Shock interactions for the Burgers-Hilbert Equation

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Abstract

This paper provides an asymptotic description of a solution to the Burgers-Hilbert equation in a neighborhood of a point where two shocks interact. The solution is obtained as the sum of a function with H^2 regularity away from the shocks plus a corrector term having an asymptotic behavior like $|x|\ln|x|$ close to each shock. A key step in the analysis is the construction of piecewise smooth solutions with a single shock for a general class of initial data.

1 Introduction

Consider the balance law obtained from Burgers' equation by adding the Hilbert transform as a source term

$$u_t + \left(\frac{u^2}{2}\right)_x \;=\; \mathbf{H}[u]\,. \tag{1.1}$$

This equation was derived in [1] as a model for nonlinear waves with constant frequency. Here the nonlocal source term

$$\mathbf{H}[f](x) \;\doteq\; \lim_{\varepsilon\to 0+} \frac{1}{\pi}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy$$

denotes the Hilbert transform of a function $f \in \mathbf{L}^2(\mathbb{R})$. It is well known [8] that $\mathbf{H}$ is a linear isometry from $\mathbf{L}^2(\mathbb{R})$ onto itself. Given any initial data

$$u(0,\cdot) \;=\; \bar u(\cdot) \tag{1.2}$$

with $\bar u \in H^2(\mathbb{R})$, the local existence and uniqueness of solutions to (1.1) was proved in [6], together with a sharp estimate on the time interval where this solution remains smooth. For a general initial data $\bar u \in \mathbf{L}^2(\mathbb{R})$, the global existence of entropy weak solutions to (1.1) was proved in [3], together with a partial uniqueness result. We remark that the well-posedness of the Cauchy problem for (1.1) remains a largely open question.

More recently, piecewise continuous solutions with a single shock have been constructed in [4]. As shown in Fig. 1, these solutions have the form

$$u(t,x) \;=\; w\big(t, x-y(t)\big) + \varphi\big(x-y(t)\big), \tag{1.3}$$

where $y(t)$ denotes the location of the shock at time t, and $w(t,\cdot) \in H^2\big(]-\infty,0[\,\cup\,]0,+\infty[\big)$ for all $t \geq 0$. Moreover, φ is a fixed function with compact support, describing the asymptotic behavior of the solution near the shock. It is smooth outside the origin and satisfies

$$\varphi(x) \;=\; \frac{2}{\pi}|x|\ln|x| \qquad \text{for} \quad |x| \leq 1\,. \tag{1.4}$$

Remarkably, this "corrector term" φ is universal, i.e., it does not depend on the particular solution of (1.1). The same analysis applies to solutions with finitely many, noninteracting shocks. In addition, the local asymptotic behavior of a solution up to the time when a new shock is formed was investigated in [9].

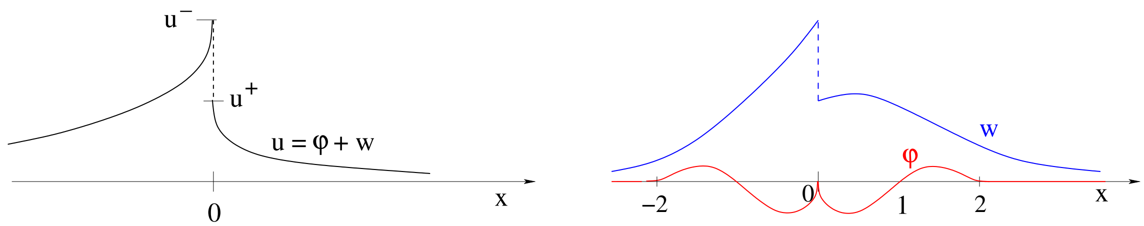

Figure 1: Decomposing a solution in the form (1.3)

The aim of the present note is to describe the asymptotic behavior of a solution in a neighborhood of a point where two shocks interact. Calling $T > 0$ the time when the interaction takes place, our analysis splits into two parts. We first describe the behavior of the solution as $t \to T-$, i.e. as the two shocks approach each other. In a second step, to construct the solution for $t > T$, we solve a Cauchy problem with initial data given at $t = T$.

As it turns out, the profile $u(T,\cdot)$ is not "well prepared", in the sense that it cannot be written in the form (1.3). To explain the difficulty, we recall that the solutions constructed in [4] had initial data of the form

$$u(0,x) \;=\; \overline{w}(x-y_0) + \varphi(x-y_0), \tag{1.5}$$

for some $\overline{w} \in H^2\big(\mathbb{R}\setminus\{0\}\big)$ and $y_0 \in \mathbb{R}$. These data are "well prepared", in the sense that they already contain the corrector term φ. A natural class of initial data, not considered in [4], is

$$u(0,x) \;=\; \overline{w}(x-y_0) \qquad \text{with} \qquad \overline{w} \in H^2\big(\mathbb{R}\setminus\{0\}\big), y_0 \in \mathbb{R}. \tag{1.6}$$

By assumption, at time $t = 0$ the derivative $u_x(0,x) \;=\; \overline{w}_x(x-y_0)$ is piecewise continuous and uniformly bounded. However, in the solution to (1.1), (1.6), at each time $t > 0$ we expect that $u_x(t,x) \to \pm\infty$ as $x \to y(t)\mp$. For this reason, the local construction of this solution requires a careful analysis. A more general class of initial data, containing both (1.5) and (1.6), as well as all profiles $u(T,\cdot)$ emerging from our shock interactions, will be studied in Section 2.

We recall here the definition of entropy weak solutions used in [3].

Definition 1.1 *By an* **entropy weak solution** *of (1.1)-(1.2) we mean a function* $u \in \mathbf{L}^1_{loc}([0,\infty[\,\times\mathbb{R})$ *with the following properties.*

(i) The map $t \mapsto u(t,\cdot)$ is continuous with values in $\mathbf{L}^2(\mathbb{R})$ and satisfies the initial condition (1.2).

(ii) For any $k \in \mathbb{R}$ and every nonnegative test function $\phi \in \mathcal{C}^1_c(]0,\infty[\times\mathbb{R})$ one has

$$\iint \Big[|u-k|\phi_t + \Big(\frac{u^2-k^2}{2}\Big)\mathrm{sign}(u-k)\phi_x + H[u(t)](x)\mathrm{sign}(u-k)\phi\Big]\,dxdt \;\geq\; 0. \quad (1.7)$$

The present paper will be concerned with a more regular class of solutions, which are piecewise continuous and can be determined by integrating along characteristics. These correspond to the "broad solutions" considered in [2, 7]. Throughout the sequel, the upper dot denotes a derivative w.r.t. time.

Definition 1.2 *An entropy weak solution $u = u(t,x)$ of (1.1)-(1.2), defined on the interval $t \in [0,T]$, will be called a* **piecewise regular solution** *if there exist finitely many shock curves $y_1(t),\ldots,y_n(t)$ such that the following holds.*

(i) For each $t \in [0,T]$, one has $u(t,\cdot) \in H^2\big(\mathbb{R}\setminus\{y_1(t),\ldots,y_n(t)\}\big)$.

(ii) For each $i = 1,\ldots,n$, the Rankine-Hugoniot conditions hold:

$$u_i^-(t) \;\doteq\; u(t,y_i(t)-) \;>\; u(t,y_i(t)+) \;\doteq\; u_i^+(t), \quad (1.8)$$

$$\dot y_i(t) \;=\; \frac{u_i^-(t)+u_i^+(t)}{2}\,. \quad (1.9)$$

(iii) Along every characteristic curve $t \mapsto x(t)$ such that

$$\dot x(t) \;=\; u\big(t,x(t)\big), \quad (1.10)$$

one has

$$\frac{d}{dt}u\big(t,x(t)\big) \;=\; \mathbf{H}[u](x(t)). \quad (1.11)$$

In the above setting, the Hilbert transform of the piecewise regular function $u(t,\cdot)$ can be computed using an integration by parts:

$$\mathbf{H}[u(t)](x) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} u_x(t,y)\,\ln|x-y|\,dy + \frac{1}{\pi}\sum_{i=1}^{n}\big[u_i^+(t)-u_i^-(t)\big]\,\ln\big|x-y_i(t)\big|\,. \quad (1.12)$$

The remainder of the paper is organized as follows. In Section 2 we state a local existence and uniqueness theorem for solutions to (1.1), valid for a class of initial data containing one single shock, but more general than in [4]. Towards the proof of Theorem 2.1, Section 3 develops various a priori estimates, while in Section 4 the local solution is constructed as a limit of a convergent sequence of approximations. As in [4], these are obtained by iteratively solving a sequence of linearized problems.

In the second part of the paper we study solutions of (1.1) with two shocks, up to the time of interaction. In Section 5 we perform some preliminary computations, motivating a particular form of the corrector term. In Section 6 we state and prove the second main result of the paper, Theorem 6.1, providing a detailed description of solutions up to the interaction time. This is achieved by a change of both time and space coordinates, so that the two shocks are located at the two points

$$x_1(t) \ = \ t \ < \ 0 \ = \ x_2(t),$$

and interact at time $t = 0$. Our analysis shows that, at the interaction time, the solution profile contains a single shock and lies within the class of initial data covered by Theorem 2.1. Combining our two theorems, one thus obtains a complete description of the solution to (1.1) in a neighborhood of the interaction time.

2 Solutions with one shock and general initial data

Consider a piecewise regular solution of the Burgers-Hilbert equation (1.1), with one single shock. By the Rankine-Hugoniot condition, the location $y(t)$ of the shock at time t satisfies

$$\dot{y}(t) \ = \ \frac{u^-(t) + u^+(t)}{2}, \qquad u^{\pm}(t) \ = \ \lim_{x \to y(t)\pm} u(t,x). \tag{2.1}$$

As in [4], we shift the space coordinate, replacing x with $x - y(t)$, so that in the new coordinate system the shock is always located at the origin. In these new coordinates, (1.1) takes the form

$$u_t + \left(u - \frac{u^-(t) + u^+(t)}{2} \right) u_x \ = \ \mathbf{H}[u]. \tag{2.2}$$

In [4], given a "well prepared" initial data (1.5), a unique piecewise smooth entropy solution to (2.2) of the form

$$u(t,x) \ = \ w(t,x) + \frac{2\eta(x)}{\pi} \cdot |x| \ln |x| \,, \qquad\qquad t > 0$$

was constructed. Here $w(t,\cdot) \in H^2\big(\mathbb{R} \setminus \{0\}\big)$, while $\eta \in C^\infty(\mathbb{R})$ is an even cut-off function, satisfying

$$\begin{cases} \eta(x) \ = \ 1 & \text{if} \quad |x| \leq 1, \\ \eta(x) \ = \ 0 & \text{if} \quad |x| \geq 2, \\ \eta'(x) \ \leq \ 0 & \text{if} \quad x \geq 0\,. \end{cases} \tag{2.3}$$

For future use, it will be convenient to introduce the function

$$\phi(x,b) \ \doteq \ \frac{2\eta(x)}{\pi} \cdot \Big[(|x| + b) \ln(|x| + b) - b \ln b \Big], \qquad\qquad x \in \mathbb{R}, \quad b \geq 0. \tag{2.4}$$

Observe that

$$\phi(0,b) \ = \ 0 \qquad\qquad \text{for all} \ \ b \geq 0. \tag{2.5}$$

Our main goal in this section is to solve the Cauchy problem for (2.2) with initial data

$$u(0,x) \ = \ \overline{w}(x) + \overline{\varphi}(x), \tag{2.6}$$

where

$$\overline{w} \in H^2\big(\mathbb{R} \backslash \{0\}\big), \qquad\qquad \overline{\varphi}(x) \ = \ \Big(c_1 \cdot \chi_{]-\infty,0[} + c_2 \cdot \chi_{]0,+\infty[} \Big) \cdot \phi(x,0), \tag{2.7}$$

for some constants $c_1, c_2 \in \mathbb{R}$. Note that this reduces to (1.5) in the case $c_1 = c_2 = 1$.

To handle the more general initial data (2.6)-(2.7), we write the solution of (2.2) in the form

$$u(t,x) \;=\; w(t,x) + \varphi^{(w)}(t,x), \tag{2.8}$$

where the corrector term $\varphi^{(w)}(t,x)$ now depends explicitly on time t and on the strength of the jump

$$\sigma^{(w)}(t) \;\doteq\; w^-(t) - w^+(t), \qquad\qquad w^{\pm}(t) \;\doteq\; w(t, 0\pm). \tag{2.9}$$

To make an appropriate guess for the function $\varphi^{(w)}$, we observe that, by (1.12), the equation (2.2) can be approximated by the simpler equation

$$u_t + \left(u - \frac{u^-(t)+u^+(t)}{2}\right) u_x \;=\; \frac{1}{\pi}(u^+(t) - u^-(t)) \ln|x|. \tag{2.10}$$

Indeed, we expect that the solutions of (2.2) and (2.10) with the same initial data will have the same asymptotic structure near the origin. Their difference will lie in the more regular space $H^2(\mathbb{R}\backslash\{0\})$. With this in mind, we thus make the *ansatz*

$$\varphi^{(w)}(t,x) \;\doteq\; \phi(x,0) + \Big((c_1-1)\cdot\chi_{]-\infty,0[} + (c_2-1)\cdot\chi_{]0,+\infty[}\Big)\cdot\phi\left(x, \frac{\sigma^{(w)}(t)t}{2}\right). \tag{2.11}$$

Inserting (2.8) into (2.2), we obtain an equation for the remaining component $w(t,\cdot)$. Namely

$$w_t + a(t,x,w)\cdot w_x \;=\; F(t,x,w), \tag{2.12}$$

where a and F are given by

$$a(t,x,w) \;=\; w(t,x) + \varphi^{(w)}(t,x) - \frac{w^-(t)+w^+(t)}{2}, \tag{2.13}$$

$$\begin{aligned} F(t,x,w) \;&=\; \mathbf{H}\left[\varphi^{(w)}\right](t,x) - \varphi^{(w)}\varphi^{(w)}_x(t,x) \\ &+\; \left(\mathbf{H}\left[w\right](t,x) - \left[\varphi^{(w)}_t(t,x) + \left(w(t,x) - \frac{w^-(t)+w^+(t)}{2}\right)\cdot\varphi^{(w)}_x(t,x)\right]\right). \end{aligned} \tag{2.14}$$

We observe that, in the present case of a solution with a single shock, by (2.5) the entropy admissibility condition (1.8) reduces to

$$w^-(t) \;>\; w^+(t). \tag{2.15}$$

Moreover, Definition 1.2 is satisfied provided that, along every characteristic curve $t \mapsto x(t;t_0,x_0) \neq 0$ obtained by solving

$$\dot{x}(t) \;=\; a(t,x,w), \qquad\qquad x(t_0) \;=\; x_0\,, \tag{2.16}$$

one has

$$w(t_0,x_0) \;=\; \overline{w}\big(x(0;t_0,x_0)\big) + \int_0^{t_0} F\Big(t, x(t;t_0,x_0), w\big(t, x(t;t_0,x_0)\big)\Big)\,dt\,. \tag{2.17}$$

The first main result of this paper provides the existence and uniqueness of an entropic solution, locally in time.

Theorem 2.1 *For every $\overline{w} \in H^2\big(\mathbb{R}\setminus\{0\}\big)$ satisfying $\overline{w}(0-)-\overline{w}(0+) > 0$ and every $c_1, c_2 \in \mathbb{R}$, the Cauchy problem for the Burgers-Hilbert equation (1.1), with initial condition as in (2.6)-(2.7), admits a unique piecewise regular solution defined for $t \in [0,T]$, for some $T > 0$ sufficiently small, depending only on M_0, δ_0, c_1, and c_2.*

The solution to the equivalent equation (2.12) will be obtained as a limit of a sequence of approximations. Namely, consider a sequence of linear approximations constructed as follows. As a first step, define

$$w_1(t,x) \;=\; \overline{w}(x) \qquad\qquad \text{for all} \;\; t \geq 0, \quad x \in \mathbb{R}\,. \tag{2.18}$$

By induction, let w_n be given. We define w_{n+1} to be the solution of the linear, non-homogeneous Cauchy problem

$$w_t + a(t,x,w_n)\cdot w_x \;=\; F(t,x,w), \qquad w(0,\cdot) \;=\; \overline{w}(x). \tag{2.19}$$

The induction argument requires three steps:

(i) Existence and uniqueness of solutions to each linear problem (2.19).

(ii) A priori bounds on the strong norm $\big\|w_n(t)\big\|_{H^2(\mathbb{R}\setminus\{0\})}$, uniformly valid for $t \in [0,T]$ and all $n \geq 1$.

(iii) Convergence in a weak norm. This will follow from the bound

$$\sum_{n\geq 2} \big\|w_n(t) - w_{n-1}(t)\big\|_{H^1(\mathbb{R}\setminus\{0\})} \;<\; \infty\,. \tag{2.20}$$

These steps will be worked out in the next two sections.

3 Preliminary estimates

To achieve the above steps (i)-(iii), we establish in this section some key estimates on the right hand side of (2.19), by splitting it into three parts:

$$F(t,x,w) \;=\; A^{(w)}(t,x) + B^{(w)}(t,x) - C^{(w)}(t,x), \tag{3.1}$$

where

$$A^{(w)} \;\doteq\; \mathbf{H}\left[\varphi^{(w)}\right] - \varphi^{(w)}\varphi^{(w)}_x, \qquad\qquad B^{(w)} \;\doteq\; \mathbf{H}[w] - \left(w - \frac{w^- + w^+}{2}\right)\cdot \phi_x(x,0)\,, \tag{3.2}$$

$$C^{(w)} \;\doteq\; \varphi^{(w)}_t + \left(w - \frac{w^- + w^+}{2}\right)\cdot \phi_x\left(x, \frac{\sigma^{(w)}(t)\,t}{2}\right)\cdot\Big((c_1-1)\cdot\chi_{]-\infty,0[} + (c_2-1)\cdot\chi_{]0,+\infty[}\Big). \tag{3.3}$$

Consider the function

$$g_b(x) \;=\; \chi_{[0,\infty[}(x)\cdot\phi(x,b), \qquad x \in \mathbb{R}, b \geq 0, \tag{3.4}$$

where $\phi(x,b)$ is given by (2.4). For every $b \in \left[0, \frac{1}{2e}\right]$ one checks that the function $g_b \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\}) \cap \mathcal{C}(\mathbb{R})$ is negative and decreasing on the open interval $\left]0, \frac{1}{2e}\right[$. Moreover, it satisfies

$$\mathrm{supp}(g_b) \ \subseteq\ [0,2], \qquad\qquad |g_b(z)| \ \leq\ \big|z \ln |z|\big| \quad \text{for all } z \in \left[0, \frac{1}{2e}\right]. \tag{3.5}$$

The next lemma provides some bounds on the Hilbert transform of g_b. As usual, by the Landau symbol $\mathcal{O}(1)$ we shall denote a uniformly bounded quantity.

Lemma 3.1 *For every $0 \leq b \leq \frac{1}{2e}$ and $|x| \leq \frac{1}{2e}$, one has*

$$\begin{cases} \big|\mathbf{H}[g_b](x)\big| \ \leq\ \mathcal{O}(1), & \left|\dfrac{d}{dx}\mathbf{H}[g_b](x)\right| \ \leq\ \mathcal{O}(1) \cdot \ln^2 |x|, \\[2ex] \left|\dfrac{d^2}{dx^2}\mathbf{H}[g_b](x)\right| \ \leq\ \mathcal{O}(1) \cdot \left|\dfrac{\ln|x|}{x}\right|, & \left|\dfrac{d^3}{dx^3}\mathbf{H}[g_b](x)\right| \ \leq\ \mathcal{O}(1) \cdot \left|\dfrac{\ln|x|}{x^2}\right|. \end{cases} \tag{3.6}$$

Moreover, for every $\delta > 0$ sufficiently small one has

$$\big\|\mathbf{H}[g_b](\cdot)\big\|_{H^2(\mathbb{R}\backslash[-\delta,\delta])} \ \leq\ \mathcal{O}(1) \cdot \delta^{-2/3}, \qquad \big\|\mathbf{H}[g_b'](\cdot)\big\|_{H^2(\mathbb{R}\backslash[-\delta,\delta])} \ \leq\ \mathcal{O}(1) \cdot \delta^{-7/4}. \tag{3.7}$$

Proof. Fix $b \in \left[0, \frac{1}{2e}\right]$. By (1.12), one has

$$\mathbf{H}[g_b](x) \ =\ \frac{1}{\pi} \cdot \int_0^2 g_b'(y) \cdot \ln|x-y| \ dy.$$

Two cases are considered:

Case 1: If $-\dfrac{1}{2e} < x < 0$ then we have the estimates

$$\begin{aligned} \big|\mathbf{H}[g_b](x)\big| \ &=\ \frac{1}{\pi} \cdot \left|\int_0^2 g_b'(y) \cdot \ln(y-x) \ dy\right| \ =\ \frac{1}{\pi} \cdot \left|\int_{|x|}^{2+|x|} g_b'(x+z) \ln z \ dz\right| \\ &=\ \frac{1}{\pi} \cdot \left|\int_{|x|}^{2+|x|} \frac{g_b(x+z)}{z} \ dz\right| \ \leq\ \mathcal{O}(1) \cdot \left(\int_{|x|}^{\frac{1}{2e}} \big|\ln z\big| dz + 1\right) \ \leq\ \mathcal{O}(1), \end{aligned}$$

$$\begin{aligned} \left|\frac{d}{dx}\mathbf{H}[g_b](x)\right| \ &=\ \frac{1}{\pi} \cdot \left|\int_0^2 \frac{g_b'(y)}{y-x} \ dy\right| \ =\ \frac{1}{\pi} \cdot \left|\int_{|x|}^{2+|x|} \frac{g_b'(x+z)}{z} \ dz\right| \\ &=\ \frac{1}{\pi} \cdot \left|\int_{|x|}^{2+|x|} \frac{g_b(x+z)}{z^2} \ dz\right| \ =\ \mathcal{O}(1) \cdot \left(\int_{|x|}^{\frac{1}{2e}} \frac{|\ln z|}{z} \ dz + 1\right) \ \leq\ \mathcal{O}(1) \cdot \ln^2 |x|, \end{aligned}$$

and similarly

$$\left|\frac{d^2}{dx^2}\mathbf{H}[g_b](x)\right| \ \leq\ \mathcal{O}(1) \cdot \left|\frac{\ln|x|}{x}\right|, \qquad\qquad \left|\frac{d^3}{dx^3}\mathbf{H}[g_b](x)\right| \ \leq\ \mathcal{O}(1) \cdot \left|\frac{\ln|x|}{x^2}\right|.$$

Case 2: If $0 < x < \dfrac{1}{2e}$, then we split $\mathbf{H}[g_b](x)$ into three parts as follows:

$$\begin{aligned}\mathbf{H}[g_b](x) &= \frac{1}{\pi}\cdot\left(\int_0^{x/2} g_b'(y)\ln(x-y)\ dy + \text{p.v.}\int_{x/2}^{3x/2} g_b'(y)\ln|y-x|\ dy\right)\\ &\qquad +\frac{1}{\pi}\cdot\int_{3x/2}^{2} g_b'(y)\ln(y-x)\ dy \ \doteq\ \frac{1}{\pi}\cdot\big(I_1+I_2+I_3\big).\end{aligned}$$

We first estimate

$$\begin{cases}|I_1(x)| &= \left|\displaystyle\int_0^{x/2} g_b'(y)\ln(x-y)\ dy\right| \ \leq\ \mathcal{O}(1)\cdot\left|\displaystyle\int_0^{x/2} g_b'(y)\ dy\right|\cdot\big|\ln x\big| \ \leq\ \mathcal{O}(1)\cdot x\ln^2 x,\\ |I_1'(x)| &= \left|\displaystyle\int_0^{x/2}\frac{g_b'(y)}{x-y}\ dy+\frac{1}{2}\cdot g_b'\left(\frac{x}{2}\right)\ln\left(\frac{x}{2}\right)\right| \ \leq\ \mathcal{O}(1)\cdot\ln^2 x,\end{cases}$$

and similarly

$$|I_1''(x)| \ \leq\ \mathcal{O}(1)\cdot\left|\frac{\ln x}{x}\right|,\qquad |I_1'''(x)| \ \leq\ \mathcal{O}(1)\cdot\left|\frac{\ln x}{x^2}\right|.$$

By a similar argument, one obtains

$$|I_3(x)| \ \leq\ \mathcal{O}(1),\qquad \big|I_3'(x)\big| \ =\ \left|\int_{3x/2}^{2}\frac{g_b'(y)}{y-x}\ dy+\frac{3}{2}\cdot g_b'\left(\frac{3x}{2}\right)\cdot\ln\left(\frac{x}{2}\right)\right| \ \leq\ \mathcal{O}(1)\cdot\ln^2 x,$$

and

$$\begin{cases}\big|I_3''(x)\big| &\leq\ \left|\displaystyle\int_{3x/2}^{2}\frac{g_b'(y)}{(y-x)^2}\ dy\right|+\mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x}\right| \ \leq\ \mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x}\right|,\\ \big|I_3'''(x)\big| &\leq\ \left|\displaystyle\int_{3x/2}^{2}\frac{g_b'(y)}{(y-x)^3}\ dy\right|+\mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x^2}\right| \ \leq\ \mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x^2}\right|.\end{cases}$$

Finally, using the fact that g_b' is concave, we obtain

$$\begin{aligned}\big|I_2(x)\big| &= \left|\lim_{\varepsilon\to0+}\int_{x/2}^{x-\varepsilon} g_b'(y)\ln(x-y)dy+\int_{x+\varepsilon}^{3x/2} g_b'(y)\ln(y-x)dy\right|\\ &= \left|\lim_{\varepsilon\to0+}\int_{\varepsilon}^{x/2}\big[g_b'(x-z)+g_b'(x+z)\big]\cdot\ln z\ dz\right| \ \leq\ 2|g_b'(x)|\cdot\left|\int_\varepsilon^{x/2}\ln z dz\right| \ \leq\ \mathcal{O}(1),\end{aligned}$$

$$\begin{aligned}\big|I_2'(x)\big| &= \left|\lim_{\varepsilon\to0+}\int_{\varepsilon}^{x/2}\big[g_b''(x-z)+g_b''(x+z)\big]\ln z dz+\frac{1}{2}[g_b'(x/2)+g_b'(3x/2)]\cdot\ln(x/2)\right|\\ &\leq\ \mathcal{O}(1)\cdot\left(|g_b''(x/2)|\cdot\left|\int_\varepsilon^{x/2}\ln z dz\right|+\ln^2 x\right) \ \leq\ \mathcal{O}(1)\cdot\ln^2 x,\end{aligned}$$

and

$$\begin{cases}\big|I_2''(x)\big| &\leq\ \left|\displaystyle\lim_{\varepsilon\to0+}\int_{\varepsilon}^{x/2}\big[g_b'''(x-z)+g_b'''(x+z)\big]\ln z dz\right|+\mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x}\right| \ \leq\ \mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x}\right|,\\ \big|I_2'''(x)\big| &\leq\ \left|\displaystyle\lim_{\varepsilon\to0+}\int_{\varepsilon}^{x/2}\big[g_b''''(x-z)+g_b''''(x+z)\big]\ln z dz\right|+\mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x^2}\right| \ \leq\ \mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x^2}\right|.\end{cases}$$

We thus achieve the same estimates as in Case 1, and this yields (3.6).

