The Vanishing Viscosity Limit for a System of H-J Equations Related to a Debt Management Problem

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Abstract

The paper studies a system of Hamilton-Jacobi equations, arising from a model of optimal debt management in infinite time horizon, with exponential discount and a bankruptcy risk. For a stochastic model with positive diffusion, the existence of an equilibrium solution is obtained by a topological argument. Of particular interest is the limit of these viscous solutions, as the diffusion parameter approaches zero. Under suitable assumptions, this (possibly discontinuous) limit can be interpreted as an equilibrium solution to a non-cooperative differential game with deterministic dynamics.

1 Introduction

This paper is concerned with a Hamilton-Jacobi equation with discontinuous coefficients

$$(r+\rho(x))V \;=\; H(x,p(x),V'), \tag{1.1}$$

together with a family of smooth viscous approximations

$$(r+\rho(x))V_n \;=\; H(x,p_n(x),V_n')+\frac{\sigma_n^2x^2}{2}V_n''. \tag{1.2}$$

The main motivation comes from a problem of optimal debt management, in infinite time horizon, with exponential discount and in the presence of a bankruptcy risk. As in [8, 9, 10, 16], this is modeled as a noncooperative game between a borrower and a pool of risk-neutral lenders. Here the independent variable x is the debt-to-income ratio, V is the value function for the borrower (i.e., his expected cost, under optimal play) while the function $\rho(x)\geq 0$ accounts for an instantaneous bankruptcy risk, which increases with the size of the debt. A distinguished feature of this model is that the discounted bond price $p(\cdot)$ is is not given a priori, but depends on the entire future evolution of the system, determined by the feedback control $u = u(x)$. Indeed, if the borrower chooses to keep the debt at a high level, then the lenders will buy the bonds at a deeply discounted price $p(x) < 1$, to compensate for the risk of losing their capital. The scalar equation (1.1) must thus be supplemented by an additional ODE for the function p. In particular, at points where the feedback control $u(\cdot)$ is discontinuous, also $p(\cdot)$ can have a jump.

In the first part of this paper we construct a solution to the second order equation (1.2).This can be interpreted as a Nash equilibrium solution to a differential game with stochastic dynamics. In the final sections, we study the limits $V_n \to V$ and $p_n \to p$ as the diffusion coefficient $\sigma_n \to 0$. Under suitable assumptions, we show that this limit yields a solution to (1.1) and determines an equilibrium solution to a differential game with deterministic dynamics.

The main technical difficulties in the analysis stem from the fact that, while all functions p_n are smooth, the limit p may well be discontinuous. Hamilton-Jacobi equations with discontinuous hamiltonian function have attracted much interest in recent years [4, 5, 6, 7, 13, 14, 15, 17]. The key difference in our equations is that, since we are modeling a game, the hamiltonian function for the borrower is not given a priori, but is determined by the optimal reply adopted by the lenders. In turn, this reply depends on the feedback control chosen by the borrower.

The remainder of the paper is organized as follows. In Section 2 we explain how the equations (1.1) arise from a problem of optimal debt management, modeled as a game between a borrower and a pool of risk-neutral lenders.

In Section 3 we prove the existence of solutions to the second order boundary value problem, for $\sigma > 0$. In the case where the bankruptcy risk is

$$\rho(x) \;=\; \begin{cases} 0 & \text{if } \; x < M, \\ +\infty & \text{if } \; x = M, \end{cases}$$

this result was proved in [9]. Here we consider more general functions $\rho(\cdot)$, using similar ideas. The main new contribution of the present paper is the analysis of the limit $\sigma \to 0$, studied in Section 4. Our main result shows the existence of a limit, for a suitable subsequence $\sigma_n \to 0$. Under a "generic" assumption on the limit functions $V(\cdot)$ and $p(\cdot)$, we prove that this limit yields a Nash equilibrium solution to a noncooperative game, where the total debt follows a deterministic evolution equation. The analysis relies on classical techniques from the theory of viscosity solutions [2, 3, 11, 12], and a number of additional arguments to handle this specific problem.

2 Derivation of the model

The model includes the following constants:

- r = discount rate,
- λ = rate at which the principal is payed back,
- B = bankruptcy cost to the borrower,

and the variables

- t = time, measured in years,
- $x(t)$ = total debt, measured as a fraction of the yearly income of the borrower,
- $u(t) \in [0,1]$ = payment rate, as a fraction of the income,
- $p(t) \in [0,1]$ = discounted bond price,
- $\rho(x)$ = instantaneous risk of bankruptcy,
- T_b = random time when bankruptcy occurs,
- $\theta(x(T_b)) \in [0,1]$ = fraction of the outstanding capital which can be recovered by the lenders in case of bankruptcy.
- $L(u)$ = cost to the borrower for implementing the control u.

We regard u as the control variable for the borrower.

Observe that, in this model, ρ is a function of x. If the total debt is large, so is the bankruptcy risk. Concerning the functions ρ, L, θ, we shall assume

(A1) *There exists a constant M, with $rM > 1$, such that the following holds. The function ρ is continuously differentiable for $x \in [0, M[$, and satisfies*

$$\rho(0) = 0 \qquad \rho'(x) \geq 0, \qquad \lim_{x \to M-} \rho(x) \;=\; +\infty. \tag{2.1}$$

(A2) *The map $\theta : [0, M] \mapsto [0,1]$ is Lipschitz continuous, nonincreasing, and strictly positive.*

(A3) *The cost function L is twice continuously differentiable for $u \in [0,1[$ and satisfies*

$$L(0) = 0, \qquad L' > 0, \quad L'' > 0, \qquad L(1) = \lim_{u \to 1-} L(u) \;\in I\!\!R \cup \{+\infty\}. \tag{2.2}$$

The constant M in **(A1)** denotes a maximum size of the debt, beyond which bankruptcy immediately occurs. To motivate **(A2)**, assume that the borrower owns an amount R_0 of collateral (real estate, gold reserves, etc...) to back up his debt. In case of bankruptcy, this will be divided among lenders. In this case the function θ will have the form

$$\theta(x) \;=\; \min\left\{1,\, \frac{R_0}{x}\right\}. \tag{2.3}$$

In the following we define

$$T^M \;\doteq\; \inf\bigl\{t > 0\,;\;\; x(t) = M\bigr\} \;\in\; I\!\!R \cup \{+\infty\} \tag{2.4}$$

the first time when the debt reaches this maximum value. Calling T_b the random time at which bankruptcy occurs, its distribution is determined as follows. If at time τ the borrower is not yet bankrupt and the total debt is $x(\tau) = y$, then the probability that bankruptcy will occur shortly after time τ is measured by

$$\text{Prob.}\Bigl\{T_b \in [\tau,\, \tau + \varepsilon] \;\Big|\; T_b > \tau\,,\; x(\tau) = y\Bigr\} \;=\; \rho(y) \cdot \varepsilon + o(\varepsilon), \tag{2.5}$$

where $o(\varepsilon)$ denotes a higher order infinitesimal as $\varepsilon \to 0$. The probability that the borrower is not yet bankrupt at time $t > 0$ is thus computed as

$$P(t) \doteq Prob.\{T_b > t\} = \begin{cases} \exp\left\{-\int_0^t \rho(x(\tau))\,d\tau\right\} & \text{if} \quad t < T^M, \\ 0 & \text{if} \quad t \geq T^M. \end{cases} \tag{2.6}$$

Notice that this depends on the function $\tau \mapsto x(\tau)$. If debt is maintained at a higher level, then the probability of bankruptcy increases. This reflects lack of confidence in the ability of the borrower to eventually repay the debt. At any random time T_b, panic may suddenly spread among the investors, leading to bankruptcy.

The discounted bond price $p = p(x)$ requires some explanation. If an investor purchases at time $t = 0$ a coupon with unit nominal value, he receives a stream of payments at all future times. The repayment rate is

$$\psi(t) = (\lambda + r)e^{-\lambda t}.$$

The primary capital is payed back at rate λ, so that the outstanding value of the loan at time $t > 0$ is $e^{-\lambda t}$. In addition, the borrower pays an interest r. If bankruptcy never occurs, then the total payoff for the lender (exponentially discounted in time) is

$$\Psi = \int_0^{+\infty} e^{-rt}(\lambda + r)e^{-\lambda t}\,dt = 1\,. \tag{2.7}$$

However, if bankruptcy occurs at a random time T_b, the payoff will be

$$\Psi = \int_0^{T_b} e^{-rt}(\lambda + r)e^{-\lambda t}\,dt + e^{-rT_b}\theta(x(T_b))e^{-\lambda T_b}\,. \tag{2.8}$$

If $\theta = 1$, so that the outstanding capital is recovered in full, then again $\Psi = 1$. In general, however, $\theta < 1$. Assuming that lenders are risk-neutral, the coupon will thus be bought at the discounted price

$$p = E\left[\int_0^{T_b} e^{-rt}(\lambda + r)e^{-\lambda t}\,dt + e^{-rT_b}\theta(x(T_b))e^{-\lambda T_b}\right]. \tag{2.9}$$

Here E denotes the expected value of the given quantity, depending on the random variable T_b.

We observe that the expected cost to the borrower can be computed by

$$J \doteq E\left[\int_0^{T_b} e^{-rt}L(u(t))\,dt + B\,e^{-rT_b}\right]. \tag{2.10}$$

To complete the model, we still need an equation describing the evolution of the debt. Assume that the yearly income $Y(t)$ of the borrower grows with rate μ and is subject to a stochastic perturbation:

$$dY(t) = \mu Y(t)\,dt + \sigma Y(t)\,dW. \tag{2.11}$$

Here W denotes standard Brownian motion. Calling $U(t)$ the repayment rate, the nominal value of the outstanding debt evolves according to the ODE

$$\dot{X}(t) = -\lambda X(t) + \frac{(\lambda + r)X(t) - U(t)}{p(t)}. \tag{2.12}$$

We define the debt-to-income ratio $x \doteq X/Y$, and set $u \doteq U/Y$. By Ito's formula, (2.11) and (2.12) yield the stochastic evolution equation

$$dx \;=\; \left[(\sigma^2-\lambda-\mu)x(t) + \frac{(\lambda+r)x(t)-u(t)}{p(t)}\right] dt + \sigma x(t)\, dW\,, \qquad\qquad x(0) \;=\; y\,. \tag{2.13}$$

We are mainly interested in controls in feedback form:

$$u \;=\; u^*(x) \quad \text{for} \quad x \in [0, M[\,.$$

Motivated by the optimality conditions, it is convenient to define

$$u^\sharp(p,\xi) \;\doteq\; \arg\min_{\omega\in[0,1]} \left\{L(\omega) - \frac{\xi}{p}\omega\right\} \;=\; \begin{cases} 0 & \text{if} \quad s \leq L'(0), \\ (L')^{-1}\left(\dfrac{\xi}{p}\right) & \text{if} \quad \dfrac{\xi}{p} > L'(0). \end{cases} \tag{2.14}$$

$$\xi^\sharp(x,p) \;\doteq\; \operatorname*{argmax}_{\xi\geq 0} \; H(x,p,\xi), \tag{2.15}$$

Definition 1. *Consider a value function $V(\cdot)$, a feedback control $u^*(\cdot)$, and a price function $p(\cdot)$, defined for $x \in [0, M[\,$. We say that the triple (V, p, u^*) yields a Nash equilibrium solution to the debt management problem if the following holds.*

(i) Given the price $p = p(x)$, one has that $V(\cdot)$ is the value function and $u^*(x) = u^\sharp(p(x), V'(x))$ is the optimal feedback control, in connection with the stochastic control problem

$$\textit{minimize:} \;\; E\left[\int_0^{T_b} e^{-rt} L(u(t))\, dt + B\, e^{-rT_b}\right] \tag{2.16}$$

subject to

$$dx \;=\; \left[(\sigma^2-\lambda-\mu)x + \frac{(\lambda+r)x-u}{p(x)}\right] dt + \sigma x\, dW\,. \tag{2.17}$$

Here the distribution of the random time T_b is determined by (2.5).

(ii) Given the feedback control $u = u^*(x)$ in (2.17), for every $x_0 \in [0, M[$ one has

$$p(x_0) \;=\; E\left[\int_0^{T_b} e^{-rt}(\lambda+r)e^{-\lambda t}\, dt + e^{-rT_b}\theta(x(T_b))e^{-\lambda T_b}\right]_{x(0)=x_0}\,. \tag{2.18}$$

The theory of stochastic optimal control yields a second order ODE satisfied by the value function V. Using the Feynman-Kac formula [18] to derive an equation for $p(.)$, we eventually obtain the system

$$(r+\rho(x))V \;=\; \min_{\omega\in[0,1]} \left\{L(\omega) - \frac{V'}{p}\,\omega\right\} + \rho(x)B + V'\cdot\left(\frac{\lambda+r}{p} + \sigma^2 - \lambda - \mu\right)x + \frac{\sigma^2x^2}{2}\cdot V''(x)\,, \tag{2.19}$$

$$\rho(x)(p-\theta(x)) \;=\; (r+\lambda)(1-p) + p'\cdot\left[(\sigma^2-\lambda-\mu)x + \frac{(\lambda+r)x - u^\sharp(p,V')}{p}\right] + \frac{\sigma^2x^2}{2}\cdot p''(x)\,. \tag{2.20}$$

This is supplemented by the boundary conditions

$$\begin{cases} V(0) &= 0, \\ V(M) &= B, \end{cases} \qquad \begin{cases} p(0) &= 1, \\ p(M) &= \theta(M)\,. \end{cases} \tag{2.21}$$

The optimal feedback control is then $u^*(x) = u^\sharp(p(x), V'(x))$, as in (2.14).

3 Solutions to the viscous equation

In this section we prove the existence of a solution to the boundary value problem (2.19)–(2.21), which in turn provides an equilibrium solution to the debt management problem.

