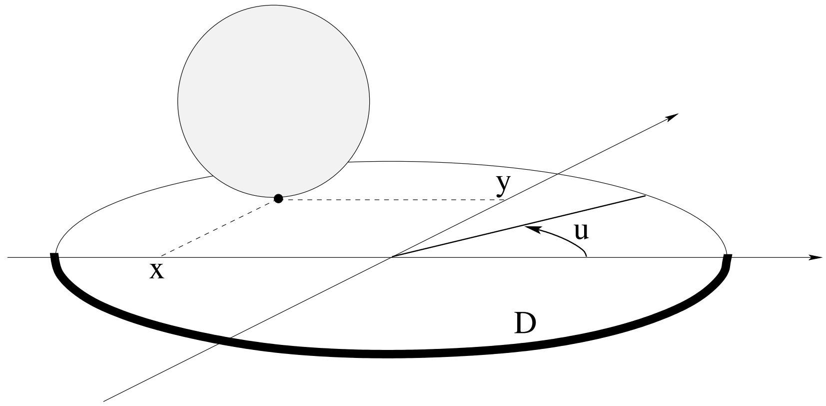

Figure 3: A ball rolling without sliding on a rotating disc.

In terms of these coordinates, if $\dot{q} = \sum_j c_j \mathbf{V}_j$, the non-holonomic constraints (8.7) take the form

$$c_4 + \frac{r}{\kappa} c_2 + q^5 c_6 \;=\; 0\,, \qquad\qquad c_5 - \frac{r}{\kappa} c_1 - q^4 c_6 \;=\; 0\,.$$

We thus obtain

$$\Delta \;=\; span\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5\}, \qquad\qquad \Delta^\perp \;=\; span\{\mathbf{V}_6\}, \tag{8.10}$$

$$\Delta \cap \Gamma \;=\; span\left\{\mathbf{V}_1 + \frac{r}{\kappa}\mathbf{V}_5,\ \mathbf{V}_2 - \frac{r}{\kappa}\mathbf{V}_4,\ \mathbf{V}_3\right\}, \tag{8.11}$$

$$(\Delta \cap \Gamma)^\perp \;=\; span\left\{\mathbf{V}_1 - \frac{\kappa}{r}\mathbf{V}_5,\ \mathbf{V}_2 + \frac{\kappa}{r}\mathbf{V}_4,\ \mathbf{V}_6\right\}. \tag{8.12}$$

Using the basis$\{\mathbf{V}_1, \ldots, \mathbf{V}_6\}$, all projections $P_J, P_J^*,\ J \in \{I, II, III\}$ are given by constant matrices, so hypothesis (iii) in Theorem 6.2 is verified. By (8.8)-(8.10) it is clear that the matrix g of the kinetic energy with respect to the basis $(\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^6})$ is independent of q^6 and has all zeros in the sixth row and the six column, except for the corner entry which is equal to 1. Therefore the same is valid for the inverse matrix g^{-1}. Denoting by p^T the transpose of the row vector p, this implies

- $H \doteq \frac{1}{2} p g^{-1} p^T$ is independent of q^6.
- If the non-holonomic constraint (8.7) is not present, then the system is fit for jumps.

By Theorem 6.2 we thus conclude that this system is fit for jumps. In particular, a rapid small oscillation of the disc around its center will not cause the ball to fly away to infinity.

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References

[1] J. Baillieul, The geometry of controlled mechanical systems, in *Mathematical Control Theory*, J. Baillieul & J. C. Willems, Eds., Springer Verlag, New York, 1998, 322-354.

[2] A. M. Bloch, *Nonholonomic Mechanics and Control*, Springer Verlag, 2003.

[3] A. M. Bloch, J. E. Marsden, and D. V. Zenkov, Nonholonomic dynamics. *Notices Amer. Math. Soc.* **52** (2005), 324–333.

[4] A. Bressan, Impulsive control of Lagrangian systems and locomotion in fluids, *Discr. Cont. Dynam. Syst.* **20** (2008), 1-35.