Finally, the function g_b is continuous with compact support and smooth outside the origin. Therefore, the Hilbert transform $\mathbf{H}[g_b]$ is smooth outside the origin. As $|x|\to\infty$, one has

$$\mathbf{H}[g_b](x) \;=\; \mathcal{O}(1)\cdot|x|^{-1}, \qquad \frac{d^k}{dx^k}\left(\mathbf{H}[g_b]\right)(x) \;=\; \mathcal{O}(1)\cdot x^{-(k+1)}, \quad k=1,2,3.$$

Thus, (3.6) yields (3.7). $\square$

Remark 3.1 For every $0<b\le\frac{1}{2e}$, one has

$$\frac{d}{db}g_b(x) \;=\; \frac{2\eta(x)}{\pi}[\ln(x+b)-\ln(b)] \qquad \text{for all } x>0.$$

Since

$$|\ln(x+b)-\ln(b)| \;\le\; \frac{x}{b},$$

the same arguments used in the proof of Lemma 3.1 yield that, for $0<|x|\le\dfrac{1}{2e}$,

$$\left|\mathbf{H}\left[\frac{d}{db}g_b\right](x)\right| \;=\; \mathcal{O}(1)\cdot\frac{1}{b}, \qquad \left|\frac{d}{dx}\mathbf{H}\left[\left(\frac{d}{db}g_b\right)\right](x)\right| \;=\; \mathcal{O}(1)\cdot\frac{|\ln(x)|}{b},$$

$$\left|\frac{d^2}{dx^2}\mathbf{H}\left[\left(\frac{d}{db}g_b\right)\right](x)\right| \;=\; \mathcal{O}(1)\cdot\left|\frac{\ln(x)}{bx}\right|.$$

Moreover, for $\delta>0$ sufficiently small,

$$\left\|\mathbf{H}\left[\frac{d}{db}g_b\right]\right\|_{H^1(\mathbb{R}\setminus[-\delta,\delta])} \;\le\; \mathcal{O}(1)\cdot\frac{1}{b}, \qquad \left\|\mathbf{H}\left[\frac{d}{db}g_b\right]\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\le\; \mathcal{O}(1)\cdot\frac{1}{b}\cdot\delta^{-2/3}.$$

The next lemma provides some a priori estimates on the function $F=F(t,x,w)$ introduced at (2.14).

Lemma 3.2 *Let $w:[0,T]\times\mathbb{R}\to\mathbb{R}$ such that $w(t,\cdot)\in H^2(\mathbb{R}\setminus\{0\})$ for all $t\in[0,T]$,*

$$\|w(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})} \;\le\; M_0, \qquad \sigma^{(w)}(t) \;\doteq\; w(t,0+)-w(t,0-) \;>\; 0.$$

Moreover, assume that $0<T<\dfrac{1}{4eM_0}$ and that $\sigma^{(w)}(\cdot)$ is locally Lipschitz on $]0,T]$.

Then there exists a constant $C_1>0$, M_0,δ_0,c_1, and c_2 such that, for a.e. $t\in[0,T]$ and $|x|<\frac{1}{2e}$, one has

$$\begin{cases} |F(t,x,w)| \;\le\; C_1\cdot\left((1+M_0)\cdot|\ln t|+\dfrac{|\dot\sigma^{(w)}(t)|}{\sigma^{(w)}(t)}\cdot|x|\right), \\[2ex] |F_x(t,x,w)| \;\le\; C_1\cdot\left((1+M_0)\cdot|x|^{-1/4}+\dfrac{|\dot\sigma^{(w)}(t)|}{\sigma^{(w)}(t)}\right). \end{cases} \tag{3.8}$$

Furthermore, for every $\delta>0$ sufficiently small

$$\|F(t,x,w)\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\le\; C_1\cdot\left[\left(1+M_0+\frac{|\dot\sigma^{(w)}(t)|}{|\sigma^{(w)}(t)|}\right)\cdot\delta^{-2/3}+(1+M_0)\cdot|\ln(t)|\right]. \tag{3.9}$$

Proof. According to (3.1), the function F can be decomposed as the sum of three terms, which will be estimated separately.

1. By the analysis in [4, Section 3], for every $(t,x) \in [0,T] \times \left[-\frac{1}{2e}, \frac{1}{2e}\right]$ one has

$$\begin{cases} \left|B^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1) \cdot M_0, & \left|B_x^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1) \cdot M_0 \cdot |\ln|x||^2, \\ \left\|B^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\leq\; \mathcal{O}(1) \cdot M_0 \cdot \delta^{-2/3}. \end{cases} \tag{3.10}$$

2. Next, we estimate $C^{(w)}(t,x)$. For every $0 < x < \dfrac{1}{2e}$, we have

$$C^{(w)}(t,x) \;=\; E_2^{(w)}(t) + (c_2 - 1) \cdot \left[\frac{\dot{\sigma}^{(w)}(t)t}{2} + (w(t,x) - w(t,0+))\right] \cdot \phi_x\left(x, \frac{\sigma^{(w)}(t)t}{2}\right), \tag{3.11}$$

where we define

$$E_2^{(w)}(t) \;\doteq\; \frac{1-c_2}{2} \cdot \phi_x\left(\frac{\sigma^{(w)}(t)t}{2}, 0\right) \cdot \left(\dot{\sigma}^{(w)}(t)t + \sigma^{(w)}(t)\right).$$

Since $\|w(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})} \leq M_0$, one has $|\sigma^{(w)}(t)| \leq 2M_0$,

$$\left|C^{(w)}(t,0+)\right| \;=\; \left|\frac{1-c_2}{2} \cdot \phi_x\left(\frac{\sigma^{(w)}(t)t}{2}, 0\right) \cdot \sigma^{(w)}(t)\right| \;\leq\; \mathcal{O}(1) \cdot (1+M_0) \cdot |\ln t|, \tag{3.12}$$

and

$$\left|C^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1)\left((1+M_0) \cdot |\ln t| + \frac{|\dot{\sigma}^{(w)}(t)|}{\sigma^{(w)}(t)} \cdot |x|\right).$$

Moreover, observing that

$$\begin{cases} C_x^{(w)}(t,x) \;=\; (c_2 - 1) \cdot \dfrac{d}{dx}\left[\left(\dfrac{\dot{\sigma}^{(w)}(t)t}{2} + (w(t,x) - w(t,0+))\right) \cdot \phi_x\left(x, \dfrac{\sigma^{(w)}(t)t}{2}\right)\right], \\ C_{xx}^{(w)}(t,x) \;=\; (c_2 - 1) \cdot \dfrac{d^2}{dx^2}\left[\left(\dfrac{\dot{\sigma}^{(w)}(t)t}{2} + (w(t,x) - w(t,0+))\right) \cdot \phi_x\left(x, \dfrac{\sigma^{(w)}(t)t}{2}\right)\right], \end{cases} \tag{3.13}$$

we estimate

$$\begin{cases} \left|C_x^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1) \cdot \left(M_0 \cdot |x|^{-1/4} + \dfrac{|\dot{\sigma}^{(w)}(t)|}{\sigma^{(w)}(t)}\right), \\ \left|C_{xx}^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1) \cdot \left[\left(M_0 + \dfrac{|\dot{\sigma}^{(w)}(t)|}{|\sigma^{(w)}(t)|}\right) \cdot |x|^{-1} + |w_{xx}(t,x)| \cdot \big|\ln|x|\big|\right]. \end{cases} \tag{3.14}$$

Similarly, for every $-\dfrac{1}{2e} < x < 0$, we have

$$C^{(w)}(t,x) \;=\; E_1^{(w)}(t) - (c_1 - 1) \cdot \left[\frac{\dot{\sigma}^{(w)}(t)t}{2} - (w(t,x)) - w(t,0-))\right] \cdot \phi_x\left(x, \frac{\sigma^{(w)}(t)t}{2}\right),$$

$$E_1^{(w)}(t) \doteq \frac{1-c_1}{2} \cdot \phi_x \left(\frac{\sigma^{(w)}(t)t}{2}, 0 \right) \cdot \left(\dot\sigma^{(w)}(t)t + \sigma^{(w)}(t) \right).$$

This yields the same bounds as in (3.11)-(3.14). We thus conclude

$$\left\| C^{(w)}(t,\cdot) \right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \leq \mathcal{O}(1) \cdot \left[\left(M_0 + \frac{|\dot\sigma^{(w)}(t)|}{|\sigma^{(w)}(t)|} \right) \cdot \delta^{-1/2} + M_0 \cdot |\ln(t)| \right]. \tag{3.15}$$

3. Finally, to obtain a bound on $A^{(w)}$ we observe that, by (3.5),

$$\left| \varphi^{(w)}(t,x) \right| \leq \mathcal{O}(1) \cdot \left| x \ln|x| \right|, \qquad (t,x) \in [0,T] \times \left[-\frac{1}{2e}, \frac{1}{2e} \right].$$

This leads to the estimates

$$\left| \varphi^{(w)} \varphi_x^{(w)} \right| \leq \mathcal{O}(1) \cdot |x|^{1/2}, \quad \left| \left(\varphi^{(w)} \varphi_x^{(w)} \right)_x \right| \leq \mathcal{O}(1) \cdot \ln^2 |x|, \quad \left| \left(\varphi^{(w)} \varphi_x^{(w)} \right)_{xx} \right| \leq \mathcal{O}(1) \cdot \left| \frac{\ln |x|}{x} \right|.$$

Thus,

$$\left\| \varphi^{(w)}(t,\cdot) \varphi_x^{(w)}(t,\cdot) \right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \leq \mathcal{O}(1) \cdot \delta^{-2/3}.$$

On the other hand, if $0 < T < \frac{1}{4eM_0}$, then $\sup_{t\in[0,T]} \frac{\sigma^{(w)}(t)t}{2} \leq \frac{1}{2e}$ and Lemma 3.1 implies for all $t \in [0,T]$ and $|x| \leq \frac{1}{2e}$, that

$$\left| \mathbf{H}[\varphi^{(w)}(t,\cdot)](x) \right| \leq \mathcal{O}(1), \qquad \left| \frac{d}{dx} \mathbf{H}[\varphi^{(w)}(t,\cdot)](x) \right| \leq \mathcal{O}(1) \cdot \ln^2 |x|, \tag{3.16}$$

$$\left| \frac{d^2}{dx^2} \mathbf{H}[\varphi^{(w)}(t,\cdot)](x) \right| \leq \mathcal{O}(1) \cdot \left| \frac{\ln|x|}{x} \right|, \qquad \left\| \mathbf{H}[\varphi^{(w)}(t,\cdot)] \right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \leq \mathcal{O}(1) \cdot \delta^{-2/3}. \tag{3.17}$$

Therefore, combining (3.10)-(3.17), we obtain (3.8)-(3.9). This completes the proof. □

Our third lemma estimates the change in the function $F = F(t,x,w)$ as $w(\cdot)$ takes different values. These estimates will play a key role in the proof of convergence of the approximations considered at (2.20).

Lemma 3.3 *Let $w_i : [0,T] \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ such that for all $t \in [0,T]$, $w_i(t,\cdot) \in H^2(\mathbb{R}\setminus\{0\})$ and*

$$\|w_i(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})} \leq M_0, \qquad \left| \sigma^{(w_i)}(t) \right| \geq \delta_0.$$

Moreover, assume that $\sigma^{(w_i)}$ is locally Lipschitz on $]0,T]$ and that there exists a function $K(t)$ such that

$$\left| \dot\sigma^{(w_i)}(t) \right| \leq K(t) \qquad a.e.\ t \in (0,T).$$

Set

$$z \doteq w_2 - w_1, \quad \sigma^{(z)} \doteq \sigma^{(w_2)} - \sigma^{(w_1)}, \quad M_1(t) \doteq \|z(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})}, \quad M_2(t) \doteq \|z(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})}.$$

Then there exists a constant $C_2 > 0$, M_0, δ_0, c_1, and c_2 such that, for every $x \in \left[-\frac{1}{2e}, \frac{1}{2e}\right]$ and a.e. $t \in [0,T]$, one has

$$\left|F(t,x,w_2) - F(t,x,w_1)\right| \ \leq\ C_2 \cdot \Big[\left|\dot{\sigma}^{(z)}(t)\right| \cdot |x| + M_1(t) \cdot \big(|\ln t| + K(t)\big)\Big]. \tag{3.18}$$

Moreover, for every $\delta > 0$ sufficiently small, it holds

$$\begin{cases} \left\|\mathbf{H}\left[\varphi^{(w_2)}(t,\cdot) - \varphi^{(w_1)}(t,\cdot)\right]\right\|_{H^1(\mathbb{R}\setminus[-\delta,\delta])} + \left\|B^{(w_2)} - B^{(w_1)}\right\|_{H^1(\mathbb{R}\setminus[-\delta,\delta])} \ \leq\ C_2 \cdot \dfrac{M_1(t)}{\delta^{1/2}}, \\ \left\|F(t,\cdot,w_2) - F(t,\cdot,w_1)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \ \leq\ C_2 \cdot \left(M_2(t) \cdot \left(\left|\ln t\right| + \dfrac{1+K(t)}{\delta^{2/3}}\right) + \dfrac{|\dot{\sigma}^{(z)}(t)|}{\delta^{1/2}}\right). \end{cases} \tag{3.19}$$

Proof. 1. For notational convenience, we set

$$\mathbf{A}^{(z)} \ \doteq\ A^{(w_2)} - A^{(w_1)}, \quad \mathbf{B}^{(z)} \ \doteq\ B^{(w_2)} - B^{(w_1)}, \quad \mathbf{C}^{(z)} \ \doteq\ C^{(w_2)} - C^{(w_1)}.$$

From [4, Section 3], for every $(t,x) \in [0,T] \times \left[-\frac{1}{2e}, \frac{1}{2e}\right]$, it holds

$$\begin{cases} \left|\mathbf{B}^{(z)}(t,x)\right| = \mathcal{O}(1) \cdot M_1(t), \qquad \left\|\mathbf{B}^{(z)}(t,\cdot)\right\|_{H^1(\mathbb{R}\setminus[-\delta,\delta])} \ \leq\ \mathcal{O}(1) \cdot \dfrac{M_1(t)}{\delta^{1/2}}, \\ \left\|\mathbf{B}^{(z)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \leq \mathcal{O}(1) \cdot \dfrac{M_2(t)}{\delta^{2/3}}. \end{cases} \tag{3.20}$$

2. We now provide bounds on $\mathbf{C}^{(z)}(t,x)$. By (3.11)-(3.13), for every $0 < x < \dfrac{1}{2e}$ one has

$$\left|\mathbf{C}^{(z)}(t,x)\right| \ \leq\ \mathcal{O}(1)\left[\frac{|\dot{\sigma}^{(z)}(t)|}{\delta_0} \cdot x + M_1(t) \cdot \left(\frac{K(t)x}{\delta_0^2} + \frac{M_0 x + M_0^2 + 1}{\delta_0} + \left|\ln t\right| + \left|x^{1/2}\ln x\right|\right)\right], \tag{3.21}$$

$$\begin{aligned} \left|\mathbf{C}_x^{(z)}(t,x)\right| &\leq \mathcal{O}(1) \cdot \left[\frac{|\dot{\sigma}^{(z)}(t)|}{\delta_0} + \left(\frac{K(t)}{\delta_0^2} + \frac{M_0}{\delta_0}\right) \cdot \left|\sigma^z(t)\right| + \frac{\left|z(t,x) - z(t,0+)\right|}{x}\right. \\ &\qquad\qquad \left. + |z_x(t,x)| \cdot \left(|\ln(t)| + \frac{1}{\delta_0} + M_0\right)\right] \\ &\leq \mathcal{O}(1) \cdot \left[\frac{|\dot{\sigma}^{(z)}(t)|}{\delta_0} + \left(\frac{K(t)}{\delta_0^2} + \frac{M_0}{\delta_0}\right) \cdot M_1(t) + \frac{M_1(t)}{x^{1/2}} + |z_x(t,x)| \cdot \left(|\ln(t)| + \frac{1}{\delta_0} + M_0\right)\right], \end{aligned} \tag{3.22}$$

$$\begin{aligned} \left|\mathbf{C}_{xx}^{(z)}(t,x)\right| &\leq \mathcal{O}(1) \cdot \left\{\frac{|\dot{\sigma}^{(z)}(t)|}{\delta_0 x} + |z_{xx}(t,x)| \cdot \left(|\ln(t)| + \frac{1}{\delta_0} + M_0\right) + \frac{\left|z_x(t,x)\right|}{x} + \frac{|z(t,x) - z(t,0+)|}{x^2}\right. \\ &\qquad \left. + \left|\sigma^{(z)}(t)\right| \cdot \left(\frac{|w_{1,xx}(t,x)|}{\delta_0} + \frac{K(t)}{\delta_0^2 x} + \frac{|w_{1,x}(t,x)|}{\delta_0 x} + \frac{\left|w_1(t,x) - w_1(t,0+)\right|}{\delta_0 x^2}\right)\right\} \\ &\leq \mathcal{O}(1) \cdot \left\{\left(\frac{|\dot{\sigma}^{(z)}(t)|}{\delta_0} + \frac{K(t)M_1(t)}{\delta_0^2} + \frac{M_1(t)M_0}{\delta_0} + M_2(t)\right) \cdot \frac{1}{x}\right. \\ &\qquad \left. + |z_{xx}(t,x)| \cdot \left(|\ln(t)| + \frac{1}{\delta_0} + M_0\right) + \left|w_{1,xx}(t,x)\right| \cdot \frac{M_1(t)}{\delta_0}\right\}. \end{aligned} \tag{3.23}$$

For every $-\frac{1}{2e} < x < 0$, by a similar argument, we obtain the same bounds as in (3.21)-(3.23). Therefore

$$\begin{cases} \left\|\mathbf{C}^{(z)}(t,\cdot)\right\|_{H^1(\mathbb{R}\setminus\{0\})} & \leq \ \mathcal{O}(1)\cdot\left(M_1(t)\cdot\left(\left|\ln t\right| + \dfrac{M_0+1}{\delta_0}+\dfrac{K}{\delta_0^2}\right)+\dfrac{|\dot\sigma^{(z)}(t)|}{\delta_0}\right), \\ \left\|\mathbf{C}^{(z)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} & \leq \mathcal{O}(1)\cdot\left(M_2(t)\cdot\left[\left|\ln t\right| + \left(\dfrac{M_0+1}{\delta_0}+\dfrac{K}{\delta_0^2}\right)\cdot\dfrac{1}{\sqrt{\delta}}\right]+\dfrac{|\dot\sigma^{(z)}(t)|}{\delta_0\sqrt{\delta}}\right). \end{cases} \tag{3.24}$$

2. To achieve bound on $\mathbf{A}^{(z)}$, for $0 < x < \frac{1}{2e}$ we compute

$$\begin{aligned} \left|\varphi^{(w_2)}\varphi_x^{(w_2)} - \varphi^{(w_1)}\varphi_x^{(w_1)}\right| \ &\leq \ \mathcal{O}(1)\cdot\frac{\left|\sigma^{(z)}(t)\right|}{\delta_0}\cdot\left|x\ln|x|\right| \ \leq \ \mathcal{O}(1)\cdot\frac{M_1(t)}{\delta_0}\cdot\left|x\ln|x|\right|, \\ \left|\left(\varphi^{(w_2)}\varphi_x^{(w_2)} - \varphi^{(w_1)}\varphi_x^{(w_1)}\right)_x\right| \ &\leq \ \mathcal{O}(1)\cdot\frac{\left|\sigma^{(z)}(t)\right|}{\delta_0}\cdot\left|\ln|x|\right| \ \leq \ \mathcal{O}(1)\cdot\frac{M_1(t)}{\delta_0}\cdot\left|\ln|x|\right|, \\ \left|\left(\varphi^{(w_2)}\varphi_x^{(w_2)} - \varphi^{(w_1)}\varphi_x^{(w_1)}\right)_{xx}\right| \ &\leq \ \mathcal{O}(1)\cdot\frac{\left|\sigma^{(z)}(t)\right|}{\delta_0}\cdot\left|\frac{\ln|x|}{x}\right| \ \leq \ \mathcal{O}(1)\cdot\frac{M_1(t)}{\delta_0}\cdot\left|\frac{\ln|x|}{x}\right|. \end{aligned}$$

This yields

$$\begin{cases} \left\|\varphi^{(w_2)}\varphi_x^{(w_2)} - \varphi^{(w_1)}\varphi_x^{(w_1)}\right\|_{H^1(\mathbb{R}\setminus\{0\})} & \leq \ \mathcal{O}(1)\cdot\dfrac{M_1(t)}{\delta_0}, \\ \left\|\varphi^{(w_2)}\varphi_x^{(w_2)} - \varphi^{(w_1)}\varphi_x^{(w_1)}\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} & \leq \ \mathcal{O}(1)\cdot\dfrac{M_1(t)}{\delta_0}\cdot\delta^{-2/3}. \end{cases}$$

On the other hand, for $0 < x < \frac{1}{2e}$ we observe that

$$\begin{aligned} \varphi^{(w_2)}(t,x) - \varphi^{(w_1)}(t,x) \ &= \ (c_2-1)\cdot\left[\phi\left(x,\frac{\sigma^{(w_2)}(t)t}{2}\right) - \phi\left(x,\frac{\sigma^{(w_1)}(t)t}{2}\right)\right] \\ &= \ \frac{c_2-1}{2}\cdot\left(\int_0^1 \phi_b\left(x,\frac{\sigma^{(w_1)}(t)t}{2}+\tau\cdot\frac{\sigma^{(z)}(t)t}{2}\right)d\tau\right)\cdot\sigma^{(z)}(t)t \\ &= \ \frac{c_2-1}{2}\cdot\left(\int_0^1 \frac{d}{db}g_{b_\tau}(x)\ d\tau\right)\cdot\sigma^{(z)}(t)t, \end{aligned} \tag{3.25}$$

with $b_\tau = \dfrac{\sigma^{(w_1)}(t)t}{2} + \tau\cdot\dfrac{\sigma^{(z)}(t)t}{2}$. Thus, by Remark 3.1 it follows

$$\begin{cases} \left|\mathbf{H}\left[\varphi^{(w_2)}(t,\cdot) - \varphi^{(w_1)}(t,\cdot)\right](x)\right| & \leq \ \mathcal{O}(1)\cdot\dfrac{M_1(t)}{\delta_0}, \\ \left|\dfrac{d}{dx}\mathbf{H}\left[\varphi^{(w_2)}(t,\cdot) - \varphi^{(w_1)}(t,\cdot)\right](x)\right| & \leq \ \mathcal{O}(1)\cdot M_1(t)\cdot\dfrac{\left|\ln|x|\right|}{\delta_0}, \end{cases}$$

and

$$\begin{cases} \left\|\mathbf{H}\left[\varphi^{(w_2)}(t,\cdot) - \varphi^{(w_1)}(t,\cdot)\right]\right\|_{H^1(\mathbb{R}\setminus[-\delta,\delta])} & \leq \ \mathcal{O}(1)\cdot M_1(t)\cdot\dfrac{1}{\delta_0}, \\ \left\|\mathbf{H}\left[\varphi^{(w_2)}(t,\cdot) - \varphi^{(w_1)}(t,\cdot)\right]\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} & \leq \ \mathcal{O}(1)\cdot\dfrac{M_1(t)}{\delta_0}\cdot\delta^{-2/3}. \end{cases}$$

Similarly, one gets the same estimate for $-\frac{1}{2e} < x < 0$. Therefore (3.2) yields

$$\begin{cases} \left|\mathbf{A}^{(z)}(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot\dfrac{M_1(t)}{\delta_0}, \qquad \left|\mathbf{A}^{(z)}_x(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot M_1(t)\cdot\dfrac{\big|\ln|x|\big|}{\delta_0}, \\ \left\|\mathbf{A}^{(z)}(t,\cdot)\right\|_{H^1(\mathbb{R}\setminus[-\delta,\delta])} \ \leq\ \mathcal{O}(1)\cdot\dfrac{M_1(t)}{\delta_0}\,, \\ \left\|\mathbf{A}^{(z)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \ \leq\ \mathcal{O}(1)\cdot\dfrac{M_1(t)}{\delta_0}\cdot\delta^{-2/3}\,. \end{cases} \tag{3.26}$$

Finally, combining the estimates (3.20)-(3.26), we obtain (3.18)-(3.19). □

4 Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1 by constructing a solution to the Cauchy problem (2.2) with general initial data of the form (2.6)-(2.7), locally in time. This solution will be obtained as limit of a Cauchy sequence of approximate solutions $w_n(t,x)$, following the steps (i)–(iii) outlined at the end of Section 2.