Consider the functions

$$H(x,p,\xi) \doteq \min_{\omega\in[0,1]} \left\{ L(\omega) - \frac{\xi}{p}\omega \right\} + \xi \left(\frac{\lambda + r}{p} - (\lambda+\mu) \right) x + \rho(x) B\,, \tag{3.1}$$

$$\mathcal{H}(x,p,\xi,\sigma) \doteq H(x,p,\xi) + \sigma^2 x\xi\,. \tag{3.2}$$

Toward the analysis of H, observe that

$$\min_{\omega\in[0,1]} \{L(\omega) - s\,\omega\} = L\big(\omega^\sharp(s)\big) - s\,\omega^\sharp(s), \tag{3.3}$$

where

$$\omega^\sharp(s) = \begin{cases} 0 & \text{if } \; s \le L'(0), \\ (L')^{-1}(s) & \text{if } \; s > L'(0). \end{cases} \tag{3.4}$$

An elementary computation yields

$$H_p(x,p,\xi) = \frac{\xi}{p^2} \left[\omega^\sharp\Big(\frac{\xi}{p}\Big) - (\lambda+r)x \right], \tag{3.5}$$

$$H_\xi(x,p,\xi) = -\frac{1}{p}\omega^\sharp\Big(\frac{\xi}{p}\Big) + \left(\frac{\lambda+r}{p} - (\lambda+\mu) \right) x\,, \tag{3.6}$$

$$H_{\xi\xi}(x,p,\xi) = \begin{cases} 0 & \text{if } \; \dfrac{\xi}{p} \le L'(0), \\ \left[p^2\, L''\left(\omega^\sharp\Big(\dfrac{\xi}{p}\Big) \right) \right]^{-1} & \text{if } \; \dfrac{\xi}{p} > L'(0). \end{cases} \tag{3.7}$$

Lemma 1. *Let the conditions (2.2) hold. Then, for all $\xi \ge 0$ and $0 < p \le 1$, the function $\mathcal{H}(x,p,\xi,\sigma)$ in (3.2) satisfies*

$$\begin{aligned} \rho(x)B + \left(\frac{(\lambda+r)x - 1}{p} + (\sigma^2 - \lambda - \mu)x \right)\xi \; &\le \; \mathcal{H}(x,p,\xi,\sigma) \\ &\le \; \rho(x)B + \left(\frac{\lambda + r}{p} + \sigma^2 - \lambda - \mu \right) x\xi, \end{aligned} \tag{3.8}$$

$$\frac{(\lambda+r)x-1}{p}+(\sigma^2-\lambda-\mu)x \;\le\; \mathcal{H}_\xi(x,p,\xi,\sigma) \;\le\; \left(\frac{\lambda+r}{p}+\sigma^2-\lambda-\mu\right)x, \tag{3.9}$$

$$-\frac{\xi}{p^2}(\lambda+r)x \;\le\; \mathcal{H}_p(x,p,\xi,\sigma) \;\le\; \frac{\xi}{p^2}\left(1-(\lambda+r)x\right). \tag{3.10}$$

Moreover, for every $x,p>0,\sigma\ge 0$, the map $\xi\mapsto\mathcal{H}(x,\xi,p,\sigma)$ is concave down and satisfies

$$\mathcal{H}(x,p,0,\sigma) \;=\; \rho(x)B, \tag{3.11}$$

$$\mathcal{H}_\xi(x,p,0,\sigma) \;=\; \left(\frac{\lambda+r}{p}+\sigma^2-\lambda-\mu\right)x, \tag{3.12}$$

$$\lim_{\xi\to+\infty}\mathcal{H}(x,p,\xi,\sigma) \;=\; \begin{cases} -\infty, & \text{if } \frac{1}{p} > \left(\frac{\lambda+r}{p}-\lambda+\sigma^2-\mu\right)x, \\ +\infty, & \text{if } \frac{1}{p} \le \left(\frac{\lambda+r}{p}-\lambda+\sigma^2-\mu\right)x. \end{cases} \tag{3.13}$$

Proof. The properties (3.8)-(3.9) and (3.11)-(3.13) follow directly from Lemma 2.1 in [9]. By (2.14), the estimate (3.10) follows from

$$\mathcal{H}_p(x,p,\xi,\sigma) \;=\; \frac{\xi}{p^2}\Big(u^\sharp(p,\xi)-(\lambda+r)x\Big).$$

□

Theorem 1. *Under the assumptions* **(A1)–(A3)***, the boundary value problem (2.19)–(2.21) admits at least one solution $(V,p)\in\mathcal{C}^2$. Moreover $V'(x)\ge 0$ and $p'(x)\le 0$, for all $0<x<M$.*

Proof. 1. In terms of the Hamiltonian function $\mathcal{H}$ at (3.2), the equations (2.19)-(2.20) can be written as

$$\begin{cases} 0 \;=\; -rV+\mathcal{H}(x,V_x,p,\sigma)+\dfrac{(\sigma x)^2}{2}V_{xx}, \\ 0 \;=\; -\rho(x)\left(p-\theta(x)\right)+(r+\lambda)(1-p)+\mathcal{H}_\xi(x,V_x,p,\sigma)p_x+\dfrac{(\sigma x)^2}{2}p_{xx}\,. \end{cases} \tag{3.14}$$

Following an argument used in [9], a solution of (3.14) satisfying the boundary conditions (2.21) will be obtained by constructing stationary solutions of a related parabolic problem. For any $\varepsilon>0$, let ρ_ε be a $\mathcal{C}^1$ function such that

$$\rho_\varepsilon(x) \;=\; \begin{cases} \rho(x) & \text{if } \rho(x)\le\varepsilon^{-1}, \\ 2\varepsilon^{-1} & \text{if } \rho(x)\ge 2\varepsilon^{-1}. \end{cases} \tag{3.15}$$

Consider the parabolic system

$$\begin{cases} V_t \;=\; -\left(r+\rho_\varepsilon(x)\right)V+\mathcal{H}(x,V_x,p,\sigma)+\left(\varepsilon+\dfrac{(\sigma x)^2}{2}\right)V_{xx}, \\ p_t \;=\; -\rho_\varepsilon(x)\left(p-\theta(x)\right)+(r+\lambda)(1-p)+\mathcal{H}_\xi(x,V_x,p,\sigma)p_x+\left(\varepsilon+\frac{(\sigma x)^2}{2}\right)p_{xx}, \end{cases} \tag{3.16}$$

with boundary data

$$\begin{cases} V(t,0) \;=\; 0, \\ V(t,M) \;=\; B, \end{cases} \qquad \begin{cases} p(t,0) \;=\; 1, \\ p(t,M) \;=\; \theta(M)\,. \end{cases} \tag{3.17}$$

Notice that these ε-approximations remove the singularities at $x = 0$ (where the diffusion coefficient would vanish) and at $x = M$ (where $\rho = +\infty$). Indeed, the system (3.16) is uniformly parabolic, with $\mathcal{C}^1$ coefficients.

2. Adopting a semigroup notation, let $t \mapsto (V(t), p(t)) = S_t(V_0, p_0)$ be the solution of the parabolic system (3.16-3.17) with initial data

$$V(0,x) \;=\; V_0(x), \qquad p(0,x) \;=\; p_0(x).$$

Consider the closed, convex set of functions

$$\begin{aligned}\mathcal{D} \;\doteq\; \Big\{ &(V,p) : [0,M] \mapsto [0,B] \times [\theta(M),1]; \quad V,p \in \mathcal{C}^2, \\ &V_x \geq 0 \;\text{ and the boundary conditions (2.21) hold} \Big\}.\end{aligned} \tag{3.18}$$

We claim that the domain $\mathcal{D}$ is positively invariant under the semigroup $\{S_t\}$.
Indeed, defining the constant functions

$$V^+(t,x) \doteq B, \qquad V^-(t,x) \doteq 0, \qquad p^+(t,x) \doteq 1, \qquad p^-(t,x) \doteq \theta(M),$$

one easily checks that V^+ is a supersolution and V^- is a subsolution of the first equation in (3.16). Namely,

$$-(r+\rho(x))V^+ + \mathcal{H}(x, V_x^+, p, \sigma) + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) V_{xx}^+ \leq 0, \quad V^+(t,0) \geq 0, \quad V^+(t,M) \geq B,$$

$$-(r+\rho(x))V^- + \mathcal{H}(x, V_x^-, p, \sigma) + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) V_{xx}^- \geq 0, \quad V^-(t,0) \leq 0, \quad V^-(t,M) \leq B.$$

Similarly, p^+ is a supersolution and p^- is a subsolution of the second equation in (3.16). For any initial data $(V_0, p_0) \in \mathcal{D}$, the solution of (3.16)-(3.17) will satisfy

$$0 \leq V(t,x) \leq B, \qquad\qquad \theta(M) \leq p(t,x) \leq 1, \tag{3.19}$$

for all $x \in [0,M]$ and $t \geq 0$. As a consequence,

$$\begin{cases} V_x(t,0) \;\geq\; 0, \\ V_x(t,M) \;\geq\; 0, \end{cases} \qquad\qquad \begin{cases} p_x(t,0) \;\leq\; 0, \\ p_x(t,M) \;\leq\; 0. \end{cases} \tag{3.20}$$

3. Differentiating the first equation in (3.16) w.r.t. x one obtains

$$V_{xt} \;=\; -(r+\rho(x))V_x + \rho'(x)(B-V) + (\mathcal{H}_x - \rho'(x)B) + \mathcal{H}_\xi V_{xx} + \mathcal{H}_p p_x + \sigma^2 x V_{xx} + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) V_{xxx}\,. \tag{3.21}$$

By (3.5),(3.11), (3.12), and (3.19), we have

$$\mathcal{H}_x(x,p,0,\sigma) \;=\; \rho'(x)B, \qquad \mathcal{H}_\xi(x,p,0,\sigma) \;=\; \mathcal{H}_p(x,p,0,\sigma) \;=\; 0, \qquad V \leq B.$$

We thus conclude that $V_x = 0$ is a subsolution of (3.21), proving the positive invariance of the domain $\mathcal{D}$.

4. We observe that the functions $\mathcal{H}$ and $\mathcal{H}_\xi$ appearing in the equation (3.14) depend Lipschitz continuously on all their arguments. Moreover, according to the bounds (3.8)–(3.10), they have sublinear growth w.r.t. the variable $\xi = V_x$. One can thus apply Theorem 3 in [1] and obtain the existence of a steady state $(V^\varepsilon, p^\varepsilon) \in \mathcal{D}$ for the parabolic problem system (3.16)-(3.17).

We remark that the analysis in [1] applies to parabolic systems in any space dimension. In this general case, by Schauder's estimates, the solution has $\mathcal{C}^{2,\alpha}$ regularity. In our case, however, the domain $[0, M]$ is 1-dimensional. Solving (3.14) for V_{xx} and p_{xx} and using the fact that $\mathcal{H}, \mathcal{H}_\xi$ are Lipschitz continuous we conclude that these functions $V^\varepsilon, p^\varepsilon$ are twice differentiable with Lipschitz continuous second derivatives.

We claim that $p^\varepsilon_x(x) \le 0$ for all $x \in [0, M]$. Indeed, assume by contradiction that $p^\varepsilon_x(x_0) > 0$ at some point x_0. Using the boundary condition (2.21) and the bounds (3.20), we can define the two points

$$x_1 \doteq \max\Big\{y;\ \ y < x_0,\ \ p^\varepsilon_x(y) = 0 \text{ and } p^\varepsilon_x(x) \ge 0 \text{ for all } \ x \in [y, x_0]\Big\} \ > \ 0,$$

$$x_2 \doteq \min\Big\{y;\ \ y > x_0,\ \ p^\varepsilon_x(y) = 0 \text{ and } p^\varepsilon_x(x) \ge 0 \text{ for all } \ x \in [x_0, y]\Big\} \ < \ M.$$

It is clear that $x_1 < x_2$, $p^\varepsilon_{xx}(x_2) \le 0 \le p^\varepsilon_{xx}(x_1)$ and $p^\varepsilon(x_1) < p^\varepsilon(x_2)$. On the other hand, by definition we have

$$\begin{aligned} & p^\varepsilon(x_1) - p^\varepsilon(x_2) \\ & = \frac{\rho(x_1)\theta(x_1) + r + \lambda + \frac{\varepsilon + (\sigma x_1)^2}{2} p^\varepsilon_{xx}(x_1)}{\rho(x_1) + r + \lambda} - \frac{\rho(x_2)\theta(x_2) + r + \lambda + \frac{\varepsilon + (\sigma x_2)^2}{2} p^\varepsilon_{xx}(x_2)}{\rho(x_2) + r + \lambda} \\ & \ge \frac{\rho(x_1)\theta(x_1) + r + \lambda}{\rho(x_1) + r + \lambda} - \frac{\rho(x_2)\theta(x_2) + r + \lambda}{\rho(x_2) + r + \lambda} \\ & = -\int_{x_1}^{x_2} \frac{\theta'(x)\rho(x)(\rho(x) + r + \lambda) + \rho'(x)(\theta(x) - 1)(r + \lambda)}{(\rho(x) + r + \lambda)^2}\, dx \ \ge \ 0, \end{aligned}$$

reaching a contradiction. We thus conclude that $p_x(x) \le 0$ for all $x \in [0, M]$.

5. In the remainder of the proof, letting $\varepsilon \to 0$ we shall obtain a limit (V, p), which provides a $\mathcal{C}^2$ solution to the original boundary value problem (2.19)–(2.21). By setting

$$\zeta \ = \ \zeta(x, p, V') \ \doteq \ \left[\frac{(\lambda + r)x - u^\sharp(p, V')}{p} - (\lambda + \mu)x\right], \tag{3.22}$$

the system satisfied by $(V^\varepsilon, p^\varepsilon)$ can be written as

$$\begin{cases} rV \ = \ (\zeta + \sigma^2 x)V' + \rho_\varepsilon(x)(B - V) + L(u^*) + \dfrac{\varepsilon + \sigma^2 x^2}{2} \cdot V''(x)\,, \\ \rho_\varepsilon(x)(p - \theta(x)) \ = \ (\zeta + \sigma^2 x)\, p' \ + (r + \lambda)(1 - p) + \dfrac{\varepsilon + \sigma^2 x^2}{2} \cdot p''(x)\,. \end{cases} \tag{3.23}$$

By (3.12)-(3.13), we can find $\delta > 0$ sufficiently small and $\xi_0 > 0$ such that

$$x \in [0, \delta], \quad p \in [\theta(M), 1], \quad \xi \ge \xi_0 \qquad \Longrightarrow \qquad \mathcal{H}(x, p, \xi, \sigma) \le 0. \tag{3.24}$$

This in turn implies that, if $V'(x) > \xi_0$ for some $x \in [0, \xi]$, then by (2.19) one has $V''(x) \geq 0$. We thus conclude that either $V'(x) \leq \xi_0$ for all $x \in [0, \delta]$, or else the global maximum of V' is attained on the subinterval $[\delta, M]$.

Notice that, since $u^\sharp < 1$ and $r > \mu$, by (3.22) one has

$$(r-\mu)x \ \geq \ 1 \quad \Longrightarrow \quad \zeta > 0.$$

Roughly speaking, this means that if the debt x is large enough it will keep increasing.

Consider the first equation in (3.23). On the region where $\zeta \geq 0$ the first three terms on the right hand side are all nonnegative. Recalling that $V' \geq 0$ and $0 \leq V \leq B$, we thus obtain

$$V''(x) \ \leq \ \frac{2rV}{\sigma^2 x^2} \ \leq \ \frac{2rB}{\sigma^2 x^2}. \tag{3.25}$$

This upper bound on the second derivative immediately yields a uniform bound on V' on $[(r-\mu)^{-1} + \delta, M]$.
Then we consider the interval $[\delta, (r-\mu)^{-1} + \delta]$. By the intermediate value theorem, there exists a point $\hat{x} \in [\delta, (r-\mu)^{-1} + \delta]$ such that

$$V'(\hat{x}) \ = \ \frac{V((r-\mu)^{-1} + \delta) - V(\delta)}{(r-\mu)^{-1} + \delta - \delta} \ \leq \ B(r-\mu). \tag{3.26}$$

The first equation in (3.23) yields an inequality of the form

$$|V''| \ \leq \ c_1 |V'| + c_2, \qquad\qquad x \in [\delta, (r-\mu)^{-1} + \delta], \tag{3.27}$$

for suitable constants c_1, c_2 independent of ε. By Gronwall's lemma, (3.27) and (3.26) yield a uniform bound of V' on $[\delta, (r-\mu)^{-1} + \delta]$. We thus conclude that V' is uniformly bounded for all $x \in [\delta, M]$ and thus is uniformly bounded for all $x \in [0, M]$, independent of ε.

