Control Problems for a Class of Set Valued Evolutions

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Abstract

The paper studies controllability problems for the reachable set of a differential inclusion. These were originally motivated by models of control of a flock of animals. Conditions are derived for the existence or nonexistence of a strategy which confines the reachable set within a given bounded region, at all sufficiently large times. Steering problems and the asymptotic shape of the reachable set are also investigated.

Key words: differential inclusion, reachable set, global confinement.

1 Introduction

The analysis and control of evolution equations on a general metric space has been the topic of several investigations [4, 5, 15]. In particular, this provides a convenient setting for the control of the evolution of a set $S(t)$, depending on time.

Aim of this paper is to analyze a specific class of control problems for a moving set. A version of this model, involving conservation laws, was first proposed by R. Colombo and M. Mercier [12] to describe the controlled motion of a flock of animals. Let $\rho = \rho(t,x)$ denote the density of individuals at time t at the location $x \in I\!R^2$. The time evolution of this density is governed by the scalar conservation law

$$\rho_t + \mathrm{div}(\rho\,\mathbf{v}) \;=\; 0. \tag{1.1}$$

It is assumed that individuals choose their velocity $\mathbf{v}$ with two goals in mind:

(i) spread out toward less crowded areas,

(ii) move away from a repelling source, located at a variable position $\xi(t) \in I\!R^2$.

To model (i), a meaningful choice is to set

$$\mathbf{v} \;=\; \mathbf{v}(\rho, \nabla\rho) \;=\; -\frac{\nabla\rho}{\sqrt{\rho^2 + \frac{1}{c^2}|\nabla\rho|^2}}\,, \tag{1.2}$$

for some constant $c > 0$ determined by the maximum speed. This yields the conservation law

$$\rho_t - \mathrm{div}\left(\frac{\rho\,\nabla\rho}{\sqrt{\rho^2 + \frac{1}{c^2}|\nabla\rho|^2}}\right) \;=\; 0. \tag{1.3}$$

The equation (1.3), which coincides with the relativistic heat equation, was studied in [1]. The Cauchy problem has globally defined, unique solutions.

To model (ii), let $\xi = \xi(t) \in \mathbb{R}^2$ be the position of the repelling source. For example, this could be the position of a dog, controlling a flock of sheep. The velocity of an individual (a sheep) located at x will be described as

$$\mathbf{v}(x,\xi) \;=\; \varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\,. \tag{1.4}$$

A natural set of assumptions on the function φ is:

(A1) *The map $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a non-increasing function, with $\varphi(s) \to 0$ as $s \to +\infty$.*

For example, one may take

$$\varphi(r) \;=\; ae^{-br}. \tag{1.5}$$

In alternative, one can also consider

$$\varphi(r) \;\doteq\; \begin{cases} \alpha & \text{if} \qquad r \leq \sigma\,,\\ 0 & \text{otherwise}\,, \end{cases} \tag{1.6}$$

or

$$\varphi(r) \;\doteq\; \min\;\{\beta,\; \alpha r^{-\gamma}\}\,. \tag{1.7}$$

It will be convenient to represent the function $\mathbf{v}$ in (1.4) as

$$\mathbf{v}(x,\xi) \;=\; \nabla_x\Phi(|x-\xi|)\,, \qquad\qquad \Phi(r) \;\doteq\; \int_0^r \varphi(s)\,ds\,, \tag{1.8}$$

where the gradient is taken w.r.t. the x-variable.

Combining the two terms in (1.2) and (1.4), the conservation law (1.1) takes the form

$$\rho_t - \mathrm{div}\left(\frac{\rho\,\nabla\rho}{\sqrt{\rho^2 + \frac{1}{c^2}|\nabla\rho|^2}}\right) \;=\; \mathrm{div}\Big(\rho\,\nabla\Phi(x,\xi(t))\Big). \tag{1.9}$$

Following [12], we regard (1.9) as a controlled PDE, where $\xi(\cdot)$ is the control function. In the present paper, instead of the density function ρ itself, we focus on the control of the *support* of ρ. This will be denoted as

$$S(t) \;\doteq\; \mathrm{Supp}(\rho(t,\cdot)) \;=\; \overline{\Big\{x \in \mathbb{R}^2\,;\;\; \rho(t,x) > 0\Big\}}\,,$$

where the overline indicates the closure of a set.

For the equation (1.3) with smooth initial data $\rho(0,x) = \rho_0(x)$, a major result proved in [2] states that the support of the density $\rho(t,\cdot)$ expands with speed c in all directions. Namely,

$$S(t) \;=\; \Big\{x \in I\!R^2\,;\;\; d(x, S(0)) \leq ct\Big\} \;=\; S(0) + \overline{B}(0, ct)\,. \tag{1.10}$$

Here and in the sequel, $B(y,r)$ and $\overline{B}(y,r)$ denote respectively the open and closed disc centered at y with radius r.