Step 1. Consider any initial profile $\overline{w} \in H^2(\mathbb{R}\setminus\{0\})$. Let $\delta_0, M_0 > 0$ be the constants defined by the identities

$$\overline{w}(0-) - \overline{w}(0+) \ =\ 6\delta_0, \qquad \|\overline{w}\|_{H^2(\mathbb{R}\setminus\{0\})} \ =\ \frac{M_0}{2}\,. \tag{4.1}$$

Given two constants $c_1, c_2 \in \mathbb{R}$, the corresponding initial data of the form (2.6)-(2.7) is

$$u(0,x) \ =\ \overline{w}(x) + \Big(c_1\cdot\chi_{]-\infty,0[} + c_2\cdot\chi_{]0,+\infty[}\Big)\cdot\phi(x,0).$$

Moreover, let $w_n : [0,T]\times\mathbb{R}\to\mathbb{R}$ be a function such that

$$|w_n(t,0\pm) - \overline{w}(0\pm)| \ \leq\ \delta_0, \qquad \|w_n(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})} \ \leq\ M_0, \qquad t\in[0,T]. \tag{4.2}$$

Set $\sigma_n(t) \doteq w_n(t,0-) - w_n(t,0+)$. As in (2.11), the correction term associated to w_n is denoted by

$$\varphi_n(t,x) \ \doteq\ \phi(x,0) + \Big((c_1-1)\cdot\chi_{]-\infty,0[} + (c_2-1)\cdot\chi_{]0,+\infty[}\Big)\cdot\phi\left(x, \frac{\sigma_n(t)t}{2}\right). \tag{4.3}$$

In this step, we will establish the existence and uniqueness of solutions to the linear problem (2.19).

We begin by observing that the speed of all characteristics for (2.19) is

$$a_n(t,x) \ \doteq\ a(t,x,w_n) \ =\ \varphi_n(t,x) + w_n(t,x) - \frac{w_n^-(t) + w_n^+(t)}{2}\,,$$

where $\varphi_n(t,x) \doteq \varphi^{(w_n)}(t,x)$, the correction term associated to w_n. From (4.3) and (4.2) it follows that $\varphi_n(t,0) = 0$ and

$$-4\delta_0 \ \leq\ a_n(t,0+) \ =\ -\frac{\sigma_n(t)}{2} \ \leq\ -2\delta_0, \qquad 2\delta_0 \ \leq\ a_n(t,0-) \ =\ \frac{\sigma_n(t)}{2} \ \leq\ 4\delta_0.$$

Furthermore, for any given $(t,x)\in[0,T]\times\,]0,\frac{1}{2e}]$, we estimate, using (3.5),

$$\begin{aligned}|a_n(t,x)-a_n(t,0+)| &\le \frac{2(2+|c_2|)|x\ln x|}{\pi}+\int_0^x |w_{n,x}(t,y)|dy\\ &\le \frac{2(2+|c_2|)|x\ln x|}{\pi}+x^{1/2}\cdot\left(\int_0^x |w_{n,x}(t,y)|^2dy\right)^{1/2} \le (2+|c_2|+M_0)\cdot\sqrt{x}.\end{aligned}$$

Similarly, we also have

$$|a_n(t,x)-a_n(t,0-)| \le (2+|c_1|+M_0)\cdot\sqrt{|x|}, \qquad (t,x)\in[0,T]\times\left[-\frac{1}{2e},0\right[.$$

In particular, setting

$$\delta_1 \doteq \frac{1}{4}\cdot\left(\frac{\delta_0}{4+|c_1|+|c_2|+M_0}\right)^2 \le \frac{1}{16}, \tag{4.4}$$

we have

$$\begin{cases}-5\delta_0 \le a_n(t,x) \le -\delta_0, & (t,x)\in[0,T]\times\,]0,2\delta_1],\\ \delta_0 \le a_n(t,x) \le 5\delta_0, & (t,x)\in[0,T]\times[-2\delta_1,0[\,.\end{cases} \tag{4.5}$$

Next, choose

$$0 < T < \min\left\{\frac{\delta_1}{10\delta_0},\frac{1}{2e}\right\}, \tag{4.6}$$

and denote by $t\mapsto x(t;t_0,x_0)$ the solution to the Cauchy problem

$$\dot{x}(t) = a_n(t,x(t)), \qquad x(t_0) = x_0. \tag{4.7}$$

By (4.5) it follows

$$\delta_0(t_0-t) \le |x(t;t_0,x_0)-x_0| \le 5\delta_0(t_0-t), \qquad |x_0|\le\delta_1,\quad 0\le t\le t_0\le T. \tag{4.8}$$

The next lemma provides the Lipschitz continuous dependence of the characteristic curves considered at (4.7).

Lemma 4.1 *Let w_n,φ_n be as in (4.2)-(4.3). Then there exists a constant $K_1>0$, depending on M_0,δ_0,c_1, and c_2, such that, for any $x_1,x_2\in[-\delta_1,0[$ or $x_1,x_2\in]0,\delta_1]$, one has*

$$\Big|x(t;\tau,x_2)-x(t;\tau,x_1)\Big| \le K_1\cdot|x_2-x_1|, \qquad \text{for all } 0\le t\le\tau\le T. \tag{4.9}$$

Proof. We shall prove (4.9) for $x_1,x_2\in[-\delta_1,0[$, the other case being entirely similar. For any $-\delta_1\le z_1<z_2<0$, it holds

$$\begin{aligned}\Big|a_n(t,z_2)-a_n(t,z_1)\Big| &\le |w_n(t,z_2)-w_n(t,z_1)|+\Big|\varphi_n(t,z_2)-\varphi_n(t,z_1)\Big|\\ &\le \left[M_0+\frac{4+2|c_1|}{\pi}\cdot\big(1+\big|\ln|z_2|\big|\big)\right]\cdot|z_2-z_1|.\end{aligned}$$

Therefore, from (4.8), it follows

$$\frac{d}{dt}\big|x(t;\tau,x_2)-x(t;\tau,x_1)\big| \le \left[M_0+\frac{4+2|c_1|}{\pi}\cdot\big(1+\big|\ln|\delta_0(\tau-t)|\big|\big)\right]\cdot\big|x(t;\tau,x_2)-x(t;\tau,x_1)\big|,$$

and this yields (4.9). □

From (4.5), by the same arguments as in [4, Lemma 4.1], one obtains:

Lemma 4.2 *Let w_n, φ_n be as in (4.2)-(4.3). There exists $T > 0$ sufficiently small, depending only on M_0, δ_0, c_1, c_2, such that, for every $\tau \in [0,T]$ and any solution v of the linear equation*

$$v_t + a_n(t,x) \cdot v_x \;= 0, \qquad v(0,\cdot) \;=\; \bar{v} \;\in\; H^2\big(\mathbb{R}\backslash[-\delta_0\tau, \delta_0\tau]\big),$$

one has

$$\|v(\tau,\cdot)\|_{H^2\left(\mathbb{R}\backslash\{0\}\right)} \;\leq\; \frac{3}{2} \cdot \|\bar{v}\|_{H^2\left(\mathbb{R}\backslash[-\delta_0\tau,\delta_0\tau]\right)}.$$

Step 2. Consider a sequence of approximate solutions $w^{(k)}$ to (2.19), inductively defined as follows.

- $w^{(1)}(t,\cdot) \;\doteq\; \overline{w}(\cdot)$ for all $t \geq 0$.
- For every $k \geq 1$, $w^{(k+1)}(t,\cdot)$ solves the linear equation

$$w_t + a_n(t,x) \cdot w_x \;=\; F^{(k)}(t,x), \qquad w(0,\cdot) \;=\; \overline{w}(\cdot)$$

  with $F^{(k)}(t,x) \doteq F\left(t,x,w^{(k)}\right)$. Equivalently, $w^{(k+1)}$ satisfies the integral identiies

$$w^{(k+1)}(t_0,x_0) \;=\; \overline{w}(x(0;t_0,x_0)) + \int_0^{t_0} F^{(k)}(t, x(t;t_0,x_0))dt. \tag{4.10}$$

The following lemma provides a priori estimates on $w^{(k)}$, uniformly valid for all $k \geq 1$.

Lemma 4.3 *Let w_n, φ_n be as in (4.2)-(4.3). Then there exists $T > 0$ sufficiently small, depending only on M_0, δ_0, c_1, c_2, and satisfying* (4.6) *so that the following holds. For every $k \geq 1$ and a.e. $\tau \in [0,T]$, one has*

$$\left|w^{(k)}(\tau, 0\pm) - \overline{w}(0\pm)\right| \;\leq\; \delta_0, \tag{4.11}$$

$$\left|\dot{\sigma}^{(k)}(\tau)\right| \;\leq\; 4C_1(1+M_0) \cdot |\ln \tau|, \tag{4.12}$$

$$\left\|w^{(k)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{0\})} \;\leq\; M_0\,, \tag{4.13}$$

for some constant $C_1 > 0$.

Proof. 1. It is clear that (4.11)-(4.13) hold for $k = 1$. By induction, assume that (4.11) holds for a given $k \geq 1$. By the assumptions (4.1) and (4.11), for all $\tau \in [0,T]$ one obtains

$$\sigma^{(k)}(\tau) \;\;\geq\;\; \overline{w}(0-) - \overline{w}(0+) - \left|w^{(k)}(\tau,0+) - \overline{w}(0+)\right| - \left|w^{(k)}(\tau,0-) - \overline{w}(0-)\right| \;\geq\; 4\delta_0.$$

For a fixed $\tau \in [0,T]$, let $x^{\pm} : [0,\tau] \mapsto \mathbb{R}$ be the characteristics which reach the origin at time τ, from the left and the right, respectively. Recalling (4.10), (3.8), (4.1), and (4.8), we estimate

$$\begin{aligned}
\left|w^{(k+1)}(\tau,0\pm)-\overline{w}(0\pm)\right| \;&\leq\; \left|\overline{w}(x^{\pm}(0))-\overline{w}(0\pm)\right|+\int_0^\tau \left|F^{(k)}(t,x^{\pm}(t))\right|dt\\
&\leq\; 3M_0\delta_0\tau+C_1\cdot\int_0^\tau\left((1+M_0)\cdot|\ln t|+\frac{|\dot\sigma^{(k)}(t)|}{|\sigma^{(k)}(t)|}\cdot\left|x^{\pm}(t)\right|\right)dt\\
&\leq\; 3M_0\delta_0\tau+C_1(1+M_0)\cdot\int_0^\tau (1+5C_1(\tau-t))\cdot|\ln(t)|dt\\
&\leq\; \mathcal{O}(1)\cdot(1+M_0)\cdot\left|\ln(\tau)\right|\cdot\tau\;\leq\;\mathcal{O}(1)\cdot(1+M_0)\cdot\left|\ln(T)\right|\cdot T
\end{aligned}$$

and this shows that $w^{(k+1)}$ satisfies (4.11), provided that $T>0$ is chosen sufficiently small, depending only on M_0,δ_0,c_1, and c_2.

2. For any $\tau\in[0,T]$ and $-\delta_1\leq\bar x_2<\bar x_1<0$, consider the characteristics

$$t\mapsto x_1(t)\;=\;x(t;\tau,\bar x_1),\qquad\qquad t\mapsto x_2(t)\;=\;x(t;\tau,\bar x_2).$$

Using (4.10), (4.9), (3.8), (4.11), and (4.8), we estimate

$$\begin{aligned}
&\left|w^{(k+1)}(\tau,\bar x_2)-w^{(k+1)}(\tau,\bar x_1)\right|\\
&\quad\leq\;\left|\overline{w}(x_2(0))-\overline{w}(x_1(0))\right|+\int_0^\tau\left|F^{(k)}(t,x_2(t))-F^{(k)}(t,x_1(t))\right|dt\\
&\quad\leq\; M_0K_1\cdot|\bar x_2-\bar x_1|+C_1\cdot\int_0^\tau\left((1+M_0)\cdot|x_1(t)|^{-1/4}+\frac{|\dot\sigma^{(k)}(t)|}{\sigma^{(k)}(t)}\right)\cdot|x_2(t)-x_1(t)|dt\\
&\quad\leq\;\; M_0K_1\cdot\left(1+\mathcal{O}(1)\cdot\left(1+\frac{1}{M_0}\right)\cdot\left[\left(\frac{\tau}{\delta_0}\right)^{1/4}+\frac{|\tau\ln\tau|}{\delta_0}\right]\right)\cdot|\bar x_2-\bar x_1|.
\end{aligned}$$

Therefore, choosing $T>0$ sufficiently small, we obtain

$$\left|w_x^{(k+1)}(\tau,x)\right|\;\leq\;3M_0K_1\qquad\text{for all }\tau\in[0,T],x\in[-\delta_1,0[\,.\tag{4.14}$$

An entirely similar estimate holds for $\tau\in[0,T],x\in]0,\delta_1]$.

3. Next, given any $0\leq\tau_1<\tau_2\leq T$, denote by $t\mapsto x_i^{\pm}(t)\doteq x(t;\tau_i,0\pm)$ the characteristic which reaches the origin at time τ_i, from the positive or negative side, respectively. Recalling (4.9)–(4.11), (3.8), and (4.14), we estimate

$$\begin{aligned}
&\left|w^{(k+1)}(\tau_2,0\pm)-w^{(k+1)}(\tau_1,0\pm)\right|\\
&\quad\leq\;\;\left|w^{(k+1)}\big(\tau_1,x_2^{\pm}(\tau_1)\big)-w^{(k+1)}(\tau_1,0\pm)\right|+\int_{\tau_1}^{\tau_2}\left|F^{(k)}(t,x_2^{\pm}(t))\right|dt\\
&\quad\leq\; 3M_0K_1\left|x_2^{\pm}(\tau_1)\right|+C_1\cdot\int_{\tau_1}^{\tau_2}\left((1+M_0)\cdot|\ln t|+\frac{|\dot\sigma^{(k)}(t)|}{|\sigma^{(k)}(t)|}\cdot\left|x_2^{\pm}(t)\right|\right)dt\\
&\quad\leq\; 15M_0K_1\delta_0(\tau_2-\tau_1)+C_1(1+M_0)\int_{\tau_1}^{\tau_2}|\ln t|(1+5C_1(\tau_2-t))dt\\
&\quad\leq\;\;(15M_0K_1\delta_0+C_1(1+M_0)|\ln(\tau_1)|)\cdot(\tau_2-\tau_1)\;\leq\;2C_1(1+M_0)|\ln(\tau_1)|\cdot(\tau_2-\tau_1),
\end{aligned}$$

provided that $T > 0$ is sufficiently small. In particular, we have

$$\left|\sigma^{(k+1)}(\tau_2) - \sigma^{(k+1)}(\tau_1)\right| \;\leq\; 4C_1(1+M_0)|\ln\tau_1|\cdot(\tau_2-\tau_1).$$

This shows that $\dot\sigma^{(k+1)}$ satisfies (4.12).

4. Finally, from Lemma 4.2, Lemma 3.2, (4.12), and Duhamel's formula, for all $\tau\in[0,T]$ we obtain

$$\begin{aligned}\left\|w^{(k+1)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{0\})} \;&\leq\; \frac{3}{2}\cdot\|\overline{w}\|_{H^2\left(\mathbb{R}\backslash[-\delta_0\tau,\delta_0\tau]\right)}+\frac{3}{2}\cdot\int_0^\tau \left\|F^{(k)}(t,\cdot)\right\|_{H^2\left(\mathbb{R}\backslash[-\delta_0(\tau-t),\delta_0(\tau-t)]\right)}\,dt\\ &\leq\; \frac{3M_0}{4}+\frac{3}{2}\cdot C_1(1+M_0)\cdot\int_0^\tau\left(1+\frac{C_1}{\delta_0}\cdot|\ln t|\right)\cdot\delta_0^{-2/3}\cdot(\tau-t)^{-2/3}+|\ln t|dt\\ &\leq\; \frac{3M_0}{4}+\frac{3}{2}\cdot C_1(1+M_0)\cdot\left[6\left(1+\frac{C_1}{\delta_0}\cdot|\ln\tau|\right)\delta_0^{-2/3}\tau^{1/3}+\left|\tau\ln\tau\right|\right]\;\leq\; M_0\,,\end{aligned}$$

provided that $T > 0$ is sufficiently small, depending only on M_0,δ_0,c_1, and c_2. This shows that (4.13) is satisfied by $w^{(k+1)}$ as well. □

Thanks to the above estimates, we can now prove that the sequence of approximations $w^{(k)}$ is Cauchy and converges to a solution w of the linear problem (2.19). This is a key step toward the proof of Theorem 2.1.

Lemma 4.4 *Let w_n,φ_n be as in (4.2)-(4.3). Then, for some $T > 0$ sufficiently small, depending only on M_0,δ_0,c_1,c_2, such that, the sequence of approximations $\big(w^{(k)}\big)_{k\geq 1}$ converges to a limit function w in $\mathbf{L}^\infty\big([0,T],H^2(\mathbb{R}\backslash\{0\})\big)$, i.e.,*

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left\|w^{(k)}(t,\cdot)-w(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{0\})} \;=\; 0.$$

The function w provides a solution to the Cauchy problem (2.19) and satisfies

$$|w(\tau,0\pm)-\overline{w}(0\pm)| \;\leq\; \delta_0,\qquad \|w(t,\cdot)\|_{H^2(\mathbb{R}\backslash\{0\})}\;\leq\; M_0,\qquad t\in[0,T] \tag{4.15}$$

Moreover, $\sigma(t)\doteq w(t,0-)-w(t,0+)$ is locally Lipschitz in $(0,T)$ and

$$|\dot\sigma(t)| \;\leq\; 4C_1(1+M_0)\cdot|\ln t|,\qquad a.e.\ t\in(0,T). \tag{4.16}$$

Proof. 1. For any $k\geq 1$, we set

$$\begin{cases} z^{(k)} \;\doteq\; w^{(k+1)}-w^{(k)}, \qquad \sigma_z^{(k)}(t)\;\doteq\; z^{(k)}(t,0-)-z^{(k)}(t,0+),\\ M_z^{(k)}(t)\;\doteq\;\left\|z^{(k)}(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{0\})},\quad \beta_k(\tau)\;\doteq\;\sup_{t\in[0,\tau]}M_z^{(k)}(t),\quad \alpha_k(\tau)\;\doteq\;\sup_{t\in[0,\tau]}\left|\dot\sigma_z^{(k)}(t)\right|.\end{cases} \tag{4.17}$$

Recalling Lemma 3.3, Lemma 4.2, and Lemma 4.3, and using Duhamel's formula, we estimate

$$\begin{aligned}M_z^{(k+1)}(\tau) \;&\leq\; \frac{3}{2}\cdot\int_0^\tau\left\|F^{(k+1)}(t,\cdot)-F^{(k)}(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash[-\delta_0(\tau-t),\delta_0(\tau-t)])}\,dt\\ &\leq\; C_3\cdot\int_0^\tau\beta_k(t)\cdot|\ln(t)|\left(1+\frac{1}{(\tau-t)^{2/3}}\right)+\frac{\alpha_k(t)}{\sqrt{\tau-t}}\,dt\;\leq\; C_4\cdot\left(\beta_k(\tau)\tau^{1/3}+\alpha_k(\tau)\tau^{1/2}\right),\end{aligned}$$

and this implies

$$\beta_{k+1}(\tau) \;\leq\; C_4 \cdot \Big(\beta_k(\tau)\tau^{1/3} + \alpha_k(\tau)\tau^{1/2}\Big), \qquad \tau \in [0,T],\ k \geq 1, \tag{4.18}$$

for some constant $C_3, C_4 > 0$ depending only on M_0, δ_0, c_1, and c_2.

2. We now establish a bound on $\big\|\dot\sigma^{(k+1)}\big\|_{\mathbf{L}^\infty([0,T])}$. Given any $0 < \tau_1 < \tau_2 \leq T$, denote by $t \mapsto x_i^\pm(t) \doteq x(t;\tau_i,0\pm)$ the characteristics, which reach the origin at time τ_i, from the positive or negative side, respectively. Using (3.18), (4.12), and (4.8) we obtain

$$\begin{aligned}
&\Big|z^{(k+1)}(\tau_2,0\pm) - z^{(k+1)}(\tau_1,0\pm)\Big| \\
&\quad\leq\; \Big|z^{(k+1)}\big(\tau_1,x_2^\pm(\tau_1)\big) - z^{(k+1)}(\tau_1,0\pm)\Big| + \int_{\tau_1}^{\tau_2}\Big|F^{(k+1)}\big(t,x_2^\pm(t)\big) - F^{(k)}\big(t,x_2^\pm(t)\big)\Big|\,dt \\
&\qquad\qquad\leq\; \beta_{k+1}\cdot\big|x_2^\pm(\tau_1)\big| + C_5\cdot\int_{\tau_1}^{\tau_2}\alpha_k(t)\cdot\big|x_2^\pm(t)\big| + \beta_k(t)\cdot\big|\ln t\big|\,dt \\
&\qquad\qquad\leq\; \big(5\beta_{k+1}(\tau_2)\delta_0 + C_6\cdot\big[\beta_k(\tau_2)\cdot\big|\ln\tau_1\big| + \alpha_k(\tau_2)(\tau_2-\tau_1)\big|\big]\big)\cdot(\tau_2-\tau_1),
\end{aligned}$$

for some constants $C_5 > 0$ and $C_6 > 0$ depending only on M_0, δ_0, c_1, and c_2. Thus, for $0 < T < \delta_0$ sufficiently small, we obtain

$$\alpha_{k+1}(\tau) \;\leq\; 10\beta_{k+1}(\tau)\delta_0 + 2C_5\beta_k(\tau)\big|\ln\tau\big| \qquad \tau \in]0,T], k \geq 1, \tag{4.19}$$

and (4.18) yields

$$\beta_{k+1}(\tau) \;\leq\; C_7\cdot(\beta_k(\tau) + \beta_{k-1}(\tau))\cdot\tau^{1/3}.$$

for some $C_7 > 0$ depending only on M_0, δ_0, c_1, and c_2. In particular, for $T > 0$ sufficiently small, one has

$$\beta_{k+1}(\tau) + \frac{1}{2}\beta_k(\tau) \;\leq\; \frac{3}{4}\cdot\left(\beta_k(\tau) + \frac{1}{2}\beta_{k-1}(\tau)\right),$$

which implies

$$\sum_{k=1}^\infty \sup_{\tau\in(0,T]}\Big\|z^{(k)}(\tau,\cdot)\Big\|_{H^2(\mathbb{R}\backslash\{0\})} \;<\; \infty.$$

We thus conclude that $\big(w^{(k)}\big)_{k\geq 1}$ is a Cauchy sequence in $\mathbf{L}^\infty([0,T],\mathbf{H}^2(\mathbb{R}\backslash\{0\}))$ and converges to a limit function $w \in \mathbf{L}^\infty([0,T],\mathbf{H}^2(\mathbb{R}\backslash\{0\}))$, which provides the solution to the linear problem (2.19), and satisfies (4.15). Moreover, since $\lim_{k\to\infty} w^{(k)}(\tau,0\pm) = w(\tau,0\pm)$, one has that $\lim_{k\to\infty}\sigma^{(k)}(\tau) = \sigma(\tau)$ for all $\tau \in [0,T]$. Thus, from (4.12), $\sigma(\cdot)$ is locally Lipscthitz in $(0,T)$ and satisfies (4.16). □

We are now ready to complete the proof of our first main result.