A similar analysis applies to p'. Since H_ξ is uniformly bounded, and $\rho_\varepsilon(x)$ is uniformly bounded on $x \in [\delta, M - \delta]$, for a given $\delta > 0$ independent of ε, we obtain an inequality of the form

$$|p''| \ \leq \ a_1 |p'| + a_2, \qquad\qquad x \in [\delta,\ M - \delta], \tag{3.28}$$

for suitable constants a_1, a_2 independent of ε.

Summarizing the previous analysis, we thus have the bounds

$$\begin{cases} |V'(x)| \ \leq \ C & \text{for all } x \in [0, M], \\ |p'(x)| \ \leq \ C_\delta & \text{for all } x \in [\delta,\ M - \delta], \end{cases} \tag{3.29}$$

for some constants C and C_δ independent of ε.

6. By (3.29) we can extract a subsequence $\varepsilon_n \to 0$ and achieve the convergence

$$\begin{cases} V^{\varepsilon_n}(x) \ \to \ V(x) & \text{uniformly for } x \in [0, M], \\ p^{\varepsilon_n}(x) \ \to \ p(x) & \text{uniformly for } x \in [\delta, M - \delta], \text{ for any } \delta > 0. \end{cases} \tag{3.30}$$

Since all solutions $(V^\varepsilon, p^\varepsilon)$ satisfy the boundary condtions (2.21), by uniform convergence it is clear that the same boundary conditions are satisfied by V.

To prove that also $p(\cdot)$ satisfies the required boundary conditions (2.21), we construct a lower solution p^- and an upper solution p^+ of the second ODE equation in (3.23), independent of ε. Following the proof of Theorem 3.1 in [9], we have

$$1 \;\geq\; p^\varepsilon(x) \;\geq\; p^-(x) \;\doteq\; 1 - cx^\gamma \qquad\qquad \text{for all } x \in [0, x_1],$$

for some suitable constants $c > 0, \gamma > 0, x_1 > 0$.

Next, to construct an upper solution p^+, we choose $\tilde{x}_2$ such that $(r-\mu)^{-1} < x_2 < M$, and take $\delta > 0$ small enough such that $M - \delta > x_2$. Moreover, we introduce the constant

$$\kappa \;\doteq\; \max\left\{\frac{R_0}{M^2},\; \frac{(r+\lambda)[1-\theta(M)]}{x_2(r-\mu)-1},\; \frac{1-\theta(M)}{M-x_2}\right\},$$

and define

$$p^+(x) \;\doteq\; \kappa(M-x) + \theta(M) \qquad \text{for } \; x \in [x_2, M].$$

By construction, $p^+(x_2) \geq 1 \geq p^\varepsilon(\tilde{x}_2)$ and

$$\begin{aligned} &\rho(x)[\theta(x) - \theta(M) - \kappa(M-x)] + (r+\lambda)(1-p^+) - \kappa\mathcal{H}_\xi(x, p^\varepsilon, V', \sigma) \\ &\qquad \leq\; \rho(x)(M-x)\left(\tfrac{R_0}{M^2} - \kappa\right) + (r+\lambda)(1-\theta(M)) - \kappa\left[x_2(r+\mu)-1\right] \;\leq\; 0, \end{aligned}$$

where we used the lower bound (3.9) on H_ξ. We thus conclude that $p^+(x)$ is a upper solution of the second equation in (3.23) on $x \in [x_2, M]$. Hence

$$\theta(M) \;\leq\; p^\varepsilon(x) \;\leq\; p^+(x) \;=\; \kappa(M-x) + \theta(M) \qquad \text{for } \; x \in [x_2, M],$$

for all $\varepsilon > 0$.

7. By the previous analysis, the limit functions V, p in (3.30) satisfy all boundary conditions in (2.21). Observing that the convergence is uniform and the coefficient $\frac{\sigma^2 x}{2}$ is uniformly positive, on every compact subinterval $[\delta,\, M-\delta]$, we now show that (V, p) satisfy the system of second order ODEs (2.19). This is accomplished in three steps.

(i) Using the bounds (3.29) in (3.23) we obtain that the functions V'', p'' are uniformly bounded on the interval $[\delta,\, M-\delta]$, by a constant independent of ε.

(ii) From the boundedness of V'', p'' it follows that the functions V', p' are uniformly Lipschitz continuous on $[\delta,\, M-\delta]$.

(iii) In turn, using the Lipschitz continuity of V', p' in (3.23) we conclude that the functions V'', p'' are uniformly Lipschitz continuous on $[\delta,\, M-\delta]$.

By possibly extracting a further subsequence, we thus obtain the convergence $(V^{\varepsilon_n}, p^{\varepsilon_n}) \to (V, p)$ in the space $\mathcal{C}^2([\delta,\, M-\delta])$. It is now clear that (V, p) provides a classical solution to the system of equations (3.14).

□

4 The vanishing viscosity limit

We now consider a sequence of solutions (V_n, p_n) to the system (2.19)–(2.21), with diffusion coefficient $\sigma = \sigma_n \to 0+$. Recalling (3.22), we write (2.19)-(2.20) in the equivalent form

$$\begin{cases} rV \;=\; (\zeta + \sigma^2 x)V' + \rho(x)(B - V) + L(u^\sharp(p, V')) + \dfrac{\sigma^2 x^2}{2} \cdot V''(x)\,, \\ \rho(x)(p - \theta(x)) \;=\; (\zeta + \sigma^2 x)\, p' \; + (r + \lambda)(1 - p) + \dfrac{\sigma^2 x^2}{2} \cdot p''(x)\,. \end{cases} \tag{4.1}$$

Our first result shows the existence of a limit, as $\sigma \to 0$.

Theorem 2. *Let the assumptions* **(A1)–(A3)** *hold. Let V_n, p_n be a sequence of solutions to the boundary value problem (2.19)–(2.21), with diffusion coefficients $\sigma_n \to 0$.*

Then there exists one point $\hat{x} \in [0, M]$ such that, by possibly extracting a subsequence, one has the uniform convergence $V_n \to V$ on every domain of the form $[0, \hat{x}-\delta] \cup [\hat{x}+\delta, M]$. In addition, one has the pointwise convergence $p_n(x) \to p(x)$ and $V_n'(x) \to V'(x)$, for a.e. $x \in [0, M]$. Also, the limit satisfies the H-J equation

$$(r + \rho(x))V(x) \;=\; H(x, V'(x), p(x)) \tag{4.2}$$

for a.e. $x \in [0, M]$.

One may think of $\hat{x}$ as a threshold for the debt size. When $x < \hat{x}$, the optimal feedback strategy keeps the debt uniformly bounded in time. On the other hand, when $x > \hat{x}$, the debt keeps increasing and reaches the value M in finite time, where bankruptcy instantly follows.

Our second result shows that, under suitable assumptions, the above limit (V, p) yields a Nash equilibrium solution to a problem with deterministic dynamics. Toward this goal, we first introduce an auxiliary function $W : [0, M[\, \mapsto [0, +\infty]$, defined by

$$W(y) \;\doteq\; \sup_{\xi \geq 0} H(y, p(y), \xi) \;=\; \frac{1}{r + \rho(y)} \Big\{ L\Big((\lambda + r)y - (\lambda + \mu)p(y)y \Big) + \rho(y)B \Big\}. \tag{4.3}$$

Recalling (3.6), we can interpret $W(y)$ as the expected cost achieved by keeping the debt at the constant value $x(t) = y$. This is well defined as long as the control needed to keep the debt constant satisfies

$$\bar{u}(y) \;\doteq\; \Big(\lambda + r - (\lambda + \mu)p(y) \Big) y \;<\; 1. \tag{4.4}$$

On the other hand, if $\bar{u}(y) \geq 1$, then $W(y) = +\infty$. Since $p(\cdot)$ is monotone decreasing, the function

$$y \;\mapsto\; \Big\{ L\Big((\lambda + r)y - (\lambda + \mu)p(y)y \Big) + \rho(y)B \Big\} \;=\; (r + \rho(y))W(y)$$

is monotone increasing. Recalling that the instantaneous bankruptcy risk $\rho(\cdot)$ is a smooth function, we conclude that W has bounded variation on any subinterval $[0, \bar{y}]$ where $W < +\infty$. Downward jumps in p correspond to upward jumps in W.

We can now state the final result of this paper:

Theorem 3. *In the same setting of Theorem 2, let $W(\cdot)$ be the function at (4.3). Then $V(x) \le W(x-)$ for all $x \in [0, M]$.*

Moreover assume that the following two conditions hold:

(i) The set of points y where $V(y) = W(y-)$ is finite.

(ii) If p has a jump at a point $\bar{y}$ where $V(\bar{y}) < W(\bar{y}-)$, and if the time derivative

$$\dot{x} \;=\; \zeta(x) \;\doteq\; \frac{(\lambda+r)x - u^\sharp(p(x), V'(x))}{p(x)} - (\lambda+\mu)x$$

has the same sign to the right and to the left of $\bar{y}$, then

$$F^-(\bar{y}, p_1, V(\bar{y})) \;<\; \xi^\sharp(\bar{y}, p_2) \;<\; F^+(\bar{y}, p_3, V(\bar{y})), \tag{4.5}$$

for all $p_1, p_2, p_3 \in [p(\bar{y}+),\, p(\bar{y}-)]$ with $\xi^\sharp$ defined in (2.15).

Then, setting $u^*(x) = u^\sharp(p(x), V'(x))$ as in (2.14), the functions (V, p, u^*) provide an equilibrium solution to the debt management problem, according to Definition 1.

A proof of Theorems 2 and 3 will be given in several steps.

1 - Existence of a pointwise limit.

Since each V_n is increasing and each p_n is decreasing, by Helly's compactness theorem we can choose a subsequence and achieve the pointwise convergence

$$V_n(x) \;\to\; V(x) \;\in\; [0, B], \qquad\qquad p_n(x) \;\to\; p(x) \;\in\; [\theta(M), 1] \tag{4.6}$$

for every $x \in [0, M]$. Clearly, V and p are monotone. The structure of this limit will be investigated in the following steps.

2 - Local Lipschitz continuity of V.

For every $n \ge 1$, since $p_n(\cdot)$ is non-increasing there exists a unique point $\hat{x}_n$ such that

$$\begin{cases} \Big(\lambda + r - (\lambda + \mu - \sigma_n^2) p_n(x)\Big) x < 1 & \text{if } x < \hat{x}_n, \\ \Big(\lambda + r - (\lambda + \mu - \sigma_n^2) p_n(x)\Big) x > 1 & \text{if } x > \hat{x}_n. \end{cases}$$

By possibly choosing a further subsequence, we can assume the convergence $\hat{x}_n \to \hat{x}$. We claim that the limit function V is Lipschitz continuous on every interval of the form $[0, \hat{x} - \varepsilon]$ or $[\hat{x} + \varepsilon, M]$. This will be proved by showing that all derivatives V_n' are uniformly bounded on these intervals.

To estimate V_n' on $[\hat{x} + \varepsilon, M]$, let ζ_n be as in (3.22), with V, p replaced by V_n, p_n. By construction, for any given $\varepsilon > 0$ there exists $\zeta^\dagger > 0$ such that

$$\zeta_n(x) + \sigma_n^2 x \;\ge\; \zeta^\dagger \qquad\qquad \text{for all } x \in [\hat{x} + \varepsilon,\, M], \tag{4.7}$$

for all $n \geq 1$ sufficiently large.

We observe that, if the maximum of V_n' on $[\hat{x}+\varepsilon,\ M]$ is attained at an interior point y, then

$$rV_n(y) \ \geq\ (\zeta_n(y) + \sigma_n^2 y)V_n'(y). \tag{4.8}$$

This yields the bound

$$V_n'(y) \ \leq\ \frac{rB}{\zeta_n(y)+\sigma_n^2 y} \ \leq\ \frac{rB}{\zeta^\dagger}\,, \tag{4.9}$$

and we are done.

If the maximum of V_n' is attained at $x = M$, then $V_n''(M^-) \geq 0$. Hence (4.8) holds at $y = M$. In turn, this implies (4.9).

It remains to study the case where V_n' attains its maximum at $\hat{x}+\varepsilon$. By the intermediate value theorem, there is a point $y^\dagger \in [\hat{x}+(\varepsilon/2),\ \hat{x}+\varepsilon]$ such that

$$V_n'(y^\dagger) \ =\ \frac{V_n(\hat{x}+\varepsilon) - V_n(\hat{x}+(\varepsilon/2))}{\varepsilon/2} \ \leq\ \frac{2B}{\varepsilon}\,. \tag{4.10}$$

Either

$$V_n'(x) \ \leq\ \frac{2B}{\varepsilon} \qquad\qquad \text{for all } x \in [y^\dagger, M], \tag{4.11}$$

and we are done. Or else V_n' attains a maximum at a point $y \in]y^\dagger, M]$. But in this case the bound (4.9) holds.

Since the bounds (4.9), (4.11) are independent of n, they imply the Lipschitz continuity of V on the interval $[\hat{x}+\varepsilon, M]$.

Next, we derive uniform bounds on V_n' on $[0, \hat{x}-\varepsilon]$. As before, there exists a point $y^\dagger \in [\hat{x}-\varepsilon,\ \hat{x}-(\varepsilon/2)]$ where (4.10) holds. By the argument at (3.24), there exists $\xi_0, \delta > 0$ such that either $V_n'(x) < \xi_0$ for all $x \in [0,\delta]$, or else the global maximum of V_n' is attained on the interval $[\delta, M]$.

It thus suffices to study local maxima of V_n' on the interval $[\delta, \hat{x}-(\varepsilon/2)]$. On this interval we have

$$\big(\lambda + r + (\sigma_n^2 - \lambda - \mu)p_n(x)\big)x \ <\ 1 - \epsilon_0\,. \tag{4.12}$$

for some $\epsilon_0 > 0$ small enough and all n suitably large.

By (4.12), the definition of Hamiltonian function $\mathcal{H}$ at (3.2) yields

$$\lim_{\xi\to+\infty} \mathcal{H}(x,p,\xi,\sigma_n) \ =\ -\infty. \tag{4.13}$$

For $x \in [\delta,\ \hat{x}-(\varepsilon/2)]$ and $p \in [\theta(M), 1]$, call $\xi = \xi_n^0(x,p) > 0$ the value where the concave function $\xi \mapsto \mathcal{H}(x,p,\xi,\sigma_n)$ vanishes, i.e.

$$\mathcal{H}(x,p,\xi_n(x,p),\sigma_n) \ =\ 0\,.$$

Observe that these values are uniformly bounded. Namely, there exists some constant C_0 independent of n such that

$$\xi_n^0(x,p) \ \leq\ C_0\,. \tag{4.14}$$

If V_n' attains a local maximum at a point $y \in [\delta, \hat{x} - (\varepsilon/2)]$, then $V_n''(y) = 0$ and the first equation in (3.23) implies

$$\mathcal{H}(y, p_n(y), V_n'(y), \sigma_n) \;=\; (\zeta_n(y) + \sigma_n^2 y) V_n'(y) + L(u_n^*(y)) + \rho(y) B \;\geq\; 0. \tag{4.15}$$

Comparing (4.15) with (4.13) and using the uniform bound (4.14), we conclude that

$$V_n'(y) \;\leq\; C_0\,. \tag{4.16}$$

Thanks to the uniform bounds on the derivatives V_n' in (4.16) and (3.24), letting $n \to \infty$ we obtain the Lipschitz continuity of the limit function V on the interval $[0, \hat{x} - \varepsilon]$.