Motivated by (1.10), we introduce a model for the evolution of the region $S(t) \subset I\!R^2$ occupied by a flock of animals at time t, formulated in terms of a differential inclusion. For $x, \xi \in I\!R^2$, it is natural to consider the set of velocities

$$G(x,\xi) \;\doteq\; \overline{B}\left(\varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\;,\;c\right).$$

However, the multifunction G defined in this way may not be upper semicontinuous (at points x, ξ such that either $x = \xi$ or else φ is discontinuous at $s = |x-\xi|$). Throughout the following, we shall thus work with the upper semicontinuous convex valued regularization of the above multifunction. This is obtained by setting

$$G(x,\xi) \;\doteq\; \left\{y\,;\;\; \left|y - \lambda\frac{x-\xi}{|x-\xi|}\right| \;\leq\; c \quad \text{for some } \lambda \in [\varphi(s+),\, \varphi(s-)]\right\} \qquad \text{if } \; x \neq \xi\,, \tag{1.11}$$

$$G(x,x) \;=\; \overline{B}(0\,,\; \varphi(0) + c)\,. \tag{1.12}$$

Under the assumptions (A1), one easily checks that the multifunction G in (1.11)-(1.12) is upper semicontinuous with compact, convex values. Moreover, it satisfies the uniform bound

$$G(x,\xi) \;\subseteq\; \overline{B}(0\,,\; \varphi(0) + c) \qquad\qquad \text{for all } x, \xi \in I\!R^2\,. \tag{1.13}$$

Calling S_0 the region occupied at time $t = 0$, we denote by $S(t)$ be the reachable sets for the differential inclusion

$$\dot{x} \;\in\; G(x, \xi(t))\,, \qquad\qquad x(0) \in S_0\,. \tag{1.14}$$

In other words, for any $t \geq 0$,

$$S(t) \;\doteq\; \Big\{x(t)\,;\quad x(0) \in S_0\,,\qquad x(\cdot) \;\text{ is absolutely continuous,} \\ \dot{x}(\tau) \;\in\; G\,(x(\tau),\,\xi(\tau)) \;\; \text{for a.e. } \tau \in [0,t]\Big\}. \tag{1.15}$$

As a first model, one may consider any continuous function $\xi : [0,\infty[\mapsto I\!R^2$ as an admissible control. If $\xi(\cdot)$ denotes the position of a dog initially located at ξ_0, that can run at a maximum speed σ, a more realistic model would include the assumption

(A2) *The control function* $t \mapsto \xi(t) \in I\!R^2$ *is Lipschitz continuous, with*

$$|\dot{\xi}(t)| \;\leq\; \sigma\,, \qquad\qquad \xi(0) = \xi_0\,. \tag{1.16}$$

As for the fire confinement problem [8, 9], this model leads to some natural questions.

1 - Global confinement. Assume that the initial set S_0 is bounded. Is it possible to keep the set $S(t)$ uniformly bounded for all positive times?

We thus seek conditions which provide the existence (or nonexistence) of a radius $R > 0$ and a control $\xi(\cdot)$ such that

$$S(t) \subseteq \overline{B}(0, R) \qquad \text{for all } t \geq 0\,. \tag{1.17}$$

A related question is the following. Let two points $P_1, P_2 \in \mathbb{R}^2$ be given, together with radii $r_1, r_2 > 0$. Assuming that $S_0 \subseteq \overline{B}(P_1, r_1)$, is it possible to find a control $\xi(\cdot)$ that, at some later time $\tau > 0$, one has $S(\tau) \subseteq \overline{B}(P_2, r_2)$?

2 - Steering with constant speed. Assume we want to steer the flock, say in the direction of the x_1-axis, with constant speed $\lambda \geq 0$. When is this possible?

More precisely, we ask whether there exists a radius $R > 0$ and a control $t \mapsto \xi(t)$ such that

$$S(t) \in B(t\lambda\mathbf{e}_1,\, R) \qquad \text{for all } t \geq 0\,. \tag{1.18}$$

By $\mathbf{e}_1$ we denote the unit vector parallel to the x_1-axis.

3 - Quasi-stationary domains. A further problem is to identify the family of compact sets S_0 for which the following stabilization property holds: For every $\varepsilon > 0$, there exists a control $\xi(\cdot)$ such that the corresponding sets $S(t)$ in (1.15) satisfy

$$d_H(S(t)\,,\, S_0) \leq \varepsilon \qquad \text{for all } t \geq 0\,. \tag{1.19}$$

In the analysis of controllability properties, a key role is played by rotating controls, where the point $\xi(t)$ rotates along a circumference with constant speed. For this reason, in the last section we study this situation in more detail. In the setting we are considering, from general results about periodic orbits of dynamical systems [14, 16] it already follows that the map $t \mapsto S(t)$ converges to a time periodic function as $t \to +\infty$, w.r.t. the Hausdorff distance. In the present case, a stronger result can be shown. Namely, in a set of rotating coordinates, for all times t sufficiently large the boundary of the set $S(t)$ is a Lipschitz curve that admits a polar coordinate representation $r = r(t, \theta)$. Moreover, as $t \to \infty$, one has the uniform convergence $r(t, \theta) \mapsto r_\infty(\theta)$, for a smooth function r_∞, characterized as the unique 2π-periodic orbit of a suitable ODE.

Section 2 of this paper contains an approximation theorem. Given a probability measure μ on $\mathbb{R}^2$ and a corresponding "averaged" multifunction $x \mapsto G(x, \mu)$, we consider the differential inclusion

$$\dot{x}(t) \in G(x(t), \mu)\,, \qquad S(0) = S_0\,. \tag{1.20}$$

One can then construct a sequence of time periodic control functions $\xi_n(\cdot)$ such that the corresponding reachable sets $S_n(t)$ for (1.14) are "almost contained" in the reachable sets for (1.20).

In Section 3 we give some necessary and some sufficient conditions in order that the global confinement problem or the steering problem admit a solution. Finally, Section 4 analyzes in more detail the case where stabilization is achieved by means of a control function $\xi(\cdot)$ rotating with constant speed. In this case, it is shown that the reachable set $S(t)$ converges in a strong sense to a periodic multifunction.

For the basic theory of multifunctions and differential inclusions we refer to [6, 13]. A survey of different models describing the motion of flocks of animals can be found in the recent paper [7].