Proof of Theorem 2.1. As outlined at the end of Section 2, we construct, by induction, a sequence of approximate solutions $(w_n)_{n\geq 1}$ where each w_n is the solution to the linear problem (2.19). For some $T > 0$ small enough, depending only on M_0, δ_0, c_1, and c_2, we claim that

$$\sum_{n\geq 2}\big\|w_n(\tau) - w_{n-1}(\tau)\big\|_{H^1(\mathbb{R}\backslash\{0\})} \;<\; \infty \qquad \text{for all } t \in [0,T]\,. \tag{4.20}$$

For a fixed $n \geq 2$, we define

$$\begin{cases} W_n \doteq w_n - w_{n-1}, \quad a_n(t,x) \doteq a(t,x,w_n), \quad A_n(t,x) \doteq a_n(t,x) - a_{n-1}(t,x), \\ v_n \doteq \varphi^{(w_n)} - \phi(\cdot,0), \quad V_n \doteq v_n - v_{n-1}, \quad \beta_n(t) \doteq \sup_{s\in[0,t]} \|W_n\|_{H^1(\mathbb{R}\backslash\{0\})}. \end{cases} \tag{4.21}$$

Set $Z_n = W_n + V_n$. From the above definitions, by (2.19) it follows

$$Z_{n+1,t} + a_n \cdot Z_{n+1,x} \;=\; -\left(A_n w_{n,x} + A_{n+1} v_{n+1,x}\right) + G_{n+1} \tag{4.22}$$

with

$$G_{n+1} \;\doteq\; B^{(w_{n+1})} - B^{(w_n)} + \mathbf{H}\left[\varphi^{(w_{n+1})} - \varphi^{(w_n)}\right] - \left(\varphi^{(w_{n+1})} - \varphi^{(w_n)}\right) \cdot \phi_x(\cdot,0).$$

Recalling the first inequality in (3.19) and (4.15), we estimate

$$\begin{cases} \left\|A_n(t,\cdot)\cdot w_{n,x}(t,\cdot)\right\|_{H^1\left(\mathbb{R}\backslash[-\delta,\delta]\right)} \;\leq\; C_7 \cdot M_n(t) \;\leq\; C_7 \cdot \beta_n(t), \\ \left\|A_{n+1}(t,\cdot)\cdot v_{n+1,x}(t,\cdot)\right\|_{H^1\left(\mathbb{R}\backslash[-\delta,\delta]\right)} \;\leq\; C_7 \cdot \dfrac{M_{n+1}(t)}{\delta^{1/2}} \;\leq\; C_7 \cdot \dfrac{\beta_{n+1}(t)}{\delta^{1/2}}, \\ \|G_{n+1}(t,\cdot)\|_{H^1(\mathbb{R}\backslash[-\delta,\delta])} \;\leq\; C_7 \cdot \dfrac{M_{n+1}(t)}{\delta^{1/2}} \;\leq\; C_7 \cdot \dfrac{\beta_{n+1}(t)}{\delta^{1/2}}. \end{cases}$$

for some constant $C_7 > 0$ depending only on M_0, δ_0, c_1, and c_2. Hence, choosing $T > 0$ sufficiently small, we have, using Duhamel's formula,

$$\begin{aligned} \|Z_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \;&\leq\; \frac{3}{2} \cdot \int_0^\tau \|G_{n+1} - A_n w_{n,x} - A_{n+1} v_{n+1,x}\|_{H^1(\mathbb{R}\backslash[-\delta_0(\tau-t),\delta_0(\tau-t)])}\, dt \\ &\leq\; \frac{3C_8}{2} \cdot \int_0^\tau \beta_n(t) + \frac{\beta_{n+1}(t)}{(\tau-t)^{1/2}}\, dt \;=\; \frac{3C_8}{2} \cdot \left(\beta_n(\tau)\cdot\tau + 2\beta_{n+1}(\tau)\tau^{1/2}\right) \end{aligned} \tag{4.23}$$

for some constant $C_8 > 0$. On the other hand, (2.11), (4.21), and (3.25) imply

$$\|V_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \;\leq\; C_9 \cdot \|W_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \cdot \tau^{1/4},$$

and (4.23) yields

$$\|W_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \;\leq\; \frac{3C_8}{2} \cdot \left(\beta_n(\tau)\cdot\tau + 2\beta_{n+1}(\tau)\tau^{1/2}\right) + C_9 \cdot \|W_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \cdot \tau^{1/4}.$$

for some constant $C_9 > 0$ depending only on M_0, δ_0, c_1, and c_2. In particular, for $T > 0$ sufficiently small, one has that

$$\beta_{n+1}(\tau) \;\leq\; \frac{1}{2} \cdot \beta_n(\tau) \qquad \text{for all } \tau \in [0,T].$$

Thus, (4.20) holds and for every $t \in [0,T]$ the sequence of approximations $w_n(t,\cdot)$ is Cauchy in the space $H^1\left(\mathbb{R}\backslash\{0\}\right)$, and hence it converges to a unique limit $w(t,\cdot)$.

It remains to check that this limit function w is an entropic solution, i.e., it satisfies, cf. (2.2), (2.8), and (2.6),

$$\left(w + \varphi^{(w)}\right)(t_0,x_0) = (\overline{w} + \overline{\varphi})(x(0;t_0,x_0)) + \int_0^{t_0} \mathbf{H}\left[w + \varphi^{(w)}\right](t, x(t;t_0,x_0))dt,$$

where $t \mapsto x(t; t_0, x_0)$ is the characteristics curve, obtained by solving (2.16). This follows from slightly rewriting (2.19), which yields

$$\left(w_{n+1} + \varphi^{(w_{n+1})}\right)(t_0, x_0) = (\overline{w} + \overline{\varphi})(x_n(0; t_0, x_0)) + \int_0^{t_0} \mathbf{H}\left[w_{n+1} + \varphi^{(w_{n+1})}\right](t, x_n(t; t_0, x_0))dt$$
$$- \int_0^{t_0} \left(Z_{n+1} - \frac{W_{n+1}^-(t) + W_{n+1}^+(t)}{2}\right) \varphi_x^{(w_{n+1})}(t, x_n(t; t_0, x_0))dt,$$

where $t \mapsto x_n(t; t_0, x_0)$ denotes the characteristic curve, obtained by solving (4.7).

Finally, to prove uniqueness, assume that $\widetilde{w}, w$ are two entropic solutions. We then define

$$W \doteq \widetilde{w} - w, \qquad \beta(\tau) \doteq \sup_{s\in[0,\tau]} \|W(s,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})}.$$

The arguments used in the previous steps now yield the inequality

$$\beta(\tau) \leq \frac{1}{2} \cdot \beta(\tau) \qquad \text{for all } \tau \in [0, T],$$

which implies $\beta(\tau) = 0$ for all $\tau \in [0, T]$ and completes the proof. □

5 Two interacting shocks

In this section, denote by $u(t, x)$ the solution to Burgers' equation

$$u_t + uu_x = 0, \qquad u(0, x) = \bar{u}(x), \tag{5.1}$$

and by $v(t, x)$ the solution to the perturbed linearized equation

$$v_t + uv_x = \mathbf{H}[u(t,\cdot)](x), \qquad v(0, x) = \bar{v}(x). \tag{5.2}$$

By the method of characteristics, at all points where u is continuous, one has

$$v(\tau, y) = \bar{v}\big(y - \tau u(\tau, y)\big) + \int_0^{\tau} \mathbf{H}[u(t,\cdot)](y - (\tau - t)\, u(\tau, y))\, dt. \tag{5.3}$$

We expect that v can provide a leading order correction term, in an *ansatz* describing the solution with two interacting shocks to the Burgers-Hilbert equation (1.1).

To fix the ideas, consider a piecewise constant solution to Burgers' equation containing two interacting shocks with initial data

$$u(0, x) = \bar{u}(x) = \begin{cases} u^\ell & \text{if } x < \bar{x}_1, \\ u^m & \text{if } \bar{x}_1 < x < \bar{x}_2, \\ u^r & \text{if } \bar{x}_2 < x. \end{cases} \tag{5.4}$$

with $u^\ell > u^m > u^r$. Setting

$$\begin{cases} \sigma_1 = u^\ell - u^m, \\ \sigma_2 = u^m - u^r, \end{cases} \qquad \begin{cases} a_1 = \dfrac{u^\ell + u^m}{2}, \\ a_2 = \dfrac{u^m + u^r}{2}, \end{cases}$$

$$x_1(t) \;=\; \bar x_1 + a_1 t, \qquad\qquad x_2(t) \;=\; \bar x_2 + a_2 t,$$

we thus have

$$u(t,x) \;=\; \begin{cases} u^\ell & \text{if } \; x < x_1(t)\,, \\ u^m & \text{if } \; x_1(t) < x < x_2(t)\,, \\ u^r & \text{if } \; x_2(t) < x\,. \end{cases} \tag{5.5}$$

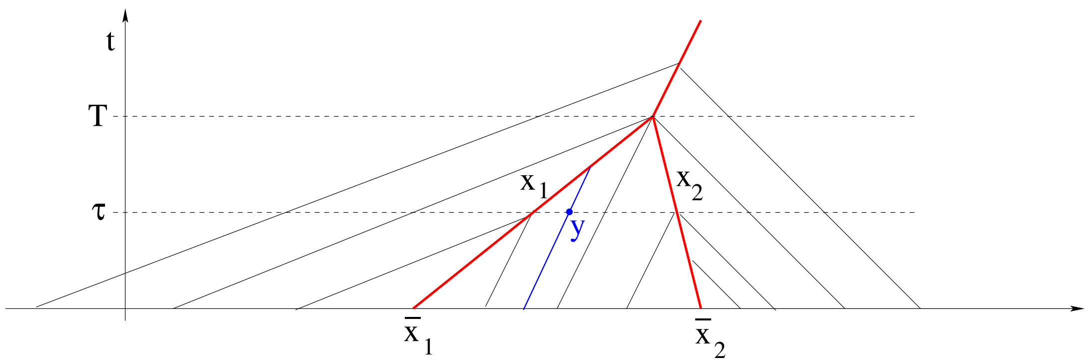

Figure 2: The characteristics for a solution to Burgers' equation with two shocks at $x_1(t) < x_2(t)$.

We now compute the corresponding solution $v = v(\tau, y)$ of (5.2). For this purpose, consider the characteristic through the point (τ, y), namely

$$x(t) \;=\; y + (t-\tau) u(\tau, y). \tag{5.6}$$

Recalling (1.12), we compute the integral

$$\begin{aligned} I(\tau,y) \;&\doteq\; \int_0^\tau \mathbf{H}[u(t,\cdot)](x(t))\,dt \;=\; -\frac{\sigma_1}{\pi}\int_0^\tau \ln|x_1(t)-x(t)|\,dt - \frac{\sigma_2}{\pi}\int_0^\tau \ln|x_2(t)-x(t)|\,dt \\ &=\; \frac{\sigma_1}{2|\dot x_1 - \dot x|}\Big(\varphi(|x_1(\tau)-y|) - \varphi\big(|x_1(0)-x(0)|\big)\Big) \\ &\qquad\qquad + \frac{\sigma_2}{2|\dot x_2 - \dot x|}\Big(\varphi(|x_2(\tau)-y|) - \varphi\big(|x_2(0)-x(0)|\big)\Big) + e(t,x). \end{aligned} \tag{5.7}$$

Here φ is given by (1.4), while $e = e(t,x)$ is an additional smooth correction term. Neglecting smooth terms, we thus consider three cases, depending on the location of the characteristic $x(t)$ w.r.t. the two shocks:

CASE 1: $y < x_1(\tau)$. We then have

$$I(\tau,y) \;\approx\; \frac{2}{\pi}\left[(x_1(\tau)-y)\ln(x_1(\tau)-y) + \frac{\sigma_2}{2\sigma_1+\sigma_2}(x_2(\tau)-y)\ln(x_2(\tau)-y)\right]. \tag{5.8}$$

CASE 2: $x_1(\tau) < y < x_2(\tau)$. We then have

$$I(\tau,y) \;\approx\; \frac{2}{\pi}\left[(y-x_1(\tau))\ln(y-x_1(\tau)) + (x_2(\tau)-y)\ln(x_2(\tau)-y)\right]. \tag{5.9}$$

CASE 3: $x_2(\tau) < y$. We then have

$$I(\tau, y) \approx \frac{2}{\pi}\left[\frac{\sigma_1}{\sigma_1+2\sigma_2}(y-x_1(\tau))\ln(y-x_1(\tau)) + (y-x_2(\tau))\ln(y-x_2(\tau))\right]. \tag{5.10}$$

Next, consider a more general piecewise smooth solution u of the Burgers–Hilbert equation (1.1) with two interacting shocks located at points $x_1(\tau) < x_2(\tau)$ with strengths

$$\begin{cases} \sigma_1(\tau) &= u\big(\tau, x_1(\tau)-\big) - u\big(\tau, x_1(\tau)+\big), \\ \sigma_2(\tau) &= u\big(\tau, x_2(\tau)-\big) - u\big(\tau, x_2(\tau)+\big), \end{cases} \tag{5.11}$$

respectively. As the interaction time T is approached, we expect that the two limits will coincide

$$\lim_{\tau\to T-} u(\tau, x_1(\tau)+) = \lim_{\tau\to T-} u(\tau, x_2(\tau)-).$$

Furthermore, as shown in Fig. 2, all characteristics located in the triangular region between the two shocks will hit one of them within time T.

To construct such solutions, we should thus try with an *ansatz* of the form

$$u(\tau, y) = w(\tau, y) + \phi(\tau, y), \tag{5.12}$$

where

$$w(\tau,\cdot) \in H^2\Big(]-\infty, x_1(\tau)[\ \cup\ ]x_1(\tau), x_2(\tau)[\ \cup\ ]x_2(\tau), +\infty[\Big). \tag{5.13}$$

Moreover, in view of (5.8)–(5.10), the correction ϕ should be defined as

$$\phi(\tau, y) = \begin{cases} \frac{2}{\pi}\left[(x_1(\tau)-y)\ln(x_1(\tau)-y) + \frac{\sigma_2(\tau)}{2\sigma_1(\tau)+\sigma_2(\tau)}(x_2(\tau)-y)\ln(x_2(\tau)-y)\right] & \text{if } y < x_1(\tau), \\ \frac{2}{\pi}\Big[(y-x_1(\tau))\ln(y-x_1(\tau)) + (x_2(\tau)-y)\ln(x_2(\tau)-y)\Big] & \text{if } x_1(\tau) < y < x_2(\tau), \\ \frac{2}{\pi}\left[\frac{\sigma_1(\tau)}{\sigma_1(\tau)+2\sigma_2(\tau)}(y-x_1(\tau))\ln(y-x_1(\tau)) + (y-x_2(\tau))\ln(y-x_2(\tau))\right] & \text{if } x_2(\tau) < y. \end{cases} \tag{5.14}$$

Note that, for each fixed time $\tau < T$, since $x_1(\tau) < x_2(\tau)$, for $y < x_1(\tau)$, the term $\ln(x_2(\tau)-y)$ remains smooth. The same is true for the term $\ln(y-x_1(\tau))$ in the region where $y > x_2(\tau)$. As a consequence, the asymptotic profile of the function $\phi(\tau,\cdot)$ near both points $x_1(\tau)$ and $x_2(\tau)$ has the same "$x\ln|x|$" singularity that we encountered before. However, these two additional terms cannot be removed from the definition of ϕ, because they are not uniformly smooth as $\tau \to T-$.

6 Constructing a solution with two interacting shocks

We consider here a solution of the Burgers-Hilbert equation (1.1), which is piecewise continuous and which has two shocks located at the points $y_1(t) < y_2(t)$. By the Rankine-Hugoniot conditions, the time derivatives satisfy

$$\dot{y}_i(t) = \frac{u_i^-(t) + u_i^+(t)}{2}, \qquad i = 1, 2. \tag{6.1}$$

Here $u_i^\pm(t) \doteq u_i\big(t, y_i(t)\pm\big)$ denote the left and the right limits of $u(t,x)$ as $x \to y_i(t)$. Throughout the following, we assume that

$$\frac{u_1^-(t)+u_1^+(t)}{2} ~=~ \dot y_1(t) ~>~ \dot y_2(t) ~=~ \frac{u_2^-(t)+u_2^+(t)}{2}\,.$$

The function $\tau \doteq y_1 - y_2$ is negative and monotone increasing. It will be useful to change the space and the time variables, so that in the new variables $\tilde t, \tilde x$ the location of one shock is fixed, while the other moves with constant speed 1. For this purpose, we set

$$\tilde x ~\doteq~ x - y_2(t), \qquad\qquad \tilde t ~\doteq~ \tau(t) ~<~ 0\,.$$

As a consequence, the two shocks, in the new coordinate system, are located at

$$y_1(\tilde t) ~=~ \tilde t, \qquad\qquad y_2(\tilde t) = 0,$$

and interact at the point $(\tilde t, \tilde x) = (0,0)$. Introducing the function

$$v\big(\tau(t), x\big) ~=~ u\big(t, x + y_2(t)\big), \tag{6.2}$$

we define the left and right values

$$v_1^\pm\big(\tau(t)\big) ~\doteq~ v\big(\tau(t), \tau(t)\pm\big) ~=~ u\big(t, y_1(t)\pm\big), \qquad v_2^\pm\big(\tau(t)\big) ~\doteq~ v\big(\tau(t), 0\pm\big) ~=~ u\big(t, y_2(t)\pm\big). \tag{6.3}$$

The change of variables (6.2) yields

$$v_x(\tau,x) ~=~ u_x\big(t, x+y_2(t)\big), \qquad v_\tau(\tau,x) ~=~ \frac{u_t\big(t,x+y_2(t)\big) + \dot y_2(t)\cdot u_x\big(t, x+y_2(t)\big)}{\dot y_1(t) - \dot y_2(t)}.$$

Therefore, (1.1) implies

$$v_\tau\big(\tau(t),x\big) + \frac{\big[v\big(\tau(t),x\big) - \dot y_2(t)\big]\cdot v_x\big(\tau(t),x\big)}{\dot y_1(t) - \dot y_2(t)} ~=~ \frac{\mathbf{H}[v\big(\tau(t),\cdot\big)](x)}{\dot y_1(t) - \dot y_2(t)}\,. \tag{6.4}$$

Thus, by (6.1), (6.3), and (6.4), we can recast the original equation (1.1) in the following equivalent form

$$u_t + \frac{1}{a_1(t) - a_2(t)}\cdot \big[u - a_2(t)\big]\cdot u_x ~=~ \frac{\mathbf{H}[u]}{a_1(t)-a_2(t)}\,. \tag{6.5}$$

Given $\tau_0 < 0$, for $t \in [\tau_0, 0]$ the two functions

$$a_1(t) ~\doteq~ \frac{u_1^-(t)+u_1^+(t)}{2}, \qquad\qquad a_2(t) ~\doteq~ \frac{u_2^-(t)+u_2^+(t)}{2}, \tag{6.6}$$

yield the speeds of the two shocks in the original coordinates, as shown in Fig. 3.

We shall construct the solution of (6.5) in the form

$$u(t,x) ~=~ w(t,x) + \varphi(t,x) \qquad \text{for all } (t,x) \in [\tau_0, 0]~\times \mathbb{R}\,. \tag{6.7}$$

Here φ is a continuous function, which satisfies $\varphi(t,t) = \varphi(t,0) = 0$, while

$$w(t,\cdot) ~\in~ H^2\Big(\,]-\infty, t[\;\cup\;]t,0[\;\cup\;]0,+\infty[\,\Big) \qquad \text{for all } t \in [\tau_0, 0]. \tag{6.8}$$

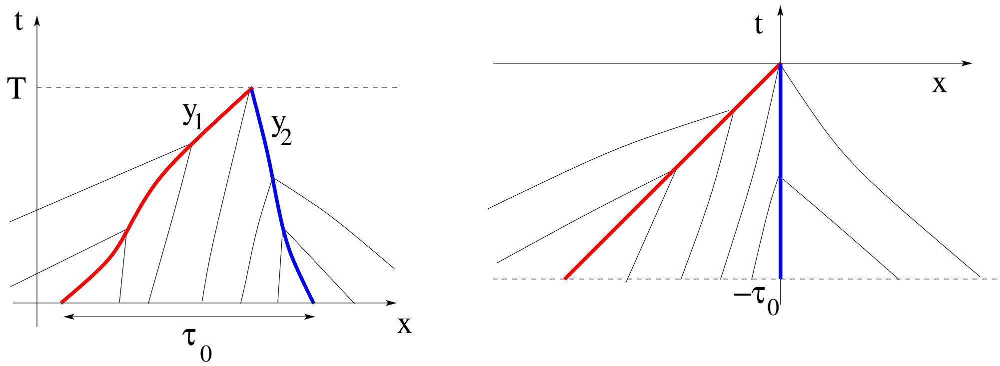

Figure 3: Positions of the two shocks in the original variables (left), and in the adapted variables (right).

According to (6.8), the function $w(t,\cdot)$ is continuously differentiable outside the two points $x=t$ and $x=0$. Moreover, the distributional derivative $D_x w(t,\cdot)$ is an $\mathbf{L}^2$ function restricted to each interval $]-\infty,t[\,,\ ]t,0[\ $ and $]0,+\infty[\,$. However, both $w(t,\cdot)$ and $w_x(t,\cdot)$ can have a jump at $x=t$ and at $x=0$. At the points (t,t) and $(t,0)$, the following traces are well defined:

$$\left\{\begin{array}{rcl} w_1^-(t) &\doteq& w(t,t-) \;=\; u_1^-(t),\\[2mm] w_1^+(t) &\doteq& w(t,t+) \;=\; u_1^+(t),\end{array}\right. \qquad \left\{\begin{array}{rcl} b_1^-(t) &\doteq& w_x(t,t-),\\[2mm] b_1^+(t) &\doteq& w_x(t,t+),\end{array}\right. \tag{6.9}$$

$$\left\{\begin{array}{rcl} w_2^-(t) &\doteq& w(t,0-) \;=\; u_2^-(t),\\[2mm] w_2^+(t) &\doteq& w(t,0+) \;=\; u_2^+(t),\end{array}\right. \qquad \left\{\begin{array}{rcl} b_2^-(t) &\doteq& w_x(t,0-),\\[2mm] b_2^+(t) &\doteq& w_x(t,0+).\end{array}\right. \tag{6.10}$$

For the shocks to be entropy admissible, the inequalities

$$w_1^-(t) \;>\; w_1^+(t), \qquad\qquad w_2^-(t) \;>\; w_2^+(t), \tag{6.11}$$

will always be assumed. Writing

$$a_1^{(w)}(t) \;\doteq\; \frac{w_1^-(t)+w_1^+(t)}{2} \qquad\qquad a_2^{(w)}(t) \;\doteq\; \frac{w_2^-(t)+w_2^+(t)}{2}\,, \tag{6.12}$$

the equation (6.5) reads

$$w_t + a(t,x,w)\cdot w_x \;=\; F(t,x,w), \tag{6.13}$$

where a and F are given by

$$a(t,x,w) \;=\; \frac{w(t,x)+\varphi^{(w)}(t,x)-a_2^{(w)}(t)}{a_1^{(w)}(t)-a_2^{(w)}(t)}\,, \tag{6.14}$$

$$F(t,x,w) \;=\; \left[\frac{\mathbf{H}[w]-(w-a_2^{(w)}(t))\cdot\varphi_x^{(w)}}{a_1^{(w)}(t)-a_2^{(w)}(t)} - \varphi_t^{(w)}\right] + \frac{\mathbf{H}[\varphi^{(w)}]-\varphi^{(w)}\varphi_x^{(w)}}{a_1^{(w)}(t)-a_2^{(w)}(t)}, \tag{6.15}$$

respectively. Here the function $\varphi^{(w)}(t,x)$ is chosen in such a way that a cancellation between leading order terms near to the location of the two shocks at $x=t$ and at $x=0$ is achieved. More precisely, in view of (5.14) and recalling (2.3) and (2.4), we set

$$\phi_0(x) \;=\; \phi(x,0) \;=\; \frac{2\eta(x)}{\pi}\cdot|x|\ln|x|, \tag{6.16}$$

and define

$$\varphi^{(w)}(t,x) \;=\; \begin{cases} \phi_0(x-t) + \dfrac{\sigma_2^{(w)}(t)}{2\sigma_1^{(w)}(t)+\sigma_2^{(w)}(t)}\cdot\big(\phi_0(x)-\phi_0(t)\big) & \text{if } \ x<t, \\ \phi_0(x-t)+\phi_0(x)-\phi_0(t) & \text{if } \ t<x<0, \\ \dfrac{\sigma_1^{(w)}(t)}{\sigma_1^{(w)}(t)+2\sigma_2^{(w)}(t)}\cdot\big(\phi_0(x-t)-\phi_0(t)\big)+\phi_0(x) & \text{if } \ 0<x\,. \end{cases} \tag{6.17}$$

The following theorem provides the existence of solutions to the Cauchy problem for (1.1) where the initial datum contains two shocks. In particular, the solution to (6.5) is constructed up to the time where the two shocks interact. Furthermore, the solution is of the form (6.7), where $\varphi = \varphi^{(w)}$, the corrector function defined in (6.17).