For every $\varepsilon > 0$, we thus conclude that on $[0, \hat{x} - \varepsilon] \cup [\hat{x} + \varepsilon, M]$ the sequence V_n converges uniformly to a Lipschitz continuous limit function V. As a consequence, V is differentiable at a.e. point $x \in [0, M]$ and satisfies the boundary conditions in (2.21).

3 - An upper bound on V.

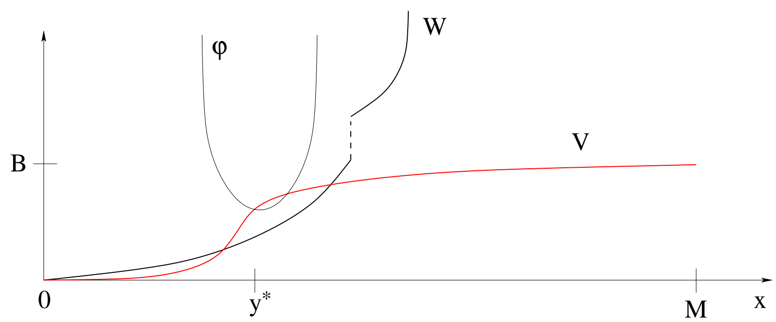

Figure 1: Proving that $V \leq W$.

We claim that, for every $y \in [0, M] \setminus \{\hat{x}\}$, one has

$$V(y) \;\leq\; W(y). \tag{4.17}$$

Intuitively, this should be obvious: V describes a minimum cost, while W is a cost achieved by one particular strategy that keeps the debt constant. For $y > \hat{x}$, recalling (4.7) we see that $W(y) = +\infty$ and there is nothing to prove.

To prove (4.17), assume on the contrary that (4.17) fails at some point $\bar{y} < \hat{x}$. Since V is continuous outside $\hat{x}$, we can find another point y^* such that $V(y^*) \;>\; W(y^*)$ and V, W, p are all continuous at y^*. We can choose $\varepsilon_0 > 0$ small enough so that

$$|x - y^*| \leq \varepsilon_0 \qquad \Longrightarrow \qquad V(x) \;>\; W(x) + \delta_0 \tag{4.18}$$

for some $\delta_0 > 0$. For the given $\delta_0 > 0$, there exists $\delta_1 > 0$ such that

$$\left| \sup_{\xi \geq 0} H(x, \xi, p) - (r + \rho(x)) W(x) \right| \;\leq\; \frac{\delta_0}{2},$$

whenever $|p - p(x)| \leq 2\delta_1$ and $|x - y^*| \leq \varepsilon_0$. Given δ_1, by the continuity of $p(\cdot)$ at y^* we can further reduce ε_0 so that $|p(x) - p(y^*)| \leq \delta_1$ for all $|x - y^*| \leq \varepsilon_0$. Define

$$\varphi(x) \;=\; \frac{1}{\varepsilon_0^2 - (x - y^*)^2}\,,$$

for all $n \geq 1$ large enough, the function $V_n - \varphi$ will have a local maximum at some point x_n, with $|x_n - y^*| < \varepsilon_0$. Recalling (2.19) we now have

$$\begin{aligned}(r+\rho(x_n))V_n(x_n) &= H(x_n, V_n'(x_n), p_n(x_n)) + \sigma^2 V_n'(x_n)\, x_n + \frac{\sigma^2 x_n^2}{2} V_n''(x_n) \\ &\leq \sup_{\xi\geq 0} H(x_n, \xi, p_n(x_n)) + \sigma^2 V_n'(x_n)\, x_n + \frac{\sigma^2 x_n^2}{2} \varphi''(x_n).\end{aligned} \tag{4.19}$$

Letting $n \to \infty$ we can choose a subsequence such that $x_n \to x^*$ such that $p_n(x_n) \to p^*$, with $|x^* - y^*| \leq \varepsilon_0$ and $|p^* - p(x^*)| \leq |p^* - p(y^*)| + |p(y^*) - p(x^*)| \leq 2\delta_1$. Thanks to the boundedness of V_n' and φ'', we obtain

$$(r+\rho(x^*))V(x^*) \;\leq\; \sup_{\xi\geq 0} H(x^*, \xi, p^*) \;<\; (r+\rho(x^*))W(x^*) + \frac{\delta_0}{2}.$$

This contradicts the assumption (4.18), proving our claim.

4 - Behavior on the region where $V < W$.

To help the reader, we first give an overview of the forthcoming analysis. By (4.7), for $x > \hat{x}$ the debt is so large that it cannot be reduced. The dynamics is monotone increasing, and reaches the maximum value M in finite time. At that time, bankruptcy immediately occurs.

We thus focus on the region where $x < \hat{x}$. In general, we expect that $V(x) < W(x)$, except for a finite number of points where equality holds. These are the debt sizes at which the optimal strategy consists in keeping the debt constant in time.

In the following we consider any interval $[a, b]$ where the strict inequality $V(x) \leq W(x) - \delta$ holds for some $\delta > 0$. We will show that on this interval the limit dynamics is piecewise smooth. Namely, there exists an intermediate point $\bar{x} \in [a, b]$ such that V', p are continuous separately on $]a, \bar{x}[$ and on $]\bar{x}, b[$, and moreover

$$V'(x) \;=\; \begin{cases} F^+(x, V(x), p(x)), & \text{if} \quad x \in]a, \bar{x}[\,, \\ F^-(x, V(x), p(x)) & \text{if} \quad x \in]\bar{x}, b[\,, \end{cases} \tag{4.20}$$

$$\dot{x} \;=\; \zeta(x, p, V') \;\doteq\; \frac{(\lambda + r)x - u^\sharp(p(x), V'(x))}{p(x)} - (\lambda + \mu)x \; \begin{cases} < \; 0 & \text{if} \quad x \in]a, \bar{x}[\,, \\ > \; 0 & \text{if} \quad x \in]\bar{x}, b[\,. \end{cases} \tag{4.21}$$

In other words, each open interval $I =]a, b[$ where $V < W$ can be decomposed into a left portion $]a, \bar{x}[$, where the dynamics is strictly decreasing, and a right portion $]\bar{x}, b[$ where the dynamics is strictly increasing (see Fig. 2).

For technical reasons, we first prove the result assuming that all jumps in $p(\cdot)$ contained inside $]a, b[$ satisfy the inequality (4.5). At step 6 we will show that the assumption (4.5) is only needed in the case described by condition (ii) in Theorem 3.

We now begin working out details. If $x < \hat{x}$ and $V(x) < W(x) < +\infty$, then as shown in [10] the equation

$$(r+\rho(x))V(x) \;=\; H(x, \xi, p(x)) \tag{4.22}$$

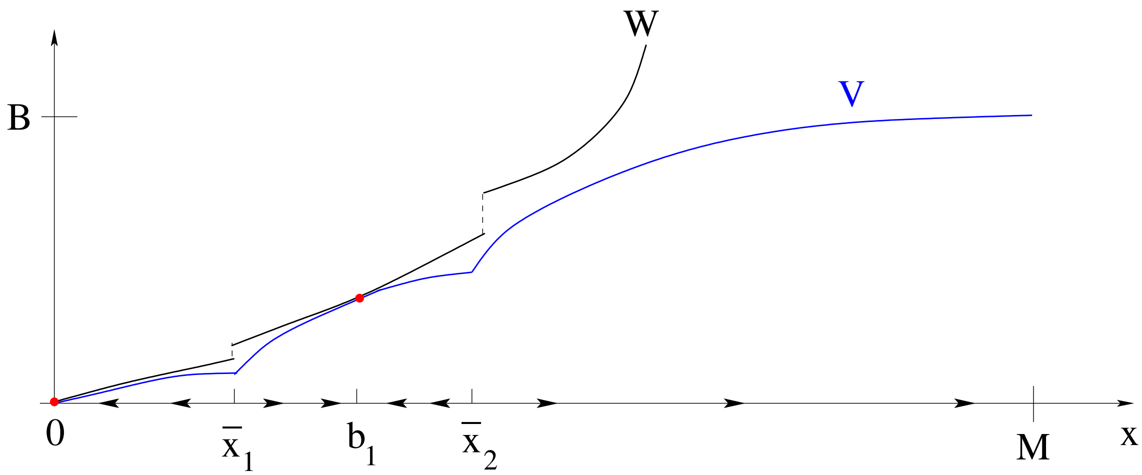

Figure 2: A limiting dynamics. Here the coincidence set, where $V = W$ and the dynamics is stationary, is $\Omega = \{0, b_1\}$. Restricted to the two open intervals $I_1 =]0, b_1[$ and $I_2 =]b_1, M[$ the dynamics is piecewise smooth. Namely, there exists intermediate points $\bar{x}_i \in I_i$ such that $\dot{x} < 0$ for $x < \bar{x}_i$ and $\dot{x} > 0$ for $x > \bar{x}_i$.

has two solutions, which we denote by

$$\xi^- \;=\; F^-(x, V(x), p(x)) \;<\; F^+(x, V(x), p(x)) \;=\; \xi^+. \tag{4.23}$$

Consider any open interval $I =]a, b[$ such that

$$V(x) \;<\; W(x) - \delta \tag{4.24}$$

for some $\delta > 0$ and all $a < x < b$. We claim that there exists $\bar{x} \in [a, b]$ such that (by possibly choosing a subsequence)

$$\begin{cases} V_n'(x) \;\to\; V'(x) \;=\; F^+(x, V(x), p(x)) & \text{for a.e. } \; x < \bar{x}, \\ V_n'(x) \;\to\; V'(x) \;=\; F^-(x, V(x), p(x)) & \text{for a.e. } \; x > \bar{x}\,. \end{cases} \tag{4.25}$$

Toward this goal, consider a point $x \in I$ where V' exists and p is continuous, hence $p_n(x) \to p(x)$ as $n \to \infty$. We claim that (4.2) holds. Indeed, assume $V'(x) = \alpha$. Then we can choose test functions $\phi_1, \phi_2 \in \mathcal{C}^1$, such that

- $\phi_1'(x) = \alpha$ and $V - \phi_1$ has a strict local maximum at x,
- $\phi_2'(x) = \alpha$ and $V - \phi_2$ has a strict local minimum at x,

Given $\varepsilon > 0$, by approximating ϕ_1, ϕ_2 with $\mathcal{C}^2$ functions φ_1, φ_2, we can assume that

- $V - \varphi_1$ has a local maximum at a point x_1, with $|x_1 - x| < \varepsilon$ and $|\varphi_1'(x_1) - \alpha| < \varepsilon$,
- $V - \varphi_2$ has a local minimum at a point x_2, with $|x_2 - x| < \varepsilon$ and $|\varphi_2'(x_2) - \alpha| < \varepsilon$,

Letting $n \to \infty$ and using the convergence $V_n \to V$, we obtain

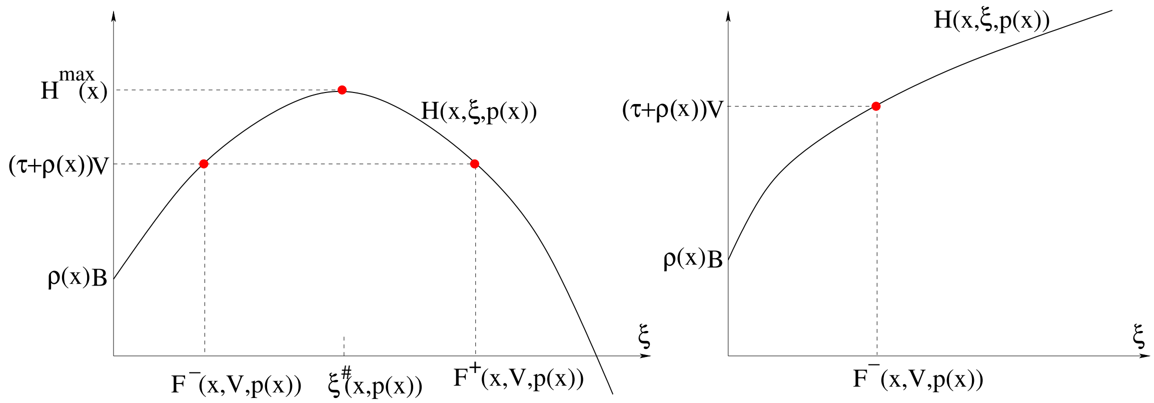

Figure 3: Left: the case where $x < \hat{x}$. For $(r + \rho(x))V > H^{max}(x, p(x))$ the equation (4.22) has no solution. At a point where $(r + \rho(x))V < H^{max}(x, p(x))$, it determines two distinct values F^-, F^+ for ξ. Right: the case where $x > \hat{x}$. For any $(r + \rho(x))V > \rho(x)B$, the equation (4.22) determines a unique solution $\xi = F^-$.

- $V_n - \varphi_1$ has a local maximum at a point $x_{1,n}$, with $|x_{1,n} - x| < \varepsilon$ and $|\varphi_1'(x_{1,n}) - \alpha| < \varepsilon$,
- $V_n - \varphi_2$ has a local minimum at a point $x_{2,n}$, with $|x_{2,n} - x| < \varepsilon$ and $|\varphi_2'(x_{2,n}) - \alpha| < \varepsilon$.

Then there exists a subsequence such that $x_{1,n} \to x_1^*$ with $|x_1^* - x| < \varepsilon$ and a similar estimate as (4.19) yields

$$(r + \rho(x_{1,n}))V_n(x_{1,n}) \geq \; H(x_n, \varphi_1'(x_{1,n}), p_n(x_{1,n})) + \sigma_n^2 \varphi_1'(x_{1,n})\, x_{1,n} + \frac{\sigma_n^2 x_{1,n}^2}{2} \varphi_1''(x_{1,n}).$$

By the convergence of $V_n \to V, p_n \to p$ and the boundedness of φ_1', φ_1'', we obtain

$$(r + \rho(x_1^*))V(x_1^*) \geq \; H(x_1^*, \varphi_1'(x_1^*), p(x_1^*)).$$

Since ε is arbitrary and $H(x, \xi, p)$ is continuous w.r.t. all variables, we have

$$(r + \rho(x))V(x) \geq \; H(x, \alpha, p(x)).$$

Using φ_2, a similar argument yields

$$(r + \rho(x))V(x) \leq \; H(x, \alpha, p(x)),$$

proving (4.2).

Next, assume that there exist two points $a < y_1 < y_2 < b$ such that

$$V'(y_1) \; = \; F^-(y_1, p(y_1), V(y_1)), \qquad\qquad V'(y_2) \; = \; F^+(y_2, p(y_2), V(y_2)).$$

Define the limit point

$$\bar{y} \; \doteq \; \sup\{y \; ; \; V'(y) = F^-(y, p(y), V(y)), \; y < y_2\}.$$

Two cases will be considered:

CASE 1: p is continuous at $\bar{y}$. In this case, V' has an upward jump at $\bar{y}$. Since $\xi \mapsto H(x, \xi, p(x))$ is strictly concave, we can then find a $\mathcal{C}^1$ function ϕ with $F^-(\bar{y}, p(\bar{y}), V(\bar{y})) < \phi'(\bar{y}) < F^+(\bar{y}, p(\bar{y}), V(\bar{y}))$ such that

- $V - \phi$ has a strict local minimum at $\bar{y}$,
- $H(\bar{y}, \phi'(\bar{y}), p(\bar{y})) > (r + \rho(\bar{y}))V(\bar{y})$.