2 Averaging and approximation results

Let $G = G(x, \xi)$ be a bounded, upper semicontinuous multifunction on $I\!R^2 \times I\!R^2$ with compact, convex values. To fix the ideas, assume

$$G(x,\xi) \subseteq \overline{B}(0,M) \qquad \text{for all} \quad x,\xi \in I\!R^2. \tag{2.1}$$

Let $\mathcal{P}$ be the family of all probability measures on $I\!R^2$. For $\mu \in \mathcal{P}$, consider the "averaged" multifunction

$$G(x,\mu) \doteq \int G(x,\xi)\, d\mu(\xi) \doteq \left\{ \int g(\xi)\, d\mu(\xi)\,; \quad g \text{ measurable}\,, \quad g(\xi) \in G(x,\xi) \text{ for all } \xi \right\}. \tag{2.2}$$

Lemma 1. *In the above setting, the multifunction $x \mapsto G(x,\mu)$ in (2.2) is bounded, upper semicontinuous, with compact convex values.*

Proof. 1. Since μ is a probability measure, by (2.1) it is clear that $G(x,\mu) \subseteq \overline{B}(0,M)$ for all $x \in I\!R^2$. Moreover, since all sets $G(x,\xi)$ are convex, $G(x,\mu)$ is convex as well.

2. To show that each set $G(x,\mu)$ is closed, consider a sequence of points

$$y_n = \int g_n(\xi)\, d\mu(\xi) \in G(x,\mu).$$

with $y_n \to y$ as $n \to \infty$. Taking a subsequence, we can assume the weak convergence $g_n \rightharpoonup g$ in $\mathbf{L}^1_\mu$, for some function g. The convexity of all sets $G(x,\xi)$ implies $g(\xi) \in G(x,\xi)$. Hence

$$y = \lim_{n\to\infty} \int g_n(\xi)\, d\mu(\xi) = \int g(\xi)\, d\mu(\xi) \in G(x,\mu),$$

proving that the set $G(x,\mu)$ is closed, hence compact.

3. Finally, we check that the map $x \mapsto G(x,\mu)$ is upper semicontinuous. Fix a point $\bar{x}$ and let $\varepsilon > 0$ be given. By the upper semicontinuity of the map $(x,\xi) \mapsto G(x,\xi)$, we can find a measurable function $\xi \mapsto r(\xi) > 0$ such that

$$G(x,\xi) \subset B(G(\bar{x},\xi)\,,\ \varepsilon/3) \qquad \text{for all } x \in B(\bar{x},\, r(\xi)). \tag{2.3}$$

Choose $\delta > 0$ such that

$$\mu\Big(\{\xi\,;\ \ r(\xi) \le \delta\}\Big) < \frac{\varepsilon M}{3}\,. \tag{2.4}$$

We claim that this choice yields

$$G(x,\mu) \subset B(G(\bar{x},\mu)\,,\ \varepsilon) \qquad \text{for all } x \in B(\bar{x},\, \delta). \tag{2.5}$$

Indeed, assume that $x \in B(\bar{x},\delta)$ and consider an arbitrary element

$$y = \int g(\xi)\, d\mu(\xi) \in G(x,\mu)\,,$$

for some function $\xi \mapsto g(\xi) \in G(x,\xi)$. Calling $\pi \circ g(\xi)$ the perpendicular projection of $g(\xi)$ on the compact convex set $G(\bar{x},\xi)$, by (2.3)-(2.4) we have

$$\begin{aligned} d(y, G(\bar{x},\mu)) &\leq \int_{r(\xi)>\delta} \Big| g(\xi) - \pi\circ g(\xi)\Big| d\mu(\xi) + \int_{r(\xi)\leq\delta} \Big| g(\xi) - \pi\circ g(\xi)\Big| d\mu(\xi) \\ &< \frac{\varepsilon}{3} + 2M\cdot \mu\Big(\{\xi;\quad r(\xi)\leq\delta\}\Big) \;<\; \varepsilon\,. \end{aligned}$$

This establishes (2.5), and hence the upper semicontinuity of the multifunction $x \mapsto G(x,\mu)$. □

We now apply the previous general result to the multifunction $G = G(x,\xi)$ in (1.11)-(1.12). Given a probability measure μ and a compact set S_0 as initial data, we denote by $S^\mu(t)$ the reachable sets for the differential inclusion (1.20). Moreover, we write $\overline{B}(A,\varepsilon)$ for the closed ε-neighborhood around a set A. In our analysis of confinement and steering problems, the following approximation result will be repeatedly used.

Theorem 1. *Let G be the multifunction in (1.11)-(1.12), assuming that φ is Lipschitz continuous and satisfies (A1). Then for any $T,\varepsilon>0$ and any compact set $S_0\subset I\!R^2$ there exists a smooth control function $\xi:[0,T]\mapsto I\!R^2$ such that the reachable sets S^ξ, S^μ for the differential inclusions (1.14), (1.20) satisfy*

$$S^\xi(t) \;\subseteq\; \overline{B}(S^\mu(t)\,,\;\varepsilon) \qquad\qquad \textit{for all } t\in[0,T]\,. \tag{2.6}$$

Proof. 1. Assume $S_0\subseteq\overline{B}(0,M_0)$ for some constant M_0. In view of (2.1) for $t\in[0,T]$, all trajectories starting in S_0 will satisfy the a priori bound

$$|x(t)| \;\leq\; M_0 + MT \qquad \text{for all} \quad t\in[0,T]\,. \tag{2.7}$$

Throughout the following, we can thus restrict our analysis to the compact disc $B^* \doteq \overline{B}(0,\, M_0 + MT)\subset I\!R^2$.