Theorem 6.1 *For any given constants $b, M_0, \delta_1, \delta_2 > 0$, there exists $\varepsilon_0 > 0$ small enough and a constant K such that the following holds.*

Consider any $\tau_0 \in [-\varepsilon_0, 0[$ and any initial condition $\overline{w} \in H^2(\mathbb{R}\backslash\{\tau_0, 0\})$ such that

$$\begin{cases} \|\overline{w}\|_{H^2(\mathbb{R}\backslash\{\tau_0,0\})} \;\le\; \dfrac{M_0}{4}, & \|\overline{w}_x\|_{\mathbf{L}^\infty(]\tau_0,0[)} \;\le\; b, \\ \overline{w}(\tau_0-)-\overline{w}(\tau_0+) \;\ge\; 8\delta_1, & \overline{w}(0-)-\overline{w}(0+) \;\ge\; 8\delta_2. \end{cases} \tag{6.18}$$

Then the Cauchy problem (6.13) with initial data

$$w(\tau_0,\cdot) \;=\; \overline{w} \;\in\; H^2\big(\mathbb{R}\setminus\{\tau_0,0\}\big) \tag{6.19}$$

admits a unique entropic solution, defined for $t \in [\tau_0, 0]$. Moreover, this solution satisfies

$$\begin{cases} \|w(t,\cdot)\|_{H^2(\mathbb{R}\backslash\{t,0\})} \;\le\; M_0, & \|w_x(t,\cdot)\|_{\mathbf{L}^\infty(]\tau_0,0[)} \;\le\; Kb, \\ w(t,t-)-w(t,t+) \;\ge\; \delta_1, & w(t,0-)-w(t,0+) \;\ge\; \delta_2 \end{cases} \tag{6.20}$$

for all $t \in [\tau_0, 0[$.

Remark 6.1 By (6.20), at the interaction time $t = 0$ the solution $u = w + \varphi^{(w)}$ is the sum of a corrector term plus a function in $H^2\big(\mathbb{R}\setminus\{0\}\big)$. This function lies within the class of initial data covered by our earlier Theorem 2.1. Thus, combining Theorems 6.1 and 2.1 yields the behavior of a solution to (1.1) across the interaction of two shocks.

Toward a proof of Theorem 6.1, solutions to (6.13) will be constructed by an iteration procedure. The main difference between this and the earlier case with a single shock is that the correction term φ now depends on time through the variable strengths σ_1, σ_2 of the two shocks. Define

$$w^{(1)}(t,x) \;=\; \begin{cases} \overline{w}(x) & \text{if } \ x \in]t,0[\cup]0,\infty[, \\ \overline{w}(x+\tau_0-t) & \text{if } \ x\in]-\infty,t[. \end{cases} \tag{6.21}$$

By induction, let $w^{(n)}$ be given and satisfy (6.8) for every $t \in [\tau_0, 0[$. Moreover, call $\sigma_1^{(n)}(t)$ and $\sigma_2^{(n)}(t)$ the strengths of the two shocks at $x = t$ and $x = 0$ of $w^{(n)}$, respectively. We construct the next iterate $w = w^{(n+1)}(t,x)$ by solving the linear equation

$$w_t + a\left(t, x, w^{(n)}\right) w_x \;=\; F(t,x,w), \tag{6.22}$$

with initial data (6.19) and a as introduced in (6.14).

The induction argument requires the following steps:

(i) Given $w^{(n)}$, the equation (6.22) with the initial data $\overline{w}$ admits a unique solution w with $w(t,\cdot) \in H^2\big(\mathbb{R} \setminus \{t,0\}\big)$ for all $t \in [\tau_0, 0[$.

(ii) A priori bounds on the strong norm $\|w^{(n)}(t,\cdot)\|_{H^2\left(\mathbb{R}\setminus\{t,0\}\right)}$ for all $t \in [\tau_0, 0[$, $n \geq 1$.

(iii) Convergence in a weak norm. This will follow from the bound

$$\sum_{n\geq 1} \|w^{(n+1)}(t,\cdot) - w^{(n)}(t,\cdot)\|_{H^2\left(\mathbb{R}\setminus\{t,0\}\right)} \;<\; +\infty.$$

6.1 Some preliminary estimates

To achieve the above steps (i)-(iii), we first establish some key estimates on the right hand side of (6.13). For any $w : [\tau_0, 0] \times \mathbb{R} \to \mathbb{R}$ such that $w(t,\cdot) \in H^2(\mathbb{R}\setminus\{t,0\})$ for all $t \in [\tau_0, 0]$, we write

$$w(t,\cdot) \;=\; v_1(t,\cdot) + v_2(t,\cdot) \tag{6.23}$$

with

$$v_2(t,x) \;\doteq\; \begin{cases} \big(w(t,0-) + x \cdot w_x(t,0-)\big) \cdot \eta(x) & \text{if} \quad x < 0\,, \\ \big(w(t,0+) + x \cdot w_x(t,0+)\big) \cdot \eta(x) & \text{if} \quad x > 0. \end{cases} \tag{6.24}$$

Recalling (6.9)-(6.12) and (6.15), we split F into the four parts:

$$F(t,x,w) \;=\; \frac{1}{a_1^{(w)}(t) - a_2^{(w)}(t)} \cdot \Big[A^{(w)}(t,x) + B^{(w)}(t,x) + C^{(w)}(t,x) + D^{(w)}(t,x)\Big]. \tag{6.25}$$

Here we take

$$A^{(w)} \;\doteq\; \mathbf{H}\left[\varphi^{(w)}\right] - \varphi^{(w)}\varphi_x^{(w)}, \qquad B^{(w)} \;\doteq\; \mathbf{H}[v_2] - \left(v_2 - \frac{v_2(t,0-) + v_2(t,0+)}{2}\right) \cdot \phi_0'(x)\,, \tag{6.26}$$

$$C^{(w)} \;\doteq\; \mathbf{H}[v_1] - \left(v_1 - \frac{v_1(t,t-) + v_1(t,t+)}{2}\right) \cdot \phi_0'(x-t), \tag{6.27}$$

$$D^{(w)} \;\doteq\; \mathbf{H}[w] - \left(w - a_2^{(w)}(t)\right) \cdot \varphi_x^{(w)} - \left(a_1^{(w)}(t) - a_2^{(w)}(t)\right) \cdot \varphi_t^{(w)} - B^{(w)} - C^{(w)}. \tag{6.28}$$

Lemma 6.1 *Let $w : [\tau_0, 0] \times \mathbb{R} \to \mathbb{R}$ be such that $w(t,\cdot) \in H^2(\mathbb{R}\setminus\{t,0\})$ for all $t \in [\tau_0, 0]$, and*

$$\|w(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{t,0\})} \;\leq\; M_0, \qquad |w_x(t,0-)| + |w_x(t,0+)| \;\leq\; b, \qquad \sigma_j^{(w)}(t) \;\geq\; \delta_0 \;>\; 0.$$

Moreover, assume that $-\frac{1}{4}\cdot\min\left\{1,\frac{\delta_0^2}{M_0^2}\right\} < \tau_0 < 0$ *and* $\sigma_j^{(w)}(\cdot)$ *is locally Lipschitz on* $[\tau_0,0[$ *for* $j=1,2$. *Then there is a constant* $C_1>0$, *depending only on* M_0,b,δ_0 *such that, for a.e.* $t\in[\tau_0,0]$ *and* $|x|<\frac{1}{2e}$, *one has*

$$\begin{cases} |F(t,x,w)| \;\le\; C_1\cdot\left[\dfrac{1+M_0+b}{\delta_0}+\big|\ln|t|\big| \right.\\ \qquad\qquad\left. +\;\chi_{\mathbb{R}\setminus]t,0[}\cdot\dfrac{\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0}\cdot\Big(|x|^{1/2}+|x-t|^{1/2}\Big)\right],\\ |F_x(t,x,w)| \;\le\; C_1\cdot\dfrac{1+M_0+b+\chi_{\mathbb{R}\setminus]t,0[}\cdot\Big(\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|\Big)}{\delta_0}\cdot\Big(|x|^{-1/2}+|x-t|^{-1/2}\Big). \end{cases}\tag{6.29}$$

Furthermore, for every $\delta>0$ *sufficiently small one has for all* $t\in[\tau_0,0]$

$$\|F(t,x,w)\|_{H^2(\mathbb{R}\setminus[-\delta,\delta]\cup[t-\delta,t+\delta])} \;\le\; C_1\cdot\left[\frac{1+M_0+b+\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0}\cdot\delta^{-2/3}+|\ln(t)|\right].\tag{6.30}$$

Proof. We observe that, for all $t\in[\tau_0,0]$, it holds

$$a_1^{(w)}(t)-a_2^{(w)}(t) \;\ge\; \delta_0+w(t,t+)-w(t,0-) \;\ge\; \delta_0-M_0\cdot|t|^{1/2} \;\ge\; \frac{\delta_0}{2},\tag{6.31}$$

$$0<\sigma_j^{(w)}(t)<\sigma_1^{(w)}(t)+\sigma_2^{(w)}(t)=w(t,t-)-w(t,t+)+w(t,0-)-w(t,0+)\le(2+\sqrt{|t|})M_0,$$
$$\|v_1(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{t\})},\|v_2(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})} \;\le\; \mathcal{O}(1)\cdot(M_0+b).$$

According to (6.25), the function F can be decomposed as the sum of four terms, which will be estimated separately.

1. Recalling (3.2) and (3.10), (6.26) and (6.27) imply that for every $(t,x)\in[\tau_0,0]\times\left[-\frac{1}{2e},\frac{1}{2e}\right]$ one has

$$\begin{cases} \left|B^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot(M_0+b), \qquad \left|B_x^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot(M_0+b)\cdot\big|\ln|x|\big|^2,\\ \left\|B^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\le\; \mathcal{O}(1)\cdot(M_0+b)\cdot\delta^{-2/3}. \end{cases}\tag{6.32}$$

and, for every $(t,x)\in[\tau_0,0]\times\left[t-\frac{1}{2e},t+\frac{1}{2e}\right]$

$$\begin{cases} \left|C^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot(M_0+b), \qquad \left|C_x^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot(M_0+b)\cdot\big|\ln|x-t|\big|^2,\\ \left\|C^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[t-\delta,t+\delta])} \;\le\; \mathcal{O}(1)\cdot(M_0+b)\cdot\delta^{-2/3}. \end{cases}\tag{6.33}$$

2. Next, we estimate $A^{(w)}$. Recalling (3.4), i.e.,

$$g_b(x) \;=\; \chi_{[0,\infty[}(x)\cdot\phi(x,b), \qquad x\in\mathbb{R}, b\ge 0,$$

(2.4), and (6.16), we can rewrite, for $(t,x)\in[\tau_0,0]\times\left[-\frac{1}{2e},\frac{1}{2e}\right]$,

$$\varphi^{(w)}(t,x) \;=\; \phi_0(x-t)+\phi_0(x)-\phi_0(t)+E_1^{(w)}(t,x)+E_2^{(w)}(t,x), \tag{6.34}$$

$$\begin{cases} E_1^{(w)}(t,x) &=& -\dfrac{2\cdot\chi_{]-\infty,t]}\cdot\sigma_1^{(w)}(t)}{2\sigma_1^{(w)}(t)+\sigma_2^{(w)}(t)}\cdot\left[\phi_0(x)-\phi_0(t)\right] &=& -\dfrac{2\sigma_1^{(w)}(t)}{2\sigma_1^{(w)}(t)+\sigma_2^{(w)}(t)}\cdot g_{|t|}(t-x), \\ E_2^{(w)}(t,x) &=& -\dfrac{2\cdot\chi_{[0,\infty[}\cdot\sigma_2^{(w)}(t)}{\sigma_1^{(w)}(t)+2\sigma_2^{(w)}(t)}\cdot\left[\phi_0(x-t)-\phi_0(t)\right] &=& -\dfrac{2\sigma_2^{(w)}(t)}{\sigma_1^{(w)}(t)+2\sigma_2^{(w)}(t)}\cdot g_{|t|}(x). \end{cases} \tag{6.35}$$

Thus, for $(t,x)\in[\tau_0,0]\times\left[-\frac{1}{2e},\frac{1}{2e}\right]$, Lemma 3.1 and [4, Section 3] imply,

$$\begin{cases} \left|\mathbf{H}\left[\varphi^{(w)}(t,\cdot)\right](x)\right| \;\leq\; \mathcal{O}(1), \qquad \left|\dfrac{d}{dx}\mathbf{H}\left[\varphi^{(w)}(t,\cdot)\right](x)\right| \leq\; \mathcal{O}(1)\cdot\left(\ln^2|x|+\ln^2|x-t|\right), \\ \left\|\mathbf{H}\left[\varphi^{(w)}(t,\cdot)\right]\right\|_{H^2(\mathbb{R}\setminus([-\delta,\delta])\cup[t-\delta,t+\delta])} \;\leq\; \mathcal{O}(1)\cdot\delta^{-2/3}. \end{cases} \tag{6.36}$$

On the other hand, given $t\in[\tau_0,0]$, for every $|x|\leq\frac{1}{2e}$, (6.34) combined with (3.5) yields

$$\left|\varphi^{(w)}(t,x)\right|\leq\min(|x\ln(|x|)|,|(x-t)\ln(|x-t|)|).$$

Furthermore, we compute for every $x\in\mathbb{R}\setminus\{t,0\}$

$$\varphi_x^{(w)}(t,x) \;=\; \left(1-\frac{2\cdot\chi_{[0,\infty[}\cdot\sigma_2^{(w)}(t)}{\sigma_1^{(w)}(t)+2\sigma_2^{(w)}(t)}\right)\phi_0'(x-t)+\left(1-\frac{2\cdot\chi_{]-\infty,t]}\cdot\sigma_1^{(w)}(t)}{2\sigma_1^{(w)}(t)+\sigma_2^{(w)}(t)}\right)\cdot\phi_0'(x), \tag{6.37}$$

which together with [4, Section 3] implies for $|x|\leq\frac{1}{2e}$ that

$$\left|\varphi_x^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1)\cdot\left(\big|\ln|x|\big|+\big|\ln|x-t|\big|\right), \qquad \left|\varphi_{xx}^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1)\cdot\left(\frac{1}{|x|}+\frac{1}{|x-t|}\right),$$

$$\left|\varphi_{xxx}^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1)\cdot\left(\frac{1}{x^2}+\frac{1}{(x-t)^2}\right).$$

A direct computation yields, for $|x|\leq\frac{1}{2e}$,

$$\begin{cases} \left|\varphi^{(w)}\varphi_x^{(w)}(t,x)\right| &\leq\; \mathcal{O}(1)\cdot\left(|x|^{1/2}+|x-t|^{1/2}\right), \\ \left|\dfrac{d}{dx}[\varphi^{(w)}\varphi_x^{(w)}](t,x)\right| &\leq\; \mathcal{O}(1)\cdot\left(\ln^2|x|+\ln^2|x-t|\right), \\ \left|\dfrac{d^2}{dx^2}[\varphi^{(w)}\varphi_x^{(w)}](t,x)\right| &\leq\; \mathcal{O}(1)\cdot\left(\left|\dfrac{\ln|x|}{x}\right|+\left|\dfrac{\ln|x-t|}{x-t}\right|\right). \end{cases}$$

and thus

$$\left\|\varphi^{(w)}\varphi_x^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[t-\delta,t+\delta]\cup[-\delta,\delta])} \;\leq\; \mathcal{O}(1)\cdot\delta^{-2/3}.$$

Recalling (6.36), we get

$$\begin{cases} \left|A^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1), \qquad \left|A_x^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1)\cdot\left(\ln^2|x|+\ln^2|x-t|\right), \\ \left\|A^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[t-\delta,t+\delta]\cup(-\delta,\delta))} \;\leq\; \mathcal{O}(1)\cdot\delta^{-2/3}. \end{cases} \tag{6.38}$$

3. Finally, to estimate $D^{(w)}$, we shall consider three cases:

Case 1: Assume that $-\frac{1}{2} < t - \frac{1}{2e} < x < t$. We have

$$D^{(w)}(t,x) \;=\; D_1^{(w)}(t,x) + D_2^{(w)}(t,x) + D_3^{(w)}(t,x) \tag{6.39}$$

with

$$\begin{cases} D_1^{(w)}(t,x) \;=\; w_x(t,0-)\cdot(t-x)\cdot\phi_0'(x-t), \\ D_2^{(w)}(t,x) \;=\; \left(\dfrac{\sigma_2^{(w)}(t)\cdot[w(t,0-)-w(t,t+)+w(t,t-)-w(t,x)]}{2\sigma_1^{(w)}(t)+\sigma_2^{(w)}(t)} + xw_x(t,0-)\right)\cdot\phi_0'(x), \\ D_3^{(w)}(t,x) \;=\; \left[a_2^{(w)}(t)-a_1^{(w)}(t)\right]\cdot\left[\left(\dfrac{\sigma_2^{(w)}(t)}{2\sigma_1^{(w)}(t)+\sigma_2^{(w)}(t)}\right)'\cdot g_{|t|}(t-x) - \dfrac{\sigma_2^{(w)}(t)\cdot\phi_0'(t)}{2\sigma_1^{(w)}(t)+\sigma_2^{(w)}(t)}\right]. \end{cases}$$

Recalling (6.16), we estimate

$$\begin{cases} \left|D_1^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot b\cdot|x-t|\big|\ln|x-t|\big| \;\le\; O(1)\cdot b, \\ \left|\dfrac{d}{dx}D_1^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot b\cdot\big|\ln|x-t|\big|, \qquad \left|\dfrac{d^2}{d^2x}D_1^{(w)}(t,x)\right| \;\le\; O(1)\cdot\dfrac{b}{|x-t|}, \end{cases} \tag{6.40}$$

$$\begin{cases} \left|D_2^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot(M_0+b)\cdot x\big|\ln|x|\big|, \quad \left|\dfrac{d}{dx}D_2^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot(M_0+b)\cdot\big|\ln|x|\big|, \\ \left|\dfrac{d^2}{dx^2}D_2^{(w)}(t,x)\right| \;\le\; |w_{xx}\cdot\phi_0'(x)| + \mathcal{O}(1)\cdot\dfrac{M_0+b}{|x|}, \end{cases} \tag{6.41}$$

and

$$\begin{cases} \left|D_3^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot\left|a_2^{(w)}(t)-a_1^{(w)}(t)\right|\cdot\left(\dfrac{\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0}|x-t|\big|\ln|x-t|\big| + \big|\ln|t|\big|\right) \\ \left|\dfrac{d}{dx}D_3^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot\left|a_2^{(w)}(t)-a_1^{(w)}(t)\right|\cdot\dfrac{\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0}\cdot\big|\ln|x-t|\big|, \\ \left|\dfrac{d^2}{dx^2}D_3^{(w)}(t,x)\right| \;\le\; \mathcal{O}(1)\cdot\left|a_2^{(w)}(t)-a_1^{(w)}(t)\right|\cdot\dfrac{\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0}\cdot\dfrac{1}{|x-t|}. \end{cases} \tag{6.42}$$

Combining (6.39)-(6.42), we obtain

$$\begin{cases} \dfrac{\left|D^{(w)}(t,x)\right|}{\left|a_2^{(w)}(t)-a_1^{(w)}(t)\right|} \;\le\; \mathcal{O}(1)\cdot\left(\dfrac{M_0+b}{\delta_0} + |\ln(t)| + \dfrac{\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0}\cdot|x-t|^{1/2}\right), \\ \dfrac{\left|\frac{d}{dx}D^{(w)}(t,x)\right|}{\left|a_2^{(w)}(t)-a_1^{(w)}(t)\right|} \;\le\; \mathcal{O}(1)\cdot\dfrac{M_0+b+\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0}\cdot\big|\ln|x-t|\big|, \\ \dfrac{\left|\frac{d^2}{dx^2}D^{(w)}(t,x)\right|}{\left|a_2^{(w)}(t)-a_1^{(w)}(t)\right|} \;\le\; \mathcal{O}(1)\cdot\left(\dfrac{M_0+b+\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0\cdot|x-t|} + \dfrac{|w_{xx}|}{\delta_0}\cdot\big|\ln|x-t|\big|\right). \end{cases} \tag{6.43}$$

Case 2: Assume that $t < x < 0$. We have

$$D^{(w)}(t,x) \;=\; w_x(t,0-)\cdot(t-x)\cdot\phi_0'(x-t)+(v_1(t,0)-v_1(t,x))\cdot\phi_0'(x)+\big[a_1^{(w)}(t)-a_2^{(w)}(t)\big]\cdot\phi_0'(t), \tag{6.44}$$

and this yields

$$\begin{cases} \dfrac{\big|D^{(w)}(t,x)\big|}{\Big|a_2^{(w)}(t)-a_1^{(w)}(t)\Big|} \;\leq\; \mathcal{O}(1)\left(\dfrac{M_0+b}{\delta_0}+\big|\ln|t|\big|\right), \\ \dfrac{\left|\dfrac{d}{dx}D^{(w)}(t,x)\right|}{\Big|a_2^{(w)}(t)-a_1^{(w)}(t)\Big|} \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_0+b}{\delta_0}\cdot\left(\big|\ln|x|\big|+\big|\ln|x-t|\big|\right), \\ \dfrac{\left|\dfrac{d^2}{dx^2}D^{(w)}(t,x)\right|}{\Big|a_2^{(w)}(t)-a_1^{(w)}(t)\Big|} \;\leq\; \mathcal{O}(1)\cdot\left(\dfrac{M_0+b}{\delta_0}\cdot\left(\dfrac{1}{|x|}+\dfrac{1}{|x-t|}\right)+\dfrac{|w_{xx}(t,x)|}{\delta_0}\cdot\big|\ln|x|\big|\right). \end{cases} \tag{6.45}$$

Case 3: Assume that $0 < x < 1/2$. As in Case 1, writing

$$D^{(w)}(t,x) \;=\; D_1^{(w)}(t,x)+D_2^{(w)}(t,x)+D_3^{(w)}(t,x) \tag{6.46}$$

with

$$\begin{cases} D_1^{(w)}(t,x) \;=\; [v_1(t,x)-v_1(t,t+)]\cdot\phi_0'(x-t) \\ \qquad\qquad +\; \left[\dfrac{\sigma_1^{(w)}(t)}{\sigma_1^{(w)}(t)+2\sigma_2^{(w)}(t)}\cdot(w(t,t+)-w(t,0-)+w(t,0+)-w(t,x))\right]\cdot\phi_0'(x-t), \\ D_2^{(w)}(t,x) \;=\; (w(t,0+)-w(t,x)+xw_x(t,0+))\cdot\phi_0'(x), \\ D_3^{(w)}(t,x) \;=\; \big[a_2^{(w)}(t)-a_1^{(w)}(t)\big]\cdot\left[\left(\dfrac{\sigma_1^{(w)}(t)}{\sigma_1^{(w)}(t)+2\sigma_2^{(w)}(t)}\right)'\cdot g_{|t|}(x)-\dfrac{\sigma_1^{(w)}(t)\cdot\phi_0'(t)}{\sigma_1^{(w)}(t)+2\sigma_2^{(w)}(t)}\right], \end{cases}$$

we estimate

$$\begin{cases} \dfrac{\big|D^{(w)}(t,x)\big|}{\Big|a_2^{(w)}(t)-a_1^{(w)}(t)\Big|} \;\leq\; \mathcal{O}(1)\cdot\left(\dfrac{M_0+b}{\delta_0}+|\ln(t)|+\dfrac{\Big|\dot\sigma_1^{(w)}(t)\Big|+\Big|\dot\sigma_2^{(w)}(t)\Big|}{\delta_0}\cdot|x|^{1/2}\right), \\ \dfrac{\left|\dfrac{d}{dx}D^{(w)}(t,x)\right|}{\Big|a_2^{(w)}(t)-a_1^{(w)}(t)\Big|} \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_0+b+\Big|\dot\sigma_1^{(w)}(t)\Big|+\Big|\dot\sigma_2^{(w)}(t)\Big|}{\delta_0}\cdot\big|\ln|x|\big|, \\ \dfrac{\left|\dfrac{d^2}{dx^2}D^{(w)}(t,x)\right|}{\Big|a_2^{(w)}(t)-a_1^{(w)}(t)\Big|} \;\leq\; \mathcal{O}(1)\cdot\left(\dfrac{M_0+b+\Big|\dot\sigma_1^{(w)}(t)\Big|+\Big|\dot\sigma_2^{(w)}(t)\Big|}{\delta_0}\cdot\dfrac{1}{|x|}+\dfrac{|w_{xx}|}{\delta_0}\cdot\big|\ln|x|\big|\right). \end{cases} \tag{6.47}$$

In summary, from (6.43), (6.45), and (6.47), given $t \in [\tau_0, 0]$, for every $x \in (-1/2, 1/2)\backslash\{t, 0\}$, it holds that

$$\begin{cases} \dfrac{\left|D^{(w)}(t,x)\right|}{\left|a_2^{(w)}(t)-a_1^{(w)}(t)\right|} \ \le\ \mathcal{O}(1)\cdot\Bigg(\dfrac{M_0+b}{\delta_0}+|\ln(t)| \\ \qquad\qquad\qquad + \chi_{\mathbb{R}\backslash]0,t[}\cdot\dfrac{\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0}\cdot\Big(|x|^{1/2}+|x-t|^{1/2}\Big)\Bigg), \\ \dfrac{\left|\dfrac{d}{dx}D^{(w)}(t,x)\right|}{\left|a_2^{(w)}(t)-a_1^{(w)}(t)\right|} \le \mathcal{O}(1)\cdot\dfrac{M_0+b+\chi_{\mathbb{R}\backslash]t,0[}\left(\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|\right)}{\delta_0}\cdot\big(\big|\ln|x|\big|+\big|\ln|x-t|\big|\big), \\ \dfrac{\left\|D^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash[t-\delta,t+\delta]\cup(-\delta,\delta))}}{\left|a_2^{(w)}(t)-a_1^{(w)}(t)\right|} \ \le\ \mathcal{O}(1)\cdot\left[\dfrac{M_0+b+\left|\dot\sigma_1^{(w)}(t)\right|+\left|\dot\sigma_2^{(w)}(t)\right|}{\delta_0\delta^{2/3}}+\big|\ln|t|\big|\right]. \end{cases} \tag{6.48}$$

To complete the proof, combining (6.32), (6.33), (6.38) and (6.48), we obtain (6.29)-(6.30). □

The next lemma estimates the change in the function $F = F(t,x,w)$ as $w(\cdot)$ takes different values. These estimates will play a key role in the proof of convergence of the approximations inductively defined by (6.22).