We then approximate ϕ by a $\mathcal{C}^2$ function φ. This yields the existence of a sequence of points y_n, all contained in a small neighborhood around $\bar{y}$, such that

- $V_n - \varphi$ has a local min at y_n,
- $H(y_n, \varphi'(y_n), p_n(y_n)) > (r + \rho(y_n))V_n(y_n) + \varepsilon_0$, for some $\varepsilon_0 > 0$ and all $n \geq 1$.

However, this implies

$$\begin{aligned} 0 &= -(r + \rho(y_n))V_n(y_n) + H(y_n, \varphi'(y_n), p_n(y_n)) + \sigma_n^2 \varphi'(y_n)\, y_n + \frac{\sigma^2 y_n^2}{2} V_n''(y_n) \\ &\geq \varepsilon_0 + \sigma_n^2 \varphi'(y_n)\, y_n + \frac{\sigma_n^2 y_n^2}{2} \varphi''(y_n). \end{aligned}$$

Letting $\sigma_n \to 0$, the right hand side converges to ε_0, providing a contradiction.

CASE 2: p has a downward jump at $\bar{y}$, with

$$F^-(\bar{y}, p_1, V(\bar{y})) \;<\; F^+(x, p_2, V(\bar{y})). \tag{4.26}$$

for any $p_1, p_2 \in [p(\bar{y}+), p(\bar{y}-)]$ such that $p_1 > p_2$. Notice that (4.26) is certainly true if (4.5) is satisfied.

As in CASE 1, using the strict concavity of $\xi \mapsto H(x, \xi, p(x))$, we can find a $\mathcal{C}^1$ function ϕ and some $\bar{\varepsilon} > 0$ such that

(i) $V - \phi$ has a strict local min at $\bar{y}$

(ii) $H(\bar{y}, \phi'(\bar{y}), p) > (r + \rho(\bar{y}))V(\bar{y})$, for every $p \in [p(\bar{y}+) - \bar{\varepsilon}\,,\; p(\bar{y}-) + \bar{\varepsilon}]$.

Using the inequalities

$$F^-(x, p_1, V) \;<\; F^-(x, p_2, V) \;<\; F^+(x, p_2, V) \tag{4.27}$$

valid for any $p_1 < p_2$, the same arguments used for CASE 1 can be applied, obtaining again a contradiction. Notice that, due to the discontinuity of $p(y)$ at $\bar{y}$, the values $p_n(y_n)$ could be any points within the interval $[p(y-) - \bar{\varepsilon}\,,\; p(y+) + \bar{\varepsilon}]$. For this reason, in this case we need the stronger assumption (ii) above.

To prove the first inequality in (4.27) we observe that, by (4.22)-(4.23), the function $F^-(x, V, p)$ is implicitly defined by the equation

$$(r + \rho(x))V \;=\; H(x, F^-, p),$$

together with the inequality

$$\dot{x} \;=\; H_\xi(x, F^-, p) \;=\; -(\lambda + \mu)x + \frac{(\lambda + r)x - \omega^\sharp\left(\frac{F^-}{p}\right)}{p} \;>\; 0.$$

Hence, for fixed x, V, differentiating w.r.t. p one finds

$$H_\xi(x, F^-, p) \cdot F^-_p(x, V, p) + H_p(x, F^-, p) \;=\; 0.$$

By (3.10) it follows $H_p < 0$, hence $F^-_p > 0$. Integrating from p_1 to p_2, the first inequality in (4.27) is proved.

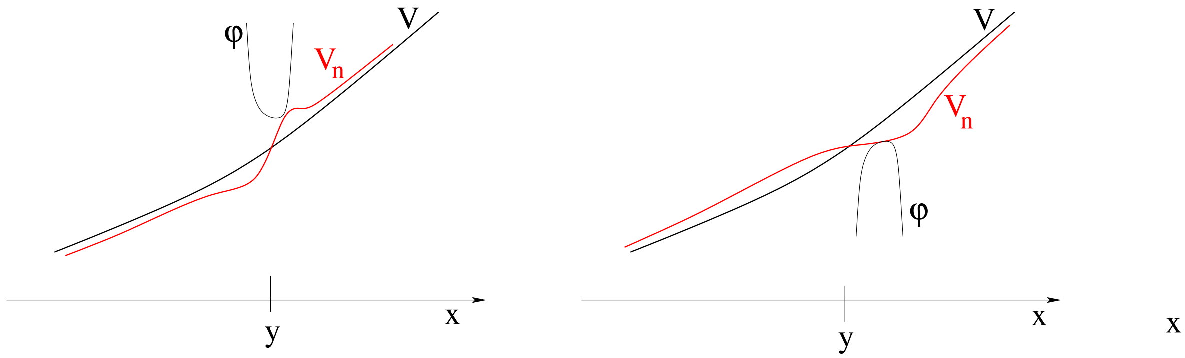

Figure 4: Proving the convergence $V_n' \to V'$.

We now analyze the convergence $V_n'(x) \to V'(x)$ in (4.25). To fix the ideas, consider a point $y \in]a, \bar{x}[$ where V' exists and p is continuous. Assume, by contradiction, that there exists a subsequence such that

$$V_n'(y) \;\to\; \alpha \;\neq\; V'(y) \;=\; F^+(y, V(y), p(y)).$$

CASE 1. $\alpha > V'(y)$ (see Fig. 4, left). Define

$$\hat{\delta} \;\doteq\; (r + \rho(y))V(y) - H\left(y, \frac{\alpha + V'(y)}{2}, p(y)\right).$$

Since $p(\cdot)$ is continuous at y, we can choose $\varepsilon, \delta > 0$ sufficiently small such that $|x - y| \leq \varepsilon$ implies $|p(x) - p(y)| < \delta$, and moreover

$$\Big|(r + \rho(y))V(y) - (r + \rho(x))V(x)\Big| \;<\; \frac{\hat{\delta}}{4}, \qquad \left|H(x, \xi, p) - H\left(y, \frac{\alpha + V'(y)}{2}, p(y)\right)\right| \;<\; \frac{\hat{\delta}}{4}, \tag{4.28}$$

whenever $|x - y| < \varepsilon$, $|\xi - \frac{\alpha + V'(y)}{2}| < \varepsilon$, and $|p - p(y)| < \delta$. By the uniform convergence $V_n \to V$ and the intermediate value theorem, for the given $\varepsilon > 0$ and all $n \geq 1$ large enough there exists a point z_n such that $0 < z_n - y < \varepsilon$ such that $|V_n'(z_n) - V'(y)| < \varepsilon$.

Using the fact that $V_n'(y) \to \alpha$, we can find a $\mathcal{C}^2$ function φ and a sequence of points x_n such that

- $\left|\varphi'(x) - \frac{\alpha + V'(y)}{2}\right| < \varepsilon$ for all $|x - y| \leq \varepsilon$,
- $V_n - \varphi$ has a local maximum at a point x_n with $y < x_n < z_n < y + \varepsilon$.

Then (4.19) yields

$$(r + \rho(x_n))V_n(x_n) \;\leq\; H(x_n, \varphi'(x_n), p_n(x_n)) + \sigma_n^2 \varphi'(x_n)\, x_n + \frac{\sigma^2 x_n^2}{2} \varphi''(x_n)\,.$$

Letting $n \to +\infty$, we can further choose a subsequence $x_n \to x^*$ with $|x^* - y| < \varepsilon$ such that $p_n(x_n) \to p^*$ with $|p^* - p(y)| < \delta$, and for the limit we have

$$(r + \rho(x^*))V(x^*) \ \le \ H(x^*, \varphi'(x^*), p^*). \tag{4.29}$$

However, (4.28) implies that

$$(r+\rho(x^*))V(x^*) \ \ge \ (r+\rho(y))V(y) - \frac{\hat{\delta}}{4} > H\left(y, \frac{\alpha + V'(y)}{2}, p(y)\right) + \frac{\hat{\delta}}{2} > H(x^*, \phi'(x^*), p^*) + \frac{\hat{\delta}}{4},$$

providing a contradiction with (4.29).

CASE 2. $\alpha < V'(y)$ (see Fig. 4, right). In this case, we can find some value $\hat{\xi} \in (\alpha, V'(y))$ defined as

$$\hat{\xi} \doteq \beta_1 \alpha + (1 - \beta_1)V'(y),$$

for some $\beta_1 \in (0, 1)$ such that

$$H(y, \xi^\sharp, p(y)) > (r + \rho(y))V(y).$$

Then the case can be handled similarly. The main difference is that now we replace $\frac{\alpha+V'(y)}{2}$ by $\hat{\xi}$ and choose $z_n < y$, so the maximum becomes a minimum.

The proof of the convergence $V_n'(y) \to V'(y)$ in the case where $V'(y) = F^-(y, V(y), p(y))$ is entirely similar.

This step completes the proof of Theorem 2. In the remaining steps we work toward a proof of Theorem 3.

5 - Convergence of the dynamics, in the region where $V < W$.

By the previous analysis, for any interval $I \ =]a, b[$ where $V < W$, there exists a point $\bar{x}$ such that (4.25) holds. In particular, the feedback dynamics (4.21) induced by the limit value function V is strictly decreasing for $x < \bar{x}$ and strictly increasing for $x > \bar{x}$.

In this step we prove that this dynamics coincides with the limit of the stochastic dynamics

$$dx_n \ = \ (\zeta_n(x_n) + \sigma_n^2 x_n)\, dt + \sigma_n x_n \, dW\,, \tag{4.30}$$

where

$$\zeta_n(x) \ \doteq \ -(\lambda + \mu)x + \frac{(\lambda + r)x - u^\sharp(p_n(x), V_n'(x))}{p_n(x)}. \tag{4.31}$$

More precisely, consider an initial datum

$$x(0) \ = \ y_0 \ \in \ ]a, b[\setminus\{\bar{x}\},$$

and let $x(\cdot)$, $x_n(\cdot)$ be the corresponding solutions to the ODE (4.21) and to the stochastic diffusion equation (4.30), respectively. We claim that, as long as $x(t) \in]a, b[$, we have

$$\lim_{n\to\infty} E\Big[|x_n(t) - x(t)|\Big] \ = \ 0. \tag{4.32}$$

Notice that, by the convergence $V_n' \to V'$ and $p_n \to p$, we have the pointwise a.e. convergence

$$\zeta_n(x) \;\to\; \zeta(x). \tag{4.33}$$

5.1 - lower bounds on the drift. To prove (4.32), we begin by showing that ζ_n remains bounded away from zero, uniformly w.r.t. n. To fix the ideas, assume $x_0 < \bar{x}$, so that the trajectory $t \mapsto x(t)$ is strictly decreasing. Notice that, if we had the uniform convergence $\zeta_n \to \zeta$, the result would be obvious.

To fix the ideas, assume that there exists $\delta > 0$ such that

$$V(x) \;\leq\; W(x) - \delta \tag{4.34}$$

for all $x \in [a,b]$ and (4.20)-(4.21) hold for a.e. $x \in I$. At this stage, we still need the assumption (4.5) on all jumps in $p(.)$, namely at every point $y \in [a,b]$ where p has a jump, there holds

$$F^-(\bar{y}, p_1, V(\bar{y})) \;<\; \xi^\sharp(\bar{y}, p_2) \;<\; F^+(\bar{y}, p_3, V(\bar{y}))\,, \tag{4.35}$$

for all $p_1, p_2, p_3 \in [p(\bar{y}+),\, p(\bar{y}-)]$. Notice that this condition trivially holds at all points where p is continuous. In turn, by continuity and compactness, this implies that there exist $\varepsilon_0, \delta_0 > 0$ such that

$$F^-(x, p, V(x)) \;\leq\; \xi^\sharp(\tilde{x}, \tilde{p}) - \delta_0, \tag{4.36}$$

$$F^+(x, p, V(x)) \;\geq\; \xi^\sharp(\tilde{x}, \tilde{p}) + \delta_0\,, \tag{4.37}$$

for any $x, \tilde{x} \in \mathcal{I}(y, \varepsilon_0)$, and $p, \tilde{p} \in \mathcal{J}(y, \varepsilon_0)$. Here we define

$$\mathcal{I}(y,\varepsilon_0) \doteq \;\Big\{x\;;\; x \in [a,b] \text{ with } |x-y| < \varepsilon_0\Big\},$$

$$\mathcal{J}(y,\varepsilon_0) \doteq \;\Big\{p\;;\; p(x_r+) \leq p \leq p(x_l-)\;,\; x_l = (y-\varepsilon_0)\vee a\;,\; x_r = (y+\varepsilon_0)\wedge b\Big\}.$$

By (4.34), there exists $\delta_1 > 0$ such that

$$\zeta(x) \;\doteq\; \frac{(\lambda+r)x - u^\sharp(p(x), V'(x))}{p(x)} - (\lambda+\mu)x \;<\; -\delta_1 \tag{4.38}$$

at every point $x \in]a, \bar{x}[$ where V', p are well defined. We claim that a similar strict inequality also holds for the sequence of approximations. Namely, there exists a constant $\delta_2 > 0$ such that

$$\zeta_n(x) \;=\; \frac{(\lambda+r)x - u^\sharp(p_n(x), V_n'(x))}{p_n(x)} - (\lambda+\mu)x \;<\; -\delta_2 \tag{4.39}$$

for all $x \in]a, \bar{x}[$ and all $n \geq 1$ sufficiently large.

Arguing by contradiction, assume that there exists a sequence of points $z_n \in]a, \bar{x}[$ such that

$$\zeta_n(z_n) \to 0.$$

By possibly taking a subsequence, we can have $z_n \to z^* \in [a, \bar{x}]$ and $p_n(z_n) \to p^* \in \mathcal{J}(z^*, \varepsilon_0)$. Meanwhile, we also have

$$V_n'(z_n) \to \xi^\sharp(z^*, p^*).$$

Since V' is defined a.e, for all n large enough, there exists $z_n < x_n < (z^* + \varepsilon_0) \wedge b$ such that $|V_n'(x_n) - \xi_1| < \frac{\delta_0}{4}$ for some

$$\xi_1 \in \left\{ F^+(x, p, V(x)) \; ; \; x \in \mathcal{I}(z^*, \varepsilon_0) \; , \; p \in \mathcal{J}(z^*, \varepsilon_0) \right\} . \tag{4.40}$$

Notice that the convex set defined in (4.40) contains the slopes of all the secant lines of $V(.)$ on the interval $\mathcal{I}(z^*, \varepsilon_0)$.

We define

$$\xi_0 = \left\{ \inf_{x \in \mathcal{I}(z^*, \varepsilon_0), \; p \in \mathcal{J}(z^*, \varepsilon_0)} F^+(x, p, V(x)) + \sup_{x \in \mathcal{I}(z^*, \varepsilon_0), \; p \in \mathcal{J}(z^*, \varepsilon_0)} \xi^\sharp(x, p) \right\} /2. \tag{4.41}$$

By the constructions of x_n and z_n with condition (4.37), for every $\bar{\varepsilon} > 0$ small, there exists a $\mathcal{C}^2$ function φ and a sequence of points y_n such that

- $\left| \varphi'(x) - \xi_0 \right| < \bar{\varepsilon}$ for all $x \in \mathcal{I}(z^*, \varepsilon_0)$.
- $V_n - \varphi$ has a local minimum at y_n with $z_n < y_n < x_n$.