2. By upper semicontinuity, there exists $\delta^\sharp>0$ small enough such that the following holds. If $G^\sharp$ is any multifunction such that

$$G^\sharp(x) \;\subseteq\; co\left(\bigcup_{|x'-x|\leq\delta^\sharp} B\Big(G(x',\mu)\,,\;\delta^\sharp\Big)\right) \qquad \text{for all } x\in B^*, \tag{2.8}$$

then the reachable sets $S^\sharp(t)$ for the differential inclusion

$$\dot{x}(t) \;\in\; G^\sharp(x(t))\,, \qquad\qquad S(0) \;=\; S_0\,, \tag{2.9}$$

satisfy

$$S^\sharp(t) \;\subseteq\; \overline{B}(S^\mu(t)\,,\;\varepsilon/2) \qquad \text{for all} \quad t\in[0,T]\,. \tag{2.10}$$

3. We now approximate μ by a purely atomic measure $\mu^\sharp$. More precisely, denote by δ_y the Dirac measure concentrating a unit mass at the point $y\in I\!R^2$. We can then find points

$y_1, \ldots, y_m \in I\!R^2$ and coefficients $\lambda_k \in [0,1]$ with $\sum_{k=1}^m \lambda_k = 1$ such that the probability measure

$$\mu^\sharp \doteq \sum_{k=1}^m \lambda_k \, \delta_{y_k} \tag{2.11}$$

yields a multifunction

$$G^\sharp(x) \doteq G(x, \mu^\sharp) = \sum_{k=1}^m \lambda_k \, G(x, y_k) \tag{2.12}$$

satisfying (2.8). In particular, this implies (2.10).

4. Using the upper semicontinuity of the multifunctions $G(\cdot, y_k)$ and $G^\sharp$ in (2.12), we can choose $\delta > 0$ small enough so that the following holds. If $G_1, \ldots, G_m$ are multifunctions such that

$$G_k(x) \subseteq \left(\bigcup_{|x'-x| \leq 2\delta} G(x', y_k) \right), \qquad k = 1, \ldots, m\,, \tag{2.13}$$

then the reachable sets $S^*(t)$ for the differential inclusion

$$\dot{x}(t) \in G^*(x(t)) \doteq \sum_{k=1}^n \lambda_k G_k(x(t))\,, \qquad S(0) = S_0\,, \tag{2.14}$$

satisfy

$$S^*(t) \subseteq \overline{B}(S^\sharp(t)\,,\ \varepsilon/4) \qquad \text{for all} \quad t \in [0, T]\,. \tag{2.15}$$

5. The control function $\xi(\cdot)$ can now be constructed as follows. Choose an integer n large enough so that $TM/n < \delta$ and divide the time interval $[0, T]$ into n equal subintervals, inserting the points $t_i = iT/n$, $0 \leq i \leq n$. Each interval $I_i = [t_{i-1}, t_i]$ is further partitioned into subintervals $I_{i,k}$ whose lengths are proportional to the coefficients λ_k, $k = 1, \ldots, m$ in (2.11). We then define

$$\xi_n(t) = y_k \qquad \text{for} \quad t \in \bigcup_{i=1}^n I_{i,k}\,. \tag{2.16}$$

The reachable sets for the differential inclusion

$$\dot{x} \in G(x, \xi_n(t))\,, \qquad x(0) \in S_0 \tag{2.17}$$

will be denoted by $S_n(t)$.

6. Let $x(\cdot)$ be any solution of (2.17). By (2.1), for $t \in [t_{i-1}, t_i]$ it follows

$$|x(t) - x(t_{i-1})| \leq M \cdot \frac{T}{n}\,. \tag{2.18}$$

Consider the polygonal approximation

$$x^*(t) \doteq x(t_{i-1}) + (t - t_{i-1}) \cdot \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \qquad \text{for all} \ \ t \in [t_{i-1}, t_i]\,.$$

By definition $x^*(t_i) = x(t_i)$ for all $i = 0, \ldots, n$. If $t \in [t_{i-1}, t_i]$, then

$$|x^*(t) - x(t)| \leq 2M(t - t_{i-1}) \leq \frac{2MT}{n}\,. \tag{2.19}$$

Moreover, our previous construction yields

$$\dot{x}^*(t) \;=\; \frac{x(t_i)-x(t_{i-1})}{t_i-t_{i-1}} \;\in\; \sum_{k=1}^m \lambda_k \left(\bigcup_{|x'-x(t_{i-1})|\leq\delta} G(x',y_k)\right) \;\subseteq\; \sum_{k=1}^m \lambda_k \left(\bigcup_{|x'-x^*(t)|\leq 2\delta} G(x',y_k)\right). \tag{2.20}$$

Recalling (2.13)-(2.15), we thus obtain

$$x^*(t) \;\in\; \overline{B}(S^\sharp(t), \varepsilon/4). \tag{2.21}$$

Choosing n so large that $\frac{2MT}{n} < \frac{\delta}{4}$, using (2.19), (2.21), and then (2.10), we obtain

$$x(t) \;\in\; \overline{B}\left(S^\sharp(t), \frac{\varepsilon}{4}+\frac{\varepsilon}{4}\right) \;\subseteq\; \overline{B}\Big(S^\mu(t),\varepsilon\Big).$$

Since $x(\cdot)$ was an arbitrary solution of the differential inclusion (2.17), we have shown that the control function $\xi(t) = \xi_n(t)$ in (2.16) yields the desired estimate (2.6).