Lemma 6.2 *Let $w_1, w_2 : [\tau_0, 0]\times\mathbb{R}\to\mathbb{R}$ be such that, for $i\in\{1,2\}$ and $t\in[\tau_0,0]$, one has $w_i(t,\cdot)\in H^2(\mathbb{R}\backslash\{t,0\})$ and*

$$\|w_i(t,\cdot)\|_{H^2(\mathbb{R}\backslash\{t,0\})}\ \le\ M_0, \qquad |w_{i,x}(t,0-)|+|w_{i,x}(t,0+)|\ \le\ b, \qquad \left|\sigma_j^{(w_i)}(t)\right|\ \ge\ \delta_0.$$

Moreover, assume that $-\frac{1}{4}\cdot\min\left\{1,\frac{\delta_0^2}{M_0^2}\right\}<\tau_0<0$ and $\sigma_j^{(w_i)}$ is locally Lipschitz on $[\tau_0,0[$ and there exists a function $K(t)$ such that

$$\max\left\{\left|\dot\sigma_1^{(w_i)}(t)\right|,\left|\dot\sigma_2^{(w_i)}(t)\right|\right\}\ \le\ K(t) \qquad a.e.\ t\in[\tau_0,0].$$

Set $z\doteq w_2-w_1$, $\sigma_i^{(z)}\doteq\sigma_i^{(w_2)}-\sigma_i^{(w_1)}$, and $\gamma^{(z)}(t)\ \doteq\ \max\left\{\left|\dot\sigma_1^{(z)}\right|,\left|\dot\sigma_2^{(z)}\right|\right\}$. Furthermore, let

$$M_2(t)\ \doteq\ \|z(t,\cdot)\|_{H^2(\mathbb{R}\backslash\{t,0\})}+|z_x(t,0-)|+|z_x(t,0+)|+|z(t,0-)|+|z(t,0+)|.$$

Then there exists a constant $C_2>0$, depending only on M_0,b,δ_0 such that, for every $x\in\left[-\dfrac{1}{2e},\dfrac{1}{2e}\right]$ and a.e. $t\in[\tau_0,0]$, one has

$$\begin{aligned}\big|F(t,x,w_2)-F(t,x,w_1)\big|\ \le\ &\frac{C_2}{\delta_0^2}\cdot(M_0+b)\cdot\gamma^{(z)}(t)\cdot\Big(|x|^{1/2}\chi_{[0,\infty[}+|x-t|^{1/2}\chi_{[-\infty,t[}\Big)\\ &+\frac{C_2}{\delta_0^2}\cdot\left[M_2(t)\cdot\left(\big|\ln|t|\big|+\frac{M_0+b}{\delta_0}+\frac{K(t)}{\delta_0}\cdot\Big(|x|^{1/2}+|x-t|^{1/2}\Big)\right)\right]\end{aligned}\tag{6.49}$$

and for every $x\in(t,0)$

$$\big|F_x(t,x,w_2)-F_x(t,x,w_1)\big|\ \le\frac{C_2}{\delta_0^2}\cdot M_2(t)\cdot\Big(1+\delta_0+M_0+b)(|x|^{-1/2}+|x-t|^{-1/2}\Big).\tag{6.50}$$

Moreover, for every $\delta > 0$ sufficiently small, it holds

$$\begin{aligned}\|F(t,\cdot,w_2) - F(t,\cdot,w_1)\|_{H^2(\mathbb{R}\setminus[-\delta,\delta]\cup[t-\delta,t+\delta])} \;&\leq\; \frac{C_2}{\delta_0^2}\cdot(M_0+b)\cdot\gamma^{(z)}(t)\cdot\delta^{-2/3}\\ &+\frac{C_2}{\delta_0^2}\cdot M_2(t)\cdot\left[\left(\frac{K(t)+M_0+b+1}{\delta_0}+M_0+b+1\right)\cdot\delta^{-2/3}+\big|\ln|t|\big|\right]. \qquad (6.51)\end{aligned}$$

Proof. 1. For notational convenience, we set

$$\mathbf{A}^{(z)} \doteq A^{(w_2)}-A^{(w_1)},\ \ \mathbf{B}^{(z)} \doteq B^{(w_2)}-B^{(w_1)},\ \ \mathbf{C}^{(z)} \doteq C^{(w_2)}-C^{(w_1)},\ \ \mathbf{D}^{(z)} \doteq D^{(w_2)}-D^{(w_1)}. \qquad (6.52)$$

Furthermore, let $z_j = v_{1,j} - v_{2,j}$ for $j = 1,\,2$, then

$$\|z_1(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{t\})} \leq \mathcal{O}(1)\cdot M_2(t) \qquad \|z_2(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})} \leq \mathcal{O}(1)\cdot M_2(t).$$

Comparing (3.2) and (6.26) and recalling (3.20), then yields, for every $(t,x) \in [\tau_0, 0] \times \left[-\frac{1}{2e},\frac{1}{2e}\right]$,

$$\begin{cases}\left|\mathbf{B}^{(z)}(t,x)\right| = \mathcal{O}(1)\cdot M_2(t), \quad \left|\mathbf{B}^{(z)}_x(t,x)\right| = \mathcal{O}(1)\cdot M_2(t)\cdot|\ln|x||^2,\\[2mm] \left\|\mathbf{B}^{(z)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \leq \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta^{2/3}}.\end{cases} \qquad (6.53)$$

Similarly, for every $(t,x) \in [\tau_0, 0] \times \left[t-\frac{1}{2e}, t+\frac{1}{2e}\right]$, it holds

$$\begin{cases}\left|\mathbf{C}^{(z)}(t,x)\right| = \mathcal{O}(1)\cdot M_2(t), \quad \left|\mathbf{C}^{(z)}_x(t,x)\right| = \mathcal{O}(1)\cdot M_2(t)\cdot|\ln|x-t||^2,\\[2mm] \left\|\mathbf{C}^{(z)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[t-\delta,t+\delta])} \leq \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta^{2/3}}.\end{cases} \qquad (6.54)$$

2. We now provide bounds on $\mathbf{A}^{(z)}(t,x)$. From (6.34) and (6.35), it follows that

$$\begin{aligned}\varphi^{(w_2)}(t,x)-\varphi^{(w_1)}(t,x) \;&=\; \left[\frac{2\sigma_1^{(w_1)}(t)}{2\sigma_1^{(w_1)}(t)+\sigma_2^{(w_1)}(t)}-\frac{2\sigma_1^{(w_2)}(t)}{2\sigma_1^{(w_2)}(t)+\sigma_2^{(w_2)}(t)}\right]\cdot g_{|t|}(t-x)\\ &+\left[\frac{2\sigma_2^{(w_1)}(t)}{\sigma_1^{(w_1)}(t)+2\sigma_2^{(w_1)}(t)}-\frac{2\sigma_2^{(w_2)}(t)}{\sigma_1^{(w_2)}(t)+2\sigma_2^{(w_2)}(t)}\right]\cdot g_{|t|}(x). \qquad (6.55)\end{aligned}$$

Since

$$\begin{cases}\left|\dfrac{2\sigma_1^{(w_1)}(t)}{2\sigma_1^{(w_1)}(t)+\sigma_2^{(w_1)}(t)}-\dfrac{2\sigma_1^{(w_2)}(t)}{2\sigma_1^{(w_2)}(t)+\sigma_2^{(w_2)}(t)}\right| \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\,,\\[3mm] \left|\dfrac{2\sigma_2^{(w_1)}(t)}{\sigma_1^{(w_1)}(t)+2\sigma_2^{(w_1)}(t)}-\dfrac{2\sigma_2^{(w_2)}(t)}{\sigma_1^{(w_2)}(t)+2\sigma_2^{(w_2)}(t)}\right| \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\,,\end{cases}$$

Lemma 3.1 implies for $x \notin \{t, 0\}$ and $|x| \leq \frac{1}{2e}$,

$$\begin{cases}\left|\mathbf{H}[\varphi^{(w_2)}(t,x)-\varphi^{(w_1)}(t,x)]\right| \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0},\\[3mm] \left|\dfrac{d}{dx}\mathbf{H}[\varphi^{(w_2)}(t,x)-\varphi^{(w_1)}(t,x)]\right| \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\left(\ln^2|x|+\ln^2|x-t|\right),\\[3mm] \left\|\mathbf{H}[\varphi^{(w_2)}(t,\cdot)-\varphi^{(w_1)}(t,\cdot)]\right\|_{H^2(\mathbb{R}\setminus[t-\delta,t+\delta]\cup(-\delta,\delta))} \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\delta^{-2/3}.\end{cases}$$

and from (3.5), we obtain for $x \notin \{t, 0\}$ and $|x| \leq \frac{1}{2e}$,

$$\begin{cases} \left|\varphi^{(w_2)}\varphi_x^{(w_2)} - \varphi^{(w_1)}\varphi_x^{(w_1)}\right| \ \leq \ \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\left(|x|^{1/2}+|x-t|^{1/2}\right), \\ \left|\dfrac{d}{dx}\left(\varphi^{(w_2)}\varphi_x^{(w_2)} - \varphi^{(w_1)}\varphi_x^{(w_1)}\right)\right| \ \leq \ \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\left(\ln^2|x|+\ln^2|x-t|\right), \\ \left|\dfrac{d^2}{dx^2}\left(\varphi^{(w_2)}\varphi_x^{(w_2)} - \varphi^{(w_1)}\varphi_x^{(w_1)}\right)\right| \ \leq \ \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\left(\dfrac{|\ln|x||}{|x|}+\dfrac{|\ln|x-t||}{|x-t|}\right), \\ \left\|\varphi^{(w_2)}\varphi_x^{(w_2)} - \varphi^{(w_1)}\varphi_x^{(w_1)}\right\|_{H^2(\mathbb{R}\backslash[t-\delta,t+\delta]\cup(-\delta,\delta))} \ \leq \ \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\delta^{-2/3}. \end{cases}$$

Thus, (6.26) yields

$$\begin{cases} \left|\mathbf{A}^{(z)}(t,x)\right| \ \leq \ \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}, \qquad \left|\mathbf{A}_x^{(z)}(t,x)\right| \ \leq \ \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\left(\ln^2|x|+\ln^2|x-t|\right), \\ \left\|\mathbf{A}^{(z)}(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash[t-\delta,t+\delta]\cup(-\delta,\delta))} \ \leq \ \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\delta^{-2/3}. \end{cases} \tag{6.56}$$

3. Finally, to achieve bound on $\mathbf{D}^{(z)}$, we consider three cases as in the proof of Lemma 6.1. As before, we define $\mathbf{D}_i^{(z)} = D_i^{(w_2)} - D_i^{(w_1)}$ for $i = 1, 2, 3$.

Case 1: Assume that $-1/2 < t - \frac{1}{2e} < x < t$. Note, the we can write

$$\begin{cases} \mathbf{D}^{(z)}(t,x) \ = \ \mathbf{D}_1^{(z)}(t,x)+\mathbf{D}_2^{(z)}(t,x)+\mathbf{D}_3^{(z)}(t,x) \\ \mathbf{D}_1^{(z)}(t,x) \ = \ I_1^{(z)}(t,x)\cdot\phi_0'(x-t), \qquad \mathbf{D}_2^{(z)}(t,x) \ = \ I_2^{(z)}(t,x)\cdot\phi_0'(x), \\ \mathbf{D}_3^{(z)}(t,x) \ = \ I_{31}^{(z)}(t)\cdot g_{|t|}(t-x)+I_{32}^{(z)}(t)\cdot\phi_0'(t). \end{cases}$$

which implies

$$\begin{cases} \left|I_1^{(z)}(t,x)\right| \ \leq \ \mathcal{O}(1)\cdot M_2(t)\cdot|t-x|, \quad \left|\partial_x I_1^{(z)}(t,x)\right| \ \leq \ \mathcal{O}(1)\cdot M_1(t), \\ \partial^2_{xx}I_1^{(z)}(t,x) \ = \ 0, \quad \left|I_2^{(z)}(t,x)\right| \ \leq \ \mathcal{O}(1)\cdot M_2(t)\cdot\dfrac{M_0+b}{\delta_0}\cdot|x|, \\ \left|\partial_x I_2^{(z)}(t,x)\right| \ \leq \ \mathcal{O}(1)\cdot\left(\dfrac{M_0+b}{\delta_0}\cdot M_2(t)+|z_x(t,x)|\right), \\ \left|\partial^2_{xx}I_2^{(z)}(t,x)\right| \ \leq \ \mathcal{O}(1)\cdot\left(\dfrac{|w_{2,xx}(t,x)|}{\delta_0}\cdot M_2(t)+|z_{xx}(t,x)|\right), \\ \left|I_{31}^{(z)}(t)\right| \ \leq \ \mathcal{O}(1)\cdot\left(\dfrac{K(t)(1+\delta_0)M_2(t)}{\delta_0^2}+\dfrac{M_0\gamma^{(z)}(t)}{\delta_0}\right), \quad \left|I_{32}^{(z)}(t)\right| \ \leq \ \mathcal{O}(1)\cdot\dfrac{M_0M_2(t)}{\delta_0}. \end{cases}$$

Thus, for $t > 0$ sufficiently small such that $|t| < e^{-M_0-b}$, it holds

$$\left|\mathbf{D}^{(z)}(t,x)\right| \ \leq \ \mathcal{O}(1)\cdot\left[\frac{M_0+b}{\delta_0}\cdot\gamma^{(z)}(t)\cdot|x-t|^{1/2} + \frac{M_2(t)}{\delta_0}\cdot\left(\frac{K(t)}{\delta_0}\cdot|x-t|^{1/2}+(1+M_0)\cdot\big|\ln|t|\big|\right)\right],$$

and

$$\begin{cases} \left|\dfrac{d}{dx}\mathbf{D}^{(z)}(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot\left[\dfrac{M_2(t)}{\delta_0}\cdot\left(\dfrac{K(t)}{\delta_0}+M_0+b+1\right)\cdot|x-t|^{-1/2}+|z_x(t,x)|\cdot\big|\ln|x|\big|\right] \\ \qquad\qquad\qquad\qquad +\mathcal{O}(1)\cdot\left(\dfrac{M_0+b}{\delta_0}\cdot\gamma^{(z)}(t)\cdot\big|\ln|t-x|\big|\right) \\ \left|\dfrac{d^2}{d^2x}\mathbf{D}^{(z)}(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\left[\left(\dfrac{K(t)}{\delta_0}+M_0+b+1\right)\cdot\dfrac{1}{|x-t|}+|w_{2xx}(t,x)|\big|\ln|x|\big|\right] \\ \qquad\qquad\qquad\qquad +\mathcal{O}(1)\cdot\left(|z_{xx}(t,x)|\cdot\big|\ln|x|\big|+\dfrac{M_0+b}{\delta_0}\cdot\gamma^{(z)}(t)\cdot\dfrac{1}{|x-t|}\right). \end{cases}$$

Case 2: Assuming $t<x<0$, we have

$$\mathbf{D}^{(z)}(t,x) \ =\ z_x(t,0-)\cdot(t-x)\cdot\phi_0'(x-t)+(z_1(t,0)-z_1(t,x))\cdot\phi_0'(x)+\left[a_1^{(z)}(t)-a_2^{(z)}(t)\right]\cdot\phi_0'(t),$$

with

$$a_j^{(z)}(t)\doteq a_j^{(w_2)}(t)-a_j^{(w_1)}(t),$$

which yields

$$\begin{cases} \left|\mathbf{D}^{(z)}(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot M_2(t)\cdot\big|\ln|t|\big|, \\ \left|\dfrac{d}{dx}\mathbf{D}^{(z)}(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot\left[M_2(t)\cdot\left(|x|^{-1/2}+|x-t|^{-1/2}\right)+|z_{1,x}(t,x)|\cdot\big|\ln|x|\big|\right], \\ \left|\dfrac{d^2}{dx^2}\mathbf{D}^{(z)}(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot\left[M_2(t)\cdot\left(\dfrac{1}{|x|}+\dfrac{1}{|x-t|}\right)+|z_{1,xx}(t,x)|\cdot\big|\ln|x|\big|\right]. \end{cases}$$

Case 3: Assume that $0<x<1/2$ and $|t|\leq e^{-M_0-b}$. As in Case 1, we estimate

$$\begin{aligned} &\left|\mathbf{D}^{(z)}(t,x)\right| \\ &\quad\leq\ \mathcal{O}(1)\cdot\left[\frac{M_2(t)}{\delta_0}\cdot\left(\frac{K(t)(M_0+\delta_0)\cdot x^{1/2}}{\delta_0}+(1+M_0)\big|\ln|t|\big|\right)+\frac{M_0+b}{\delta_0}\cdot\gamma^{(z)}(t)\cdot x^{1/2}\right] \end{aligned}$$

and

$$\begin{cases} \left|\dfrac{d}{dx}\mathbf{D}^{(z)}(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot\left[\dfrac{M_2(t)}{\delta_0}\left(\dfrac{K(t)(M_0+\delta_0)}{\delta_0}+M_0+b+1\right)\cdot|x|^{-1/2}\right] \\ \qquad\qquad\qquad\qquad +\mathcal{O}(1)\cdot\left(|z_x(t,x)|+|z_{1,x}(t,x)|+\dfrac{M_0+b}{\delta_0}\cdot\gamma^{(z)}(t)\right)\cdot\big|\ln x\big|, \\ \left|\dfrac{d^2}{dx^2}\mathbf{D}^{(z)}(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot\left[\big(|z_{xx}(t,x)|+|z_{1,xx}(t,x)|\big)\cdot\big|\ln|x|\big|+\dfrac{M_0+b}{\delta_0}\cdot\gamma^{(z)}(t)\cdot\dfrac{1}{|x|}\right] \\ \qquad +\mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\left[\left(\dfrac{K(t)(M_0+\delta_0)}{\delta_0}+M_0+b+1\right)\cdot\dfrac{1}{|x|}+|w_{2,xx}(t,x)|\big|\ln|x-t|\big|\right]. \end{cases}$$

In summary, given $t \in [\tau_0, 0]$ sufficiently small, for every $x \in (-1/2, 1/2)\backslash\{t, 0\}$, it holds that

$$\begin{cases} \left|\mathbf{D}^{(z)}(t,x)\right| \ \le\ \mathcal{O}(1)\cdot\Big[\dfrac{M_2(t)}{\delta_0}\cdot(1+M_0)\cdot\big|\ln|t|\big| \\ \qquad + \left(\dfrac{M_2(t)K(t)}{\delta_0^2} + M_2(t) + \dfrac{M_0+b}{\delta_0}\gamma^{(z)}(t)\right)\cdot\left(|x|^{1/2}\chi_{[0,\infty[} + |x-t|^{1/2}\chi_{]-\infty,t]}\right)\Big], \\ \left|\dfrac{d}{dx}\mathbf{D}^{(z)}(t,x)\right| \ \le\ \mathcal{O}(1)\cdot\frac{M_0+b}{\delta_0}\cdot\gamma^{(z)}(t)\cdot\left(|x|^{-1/4}\chi_{[0,\infty[} + |x-t|^{-1/4}\chi_{]-\infty,t]}\right) \\ \qquad + \mathcal{O}(1)\cdot\left[\left(\dfrac{M_2(t)}{\delta_0}\left(\dfrac{K(t)}{\delta_0} + M_0 + b + 1\right) + |z_x(t,x)|\right)\cdot\left(|x|^{-1/2} + |x-t|^{-1/2}\right)\right] \\ \left\|\mathbf{D}^{(z)}(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash[t-\delta,t+\delta]\cup(-\delta,\delta))} \ \le\ \mathcal{O}(1)\cdot\dfrac{M_2(t)}{\delta_0}\cdot\left[\left(\dfrac{K(t)}{\delta_0} + M_0 + b + 1\right)\cdot\delta^{-2/3} + \big|\ln|t|\big|\right] \\ \qquad + \mathcal{O}(1)\cdot\dfrac{M_0+b}{\delta_0}\cdot\gamma^{(z)}(t)\cdot\delta^{-2/3}. \end{cases} \tag{6.57}$$

Finally, combining (6.25)-(6.28), Lemma 6.1, and (6.52)-(6.57), we obtain (6.49)-(6.51) □

6.2 Proof of Theorem 6.1

We are now ready to give a proof of Theorem 6.1. Given $\tau_0 \in [-\varepsilon_0, 0[$ sufficiently small and some initial data $w(\tau_0, \cdot) = \overline{w}$ satisfying (6.18), we construct a solution to the Cauchy problem (6.13). This solution will be obtained as the limit of a Cauchy sequence of approximate solutions $w^{(n)}(t,x)$, following the steps (i)–(iii) outlined in the beginning of Section 6.

Step 1. Let $b, M_0, \delta_1, \delta_2 > 0$ and $\overline{w} \in H^2(\mathbb{R}\backslash\{\tau_0, 0\})$ such that

$$\begin{cases} \|\overline{w}\|_{H^2(\mathbb{R}\backslash\{\tau_0,0\})} \ \le\ \dfrac{M_0}{4}, \qquad \|\overline{w}_x\|_{\mathbf{L}^\infty(]t,0[)} \ \le\ b, \\ \overline{w}(\tau_0-) - \overline{w}(\tau_0+) \ =\ 8\delta_1, \qquad \overline{w}(0-) - \overline{w}(0+) \ =\ 8\delta_2. \end{cases} \tag{6.58}$$

We first establish the existence and uniqueness of solutions to the linear problem (6.22) with initial data $\overline{w}$ and a given function $w^{(n)}$ with $w^{(n)}(t,\cdot) \in H^2(\mathbb{R}\backslash\{t,0\})$ for all $t \in [\tau_0, 0[$ and such that for all $t \in [\tau_0, 0[$,

$$\begin{cases} \left\|w^{(n)}(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{t,0\})} \ \le\ M_0, \qquad \left\|w_x^{(n)}(t,x)\right\|_{\mathbf{L}^\infty(]t,0[)} \ \le\ Kb, \\ \left|w^{(n)}(t,t\pm) - \overline{w}(\tau_0\pm)\right| \ \le\ \delta_1, \qquad \left|w^{(n)}(t,0\pm) - \overline{w}(0\pm)\right| \ \le\ \delta_2\,, \end{cases} \tag{6.59}$$

for some constant $K > 0$ depending only on $b, M_0, \delta_1, \delta_2$. Note that $w^{(1)}$, defined in (6.21), satisfies all of these assumptions.