Then (4.19) yields that

$$(r + \rho(y_n)) V_n(y_n) \geq H(y_n, \varphi'(y_n), p_n(y_n)) + \sigma_n^2 \varphi'(y_n) \, y_n + \frac{\sigma^2 y_n^2}{2} \varphi''(y_n) \, .$$

Again by possibly passing to the subsequence, $y_n \to y^* \in \mathcal{I}(z^*, \varepsilon_0)$ and $p_n(y_n) \to \hat{p} \in \mathcal{J}(z^*, \varepsilon_0)$, so the limit satisfies

$$(r + \rho(y^*)) V(y^*) \;\geq\; H(y^*, \varphi'(y^*), \hat{p}). \tag{4.42}$$

Furthermore, (4.37) implies that

$$\xi^\sharp(y^*, \hat{p}) + \delta_0 \;\leq\; F^+(y^*, \hat{p}, V(y^*)).$$

Choosing $\delta_1 < \delta_0/4$, by the construction in (4.41) of ξ_0, we further obtain

$$\xi^\sharp(y^*, \hat{p}) + \frac{\delta_0}{4} \;\leq\; \varphi'(y^*) \;\leq\; F^+(y^*, \hat{p}, V(y^*)) - \frac{\delta_0}{4},$$

and thus there exists $\varepsilon_1 > 0$ such that

$$(r + \rho(y^*)) V(y^*) \;\leq\; H(y^*, \varphi'(y^*), \hat{p}) - \varepsilon_1,$$

against (4.42). So we conclude that there exists $\delta_2 > 0$ depending on δ_0 such that (4.39) holds for any $x \in]a, \bar{x}[$ and all $n \geq 1$ sufficiently large.

By possibly reducing the value of $\delta_2 > 0$, an entirely similar argument yields

$$\zeta_n(x) \;=\; \frac{(\lambda + r)x - u^\sharp(p_n(x), V_n'(x))}{p_n(x)} - (\lambda + \mu)x \;>\; \delta_2 \, , \tag{4.43}$$

for every $x \in]\bar{x}, b[$ and all n sufficiently large.

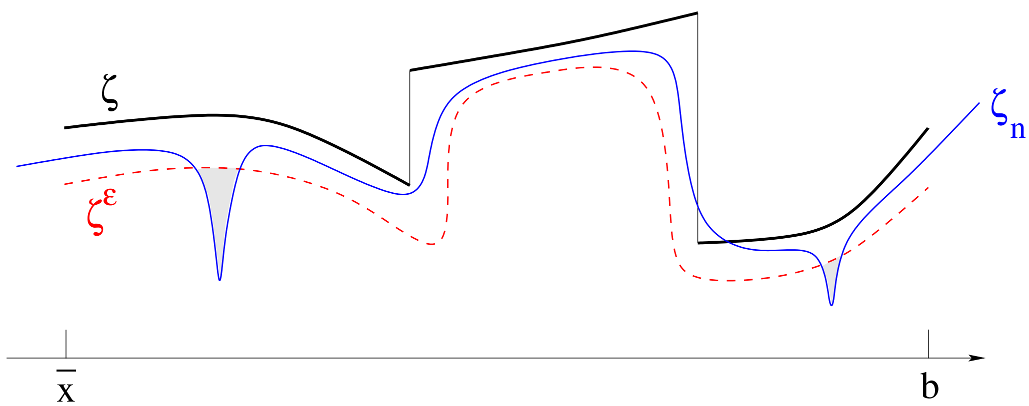

Figure 5: The functions ζ, ζ_n in (4.31)-(4.33) and the smooth approximation ζ^ε considered at (4.54). Setting $\zeta_n^\varepsilon = \min\{\zeta^\varepsilon, \zeta_n\}$, one has $\|\zeta^\varepsilon - \zeta_n^\varepsilon\|_{\mathbf{L}^1} \to 0$ as $n \to \infty$.

5.2 - BV bounds on the drift. Next, we claim that $\zeta(\cdot)$ has bounded variation on $]\bar{x}, b[$. Indeed, ζ is defined by (4.38), where $V' = F^-(x, p(x), V(x))$. Since $u^\sharp(p, \xi)$ defined in (2.14) is Lipschitz continuous w.r.t. p and ξ, while F^- is Lipschitz continuous w.r.t. x, p and V, by suitably choosing constants $\alpha_1, \alpha_2, \alpha_3$ we obtain the bound

$$\big|\zeta(x) - \zeta(y)\big| \;\leq\; \alpha_1|x - y| + \alpha_2\big|p(x) - p(y)\big| + \alpha_3\big|V(x) - V(y)\big|.$$

Since $V(\cdot)$ and $p(\cdot)$ are monotone, this implies that $\zeta(\cdot)$ has bounded variation.

5.3 - the distribution functions. Using the uniform positivity of the functions ζ_n and the pointwise a.e. convergence $\zeta_n \to \zeta$, where ζ has bounded variation, we can now prove the convergence (4.32).

To fix the ideas, assume that, for some $\delta > 0$,

$$\bar{x} + 2\delta \;<\; y_0 \;<\; b - 2\delta. \tag{4.44}$$

By the previous analysis, this implies

$$\zeta(x) \;>\; \delta_2\,, \qquad\qquad \zeta_n(x) \;>\; \delta_2\,, \tag{4.45}$$

for some $\delta_2 > 0$ and all $x \in [\bar{x} + 2\delta, \bar{x} - 2\delta]$ and all n sufficiently large.

Let $t \mapsto y(t)$ the solution to the Cauchy problem for the discontinuous ODE

$$\dot{y} \;=\; \zeta(y), \qquad\qquad y(0) = y_0. \tag{4.46}$$

Even if the function $\zeta(\cdot)$ is only measurable, the solution y is well defined. Indeed, since $\zeta(x) \geq \delta_2 > 0$, the value $y(t)$ is implicitly defined by

$$\int_{y_0}^{y(t)} \frac{dy}{\zeta(y)} \;=\; t\,. \tag{4.47}$$

Let $[0, T]$ be an interval of time such that

$$y(t) \;\leq\; b - 2\delta \qquad\qquad \text{for all } t \in [0, T]\,. \tag{4.48}$$

We will show that the convergence (4.32) holds, for all $t \in [0,T]$. This will be proved by establishing upper an lower bounds on the distribution functions of the random variables $x_n(t)$.

For each $n \geq 1$, consider the distribution function

$$Z_n(t,x) \doteq Prob.\Big\{x_n(t) < x \;\Big|\; x(0) = y_0\Big\}. \tag{4.49}$$

This function satisfies the parabolic equation

$$Z_{n,t} + \zeta_n(x) Z_{n,x} = \frac{\sigma_n^2 x^2}{2} Z_{n,xx} \tag{4.50}$$

with initial data

$$Z_n(0,x) = \begin{cases} 0 & \text{if} \quad x < y_0, \\ 1 & \text{if} \quad x > y_0\,, \end{cases} \tag{4.51}$$

We consider (4.50) restricted to the interval $[\bar{x}+\delta,\, b-\delta]$. The boundary conditions for this solution are not exactly known, but we can certainly say that

$$Z_x(t, \bar{x}+\delta) \geq 0, \qquad\qquad Z(t, b-\delta) \leq 1. \tag{4.52}$$

We claim that, for every $t > 0$

$$\lim_{n\to\infty} Z_n(t,x) = Z(t,x) \doteq \begin{cases} 0 & \text{if} \quad x < y(t), \\ 1 & \text{if} \quad x > y(t)\,, \end{cases} \tag{4.53}$$

To prove (4.53), since the map $x \mapsto \zeta(x)$ has bounded variation, for any given $\varepsilon > 0$ we can find a smooth function ζ^ε and a constant $\eta = \eta(\varepsilon) > 0$ such that (see Fig. 5)

$$\delta_2 \leq \zeta^\varepsilon(x) < \zeta(x), \qquad \text{for} \;\; x \in [\bar{x}+\delta, b-\delta], \tag{4.54}$$

and such that, calling $y^\varepsilon(\cdot)$ the unique solution to

$$\dot{y} = \zeta^\varepsilon(y), \qquad\qquad y(0) = y_0 - 2\eta\,. \tag{4.55}$$

one has

$$y^\varepsilon(t) \in [y(t)-\varepsilon,\; y(t)] \qquad\qquad \text{for all } t \in [0,T]. \tag{4.56}$$

We recall that, by the definition of T at (4.48), this holds as long as $y(t) \leq b - 2\delta$.

For every $n \geq 1$, define

$$\zeta_n^\varepsilon(x) \doteq \min\big\{\zeta^\varepsilon(x),\, \zeta_n(x)\big\}.$$

By (4.54) and the pointwise convergence $\zeta_n \to \zeta$, we have

$$\delta_2 \leq \zeta_n^\varepsilon(x) \leq \zeta^\varepsilon(x), \qquad\qquad \lim_{n\to\infty} \big\|\zeta^\varepsilon - \zeta_n^\varepsilon\big\|_{\mathbf{L}^1([\bar{x}+\delta, b-\delta])} = 0. \tag{4.57}$$

As shown in Fig. 6, consider the auxiliary functions Z_n^ε, $\widehat{Z}_n^\varepsilon$, defined as the solutions to the parabolic equations

$$Z_{n,t}^\varepsilon + \zeta^\varepsilon(x) Z_{n,x}^\varepsilon = \frac{\sigma_n^2 x^2}{2} Z_{n,xx}\,, \tag{4.58}$$

$$\widehat{Z}^\varepsilon_{n,t} + \zeta^\varepsilon_n(x)\widehat{Z}^\varepsilon_{n,x} ~=~ \frac{\sigma_n^2 x^2}{2}\widehat{Z}_{n,xx}\,, \tag{4.59}$$

on the interval $x \in [\bar{x}+\delta,\, b-\delta]$, with smooth initial data

$$Z^\varepsilon_n(0,x) ~=~ \widehat{Z}^\varepsilon_n(0,x) ~=~ \varphi_0(x), \tag{4.60}$$

where φ_0 is a $\mathcal{C}^\infty$ function such that

$$\varphi_0(x) ~=~ \left\{ \begin{array}{ll} 0 & \text{if} \quad x < y_0 - 2\eta, \\ 1 & \text{if} \quad x > y_0\,, \end{array}\right. \qquad\qquad 0 \le \varphi_0'(x) \le \eta\,. \tag{4.61}$$

Moreover, we impose the boundary conditions

$$Z^\varepsilon_{n,x}(t,\bar{x}+\delta) ~=~ \widehat{Z}^\varepsilon_{n,x}(t,\bar{x}+\delta) ~=~ 0, \tag{4.62}$$

$$Z^\varepsilon_n(t,b-\delta) ~=~ \widehat{Z}^\varepsilon_n(t,b-\delta) ~=~ 1. \tag{4.63}$$

Since $\zeta^\varepsilon(\cdot)$ is smooth, by the standard theory of parabolic equations it follows the existence of the limit

$$\lim_{n\to\infty} Z^\varepsilon_n(t,x) ~=~ Z^\varepsilon(t,x) \qquad \text{for all } t \in [0,T], \tag{4.64}$$

where Z^ε is the solution to the linear first order equation

$$Z^\varepsilon_t + \zeta^\varepsilon(x) Z^\varepsilon_x ~=~ 0, \tag{4.65}$$

with the same initial data (4.60). Moreover,

$$\lim_{n\to\infty} Z^\varepsilon_{n,x}(t,b-\delta) ~=~ 0, \qquad \lim_{n\to\infty} Z^\varepsilon_n(t,\bar{x}+\delta) ~=~ 0 \qquad \text{for all } t\in[0,T]. \tag{4.66}$$

By a comparison argument, we have the pointwise bound

$$Z^\varepsilon_n(t,x) ~\le~ \widehat{Z}^\varepsilon_n(t,x)\,. \tag{4.67}$$

In turn, this implies

$$0 ~\le~ \widehat{Z}^\varepsilon_{n,x}(t,b-\delta) ~\le~ Z^\varepsilon_{n,x}(t,b-\delta)\,. \tag{4.68}$$

Hence, by (4.66),

$$\lim_{n\to\infty} \widehat{Z}^\varepsilon_{n,x}(t,b-\delta) ~=~ 0 \qquad \text{for all } t\in[0,T]. \tag{4.69}$$

5.4 - upper bounds on the probability densities. Toward the main convergence proof, we still need to estimate the difference $\widehat{Z}^\varepsilon_n - Z^\varepsilon_n$. In this subsection, we show that the derivatives satisfy a uniform bound, independent of n:

$$\widehat{Z}^\varepsilon_{n,x} ~\le~ K_\varepsilon, \qquad\qquad Z^\varepsilon_{n,x} ~\le~ K_\varepsilon\,, \tag{4.70}$$

for some constant K_ε depending on ε but not on n. This is proved by a comparison argument. The probability densities $\phi^\varepsilon_n = Z^\varepsilon_{n,x}$ and $\widehat{\phi}^\varepsilon_n = \widehat{Z}^\varepsilon_{n,x}$ satisfy the parabolic equations

$$\phi^\varepsilon_{n,t} + (\zeta^\varepsilon(x)\phi^\varepsilon_n)_x ~=~ \left(\frac{\sigma_n^2 x^2}{2}\phi^\varepsilon_{n,x}\right)_x, \qquad\qquad \widehat{\phi}^\varepsilon_{n,t} + (\zeta^\varepsilon_n(x)\widehat{\phi}^\varepsilon_n)_x ~=~ \left(\frac{\sigma_n^2 x^2}{2}\widehat{\phi}^\varepsilon_{n,x}\right)_x. \tag{4.71}$$

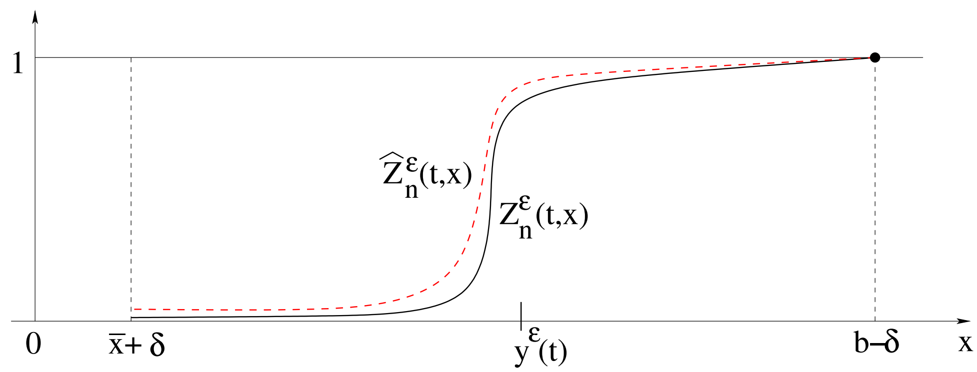

Figure 6: The functions Z_n^ε and $\widehat{Z}_n^\varepsilon$ in (4.67)-(4.68).