7. To achieve the proof, we need to modify the piecewise constant function $\xi_n(\cdot)$, making it smooth on the entire interval $[0,T]$. Relying again on the upper semicontinuity of the reachable sets, if the new function $\xi(\cdot)$ coincides with the $\xi_n(\cdot)$ on a set of times having sufficiently small Lebesgue measure, after this modification the bound (2.6) will be replaced by

$$S^\xi(t) \;\subseteq\; \overline{B}(S^\mu(t)\,,\; 2\varepsilon) \qquad\qquad \text{for all } t\in[0,T]\,.$$

Since $\varepsilon>0$ was arbitrary, this completes the proof. □

3 Confinement strategies

3.1 A necessary condition

We start by deriving a necessary condition for the existence of a confining strategy. The following result shows that, if the initial set S_0 is already large, then, no matter how fast the controller ξ can move, it cannot prevent the reachable sets $S(t)$ from becoming arbitrarily large as $t\to+\infty$. Consider the variables

$$A \;\doteq\; \pi r^2\,, \qquad\qquad \Phi \;\doteq\; \int_{B_r} \mathrm{div}\,\mathbf{v} \;=\; 2\pi r\,\varphi(r).$$

We regard Φ as a function of A, so that

$$\Phi(A) \;=\; 2\pi\sqrt{\frac{A}{\pi}}\,\varphi\left(\sqrt{\frac{A}{\pi}}\right) \;=\; \int_0^A \left[\frac{d}{dA}\Phi(A)\right].$$

Motivated by the previous computations, in the following theorem, we let $s\mapsto\hat\varphi(s)$ be the non-decreasing rearrangement of the function

$$A \;\mapsto\; \frac{d}{dA}\Phi(A) \;=\; \sqrt{\frac{\pi}{A}}\varphi\left(\sqrt{\frac{A}{\pi}}\right) + \varphi'\left(\sqrt{\frac{A}{\pi}}\right)$$

In other words, $\hat{\varphi} : [0, \infty[\mapsto I\!R$ is the unique (up to a set of zero measure) non-decreasing function such that, for every $k \leq 0$,

$$meas\Big(\{A \geq 0\,;\ \ \hat{\varphi}(A)\ \leq\ k\}\Big)\ =\ meas\left(\left\{A \geq 0\,;\ \ \sqrt{\frac{\pi}{A}}\varphi\Big(\sqrt{\frac{A}{\pi}}\Big) + \varphi'\Big(\sqrt{\frac{A}{\pi}}\Big)\ \leq\ k\right\}\right). \tag{3.1}$$

Theorem 2. *Assume that, for some constant A_0, the non-decreasing rearrangement $\hat{\varphi}$ satisfies*

$$g(A)\ \doteq\ 2c\sqrt{\pi A} + \int_0^A \hat{\varphi}(s)\,ds\ >\ 0 \qquad \textit{for every}\ \ A > A_0\,. \tag{3.2}$$

Then, if the initial set S_0 has measure $m_2(S_0) > A_0$, uniform confinement is not possible. Indeed, for any choice of the control $\xi(\cdot)$ one has

$$m_2(S(t))\ \to\ \infty \qquad as \quad t \to \infty. \tag{3.3}$$

Proof. The area of the set $S(t)$ evolves in time according to

$$\frac{d}{dt}m_2(S(t))\ =\ c \cdot m_1(\partial S(t)) + \int_{S(t)} \mathrm{div}\,\mathbf{v}\,, \tag{3.4}$$

where $\mathbf{v}$ is the vector field in (1.4). Here ∂S denotes the boundary of the set S, while m_1 is the one-dimensional Hausdorff measure, normalized so that $m_1(\gamma)$ gives the usual length of a smooth curve γ.

If $m_2(S(t)) = \pi r^2(t)$ for some $r(t)$, then the isoperimetric inequality yields

$$m_1(\partial S(t))\ \geq\ 2\pi r(t)\ =\ 2\sqrt{\pi\, m_2(S(t))}. \tag{3.5}$$

This provides a lower bound on the first term on the right hand side of (3.4).

To achieve a bound on the second term we observe that, by the definition of $\hat{\varphi}$,

$$\int_S \mathrm{div}\,\mathbf{v}\ \geq\ \inf\left\{\int_{S'} \mathrm{div}\,\mathbf{v}\ \ ;\quad S' \subset I\!R^2\,,\quad m_2(S') = m_2(S)\right\}\ =\ \int_0^{m_2(S)} \hat{\varphi}(\zeta)\,d\zeta\,. \tag{3.6}$$

Using (3.5) and (3.6) in (3.4), we obtain

$$\frac{d}{dt}m_2(S(t))\ \geq\ g\Big(m_2(S(t))\Big),$$

where $g :]A_0, \infty[\mapsto I\!R_+$ is the continuous, strictly positive function introduced at (3.2).

A standard comparison argument for ODEs now yields (3.3). □

Example 1. Consider the case where $\varphi(r) = ae^{-br}$, as in (1.5). Calling $r = |x|$, we have

$$\operatorname{div} \mathbf{v}(x) \;=\; \left(\frac{\varphi(r)}{r} + \varphi'(r)\right) \;=\; \left(\frac{1}{r} - b\right) ae^{-br} \;=\; \left(\sqrt{\frac{\pi}{A}} - b\right) ae^{-b\sqrt{A/\pi}}$$

Hence $\operatorname{div} \mathbf{v}(x) \leq 0$ if and only if $|x| \geq b{-}1$. For every set $S \subset I\!\!R^2$ we thus have

$$\int_S \operatorname{div} \mathbf{v} \;\geq\; \int_{|x|\geq 1/b} \operatorname{div} \mathbf{v} \;=\; -\,2\pi\varphi(1/b) \;=\; -\,\frac{2\pi a}{e}\,.$$

In particular, if the initial set S_0 has area $m_2(S_0) > \pi a^2/c^2e^2$, then its perimeter satisfies $c \cdot m_1(\partial S_0) \geq 2c\sqrt{\pi\, m_2(S_0)} > 2\pi a/e$. The corresponding sets $S(t)$ become arbitrarily large: $m_2(S(t)) \to \infty$ as $t \to \infty$.