Note that if such a sequence exist, then the constant δ_0 in Lemma 6.1 and Lemma 6.2 can be chosen as $\min(\delta_1, \delta_2)$. Accordingly, we define

$$\delta_0 \ \doteq\ \min(\delta_1, \delta_2).$$

Assume

$$-\frac{1}{4}\cdot\frac{\min\{\delta_1^2, \delta_2^2\}}{M_0^2} \ <\ \tau_0 \ <\ 0, \tag{6.60}$$

and denote by $t \mapsto x(t; t_0, x_0)$ the solution to the Cauchy problem

$$\dot{x}(t) \;=\; a_n(t, x(t)), \qquad x(t_0) \;=\; x_0, \tag{6.61}$$

where

$$a_n(t,x) \;\doteq\; \frac{w^{(n)}(t,x) + \varphi^{(n)}(t,x) - a_2^{(n)}(t)}{a_1^{(n)}(t) - a_2^{(n)}(t)}. \tag{6.62}$$

Here,

$$\varphi^{(n)}(t,x) \;\doteq\; \varphi^{\left(w^{(n)}\right)}(t,x), \quad a_j^{(n)}(t) = a_j^{(w^{(n)})}(t), \quad \sigma_j^{(n)}(t) = \sigma_j^{(w^{(n)})}(t) \quad \text{for } j = 1,2. \tag{6.63}$$

To begin with we study the travel direction of $x(t)$, which depends on the sign of a_n. Therefore observe that (6.58) and (6.59) imply

$$6\delta_i \;\le\; \sigma_i^{(n)}(t) \;\le\; 10\delta_i \qquad t \in [\tau_0, 0[, i \in \{1,2\}. \tag{6.64}$$

Furthermore,

$$\left| a_1^{(n)}(t) - a_2^{(n)}(t) - \frac{1}{2} \cdot \left(\sigma_1^{(n)}(t) + \sigma_2^{(n)}(t) \right) \right| \;=\; \left| w^{(n)}(t, t+) - w^{(n)}(t, 0-) \right| \;\le\; M_0 \sqrt{|t|},$$

and

$$3(\delta_1 + \delta_2) - M_0 \sqrt{|t|} \;\le\; \left| a_1^{(n)}(t) - a_2^{(n)}(t) \right| \;\le\; 5(\delta_1 + \delta_2) + M_0 \sqrt{|t|}.$$

Recalling (6.60) we end up with

$$2(\delta_1 + \delta_2) \;\le\; \left| a_1^{(n)}(t) - a_2^{(n)}(t) \right| \;\le\; 6(\delta_1 + \delta_2). \tag{6.65}$$

For every $(t,x) \in [\tau_0, 0[\times]0, \frac{1}{2}[$, one has, using (6.17), (6.35), and (3.5),

$$\begin{aligned} \left| a_n(t,x) + \frac{\sigma_2^{(n)}(t)}{2(a_1^{(n)}(t) - a_2^{(n)}(t))} \right| \;&=\; \left| \frac{w^{(n)}(t,x) - w^{(n)}(t, 0+) + \varphi^{(n)}(t,x)}{a_1^{(n)}(t) - a_2^{(n)}(t)} \right| \\ &\le\; \frac{1}{2(\delta_1 + \delta_2)} \cdot \left(\left| w^{(n)}(t,x) - w^{(n)}(t, 0+) \right| + \left| \varphi^{(n)}(t,x) \right| \right) \\ &\le\; \frac{1}{2(\delta_1 + \delta_2)} \cdot \left(M_0 x^{1/2} + 2 \left| x \ln x \right| \right). \end{aligned} \tag{6.66}$$

Similarly, for every $(t,x) \in [\tau_0, 0[\times] -1/2, t[$,

$$\begin{aligned} \left| a_n(t,x) - 1 - \frac{\sigma_1^{(n)}(t)}{2\left(a_1^{(n)}(t) - a_2^{(n)}(t) \right)} \right| \;&=\; \left| \frac{w^{(n)}(t,x) - w^{(n)}(t, t-) + \varphi^{(n)}(t,x)}{a_1^{(n)}(t) - a_2^{(n)}(t)} \right| \\ &\le\; \frac{1}{2(\delta_1 + \delta_2)} \cdot \left(M_0 \cdot |x - t|^{1/2} + 2 \cdot \left| |x - t| \ln |x - t| \right| \right), \end{aligned} \tag{6.67}$$

and for any $(t,x) \in [\tau_0, 0[\times]t, 0[$,

$$\begin{aligned} \left| a_n(t,x) - 1 + \frac{\sigma_1^{(n)}(t)}{2\left(a_1^{(n)}(t) - a_2^{(n)}(t) \right)} \right| \;&=\; \left| \frac{w^{(n)}(t,x) - w^{(n)}(t, t+) + \varphi^{(n)}(t,x)}{a_1^{(n)}(t) - a_2^{(n)}(t)} \right| \\ &\le\; \frac{1}{2(\delta_1 + \delta_2)} \cdot \left(M_0 \cdot |x - t|^{1/2} + 2 \cdot \left(\left| |x - t| \ln |x - t| \right| + \left| x \ln |x| \right| \right) \right). \end{aligned} \tag{6.68}$$

Since

$$\begin{cases} \dfrac{\sigma_1^{(n)}(t)}{\sigma_1^{(n)}(t)+10\delta_2+2M_0\sqrt{|t|}} \;\leq\; \dfrac{\sigma_1^{(n)}(t)}{2(a_1^{(n)}(t)-a_2^{(n)}(t))} \;\leq\; \dfrac{\sigma_1^{(n)}(t)}{\sigma_1^{(n)}(t)+6\delta_2-2M_0\sqrt{|t|}}\,, \\ \dfrac{\sigma_2^{(n)}(t)}{\sigma_2^{(n)}(t)+10\delta_1+2M_0\sqrt{|t|}} \;\leq\; \dfrac{\sigma_2^{(n)}(t)}{2(a_1^{(n)}(t)-a_2^{(n)}(t))} \;\leq\; \dfrac{\sigma_2^{(n)}(t)}{\sigma_2^{(n)}(t)+6\delta_1-2M_0\sqrt{|t|}}\,, \end{cases}$$

by (6.64), we conclude, using (6.64) and (6.60) one more, that

$$\begin{cases} \dfrac{\delta_1}{\delta_1+2\delta_2} \;\leq\; \dfrac{\sigma_1^{(n)}(t)}{2\big(a_1^{(n)}(t)-a_2^{(n)}(t)\big)} \;\leq\; \dfrac{2\delta_1}{2\delta_1+\delta_2}\,, \\ \dfrac{\delta_2}{2\delta_1+\delta_2} \;\leq\; \dfrac{\sigma_2^{(n)}(t)}{2\big(a_1^{(n)}(t)-a_2^{(n)}(t)\big)} \;\leq\; \dfrac{2\delta_2}{\delta_1+2\delta_2}\,, \end{cases} \qquad \text{for all } t\in[\tau_0,0].$$

Therefore, by (6.66)-(6.68) there exists $\bar{\delta}>0$ such that

$$\begin{cases} -\dfrac{5\delta_2}{2\delta_1+4\delta_2} \;\leq\; a_n(t,x) \;\leq\; -\dfrac{\delta_2}{4\delta_1+2\delta_2} & \text{for all } (t,x)\in[\tau_0,0[\times[0,\bar{\delta}], \\ 1-\dfrac{4\delta_1+\delta_0}{4\delta_1+2\delta_2} \;\leq\; a_n(t,x) \;\leq\; 1-\dfrac{2\delta_1-\delta_0}{2\delta_1+4\delta_2} & \text{for all } t\in[\tau_0,0], x\in[t,0[\cap([t,t+\bar{\delta}[\cup]-\bar{\delta},0]), \\ 1+\dfrac{\delta_1}{2\delta_1+4\delta_2} \;\leq\; a_n(t,x) \;\leq\; 1+\dfrac{5\delta_1}{4\delta_1+2\delta_2} & \text{for all } t\in[\tau_0,0], x\in[t-\bar{\delta},t]. \end{cases} \tag{6.69}$$

The next lemma provides the Lipschitz continuous dependence of the characteristic curves (6.61).

Lemma 6.3 *Let $w^{(n)}$ and $\varphi^{(n)}$ be as in (6.59) and (6.63). Given $\tau\in[\tau_0,0[$, let $x_1,x_2\in\mathbb{R}\backslash\{\tau,0\}$ with $\bar{x}_2<\bar{x}_1$ such that both $\bar{x}_1$ and $\bar{x}_2$ belong to $]-\frac{1}{2e},\tau[$, $]\tau,0[$ or $]0,\frac{1}{2e}[$. Then*

$$|x(t;\tau,\bar{x}_1)-x(t;\tau,\bar{x}_2)| \;\leq\; K_1\cdot|\bar{x}_2-\bar{x}_1| \qquad \text{for all } t\in[\tau_0,\tau[\tag{6.70}$$

for some $K_1>0$ depending only on M_0, δ_1, δ_2, K and b.

Proof. We shall prove (6.70) for $\bar{x}_1,\ \bar{x}_2\in]\tau,0[$. The other cases follow the same lines as the proof of (4.9).

For any $t<x_2<x_1<0$, (6.67) and (6.65) imply

$$\begin{aligned} |a_n(t,x_2)-a_n(t,x_1)| \;&\leq\; \frac{\big|w^{(n)}(t,x_2)-w^{(n)}(t,x_1)\big|+\big|\varphi^{(n)}(t,x_2)-\varphi^{(n)}(t,x_1)\big|}{\big|a_1^{(n)}(t)-a_2^{(n)}(t)\big|} \\ &\leq\; \frac{1}{2(\delta_1+\delta_2)}\cdot\big(Kb+\big|\ln|x_2-t|\big|+\big|\ln|x_1|\big|\big)\cdot(x_1-x_2). \end{aligned}$$

Setting $0\leq z(t)\doteq x(t;\tau,\bar{x}_1)-x(t;\tau,\bar{x}_2)$, we obtain

$$\dot{z}(t) \;\;\geq\;\; -\frac{1}{2(\delta_1+\delta_2)}\cdot\big(Kb+\big|\ln|x(t;\tau,\bar{x}_2)-t|\big|+\big|\ln|x(t;\tau,\bar{x}_1)|\big|\big)\cdot z(t).$$

Since (6.69) implies for any $\bar{x} \in (0,t)$ that

$$\left(1 - \frac{4\delta_1 + \delta_0}{2(2\delta_1 + \delta_2)}\right)(\tau - t) \leq x(\tau;\tau,\bar{x}) - x(t;\tau,\bar{x}) \leq \left(1 - \frac{2\delta_1 - \delta_0}{2(\delta_1 + 2\delta_2)}\right)(\tau - t),$$

one ends ups with

$$\frac{\dot{z}(t)}{z(t)} \geq -\frac{1}{2(\delta_1 + \delta_2)} \cdot \left(Kb + 2 \cdot \left|\ln\left(\min\left(\frac{\delta_0}{2(2\delta_1 + \delta_2)}, \frac{\delta_0}{2(\delta_1 + 2\delta_2)}\right)(\tau - t)\right)\right|\right),$$

which yields (6.70). □

Next, consider the constants

$$\gamma_0 \doteq \min\left\{\frac{\delta_0}{2(\delta_1 + 2\delta_2)}, \frac{\delta_0}{2(2\delta_1 + \delta_2)}\right\}, \qquad \gamma_1 \doteq \max\left\{\frac{5\max(\delta_1,\delta_2)}{2(\delta_1 + 2\delta_2)}, \frac{5\max(\delta_1,\delta_2)}{2(2\delta_1 + \delta_2)}\right\}, \tag{6.71}$$

and define

$$I_t^\tau \doteq [t - \gamma_0(\tau - t), t + \gamma_0(\tau - t)] \cup [-\gamma_0(\tau - t), \gamma_0(\tau - t)]. \tag{6.72}$$

From (6.69), one has

$$x(t;\tau,\bar{x}) \notin I_t^\tau \qquad \text{for all } \tau_0 \leq t < \tau \leq 0, \bar{x} \in [-1/2, 1/2]\backslash\{0,\tau\}. \tag{6.73}$$

Furthermore, for all $\tau_0 \leq t < \tau \leq 0$, one has

$$\begin{cases} |(x(t;\tau,\bar{x}) - t) - (\bar{x} - \tau)| \leq \gamma_1(\tau - t), & \bar{x} \in [-\frac{1}{2}, 0[\backslash\{\tau\}, \\ |x(t;\tau,\bar{x}) - \bar{x}| \leq \gamma_1(\tau - t), & \bar{x} \in]-\tau, \frac{1}{2}]\backslash\{0\}. \end{cases} \tag{6.74}$$

By the same arguments used in [4, Lemma 4.1], we now obtain

Lemma 6.4 *Let $w^{(n)}$ and $\varphi^{(n)}$ be as in (6.59) and (6.63). There exists $\varepsilon_0 > 0$ small enough, so that for any $-\varepsilon_0 \leq t < \tau < 0$ and any solution v of the linear equation*

$$v_t + a_n(t,x) \cdot v_x = 0, \qquad v(\tau_0, \cdot) = \bar{v} \in H^2(\mathbb{R}\backslash I_t^\tau),$$

one has

$$\|v(t,\cdot)\|_{H^2(\mathbb{R}\backslash\{t,0\})} \leq \frac{3}{2} \cdot \|\bar{v}\|_{H^2(\mathbb{R}\backslash I_t^\tau)}.$$

Step 2. Let us now consider a sequence of approximate solutions $w^{(k)}$ to (6.22) inductively defined as follows.

- $w^{(1)} : [\tau_0, 0[\times\mathbb{R} \to \mathbb{R}$ such that for all $t \in [\tau_0, 0[$,

$$w^{(1)}(t,x) = \begin{cases} \overline{w}(x) & \text{if } x \in (t,0) \cup (0,\infty), \\ \overline{w}(x + \tau_0 - t) & \text{if } x \in (-\infty, t), \end{cases}$$

  where $\overline{w}$ satisfies (6.58).

- For every $k \geq 1$, $w^{(k+1)}(t,\cdot)$ solves the linear equation

$$w_t + a_n(t,x)\cdot w_x \;=\; F^{(k)}(t,x), \qquad w(\tau_0,\cdot) \;=\; \overline{w}(\cdot)$$

with $F^{(k)}(t,x) \doteq F\left(t,x,w^{(k)}\right)$ and $\bar{w}$ as in (6.58). This can be rephrased as

$$w^{(k+1)}(t_0,x_0) \;=\; \overline{w}(x(\tau_0;t_0,x_0)) + \int_{\tau_0}^{t_0} F^{(k)}(t,x(t;t_0,x_0))dt. \tag{6.75}$$

The following lemma provides a priori estimates on $w^{(k)}$, uniformly valid for all $k \geq 1$.

Lemma 6.5 *Let $w^{(n)}$ and $\varphi^{(n)}$ be as in (6.59) and (6.63). Then there exists $\varepsilon_0 > 0$ sufficiently small so that the following holds. If $\tau_0 \in [-\varepsilon_0, 0[$, then for every $k \geq 0$ and a.e. $\tau \in [\tau_0, 0[$, one has*

$$\left|w^{(k)}(\tau,\tau\pm) - \overline{w}(\tau_0\pm)\right| \;\leq\; \delta_1, \qquad \left|w^{(k)}(\tau,0\pm) - \overline{w}(0\pm)\right| \;\leq\; \delta_2\,, \tag{6.76}$$

$$\max\left\{\left|\dot{\sigma}_1^{(k)}(\tau)\right|,\; \left|\dot{\sigma}_2^{(k)}(\tau)\right|\right\} \;\leq\; 4C_1\big|\ln|\tau|\big|, \tag{6.77}$$

$$\left\|w_x^{(k)}(\tau,x)\right\|_{\mathbf{L}^\infty(]\tau,0[)} \;\leq\; 2K_2b, \qquad \left\|w^{(k)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\setminus\{\tau,0\})} \;\leq\; M_0, \tag{6.78}$$

for some positive constants C_1 and K_2.

Proof. It is clear that (6.76)-(6.78) hold for $k = 1$. By induction, assume that (6.76)-(6.78) hold for a given $k \geq 1$.

1. We shall establish the first inequality in (6.78). Given $\tau \in [\tau_0, 0[$ and $\tau < \bar{x}_2 < \bar{x}_1 < 0$, consider the characteristics $t \mapsto x_i(t) = x(t;\tau,\bar{x}_i)$ for $i \in \{1,2\}$, which satisfy, cf. (6.73),

$$\min\{|x_i(t)|, |x_i(t)-t|\} \;\geq\; \gamma_0\cdot(\tau - t) \qquad \text{for all } t\in[\tau_0,\tau], i\in\{1,2\}. \tag{6.79}$$

Recalling (6.75), (6.70), and (6.29), we estimate

$$\begin{aligned}
&\left|w^{(k+1)}(\tau,\bar{x}_2) - w^{(k+1)}(\tau,\bar{x}_1)\right| \\
&\qquad\leq\; |\overline{w}(x_2(\tau_0)) - \overline{w}(x_1(\tau_0))| + \int_{\tau_0}^{\tau}\left|F^{(k)}(t,x_2(t)) - F^{(k)}(t,x_1(t))\right|dt \\
&\qquad\leq\; K_1\cdot\left(b + \frac{C_1(1+2M_0+2K_2b)}{\gamma_0^{1/2}\delta_0}\cdot\int_{\tau_0}^{\tau}\frac{1}{(\tau-t)^{1/2}}dt\right)\cdot|\bar{x}_2-\bar{x}_1| \\
&\qquad\qquad\leq\; K_1\cdot\left(b + \frac{2C_1(1+2M_0+2K_2b)(\tau-\tau_0)^{1/2}}{\gamma_0^{1/2}\delta_0}\right)\cdot|\bar{x}_2-\bar{x}_1|.
\end{aligned}$$

Thus, if $0 \leq -\tau_0 \leq \left(\dfrac{b\gamma_0^{1/2}\delta_0}{2C_1(1+2M_0+2K_2b)}\right)^2$, then

$$\left|w^{(k+1)}(\tau,\bar{x}_2) - w^{(k+1)}(\tau,\bar{x}_1)\right| \;\leq\; 2K_2b\cdot|\bar{x}_2-\bar{x}_1|$$

and (6.78) is satisfied by $w^{(k+1)}$.

2. We shall establish (6.77) for $i=2$ and the second inequality in (6.76). The other ones are quite similar. Given any $\tau_0 \le \tau_1 < \tau_2 \le 0$, let $t \mapsto x_2^{\pm}(t) \doteq x(t;\tau_2,0\pm)$ be the characteristics,which reach the origin at time τ_2 from the positive and negative side, respectively. From (6.74), it follows that

$$\left|x_2^{\pm}(t)\right| \ \le\ \gamma_1 \cdot |\tau_2 - t| \qquad \text{for all } t\in[\tau_1,\tau_2].$$

Furthermore, recalling (6.77), (6.78), and (6.29), we have

$$\begin{aligned}
\left|w^{(k+1)}(\tau_2,0\pm) - w^{(k+1)}(\tau_1,0\pm)\right| \ &\le\ \left|w^{(k+1)}(\tau_1,x_2^{\pm}(\tau_1)) - w^{(k+1)}(\tau_1,0\pm)\right| \\
&+ \int_{\tau_1}^{\tau_2} \left|F^{(k)}(t,x_2^{\pm}(t))\right| dt \ \le\ 2(K_2 b + M_0)\gamma_1\cdot(\tau_2-\tau_1) \\
+ C_1\cdot \int_{\tau_1}^{\tau_2} &\frac{1+2M_0+2K_2 b}{\delta_0} + \left(1+\frac{8C_1}{\delta_0}(|t|^{1/2} + 2\gamma_1^{1/2}(\tau_2-t)^{1/2})\right)|\ln|t||\ dt \\
\le\ \Bigg[2(K_2b+M_0)\gamma_1 + \frac{C_1(1+2M_0+2K_2b)}{\delta_0} &+ C_1\left(1+\frac{8C_1(1+2\gamma_1^{1/2})|\tau_0|^{1/2}}{\delta_0}\right)\big|\ln|\tau_2|\big|\Bigg]\cdot(\tau_2-\tau_1).
\end{aligned}$$

Thus, for $|\tau_0|$ is sufficiently small, we then obtain (6.77) for $(k+1)$ and $i=2$ by

$$\left|\sigma_2^{(k+1)}(\tau_2) - \sigma_2^{(k+1)}(\tau_1)\right| \ \le\ 4C_1\big|\ln|\tau_2|\big|\cdot(\tau_2-\tau_1).$$

Moreover, for the second inequality in (6.76), choose $\tau_1=\tau_0$ in the above estimate, i.e.,

$$\begin{aligned}
&\left|w^{(k+1)}(\tau_2,0\pm) - \overline{w}(0\pm)\right| \\
&\qquad \le\ \left[2(K_2b+M_0)\gamma_1 + \frac{C_1(1+2M_0+2K_2b)}{\delta_0} + C_1\left(1+\frac{8C_1}{\delta_0}\right)\cdot\big|\ln|\tau_0|\big|\right]\cdot|\tau_0|.
\end{aligned}$$

and this yields (6.76).

3. Finally, from Duhamel' formula, Lemma 6.4, (6.58), Lemma 6.1, (6.77), and (6.72), we obtain, for all $\tau\in[\tau_0,0]$,

$$\begin{aligned}
\left\|w^{(k+1)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{\tau,0\})} \ &\le\ \frac{3}{2}\|\overline{w}\|_{H^2\left(\mathbb{R}\backslash I_{\tau_0}^{\tau}\right)} + \frac{3}{2}\cdot\int_{\tau_0}^{\tau}\left\|F^{(k)}(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash I_t^{\tau})} dt \\
&\le\ \frac{3M_0}{8} + \frac{3}{2}C_1\cdot\int_{\tau_0}^{\tau}\frac{1+2M_0+2K_2b+8C_1\big|\ln|t|\big|}{\delta_0\gamma_0^{2/3}}(\tau-t)^{-2/3} + \big|\ln|t|\big|\ dt \\
&\le\ \frac{3M_0}{8} + \left(\frac{9C_1(25+2M_0+2K_2b+8C_1\cdot\big|\ln|\tau_0|\big|)}{2\delta_0\gamma_0^{2/3}}\right)\cdot|\tau_0|^{1/3} + 3C_1\cdot\big|\tau_0\ln|\tau_0|\big|.
\end{aligned}$$

Identifying an upper bound on τ_0 such that the right hand side is less or equal than M_0, shows that the second bound in (6.78) is satisfied by $w^{(k+1)}$ as well. □

Thanks to the above estimates, we can now prove that the sequence of approximations $w^{(k)}$ is Cauchy, and converges to a solution w of the linear problem (6.22). This will accomplish the inductive step, toward the proof of Theorem 6.1.

Lemma 6.6 *There exists $\varepsilon_0 > 0$ sufficiently small so that, for all $\tau_0 \in [-\varepsilon_0, 0[$ the following holds: Let $w^{(n)}, \varphi^{(n)}$ as in (6.59) and (6.63). Then the sequence of approximations $\big(w^{(k)}(t,\cdot)\big)_{k\geq 1}$ converges uniformly for all $t \in [\tau_0, 0[$ to a limit function $w(t,\cdot)$ in $H^2(\mathbb{R}\backslash\{t,0\})$. Namely,*

$$\lim_{k\to\infty} \sup_{t\in[\tau_0,0]} \left\| w^{(k)}(t,\cdot) - w(t,\cdot) \right\|_{H^2(\mathbb{R}\backslash\{t,0\})} \; = \; 0.$$

The function w provides a solution to the Cauchy problem (6.22) and satisfies for all $\tau \in [\tau_0, 0]$

$$|w(\tau,\tau\pm) - \overline{w}(\tau_0\pm)| \;\leq\; \delta_1, \qquad |w(\tau,0\pm) - \overline{w}(0\pm)| \;\leq\; \delta_2\,, \tag{6.80}$$

$$\|w_x(\tau,x)\|_{\mathbf{L}^\infty(]\tau,0[)} \;\leq\; 2K_2 b, \qquad \|w(\tau,\cdot)\|_{H^2(\mathbb{R}\backslash\{\tau,0\})} \;\leq\; M_0. \tag{6.81}$$

Moreover, $\sigma_1(t) \doteq w(t,t-) - w(t,t+)$ and $\sigma_2(t) \doteq w(t,0-) - w(t,0+)$ are locally Lipscthitz in $(\tau_0, 0)$ and

$$\max\left\{|\dot\sigma_1(\tau)|\,,\;\; |\dot\sigma_2(\tau)|\right\} \;\leq\; 4C_1\big|\ln\tau\big| \quad a.e.\ \tau\in[\tau_0,0]. \tag{6.82}$$

Proof. 1. For any $k \geq 1$, we set

$$\begin{cases} z^{(k)} &\doteq\; w^{(k+1)} - w^{(k)}, \qquad M_z^{(k)}(\tau) \;\doteq\; \left\|z^{(k)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{\tau,0\})}\,, \\ \sigma_1^{(k,z)}(\tau) &\doteq\; z^{(k)}(\tau,\tau-) - z^{(k)}(\tau,\tau+), \qquad \sigma_2^{(k,z)}(\tau) \;\doteq\; z^{(k)}(\tau,0-) - z^{(k)}(\tau,0+), \\ \alpha_k(\tau) &\doteq\; \sup\limits_{t\in[\tau_0,\tau]} \max\limits_{i\in\{1,2\}} \left|\dot\sigma_i^{(k,z)}(t)\right|\,, \\ \beta_k(\tau) &\doteq\; \sup\limits_{t\in[\tau_0,\tau]} \left(M_z^{(k)}(t) + \left|z_x^{(k)}(t,0-)\right| + \left|z_x^{(k)}(t,0+)\right| + \left|z^{(k)}(t,0-)\right| + \left|z^{(k)}(t,0+)\right|\right). \end{cases} \tag{6.83}$$

Recalling Duhamel's formula, Lemma 6.4, Lemma 6.2, (6.72), and Lemma 6.5, for all $\tau \in [\tau_0, 0]$ we estimate

$$\begin{aligned} M_z^{(k+1)}(\tau) \;&\leq\; \frac{3}{2}\cdot\int_{\tau_0}^{\tau} \left\|F^{(k+1)}(t,\cdot) - F^{(k)}(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash I_t^\tau)} dt \\ &\leq\; \frac{3C_2}{2\delta_0^2}\cdot\int_{\tau_0}^{\tau} \beta_k(t)\cdot\left[\left(\frac{4C_1\big|\ln|t|\big|}{\delta_0\gamma_0^{2/3}} + \frac{(2M_0+2K_2b+1)(1+\delta_0)}{\delta_0\gamma_0^{2/3}}\right)\cdot(\tau-t)^{-2/3} + \big|\ln|t|\big|\right] dt \\ &\qquad\qquad + \frac{3C_2}{2\delta_0^2}\cdot\int_{\tau_0}^{\tau} \frac{2M_0+2K_2b}{\gamma_0^{2/3}}\cdot\alpha_k(t)\cdot(\tau-t)^{-2/3}\; dt \\ &\leq\; C_3\cdot\left(|\tau_0-\tau|^{1/3}\cdot\big|\ln|\tau_0-\tau|\big|\cdot\beta_k(\tau) + \int_{\tau_0}^{\tau}\alpha_k(t)(\tau-t)^{-2/3}dt\right) \end{aligned} \tag{6.84}$$

for some constant C_3 depending only on M_0, b, K_2, δ_1, and δ_2.