By (4.60), all these functions have the same initial data

$$\phi_n^\varepsilon(0,x) \;=\; \widehat{\phi}_n^\varepsilon(0,x) \;=\; \begin{cases} \eta^{-1} & \text{if} \quad x \in [y_0 - \eta,\, y_0]\,, \\ 0 & \text{if} \quad x \notin [y_0 - \eta,\, y_0]\,, \end{cases} \tag{4.72}$$

while from (4.62), and (4.66), (4.69) we deduce the boundary conditions

$$\phi_n^\varepsilon(t, \bar{x}+\delta) \;=\; \widehat{\phi}_n^\varepsilon(t, \bar{x}+\delta) \;=\; 0, \tag{4.73}$$

$$\lim_{n\to\infty} \phi_n^\varepsilon(t, b-\delta) \;=\; \lim_{n\to\infty} \widehat{\phi}_n^\varepsilon(t, b-\delta) \;=\; 0, \tag{4.74}$$

uniformly for $t \in [0,T]$. A uniform upper bound for $\phi_n^\varepsilon, \widehat{\phi}_n^\varepsilon$ will be proved by constructing suitable upper solutions, independent of time.

The equations in (4.71) admit infinitely many stationary solutions $\psi(x)$. These are found by solving the equations

$$\frac{\sigma_n^2 x^2}{2}\psi_x \;=\; \zeta^\varepsilon(x)\psi + C_0 \qquad \text{or} \qquad \frac{\sigma_n^2 x^2}{2}\psi_x \;=\; \zeta_n^\varepsilon(x)\psi + C_0\,. \tag{4.75}$$

where C_0 is an arbitrary constant. Define

$$\zeta^{max} \;\doteq\; \sup_{\bar{x}+\delta \le x \le b-\delta} \zeta^\varepsilon(x), \qquad\qquad C_0 \;=\; -\,\eta^{-1}\zeta^{max}\,.$$

To construct a suitable upper bound, we consider the backward Cauchy problem on $[\bar{x}, b]$

$$\psi'(x) \;=\; \frac{2}{\sigma_n^2 x^2}\big(\zeta^\varepsilon(x)\psi - \eta^{-1}\zeta^{max}\big)\,, \qquad\qquad \psi(b) \;=\; \eta^{-1}. \tag{4.76}$$

Observe that the solution of (4.76) satisfies

$$\frac{1}{\eta} \;\le\; \psi(x) \;\le\; \frac{\zeta^{max}}{\eta\delta_2} \qquad\qquad \text{for all } x \in\,]\bar{x}, b]. \tag{4.77}$$

Indeed, this follows immediately from the implications

$$\psi \le \eta^{-1} \qquad \Longrightarrow \qquad \psi' \le 0,$$

$$\psi \ge (\eta\delta_2)^{-1}\zeta^{max} \qquad \Longrightarrow \qquad \psi' \ge 0\,.$$

Thanks to the boundary conditions (4.73)-(4.74), using a comparison argument together with (4.77) we deduce

$$Z^\varepsilon_{n,x}(t,x) \;=\; \phi^\varepsilon_n(t,x) \;\le\; \frac{\zeta^{max}}{\eta\delta_2}\,. \tag{4.78}$$

The same argument applies to $\widehat{Z}^\varepsilon_{n,x}$, yielding

$$\widehat{Z}^\varepsilon_{n,x}(t,x) \;=\; \widehat{\phi}^\varepsilon_n(t,x) \;\le\; \frac{\zeta^{max}}{\eta\delta_2}\,. \tag{4.79}$$

5.5 - convergence estimates. Comparing (4.58) with (4.59), integrating by parts, and using (4.68), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\bar x+\delta}^{b-\delta}\big(\widehat{Z}^\varepsilon_n(t,x)-Z^\varepsilon_n(t,x)\big)\,dx\\
&\quad= \int_{\bar x+\delta}^{b-\delta}\Big(\zeta^\varepsilon(x)Z^\varepsilon_{n,x}(t,x)-\zeta^\varepsilon_n(x)\widehat{Z}^\varepsilon_{n,x}(t,x)+\frac{\sigma_n^2x^2}{2}(\widehat{Z}^\varepsilon_{n,xx}-Z^\varepsilon_{n,xx})\Big)\,dx\\
&\quad= \int_{\bar x+\delta}^{b-\delta}\big(\zeta^\varepsilon(x)-\zeta^\varepsilon_n(x)\big)\widehat{Z}^\varepsilon_{n,x}(t,x)\,dx+\int_{\bar x+\delta}^{b-\delta}(\zeta^\varepsilon(x)+\sigma_n^2x)\Big(Z^\varepsilon_{n,x}(t,x)-\widehat{Z}^\varepsilon_{n,x}(t,x)\Big)\,dx\\
&\qquad+\sigma_n^2x\left(\widehat{Z}^\varepsilon_{n,x}(t,x)-Z^\varepsilon_{n,x}(t,x)\right)\Big|_{x=b-\delta}\\
&\quad\le \|\zeta^\varepsilon-\zeta^\varepsilon_n\|_{\mathbf{L}^1}\|\widehat{Z}_{n,x}\|_{\mathbf{L}^\infty}+(\|\zeta^\varepsilon_x\|_{\mathbf{L}^\infty}+1)\int_{\bar x+\delta}^{b-\delta}\big(\widehat{Z}^\varepsilon_n(t,x)-Z^\varepsilon_n(t,x)\big)\,dx\\
&\qquad+(\zeta^\varepsilon(x)+\sigma_n^2x)\big(\widehat{Z}^\varepsilon_n(t,x)-Z^\varepsilon_n(t,x)\big)\Big|_{x=\bar x+\delta}\,.
\end{aligned} \tag{4.80}$$

Here and in the following formulas all the norms $\mathbf{L}^1,\mathbf{L}^\infty$ refer to the interval $[\bar x+\delta,\, b-\delta]$. Using the bound

$$\widehat{Z}^\varepsilon_n(t,\bar x+\delta) \;\le\; Z^\varepsilon_n(t,\bar x+2\delta)+\delta^{-1}\,\big\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\|_{\mathbf{L}^1}$$

to estimate the last term in (4.80), and recalling (4.70), for all n sufficiently large we obtain

$$\begin{aligned}
&\frac{d}{dt}\big\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\big\|_{\mathbf{L}^1}\\
&\quad\le \|\zeta^\varepsilon-\zeta^\varepsilon_n\|_{\mathbf{L}^1}K_\varepsilon+(\|\zeta^\varepsilon_x\|_{\mathbf{L}^\infty}+1)\big\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\big\|_{\mathbf{L}^1}\\
&\qquad+(\zeta^{max}+1)\Big(Z^\varepsilon_n(t,\bar x+2\delta)+\delta^{-1}\,\big\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\|_{\mathbf{L}^1}\Big)\\
&\quad= A\cdot\big\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\big\|_{\mathbf{L}^1}+B_n\,,
\end{aligned} \tag{4.81}$$

with

$$\begin{aligned}
A &= (\|\zeta^\varepsilon_x\|_{\mathbf{L}^\infty}+1)+\delta^{-1}(\zeta^{max}+1),\\
B_n &= \|\zeta^\varepsilon-\zeta^\varepsilon_n\|_{\mathbf{L}^1}K_\varepsilon+(\zeta^{max}+1)\,Z^\varepsilon_n(t,\bar x+2\delta).
\end{aligned}$$

By Gronwall's lemma, for all $t\le T$ it follows

$$\big\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\big\|_{\mathbf{L}^1} \;\le\; \big(e^{At}-1\big)A^{-1}B_n\,. \tag{4.82}$$

letting $n \to \infty$, one has $B_n \to 0$, hence the right hand side of (4.82) approaches zero as well.

Summarizing the previous analysis, for any $\varepsilon > 0$ we have shown that

$$\begin{aligned}\lim_{n\to\infty} \int_{\bar{x}+\delta}^{b-\delta} \Big[Z_n(t,x) - Z(t,x)\Big]_+ dx \ &\leq \ \int_{\bar{x}+\delta}^{b-\delta} \Big(Z^\varepsilon(t,x) - Z(t,x)\Big)\, dx \\ &\quad + \lim_{n\to\infty} \int_{\bar{x}+\delta}^{b-\delta} \Big|Z_n^\varepsilon(t,x) - Z^\varepsilon(t,x)\Big|\, dx + \lim_{n\to\infty} \int_{\bar{x}+\delta}^{b-\delta} \Big(\widehat{Z}_n^\varepsilon(t,x) - Z_n^\varepsilon(t,x)\Big)\, dx \\ &\leq \ \varepsilon + 0 + 0,\end{aligned} \tag{4.83}$$

where $Z(t,x)$ is defined in (4.53) and we used the notation $[s]_+ \doteq \max\{s, 0\}$.

A similar estimate on the distribution of Z_n from below yields the complementary upper bound on the distribution of $x_n(t)$. We only sketch the argument, because it is almost identical to the previous one. Given $\varepsilon > 0$, we can find a smooth function ζ^ε and a constant $\eta = \eta(\varepsilon)$ such that

$$\delta_2 \ \leq \ \zeta(x) \ < \ \zeta^\varepsilon(x) \ \leq \ 1 + \sup_{\bar{x}+\delta \leq x \leq b-\delta} \zeta(x)$$

and such that the solution $y^\varepsilon(\cdot)$ of the Cauchy problem

$$\dot{y} \ = \ \zeta(y), \qquad\qquad y(0) \ = \ y_0 + 2\eta$$

satisfies

$$y^\varepsilon(t) \ \in \ [y(t),\, y(t) + \varepsilon] \qquad\qquad \text{for all } t \in [0, T].$$

We define the corresponding drifts as $\zeta_n^\varepsilon = \max\{\zeta^\varepsilon(x), \zeta_n(x)\}$. Then we consider the same parabolic equations (4.58)-(4.59) replacing the boundary conditions (4.62)-(4.63) with

$$Z_n^\varepsilon(t, \bar{x}+\delta) \ = \ \widehat{Z}_n^\varepsilon(t, \bar{x}+\delta) \ = \ 0, \tag{4.84}$$

$$Z_{n,x}^\varepsilon(t, b-\delta) \ = \ \widehat{Z}_{n,x}^\varepsilon(t, b-\delta) \ = \ 0. \tag{4.85}$$

As initial conditions we take (4.60), replacing (4.61) with

$$\varphi_0(x) \ = \ \begin{cases} 0 & \text{if} \quad x < y_0, \\ 1 & \text{if} \quad x > y_0 + 2\eta\,, \end{cases} \qquad\qquad 0 \leq \varphi_0'(x) \leq \eta\,. \tag{4.86}$$

In place of (4.83) we now obtain

$$\begin{aligned}\lim_{n\to\infty} \int_{\bar{x}+\delta}^{b-\delta} \Big[Z(t,x) - Z_n(t,x)\Big]_+ dx \ &\leq \ \int_{\bar{x}+\delta}^{b-\delta} \Big(Z(t,x) - Z^\varepsilon(t,x)\Big)\, dx \\ &\quad + \lim_{n\to\infty} \int_{\bar{x}+\delta}^{b-\delta} \Big|Z_n^\varepsilon(t,x) - Z^\varepsilon(t,x)\Big|\, dx + \lim_{n\to\infty} \int_{\bar{x}+\delta}^{b-\delta} \Big(Z_n^\varepsilon(t,x) - \widehat{Z}_n^\varepsilon(t,x)\Big)\, dx \\ &\leq \ \varepsilon + 0 + 0.\end{aligned} \tag{4.87}$$

Combining (4.83) with (4.87) we obtain (4.32), in the case where $y_0 \in\,]\bar{x}, b[\,$.

The case $y_0 \in\,]a, \bar{x}[\,$ can be handled in an entirely similar way.

6 - Analysis of the jumps in p, in the region where $V < W$.

The characterization of the limit dynamics at (4.25) was derived under the assumption that all jumps in $p(\cdot)$ contained inside the open interval $]a, b[$ where $V < W$ satisfy (4.5).

Indeed, assume that p has large jump at a point $y \in]a, b[$. Being non-increasing, the function p has bounded variation. Hence there exists $\varepsilon > 0$ such that all the jumps of $p(\cdot)$ contained inside $]y - \varepsilon, y[\cup]y, y + \varepsilon[$ are suitably small, so that (4.26) holds. By the previous steps, the function $\zeta(\cdot)$ at (4.38) has a constant sign on $]y - \varepsilon, y[$ and on $]y, y + \varepsilon[$.

Here we relax the smallness assumption (4.5) on all the jumps of p to just the jumps under condition (ii) in Theorem 3, which assumes that large jumps in p can only happen at point $y \in]a, b[$ where the dynamics separates or converges in its neighbourhood, namely, $\zeta(x) < 0$ on $]y - \varepsilon, y[$ and $\zeta(x) > 0$ on $]y, y + \varepsilon[$ or $\zeta(x) > 0$ on $]y - \varepsilon, y[$ and $\zeta(x) < 0$ on $]y, y + \varepsilon[$. All the other jumps should satisfy the condition (4.5).

Various cases must be considered.

CASE 1. The speed $\zeta(x)$ has the same sign on the intervals $]y-\varepsilon, y[$ and $]y, y+\varepsilon[$. Due to the smallness assumption in p, in this case, for initial points x_0 close y, the solution $t \mapsto x(t, x_0)$ of the Cauchy problem

$$\dot{x} \;=\; \zeta(x), \qquad\qquad x(0) = x_0 \tag{4.88}$$

depends continuously on x_0. By the convergence (4.32), the same is true of the limit of the random variables x_n in (4.30).

Recalling that $p(x_0)$ is defined at (2.9) as the expected return for the investors, when the initial debt size is x_0, the continuous dependence on x_0 implies that p is continuous near y. Therefore, no jump occurs.

CASE 2. $\zeta(x) > 0$ on $]y - \varepsilon, y[$ and $\zeta(x) < 0$ on $]y, y + \varepsilon[$. Though p might have large jumps at y, this still implies that the solutions of (4.88) depend continuously on the initial point x_0. As in the previous case, by (4.32) and (2.9) we conclude that p is continuous at the point y.

CASE 3. The remaining case is when $\zeta(x) < 0$ on $]y - \varepsilon[$ and $\zeta(x) > 0$ on $]y + \varepsilon[$. This is the only possible case where p can have a jump. Indeed, $\bar{x} = y$ is precisely the point inside $[a, b]$ where the dynamics changes sign. This achieves a proof of (4.20), removing the smallness assumption on the jumps of $p(\cdot)$.

7 - Behavior on the region where $V = W$.

We now consider the coincidence set

$$\Omega \;\doteq\; \{x \in [0, M]\,;\;\; V(x) = W(x-)\}, \tag{4.89}$$

and make the assumption that Ω contains finitely many points. We claim that $p(y) = p^\sharp(y)$ for all $y \in \Omega$.

From the previous analysis, on the finitely many open intervals where $V < W$ we either have

$$V'(x) \;=\; F^+(x, V(x), p(x)), \qquad\qquad \frac{(\lambda + r)x - u^\sharp(p(x), V'(x))}{p(x)} - (\lambda + \mu)x \;<\; 0, \tag{4.90}$$

or

$$V'(x) \;=\; F^-(x, V(x), p(x)), \qquad\qquad \frac{(\lambda+r)x - u^\sharp(p(x), V'(x))}{p(x)} - (\lambda+\mu)x \;>\; 0. \tag{4.91}$$

We again consider various cases.