Example 2. The function $\varphi(r)$ defined at (1.6) is not continuous. However, it can be approximated by the piecewise affine functions

$$\varphi_n(r) \;\doteq\; \begin{cases} \alpha & \text{if} \quad r \leq \sigma - 1/n\,, \\ \alpha \cdot n(\sigma - x) & \text{if} \quad \sigma - 1/n < r < \sigma\,, \\ 0 & \text{if} \quad r \geq \sigma\,. \end{cases} \tag{3.7}$$

For any set $S \subset I\!\!R^2$, the corresponding vector field $\mathbf{v}_n$ in (1.4) satisfies

$$\int_S \operatorname{div} \mathbf{v}_n \;\geq\; -\,2\pi\alpha\sigma\,.$$

Taking the limit as $n \to \infty$, by Theorem 2 we conclude that, if the initial set S_0 has area $m_2(S_0) > \pi\alpha^2\sigma^2/c^2$, then its perimeter satisfies $c \cdot m_1(\partial S_0) \geq 2c\sqrt{\pi\, m_2(S_0)} > 2\pi\alpha\sigma$, and $m_2(S(t)) \to \infty$ as $t \to \infty$.

Example 3. In the case $\varphi(r) = \min\{\beta,\, \alpha r^{-\gamma}\}$, setting $r^* \doteq (\alpha/\beta)^{1/\gamma}$ we compute

$$\operatorname{div} \mathbf{v}(x) \;=\; \begin{cases} \beta/r & \text{if} \quad |x| < r^*\,, \\ \alpha(1-\gamma)|x|^{-\gamma-1} & \text{if} \quad |x| > r^*\,. \end{cases}$$

Notice that, if $\gamma \leq 1$, then $\operatorname{div} \mathbf{v}(x) > 0$ for all $x \in I\!\!R^2$. In this case the measure $m_2(S(t))$ will be always increasing in time, and uniform confinement is impossible. On the other hand, if $\gamma > 1$, then

$$\int_S \operatorname{div}\mathbf{v} \;\geq\; \int_{|x|\geq r^*} \operatorname{div}\mathbf{v} \;=\; -\,2\pi r^*\,\varphi(r^*) \;=\; -\,2\pi\alpha^{1/\gamma}\beta^{1-1/\gamma}.$$

If the initial set S_0 has area $m_2(S_0) > \pi\alpha^2\beta^{2-2/\gamma}/c^2$, then its perimeter satisfies $c \cdot m_1(\partial S_0) \geq 2c\sqrt{\pi\, m_2(S_0)} > 2\pi\alpha^{1/\gamma}\beta^{1-1/\gamma}$, and $m_2(S(t)) \to \infty$ as $t \to \infty$.

3.2 A steering problem

Next, we consider the problem of steering the set $S(t)$, initially inside a disc $\overline{B}(P_1, r_1)$, to another disc $\overline{B}(P_2, r_2)$. To state a positive result in this direction, an auxiliary function needs to be introduced.

Fix a radius $r_0 \geq 0$, and consider a probability distribution μ uniformly distributed along the circumference centered at the origin with radius r_0. Consider the "averaged" vector field

$$\mathbf{w}(x) \doteq \int_{|\xi|=r_0} \varphi(|x-\xi|) \frac{x-\xi}{|x-\xi|} \, d\mu(\xi) . \tag{3.8}$$

Clearly this vector field is radially symmetric, having the form

$$\mathbf{w}(x) = \phi(|x|, r_0) \frac{x}{|x|} . \tag{3.9}$$

The function ϕ can be computed using the divergence theorem. Indeed, for every $0 < r < r_0$ we have

$$2\pi r \, \phi(r, r_0) = \int_{|\xi|=r_0} \left(\int_{B(0,r)} \operatorname{div} \mathbf{v}(x,\xi) \right) d\mu(\xi) = \int_{B(0,r)} \operatorname{div} \mathbf{v}(x, P) , \tag{3.10}$$

where P is any point having distance r_0 from the origin. Taking $P = (r_0, 0) \in I\!R^2$, the right hand side of (3.8) is computed by

$$\phi(r, r_0) = \frac{1}{2\pi r} \int_{r_0-r}^{r_0+r} 2s \left(\frac{\varphi(s)}{s} + \varphi'(s) \right) \arccos \left(\frac{s^2 + r_0^2 - r^2}{2r_0 s} \right) ds . \tag{3.11}$$

Theorem 3. *Assume that $\varphi : I\!R_+ \mapsto I\!R_+$ satisfies (A1) and let ϕ be the function defined at (3.8)-(3.9). Let $0 < r_2 \leq r_1$ and assume that*

$$\inf_{\rho > r} \phi(r, \rho) < -c \qquad \text{for all } r \in [r_2, r_1] . \tag{3.12}$$

Let the initial condition satisfy $S_0 \subseteq \overline{B}(P_1, r_1)$ for some point P_1. Then, for any point $P_2 \in I\!R^2$ there exists a smooth control function $t \mapsto \xi(t)$ and $T > 0$ such that the corresponding set satisfies $S(T) \subseteq \overline{B}(P_2, r_2)$.