2. We now establish a bound on $z_x^{(k+1)}(\tau,0\pm)$. Since

$$\left|z_x^{(k+1)}(\tau,0+)\right| \;\;\leq\;\; \left\|z^{(k+1)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{t,0\})} \leq M_z^{(k+1)}(\tau), \tag{6.85}$$

it suffices to have a closer look at $z_x^{(k+1)}(\tau,0-)$. Given $\tau \in [\tau_0, 0[$ and $\tau < \bar x_2 < \bar x_1 < 0$, consider the characteristics $t \mapsto x_i(t) = x(t;\tau,\bar x_i)$ for $i \in \{1,2\}$. Recalling (6.75), (6.50), (6.79), and (6.70), we estimate

$$\begin{aligned}
&\left|z^{(k+1)}(\tau,\bar{x}_2)-z^{(k+1)}(\tau,\bar{x}_1)\right| \\
&\quad= \left|\int_{\tau_0}^{\tau}\Big[F^{(k+1)}(t,x_2(t))-F^{(k+1)}(t,x_1(t))\Big]-\Big[F^{(k)}(t,x_2(t))-F^{(k)}(t,x_1(t))\Big]\,dt\right| \\
&\quad\le \int_{\tau_0}^{\tau}\int_0^1\left|\left(F_x^{(k+1)}-F_x^{(k)}\right)(t,(1-s)x_1(t)+sx_2(t))\right|\cdot|x_1(t)-x_2(t)|\;ds\;dt \\
&\quad\le \frac{2C_2K_1}{\delta_0^2\gamma_0^{1/2}}\cdot\int_{\tau_0}^{\tau}\beta_k(t)\cdot(1+\delta_0+2M_0+2K_2b)\cdot(\tau-t)^{-1/2}\;dt\cdot|\bar{x}_2-\bar{x}_1| \\
&\qquad\qquad\le C_4\cdot\beta_k(\tau)\cdot|\tau-\tau_0|^{1/2}\cdot|\bar{x}_2-\bar{x}_1|
\end{aligned}$$

for some constant C_4 depending only on M_0, b, K_1, δ_1, and δ_2. This implies

$$\left|z_x^{(k+1)}(\tau,0-)\right| \;\le\; C_4\cdot\beta_k(\tau)\cdot|\tau-\tau_0|^{1/2}. \tag{6.86}$$

3. Finally, we establish a bound on $\alpha_{k+1}(\tau)$ for $\tau\in[\tau_0,0]$. We only present here the details for $\dot{\sigma}_2^{(k+1,z)}(t)$, since $\dot{\sigma}_1^{(k+1,z)}(t)$ can estimated in the same way. Given any $\tau_0\le\tau_1<\tau_2<0$, denote by $t\mapsto x_2^{\pm}(t)\doteq x(t;\tau_2,0\pm)$ the characteristics which reach the origin at time τ_2 from the positive and negative side, respectively. Using (6.75), (6.49), (6.74), and (6.79), we estimate

$$\begin{aligned}
&\left|z^{(k+1)}(\tau_2,0\pm)-z^{(k+1)}(\tau_1,0\pm)\right| \\
&\le \left|z^{(k+1)}\big(\tau_1,x_2^{\pm}(\tau_1)\big)-z^{(k+1)}(\tau_1,0\pm)\right|+\int_{\tau_1}^{\tau_2}\left|F^{(k+1)}\left(t,x_2^{\pm}(t)\right)-F^{(k)}\left(t,x_2^{\pm}(t)\right)\right|dt \\
&\le 2\beta_{k+1}(\tau_1)\cdot\left|x_2^{\pm}(\tau_1)\right|+\frac{C_2}{\delta_0^2}\cdot\int_{\tau_1}^{\tau_2}(2M_0+2K_1b)\cdot\alpha_k(t)\cdot\gamma_1^{1/2}|\tau_2-t|^{1/2}dt \\
&+\frac{C_2}{\delta_0^2}\cdot\int_{\tau_1}^{\tau_2}\beta_k(t)\cdot\left(\frac{4C_1\sqrt{|t|}+\delta_0}{\delta_0}\big|\ln|t|\big|+\frac{2M_0+2K_1b}{\delta_0}+\frac{8C_1\gamma_1^{1/2}}{\delta_0}\cdot\big|\ln|t|\big||\tau_2-t|^{1/2}\right)dt \\
&\le \left(2\beta_{k+1}(\tau_2)\gamma_1+C_5\cdot\Big[\beta_k(\tau_2)\cdot\big|\ln|\tau_2|\big|+\alpha_k(\tau_2)\cdot|\tau_1-\tau_2|^{1/2}\Big]\right)\cdot(\tau_2-\tau_1)
\end{aligned}$$

for some constant C_5 depending only on M_0, b, K_1, δ_1, and δ_2. Thus, for $\tau\in[\tau_0,0]$,

$$\alpha_{k+1}(\tau)\;\le\;2\beta_{k+1}(\tau)\gamma_1+C_5\cdot\beta_k(\tau)\cdot\big|\ln|\tau|\big|. \tag{6.87}$$

Moreover, by choosing $\tau_1=\tau_0$ and $\tau_2=\tau\in[\tau_0,0]$, we also get

$$\left|z^{(k+1)}(\tau,0\pm)\right|\;\le\;C_6\cdot\left(\beta_k(\tau)\cdot\big||\tau-\tau_0|\cdot\ln|\tau-\tau_0|\big|+\int_{\tau_0}^{\tau}\alpha_k(t)\cdot|t-\tau|^{1/2}dt\right),$$

and (6.84)-(6.87) imply that

$$\begin{aligned}
\beta_{k+1}(\tau)\;&\le\;C_7\cdot\left(|\tau-\tau_0|^{1/3}\cdot\big|\ln|\tau-\tau_0|\big|\cdot\beta_k(\tau)+\int_{\tau_0}^{\tau}\alpha_k(t)\cdot|\tau-t|^{-2/3}dt\right) \\
&\qquad\le\;C_8\cdot|\tau-\tau_0|^{1/3}\cdot\big|\ln|\tau-\tau_0|\big|\cdot\left(\beta_k(\tau)+\beta_{k-1}(\tau)\right).
\end{aligned}$$

In particular, for $\tau_0 < 0$ sufficiently close to 0, we get

$$\beta_{k+1}(\tau) + \frac{1}{2}\cdot\beta_k \;\leq\; \frac{3}{4}\cdot\left(\beta_k(\tau) + \frac{1}{2}\cdot\beta_{k-1}\right),$$

which implies

$$\sum_{k=1}^{\infty}\sup_{\tau\in[\tau_0,0]}\left\|z^{(k)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\setminus\{t,0\})} \;<\; \sum_{k=1}^{\infty}\beta_k(\tau) \;<\; \infty.$$

We thus conclude that $\left(w^{(k)}(\tau,\cdot)\right)_{k\geq 1}$ converges uniformly for all $\tau\in[\tau_0,0[$ to a limit function $w(\tau,\cdot)$ in $H^2(\mathbb{R}\setminus\{t,0\})$, which provides the solution to the linear problem (6.22). Moreover, since $\lim_{k\to\infty} w^{(k)}(\tau,0\pm) = w(\tau,0\pm)$ and $\lim_{k\to\infty} w^{(k)}(\tau,\tau\pm) = w(\tau,\tau\pm)$, one has that $\lim_{k\to\infty}\sigma_i^{(k)}(\tau) = \sigma_i(\tau)$ for all $\tau\in[0,\tau]$. Furthermore, $\lim_{k\to\infty} w_x^{(k)}(\tau,0\pm) = w_x(\tau,0\pm)$ and $\lim_{k\to\infty} w_x^{(k)}(\tau,\tau\pm) = w_x(\tau,\tau\pm)$ and hence Lemma 6.5 implies that w satisfies (6.80)-(6.82). □

We are now ready to complete the proof of our second main theorem, describing the asymptotic behavior of solutions up to the time when two shocks interact.

Proof of Theorem 6.1. 1. By induction, we construct a sequence of approximate solutions $\left(w^{(n)}\right)_{n\geq 1}$ where each $w^{(n+1)}$ is the solution to the linear problem (6.22). Assuming that $\tau_0\in[-\varepsilon_0,0[$ is sufficiently close to 0, we claim that

$$\sum_{n\geq 1}\left\|w^{(n+1)}(t,\cdot) - w^{(n)}(t,\cdot)\right\|_{H^1(\mathbb{R}\setminus\{t,0\})} \;<\; \infty \qquad \text{for all } t\in[\tau_0,0]. \tag{6.88}$$

For a fixed $n\geq 2$, recalling that $a_n(t,x) = \dfrac{w^{(n)}(t,x)+\varphi^{(n)}(t,x)-a_2^{(n)}(t)}{a_1^{(n)}(t)-a_2^{(n)}(t)}$, we define

$$\begin{cases} W^{(n)} \;\doteq\; w^{(n)} - w^{(n-1)}, \quad A^{(n)}(\tau,x) \;\doteq\; a_n(\tau,x) - a_{n-1}(\tau,x),\\[2mm] \sigma_1^{(n)}(\tau) \;=\; W^{(n)}(\tau,\tau-) - W^{(n)}(\tau,\tau+), \quad \sigma_2^{(n)}(\tau) \;=\; W^{(n)}(\tau,0-) - W^{(n)}(\tau,0+),\\[2mm] \beta^{(n)}(\tau) \;\doteq\; \sup\limits_{t\in[\tau_0,\tau]}\left[\left\|W^{(n)}(\tau,\cdot)\right\|_{H^1(\mathbb{R}\setminus\{t,0\})} + \left|W^{(n)}(t,0-)\right| + \left|W^{(n)}(t,0+)\right|\right]. \end{cases}$$

Set $Z^{(n)} = W^{(n)} + V^{(n)}$ with $V^{(n)} = v^{(n)} - v^{(n-1)}$ and $v^{(n)} = \varphi^{(n)} - \phi_0(x-t) - \phi_0(x)$. From the above definitions, by (6.22), we deduce

$$Z_t^{(n+1)} + a_n\cdot Z_x^{(n+1)} \;=\; -\left(A^{(n)}w_x^{(n)} + A^{(n+1)}v_x^{(n+1)}\right) + G^{(n+1)} - G^{(n)} \tag{6.89}$$

with

$$G^{(n)}(t,x) \;=\; \frac{\mathbf{H}\left[w^{(n)}(t,\cdot)+\varphi^{(n)}(t,\cdot)\right](x)}{a_1^{(n)}(t)-a_2^{(n)}(t)} - a_n(t,x)\cdot\left[\phi_0'(x-t)+\phi_0'(x)\right].$$

We split

$$w^{(n)} \;=\; v_{1,n} + v_{2,n}, \qquad v_{2,n}(t,x) = \begin{cases} w(t,0-)\cdot\eta(x), & x<0,\\[2mm] w(t,0+)\cdot\eta(x), & 0<x. \end{cases}$$

Recalling the definition of $B^{(w)}$ and $C^{(w)}$ in (6.26)-(6.27), we write

$$\begin{aligned} G^{(n)}(t,x)+\phi_0'(x-t) &= -\frac{[v_{2,n}(t,x)-v_{2,n}(t,t)]\cdot\phi_0'(x-t)+[v_{1,n}(t,x)-v_{1,n}(t,0)]\cdot\phi_0'(x)}{a_1^{(n)}(t)-a_2^{(n)}(t)} \\ &\quad + \frac{B^{(v_{2,n})}(t,x)+C^{(v_{1,n})}(t,x)+\mathbf{H}[\varphi^{(n)}(t,\cdot)](x)-\varphi^{(n)}(t,x)\cdot[\phi_0(x-t)+\phi_0(x)]'}{a_1^{(n)}(t)-a_2^{(n)}(t)}. \end{aligned}$$

Here it is important to note that $W_j^{(n)}=v_j^{(n)}-v_j^{(n-1)}$ satisfies

$$\|W_j^{(n)}(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{t,0\})}\le\mathcal{O}(1)\cdot\left(M^{(n)}(t)+\left|W^{(n)}(t,0-)\right|+\left|W^{(n)}(t,0+)\right|\right)\le\mathcal{O}(1)\cdot\beta^{(n)}(t),$$

while, (6.31) implies,

$$\|v_j^{(n)}(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{t,0\})}\le\mathcal{O}(1)\|w^{(n)}\|_{H^1(\mathbb{R}\setminus\{t,0\})}\le\mathcal{O}(1)M_0.$$

Recalling (6.31), (3.10), (3.20), (6.34), (3.6), (3.5), and (6.36), we get

$$\begin{cases} \left|\left(G^{(n+1)}-G^{(n)}\right)(\tau,x)\right| \le \Gamma_1\cdot\beta^{(n+1)}(\tau), \quad \tau<x<0, \\ \left\|(G^{(n+1)}-G^{(n)})(\tau,\cdot)\right\|_{H^1(\mathbb{R}\setminus[-\delta,\delta]\cup[\tau-\delta,\tau+\delta])} \le \Gamma_1\cdot\dfrac{\beta^{(n+1)}(\tau)}{\delta^{1/2}}, \end{cases} \tag{6.90}$$

for some positive constant Γ_1. Furthermore, we have for all $x[-\frac{1}{2},\frac{1}{2}]\setminus\{\tau,0\}$ that

$$\begin{cases} \left|A^{(n)}w_x^{(n)}(\tau,x)\right|\le\Gamma_2\beta^{(n)}(\tau), \quad \left|A^{(n)}v_x^{(n)}(\tau,x)\right|\le\Gamma_2\beta^{(n)}(\tau)\cdot\left(|x|^{-1/2}+|x-\tau|^{-1/2}\right), \\ \left\|A^{(n)}w_x^{(n)}(\tau,\cdot)\right\|_{H^1(\mathbb{R}\setminus\{\tau,0\})}\le\Gamma_2\beta^{(n)}(\tau), \quad \left\|A^{(n)}v_x^{(n)}(\tau,\cdot)\right\|_{H^1(\mathbb{R}\setminus[-\delta,\delta]\cup[\tau-\delta,\tau+\delta])}\le\dfrac{\Gamma_2\beta^{(n)}(\tau)}{\delta^{1/2}}, \end{cases} \tag{6.91}$$

for some constant $\Gamma_2>0$, dependent on M_0, b ,δ_1, and δ_2. Hence, if $\tau_0<0$ is sufficiently close to 0, we have, using Duhamel's formula and (6.71), for all $\tau\in[\tau_0,0[$ that

$$\begin{aligned} \left\|Z^{(n+1)}(\tau,\cdot)\right\|_{H^1(\mathbb{R}\setminus\{\tau,0\})} &\le \frac{3}{2}\int_{\tau_0}^{\tau}\left\|\left[G^{(n+1)}-G^{(n)}-A^{(n)}w_x^{(n)}-A^{(n+1)}v_x^{(n+1)}\right](t,\cdot)\right\|_{H^1(\mathbb{R}\setminus I_t^\tau)}dt \\ &\le \frac{3}{2}(\Gamma_1+\Gamma_2)\cdot\int_{\tau_0}^{\tau}\beta^{(n)}(t)+\beta^{(n+1)}(t)\cdot\gamma_0^{-1/2}(\tau-t)^{-1/2}dt. \end{aligned}$$

Thus, there exists a constant $\Gamma_3>0$ dependent on M_0, b, δ_1, and δ_2 such that

$$\left\|Z^{(n+1)}(\tau,\cdot)\right\|_{H^1(\mathbb{R}\setminus\{\tau,0\})} \le \Gamma_3\cdot\left(\beta^{(n)}(\tau)\cdot|\tau_0-\tau|+\beta^{(n+1)}(\tau)\cdot|\tau_0-\tau|^{1/2}\right). \tag{6.92}$$

2. We establish a bound on $\left|Z^{(n+1)}(\tau,0\pm)\right|$. Given any $\tau_0\le\tau\le0$, let $t\mapsto x_2(t)\doteq x(t;\tau,0-)$ be the characteristic, which reaches the origin at time τ from the negative side. Since

$$Z^{(n+1)}(\tau_0,x)\equiv 0, \qquad \left|Z^{(n+1)}(\tau,0+)\right|\le\|Z^{(n+1)}(\tau,\cdot)\|_{H^1(\mathbb{R}\setminus\{\tau,0\})}, \tag{6.93}$$

we have

$$\begin{aligned} \left|Z^{(n+1)}(\tau,0-))\right| &\le \int_{\tau_0}^{\tau}\left|(-A^{(n)}w_x^{(n)}-A^{(n+1)}v_x^{(n+1)}+G^{(n+1)}-G^{(n)})(t,x_2(t))\right|dt \\ &\le (\Gamma_1+\Gamma_2)\cdot\int_{\tau_0}^{\tau}\beta^{(n)}(t)+\beta^{(n+1)}(t)\cdot\left(1+2\gamma_0^{-1/2}(\tau-t)^{-1/2}\right)dt \\ &\le \Gamma_4\cdot\left(\beta^{(n)}(\tau)\cdot|\tau-\tau_0|+\beta^{(n+1)}(\tau)\cdot|\tau-\tau_0|^{1/2}\right), \end{aligned} \tag{6.94}$$

where we used (6.74) and Γ_4 denotes a positive constant dependent on M_0, b, δ_1, and δ_2. Combining (6.92) -(6.94), we end up with

$$\begin{aligned}\left\|Z^{(n+1)}(\tau,\cdot)\right\|_{H^1(\mathbb{R}\setminus\{\tau,0\})}&+\left|Z^{(n+1)}(\tau,0-)\right|+\left|Z^{(n+1)}(\tau,0+)\right|\\ &\leq \Gamma_5\cdot\left(\beta^{(n)}(\tau)\cdot|\tau-\tau_0|+\beta^{(n+1)}(\tau)\cdot|\tau-\tau_0|^{1/2}\right),\end{aligned}$$

where $\Gamma_5>0$ denotes a constant dependent on M_0, b, δ_1, and δ_2.

3. From (6.34), it holds that $W^{(n+1)}(\tau,\tau\pm)=Z^{(n+1)}(\tau,\tau\pm), W^{(n+1)}(\tau,0\pm)=Z^{(n+1)}(\tau,0\pm)$, and

$$\begin{aligned}&\left\|V^{(n+1)}(\tau,\cdot)\right\|_{H^1(\mathbb{R}\setminus\{\tau,0\})}\\ &\quad\leq \Gamma_6\cdot\left(\left|W^{(n+1)}(\tau,\tau-)\right|+\left|W^{(n+1)}(\tau,\tau+)\right|+\left|W^{(n+1)}(\tau,0-)\right|+\left|W^{(n+1)}(\tau,0+)\right|\right)\\ &\quad= \Gamma_6\cdot\left(\left|Z^{(n+1)}(\tau,\tau-)\right|+\left|Z^{(n+1)}(\tau,\tau+)\right|+\left|Z^{(n+1)}(\tau,0-)\right|+\left|Z^{(n+1)}(\tau,0+)\right|\right)\\ &\quad\leq 3\Gamma_6\cdot\left(\|Z^{(n+1)}(\tau,\cdot)\|_{H^1(\mathbb{R}\setminus\{\tau,0\})}+\left|Z^{(n+1)}(\tau,0-)\right|+\left|Z^{(n+1)}(\tau,0+)\right|\right)\end{aligned}$$

for some positive constant Γ_6. Thus, we end up with

$$\begin{aligned}\beta^{(n+1)}(\tau)&\leq(1+3\Gamma_6)\left(\|Z^{(n+1)}(\tau,\cdot)\|_{H^1(\mathbb{R}\setminus\{\tau,0\})}+\left|Z^{(n+1)}(\tau,0-)\right|+\left|Z^{(n+1)}(\tau,0+)\right|\right)\\ &\leq\Gamma_7\left(\beta^{(n)}(\tau)\cdot|\tau-\tau_0|+\beta^{(n+1)}(\tau)\cdot|\tau-\tau_0|^{1/2}\right).\end{aligned}$$

Provided that $\tau_0<0$ is sufficiently close to 0, we obtain that

$$\beta^{(n+1)}(\tau)\ \leq\ \beta^{(n)}(\tau)/2\qquad\text{for all }\tau\in[\tau_0,0[.$$

Thus, (6.88) holds for all $\tau\in[\tau_0,0]$, and the sequence of approximations $w^{(n)}(\tau,\cdot)$ is Cauchy in the space $H^1(\mathbb{R}\setminus\{\tau,0\})$, and hence it converges to a unique limit $w(\tau,\cdot)$.

It remains to check that this limit function w is an entropic solution, i.e., it satisfies, cf. (6.5), (6.7), and (6.13),

$$\left(w+\varphi^{(w)}\right)(t_0,x_0)=(\overline{w}+\overline{\varphi})(x(\tau_0))+\int_{\tau_0}^{t_0}\frac{\mathbf{H}\left[w+\varphi^{(w)}\right](t,x(t))}{a_1(t)-a_2(t)}dt,$$

where $t\mapsto x(t;t_0,x_0)$ denotes the characteristics curve, obtained by solving $\dot{x}=a(t,x,w)$ with $x(t_0)=x_0$. This follows from slightly rewriting (2.19), which yields

$$\begin{aligned}\left(w^{(n+1)}+\varphi^{n+1}\right)(t_0,x_0)\ &=\ (\overline{w}+\overline{\varphi})(x_n(\tau_0))+\int_{\tau_0}^{t_0}\frac{\mathbf{H}\left[w^{(n+1)}+\varphi^{(n+1)}\right](t,x_n(t))}{a_1^{(n+1)}(t)-a_2^{(n+1)}(t)}dt\\ &-\int_{\tau_0}^{t_0}\left(Z^{(n+1)}-\frac{W^{(n+1),-}(t)+W^{(n+1),+}(t)}{2}\right)\frac{\varphi_x^{(n+1)}(t,x_n(t))}{a_1^{(n+1)}(t)-a_2^{(n+1)}(t)}dt\\ +\int_{\tau_0}^{t_0}&\left[\frac{\varphi_x^{(n+1)}(t,x_n(t))}{a_1^{(n+1)}(t)-a_2^{(n+1)}(t)}-\frac{\varphi_x^{(n+1)}(t,x_n(t))}{a_1^{(n)}(t)-a_2^{(n)}(t)}\right]\cdot\left(w^{(n)}-\frac{w^{(n),-}(t)+w^{(n),+}(t)}{2}+\varphi^{(n)}\right)dt\end{aligned}$$

where $t \mapsto x_n(t)$ denotes the characteristic curve, obtained by solving (6.61) with $x_n(t_0) = x_0$.

Finally, to prove uniqueness, assume that $\widetilde{w}$ and w are two entropic solutions. We define

$$W \doteq \widetilde{w} - w, \qquad \beta(\tau) \doteq \sup_{t \in [0,\tau]} \Big[\|W(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{t,0\})} + |W(t,0-)| + |W(t,0+)| \Big].$$

The arguments used in the previous steps now yield the inequality

$$\beta(\tau) \leq \beta(\tau)/2,$$

and this implies $Z(\tau) = 0$ for all $\tau \in [\tau_0, 0]$, completing the proof. □

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