CASE 1. There exists $\delta > 0$ such that (4.91) holds on $]y-\delta, y[$ and (4.90) holds on $]y, y+\delta[$.

Notice that, for any open interval I where (4.90) holds, p satisfies the ODE

$$p'(x) \;=\; \frac{(\rho(x)+r+\lambda)p - (r+\lambda+\theta(x)\rho(x))}{\zeta(x)} \;\leq\; 0. \tag{4.92}$$

Since $\zeta(x) < 0$ on I, (4.92) implies

$$p(x) \;\geq\; p^\sharp(x) \doteq \frac{r+\lambda+\theta(x)\rho(x)}{r+\lambda+\rho(x)}. \tag{4.93}$$

Similarly, for any I where (4.91) holds, $p(x) \leq p^\sharp(x)$. Since $p(\cdot)$ is nonincreasing, we conclude that p is continuous at y and $p(y) = p^\sharp(y)$.

CASE 2. There exists $\delta > 0$ such that (4.90) holds on $]y-\delta, y[\,\cup\,]y, y+\delta[$.
If $p(\cdot)$ is continuous at y, then

$$\zeta(x) \;\to\; 0$$

as $x \to y$. If $p(y) \neq p^\sharp(y)$, since (4.92) is satisfied on $]y, y+\delta[$, we must have $p(y) > p^\sharp(y)$. This implies that $p'(x) \to +\infty$ as $x \to y$, and thus $W'(x) \to +\infty$ as $x \to y$. However, $V(x)$ is Lipschitz continuous and $V(\cdot) \leq W(\cdot)$, reaching a contradiction.
If $p(\cdot)$ is discontinuous at y, by construction we have $\zeta(x) \to 0$ as $x \to y-$. Then the similar analysis shows that $p(y-) = p^\sharp(y)$ and $p^\sharp(y+) \geq p^\sharp(y)$, providing again a contradiction.
We conclude that in this case $p(\cdot)$ must be continuous at y and $p(y) = p^\sharp(y)$.

CASE 3. There exists $\delta > 0$ such that (4.91) holds on $]y-\delta, y[\,\cup\,]y, y+\delta[$.
If $p(\cdot)$ is continuous at y, we obtain that

$$\zeta(x) \to\; 0$$

as $x \to y$. If $p(y) \neq p^\sharp(y)$, since (4.92) is satisfied on $]y, y+\delta[$, we must have $p(y) < p^\sharp(y)$. A similar contradiction is obtained as in CASE 2.
If $p(\cdot)$ is discontinuous at y, by construction we have $\zeta(x) \to 0$ as $x \to y-$. A similar analysis now shows that $p(y-) = p^\sharp(y)$.
We conclude that in this case $p(y-) = p^\sharp(y)$ and p can have a downward jump at y.

8 - Behavior near $\hat{x}$.

By the previous analysis, V is continuous everywhere on $[0, M]$ except for a possible jump at $\hat{x}$. By the definition of $\hat{x}$, for any $x > \hat{x}$ one has $\big(\lambda + r - (\lambda+\mu)p(x)\big)x > 1$. In other words, if the debt becomes larger than $\hat{x}$, then it must keep increasing, reaching the bankruptcy level M in finite time.

On the other hand, for x close to $\hat{x}$ but smaller that this threshold, various cases can arise.

CASE 1. $\zeta(x) < 0$ for $x \in]\hat{x}-\varepsilon, \hat{x}[$. In this case, the evolution generated by the optimal feedback is discontinuous at $\hat{x}$. As a result, the discounted price p will have a jump at $\hat{x}$ so that $\big(\lambda + r - (\lambda+\mu)p(\hat x-)\big)\hat x < 1$. This guarantees that it is possible to decrease the debt, starting from $\hat x$. Notice that in this case, V may also have a jump at $\hat x$.

CASE 2. $\zeta(x) > 0$ on $]\hat x - \varepsilon, \hat x[$ and $V(\hat x-) = W(\hat x-)$. In this case, we claim that V is Lipschitz continuous on $]\hat x - \varepsilon, \hat x[$ and has a jump at $\hat x$. Moreover, one has

$$p(\hat x-) \;=\; p^\sharp(\hat x), \qquad\qquad \big(\lambda + r - (\lambda+\mu)p^\sharp(\hat x)\big)\hat x \;<\; 1.$$

Notice that the assumption $V(\hat x-) = W(\hat x-)$ implies $W(\hat x-) < +\infty$. In turn this implies

$$\big(\lambda + r - (\lambda+\mu)p(\hat x-)\big)\hat x < 1, \qquad V'(\hat x^-) = F^-(\hat x, V(\hat x-), p(\hat x-)) < \xi^\sharp(\hat x, p(\hat x-)) < +\infty,$$

so V is Lipschitz continuous on $]\hat x - \varepsilon, \hat x[$. Then we show that $p(\hat x-) = p^\sharp(\hat x)$. Using the similar argument in the previous analysis, if $p(\hat x-) \neq p^\sharp(\hat x)$, then we have $p'(x) \to +\infty$ as $x \to \hat x-$, and thus $W'(x) \to +\infty$ as $x \to \hat x-$. However, $V(x)$ is Lipschitz continuous on $]\hat x - \varepsilon, \hat x[$ and $V(\hat x-) = W(\hat x-)$, reaching a contradiction.

CASE 3. $\zeta(x) > 0$ on $]\hat x - \varepsilon, \hat x[$ and $V(\hat x-) < W(\hat x-)$. By the previous analysis, this implies that there exists $\delta > 0$ such that $V(x) < W(x) - \delta$ on $]\hat x - \varepsilon, \hat x[$. By the assumption as in (4.36) that the jump of p at $\hat x$ is sufficiently small, we obtain that the solutions of (4.88) depend continuously on the initial point x_0 for any $x_0 \in]\hat x - \varepsilon, \hat x + \varepsilon[$. By (2.9) and (2.10) we conclude that both p and V are continuous at the point $\hat x$. In this case, the evolution generated by the optimal feedback control is continuous at $\hat x$ and there exists some $\hat x^* < \hat x$ such that for any $x > \hat x^*$, the debt must keep increasing, reaching the bankruptcy level M in finite time.

9 - A limit semigroup.

By the previous analysis, if $]a, b[$ is an interval where $V < W$, then it can be partitioned into two subintervals, where either (4.90) or (4.91) hold.

Given the limit functions V, p, and $u^*(x) = u^\sharp(p(x), V'(x))$, we consider the ODE

$$\dot x \;=\; \frac{(\lambda + r)x - u^*(x)}{p(x)} - (\lambda+\mu)x\,. \tag{4.94}$$

Since the control u^* is not continuous in general, for any initial datum x_0 several Carathéodory solutions may exist. We can define a semigroup of solutions by first setting

$$S_t x_0 \;=\; x_0 \qquad\qquad \text{for all } x_0 \in \Omega \;\doteq\; \{x\,;\;\; V(x) = W(x-)\}\,. \tag{4.95}$$

On the other hand, if $x_0 \notin \Omega$, let $]a, b[$ be the connected component of $[0, M[\,\setminus \Omega$ which contains x_0. Let $\bar x \in [a, b]$ and $\{V'(x) | x \neq \bar x\}$ be as in (4.25). We then define

$$t \;\mapsto\; x(t) \;=\; S_t x_0$$

to be the unique solution of

$$\dot x \;=\; \left[\frac{(\lambda + r)x - u^\sharp(p, V'(x-))}{p(x-)} - (\lambda+\mu)x\right], \tag{4.96}$$

as long as $x(t) \notin \Omega$. We then extend the definition of S by setting

$$\tau(x_0) \;\doteq\; \sup\Big\{t \geq 0\,;\quad S_t x_0 \notin \Omega\Big\},$$

$$S_t x_0 \doteq S_{\tau(x_0)} x_0 \qquad \text{for all } t \geq \tau(x_0).$$

In other words, the evolution stops as soon as the trajectory hits Ω.

10 - A Nash equilibrium solution.

Finally, we claim that the limit values V, p, u^* provide a Nash equilibrium solution to the debt management problem. More precisely:

(i) The discounted price $p(\cdot)$ is consistent with the dynamics defined at (4.95)-(4.96). Indeed, for every initial point x_0 (with the possible exception of points $\bar{x}$ in (4.21) where the dynamics is discontinuous), one has

$$\begin{aligned} p(x_0) \;&=\; 1 - \int_0^{T^M} \big[1-\theta(S_t(x_0))\big]\, \rho(S_t(x_0)) \exp\left\{ - \int_0^t \big[r+\lambda+\rho(S_s(x_0))\big]\, ds \right\} dt \\ &\qquad - \big[1-\theta(M)\big] \exp\left\{ - \int_0^{T^M} [r+\lambda+\rho(S_t(x_0))] dt \right\}. \end{aligned} \tag{4.97}$$

(ii) Given $p(\cdot)$, the function V is the value function for the corresponding deterministic optimal control problem, namely

$$\text{minimize:} \;\; E\left[\int_0^{T_b} e^{-rt} L(u(t))\, dt + B\, e^{-rT_b} \right] \tag{4.98}$$

subject to

$$dx \;=\; \left[-(\lambda+\mu)x + \frac{(\lambda+r)x - u(t)}{p(x)} \right] dt\,, \qquad x(0) = x_0, \tag{4.99}$$

and $u^*(x) = u^\sharp(p(x), V'(x))$ is an optimal feedback control.

To prove (i), we observe that by construction the result is true for all x_0 in the finite set Ω where V and W coincide. The boundary conditions imply it is also true for $x_0 = M$.

To show that it remains true on any connected component $]a,b[\; \subset \; [0,M] \setminus \Omega$, with $a, b \in \Omega$, we observe that $p(x)$ satisfies the ODE

$$p'(x) \;=\; \frac{(\rho(x)+r+\lambda)p - (r+\lambda+\theta(x)\rho(x))}{\zeta(x)} \tag{4.100}$$

separately on the subintervals $]a,\bar{x}[$ and $]\bar{x},b[$, where $\bar{x}$ is as in (4.25). For any point $x_0 \in]a,\bar{x}[$, we have

$$\frac{d}{dt} p(S_t(x_0)) \;=\; p'(S_t(x_0)) \cdot \frac{d}{dt} S_t(x_0) \;=\; (\rho(x)+r+\lambda) p(S_t(x_0)) - (r+\lambda+\theta(x)\rho(S_t(x_0))). \tag{4.101}$$

Since $\lim_{t\to+\infty} S_t(x_0) = a \in \Omega$, we have

$$\lim_{t\to+\infty} p(S_t(x_0)) \;=\; p(a) \;=\; \frac{r+\lambda+\theta(a)\rho(a)}{r+\lambda+\rho(a)}. \tag{4.102}$$

Solving (4.100) with boundary condition (4.102) we achieve (4.97). Notice that in (4.102), the continuity of $p(\cdot)$ at a is shown under CASE 1 and CASE 2 in step 7. The analysis of $p(\cdot)$ on $]\bar{x}, b[$ is entirely similar. The only difference is that according to the analysis under CASE 1 and CASE 3 in step 7, $p(\cdot)$ is either continuous at b or $p(b-) = p^\sharp(b)$.

Finally, we prove (ii). For the given function $p(\cdot)$, call V^* the value function for the optimal control problem (4.98)-(4.99). For any initial value $x_0 \in [0, M]$, the feedback control $u^*(x) \doteq u^\sharp(p(x-), V'(x-))$ yields the cost $V(x_0)$. Thus we have

$$V^*(x_0) \le V(x_0).$$

To prove optimality we need to show that $V^*(x_0) \le V(x_0)$. Consider any measurable control function $u(t) : [0, +\infty[\to [0, 1]$. Calling $t \to x(t)$ the solution to

$$\dot{x} \;=\; \left(-\lambda - \mu + \frac{\lambda + r}{p(x)}\right) x - \frac{u(t)}{p(x)}, \qquad x(0) = x_0,$$

a standard computation yields that

$$E\left[\int_0^{T_b} e^{-rt} L(u(t))\, dt + B\, e^{-rT_b}\right] = \int_0^{T^M} \gamma(t) \Big\{\rho(x(t))\, B + L(u(t))\Big\}\, dt \;+\; \gamma(T^M)\, B,$$

where $T^M = \inf\big\{t \ge 0; x(t) = M\big\} \in]0, +\infty]$ is the bankruptcy time and

$$\gamma(t) \;\doteq\; e^{-rt} \exp\left\{-\int_0^t \rho(x(s))\, ds\right\}.$$

Then we need to show that

$$\int_0^{T^M} \gamma(t) \Big\{\rho(x(t))\, B + L(u(t))\Big\}\, dt \;+\; \gamma(T^M)\, B \ge V(x_0). \tag{4.103}$$

For $t \in [0, T^M]$, consider the function

$$\phi^u(t) \doteq \int_0^t \gamma(s) \Big\{\rho(x(s))\, B + L(u(s))\Big\}\, ds \;+\; \gamma(t) V^*(x(t)).$$

Notice that ϕ^u is absolutely continuous except for a possible upward jump at $\hat{t} \doteq \sup_{t<T^M}\{x(t) < \hat{x}\}$. By the definition of $\hat{x}$, for any control $u(\cdot)$ we have $\dot{x}(t) > 0$ for all $t \ge \hat{t}$. Indeed, $\hat{x}$ is a "point of no return": when the debt crosses $\hat{x}$, there is no way to reduce it.

At any Lebesgue point t of $u(\cdot)$, with $t \ne \hat{t}$, we compute

$$\begin{aligned}
\tfrac{d}{dt}\phi^u(t) \;&=\; \gamma(t)\left[\rho(x(t))B + L(u(t)) - (r + \rho(x(t)))V(x(t)) + V'(x(t))\dot{x}(t)\right] \\
&=\; \gamma(t)\left[\rho(x(t))B + L(u(t)) - (r + \rho(x(t)))V(x(t)) + V'(x(t))\left(\left(-\lambda - \mu + \frac{\lambda + r}{p(x)}\right) x - \frac{u(t)}{p(x)}\right)\right] \\
&\ge\; \gamma(t)\left[\min_{\omega \in [0,1]}\{L(\omega) - \frac{V'(x(t))}{p(x)}\omega\} - (r + \rho(x(t)))V(x(t)) + \rho(x(t))B\right. \\
&\qquad \left. + V'(x(t))\left(\left(-\lambda - \mu + \frac{\lambda + r}{p(x(t))}\right) x\right)\right] \\
&=\; \gamma(t)\left[H(x(t), V'(x(t)), p(x(t))) - (r + \rho(x(t)))V(x(t)) + \rho(x(t))B\right] \;=\; 0.
\end{aligned}$$

Since $V(\hat{x}+) \geq V(\hat{x}-)$, we also have $\phi^u(\hat{t}-) \leq \phi^u(\hat{t}+)$. Therefore

$$V(x_0) \;=\; \phi^u(0) \;\leq\; \lim_{t\to T^M-} \phi^u(t) \;=\; \int_0^{T^M} \gamma(t)\Big\{\rho(x(t))\,B + L(u(t))\Big\}\,dt \;+\; \gamma(T^M)\,B.$$

This completes the proof of Theorem 3. □

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