Proof. 1. Let the assumption (3.12) hold. Then for every $r \in [r_2, r_1]$ there exists $\varepsilon > 0$ and a probability measure $\mu_{\rho(r)}$, uniformly distributed along the circumference $\partial B(P_1, \rho(r))$ centered at P_1 with radius $\rho(r) > r + \varepsilon$, such that the following holds. At some time $\tau > 0$, every solution to the differential inclusion

$$\dot{x} \in G(x, \mu_{\rho(r)}) , \qquad x(0) \in \overline{B}(P_1, r + \varepsilon)$$

satisfies

$$x(\tau) \in \overline{B}(P_1, r - \varepsilon).$$

2. Since the interval $[r_2, r_1]$ is compact, by a covering argument we can find $\tau, \varepsilon > 0$ and radii R_k with

$$r_1 = R_0 > R_1 > \cdots > R_N = r_2$$

such that the following holds. For every $k = 1, \ldots, N$, every solution to the differential inclusion

$$\dot{x} \in G(x, \mu_{\rho(R_k)}) , \qquad x(0) \in \overline{B}(P_1, R_k)$$

satisfies

$$x(\tau) \;\in\; \overline{B}(P_1,\, R_{k+1}-\varepsilon).$$

By Theorem 1, for every k there exists a control function $\xi_k : [0,\tau] \mapsto I\!R^2$ such that every solution to the differential inclusion

$$\dot x \;\in\; G(x, \xi_k(t))\,, \qquad\qquad x(0)\in \overline{B}(P_1,\, R_k)$$

satisfies

$$x(\tau) \;\in\; \overline{B}(P_1,\, R_{k+1}).$$

Consider the control function $\xi : [0, N\tau] \mapsto I\!R^2$ defined as the concatenation

$$\xi(t) \;\doteq\; \xi_k(t-(k-1)\tau) \qquad\qquad t\in [(k-1)\tau,\, k\tau]\,.$$

Then every solution of the differential inclusion

$$\dot x \;\in\; G(x, \xi(t))\,, \qquad\qquad x(0)\in \overline{B}(P_1,\, r_1)$$

satisfies

$$x(N\tau) \;\in\; \overline{B}(P_1,\, r_2).$$

This already proves the theorem in the case $P_2 = P_1$.

3. Next, consider the unit vector $\mathbf{e} = (P_2 - P_1)/|P_2 - P_1|$ and choose an integer m large enough so that

$$\delta \;=\; \frac{|P_2-P_1|}{m} \;<\; r_1 - r_2\,.$$

By the previous step, for every $j\geq 1$ there exists a control function $\xi_j : [0, N\tau] \mapsto I\!R^2$ such that every solution of

$$\dot x \;\in\; G(x, \xi_j(t))\,, \qquad\qquad x(0)\in \overline{B}(P_1+(j-1)\delta\mathbf{e},\, r_1)$$

satisfies

$$x(N\tau) \;\in\; \overline{B}(P_1+(j-1)\delta\mathbf{e},\, r_2) \;\subset\; \overline{B}(P_1+j\delta\mathbf{e},\, r_1).$$

After $m+1$ steps, the concatenation of these controls $\xi_j(\cdot)$ yields a control $\xi(\cdot)$ satisfying the requirements of the theorem, with $T = (m+1)N\tau$. □

Next, we provide a sufficient condition for the solvability of the steering problem. Given a probability measure μ on $I\!R^2$, we recall that $G(x,\mu)$ denotes the averaged velocity set defined at (2.2).

Theorem 4. *Assume that $\varphi : I\!R_+ \mapsto I\!R_+$ satisfies (A1). Consider any bounded open set $\Omega\subset I\!R^2$ with $\mathcal{C}^1$ boundary, and any velocity vector $\mathbf{w}$. Assume that there exits a probability distribution μ such that, calling $\mathbf{n}(x)$ the unit outer normal to Ω at the boundary point x, one has*

$$\Big\langle \mathbf{n}(x)\,,\ \mathbf{v}-\mathbf{w}\Big\rangle \;<\; -c \qquad\qquad \textit{for all } x\in\partial\Omega\,,\quad \mathbf{v}\in G(x,\mu)\,. \tag{3.13}$$

If S_0 is any compact set contained in Ω, then there exists a continuous control function $t\mapsto \xi(t)$ such that the corresponding reachable set in (1.15) satisfies

$$S(t) \;\subset\; \Omega + t\mathbf{w} \qquad\qquad \textit{for all } t\geq 0\,. \tag{3.14}$$

Proof. **1.** Define the multifunction $G^-(x,\xi) \doteq G(x,\xi) - \mathbf{w}$. Assume that there exists a control function $\xi^-(\cdot)$ such that every solution of

$$\dot{x} \in G^-(x, \xi^-(t)), \qquad x(0) \in S_0 \tag{3.15}$$

satisfies

$$x(t) \in \Omega \qquad \text{for all} \quad t \geq 0. \tag{3.16}$$

Since G is translation invariant, i.e. $G(x,\xi) = G(x+t\mathbf{w},\ \xi+t\mathbf{w})$, the control $\xi(t) = \xi^-(t)+t\mathbf{w}$ then satisfies the conclusion of the theorem. To prove Theorem 4, it thus suffices to construct a control function ξ^- such that (3.16) holds, for every solution of (3.15).