[5] A. Bressan and F. Rampazzo, On differential systems with vector-valued impulsive controls, *Boll. Un. Matem. Italiana* **2-B**, (1988), 641-656.

[6] A. Bressan and F. Rampazzo, Impulsive control systems with commutative vector fields, *J. Optim. Theory & Appl.* **71** (1991), 67-84.

[7] A. Bressan and F. Rampazzo, On systems with quadratic impulses and their application to Lagrangean mechanics, *SIAM J. Control* **31** (1993), 1205-1220.

[8] A. Bressan and F. Rampazzo, Moving constraints as stabilizing controls in classical mechanics, *Arch. Rational Mech. Anal.* **196** (2010), 97–141.

[9] Aldo Bressan, Hyper-impulsive motions and controllizable coordinates for Lagrangean systems *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur. Mem.*, Serie VIII, Vol. XIX (1990), 197–246.

[10] Aldo Bressan, On some control problems concerning the ski or swing, *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur. Mem.*, Serie IX, Vol. I, (1991), 147-196.

[11] Aldo Bressan and M. Motta, A class of mechanical systems with some coordinates as controls. A reduction of certain optimization problems for them. Solution methods. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Mem.* **9** (1993), 5–30.

[12] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems*, Springer Verlag, 2004.

[13] F. Cardin and M. Favretti, Hyper-impulsive motion on manifolds. *Dynam. Cont. Discr. Imp. Systems* **4** (1998), 1-21.

[14] E. Cartan, Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien, *Ann. Soc. Polonaise Math.*, **6** (1927), 1-7.

[15] P. S. Krishnaprasad and D. P. Tsakiris, Oscillations, SE(2)-snakes and motion control: A study of the roller racer, *Dynamical Systems*, bf 16 (2001), 347-397.

[16] M. Levi, Geometry of vibrational stabilization and some applications, *Int. J. Bifurc. Chaos* **15** (2005), 2747-2756.

[17] M. Levi and Q. Ren, Geodesics on vibrating surfaces and curvature of the normal family, *Nonlinearity* **18** (2005), 2737-2743.

[18] W. S. Liu and H. J. Sussmann, Limits of highly oscillatory controls and the approximation of general paths by admissible trajectories, in *Proc. 30-th IEEE Conference on Decision and Control,* IEEE Publications, New York, 1991, pp. 437–442.

[19] C. Marle, Géométrie des systèmes mécaniques à liaisons actives, in *Symplectic Geometry and Mathematical Physics*, 260–287, P. Donato, C. Duval, J. Elhadad, and G. M. Tuynman Eds., Birkhäuser, Boston, 1991.

[20] B.M. Miller, The generalized solutions of ordinary differential equations in the impulse control problems, *J. Math. Syst. Estim. Control* **4** (1994), 385–388.

[21] J. Nash, The imbedding problem for Riemannian manifolds, *Annals of Math.* **63** (1956), 2063.

[22] H. Nijmejer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer Verlag, New York, 1990.

[23] F. Rampazzo, On Lagrangian systems with some coordinates as controls. *Atti Accad. Naz. Lincei, Classe di Scienze Mat. Fis. Nat.* Serie 8, **82** (1988) 685-695.

[24] F. Rampazzo, On the Riemannian structure of a Lagrangean system and the problem of adding time-dependent coordinates as controls *European J. Mechanics A/Solids* **10** (1991), 405-431.

[25] F. Rampazzo, Lie brackets and impulsive controls: an unavoidable connection, in *Differential Geometry and Control, Proc. Sympos. Pure Math.*, (AMS, Providence)(1999), 279-296.

[26] B. L. Reinhart, Foliated manifolds with bundle-like metrics. *Annals of Math.* **69** (1959), 119-132.

[27] B. L. Reinhart, *Differential geometry of foliations. The fundamental integrability problem.* Springer-Verlag, Berlin, 1983.

[28] H. J. Sussmann, On the gap between deterministic and stochastic ordinary differential equations, *Ann. Prob.* **6** (1978), 17-41.