2. Call $d(x,\partial\Omega)$ the signed distance of a point x to the boundary of Ω. By assumption, $d(\cdot,\partial\Omega)$ is smooth in a neighborhood of $\partial\Omega$ and satisfies

$$\begin{cases} d(x,\partial\Omega) < 0 & \text{if} \quad x \in \Omega, \\ d(x,\partial\Omega) > 0 & \text{if} \quad x \notin \overline{\Omega}. \end{cases}$$

Consider the sublevel sets

$$\Lambda_{-c} \doteq \{x;\ \ d(x,\partial\Omega) \leq -c\}.$$

Thanks to the assumption (3.13), we can find $c > 0$ such that the set Λ_{-c} is strongly invariant for the differential inclusion

$$\dot{x} \in G^-(x,\mu). \tag{3.17}$$

In fact, choosing a sufficiently small constant c, satisfying

$$\max_{x\in S_0} d(x,\partial\Omega) < -c < 0,$$

the following stronger statement is true. Every solution $t \mapsto x(t)$ of (3.17) with $x(0) \in \Lambda_{-c/2}$ satisfies

$$x(t) \in \Lambda_{-c/2} \qquad \text{for all} \quad t \geq 0, \qquad x(1) \in \Lambda_{-c}.$$

By Theorem 1, there exists a control function $\xi^- : [0,1] \mapsto I\!R^2$ such that every solution of

$$\dot{x} \in G^-(x,\xi^-(t)), \qquad x(0) \in \Lambda_{-c/2}$$

satisfies

$$x(t) \in \Lambda_{-c/4} \qquad \text{for all} \quad t \in [0,1], \qquad x(1) \in \Lambda_{-c/2}.$$

Extending $\xi^-(\cdot)$ by periodicity, so that $\xi^-(t+1) = \xi^-(t)$, we obtain a $\xi^- : I\!R_+ \mapsto I\!R^2$ with the desired property. Namely, every solution of (3.15) satisfies (3.16). This achieves the proof. □

4 Asymptotic shape of a rotating solution

In this section we study in more detail the evolution of the set $S(t)$, in case where the point $\xi(t)$ moves along a circumference, with constant angular speed ω.

To fix the ideas, assume that

$$\xi(t) \;=\; R(\cos\omega t\,,\;\; -\sin\omega t),$$

with $\omega = \varepsilon^{-1}$ very large. Consider a set of rotating coordinates, determined by the orthonormal frame $\{\mathbf{e}_1(t), \mathbf{e}_2(t)\}$, with $\mathbf{e}_1(t) = (\cos\omega t\,,\;\; -\sin\omega t)$.

Consider the vector fields

$$\mathbf{w}(x) \;\doteq\; \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{4.1}$$

$$\mathbf{v}(x) \;\doteq\; \varphi(|x-\xi|)\cdot \frac{x-\xi}{|x-\xi|}\,, \qquad\qquad \xi \;=\; (\xi_1, \xi_2) \;=\; (R, 0). \tag{4.2}$$

In the above system of rotating coordinates, the sets $S(t)$ are determined as the reachable sets for the differential inclusion

$$\dot x \;\in\; B\left(\frac{1}{\varepsilon}\mathbf{w}(x) + \mathbf{v}(x)\;,\; c\right). \tag{4.3}$$

For a suitable class of initial data S_0, as $t\to\infty$ we expect that $S(t)\to \overline{S}$, for some invariant set $\overline{S}$. In polar coordinates, this set has the representation

$$\overline{S} \;=\; \Big\{(r\cos\alpha,\, r\sin\alpha)\,;\;\;\; r\leq \rho_\varepsilon(\alpha)\Big\},$$

where $\alpha\mapsto \rho_\varepsilon(\alpha)$ provides a periodic solution to the ODE

$$\frac{d\rho}{d\alpha} \;=\; g_\varepsilon(\alpha,\rho)\,. \tag{4.4}$$

With reference to Fig. 1, left, if the point $x = (\rho\cos\alpha,\, \rho\sin\alpha)$ moves according to $\dot x = \mathbf{v}^+(x)$, then its polar coordinates satisfy (4.4). The function g_ε is determined by

$$g_\varepsilon(\alpha,\rho) \;=\; \frac{\cos\theta}{|x|} \;=\; \sup_{|\mathbf{y}|\leq c} \frac{\Big\langle x\;,\; \varepsilon^{-1}\mathbf{w}(x) + \mathbf{v}(x) + \mathbf{y}\Big\rangle}{|x|^2\cdot|\varepsilon^{-1}\mathbf{w}(x) + \mathbf{v}(x) + \mathbf{y}|} \;=\; \varepsilon\cdot \sup_{|\mathbf{y}|\leq c} \frac{\Big\langle x\;,\; \mathbf{v}(x) + \mathbf{y}\Big\rangle}{|x|^2\cdot|\mathbf{w}(x) + \varepsilon\mathbf{v}(x) + \varepsilon\mathbf{y}|}\,. \tag{4.5}$$

For $\varepsilon = 0$ we have $g_0(\alpha,\rho)\equiv 0$ and every constant function is a periodic solution. To find periodic solutions for $\varepsilon > 0$ we use a bifurcation technique. Call $r\mapsto F_\varepsilon(r)$ the return map for (4.4). Calling $\alpha\mapsto\rho_\varepsilon(\alpha, r)$ the solution of (4.4) with initial data $\rho(0) = r$, consider the function

$$F(r,\varepsilon) \;\doteq\; \rho_\varepsilon(2\pi, r) - r\,. \tag{4.6}$$

Zeroes of F correspond to periodic solutions. Since $F(r,0)\equiv 0$ for all r, by standard bifurcation theory we have

(i) If $\frac{\partial F}{\partial\varepsilon}(\bar r, 0) \;\neq\; 0$, then in a neighborhood of the point $(\bar r, 0)$ the only solutions of (4.6) are those with $\varepsilon = 0$.

(ii) If $\frac{\partial F}{\partial \varepsilon}(\bar{r},0) = 0$ and $\frac{\partial^2 F}{\partial r \partial \varepsilon}(\bar{r},0) \neq 0$, then there exists a nontrivial branch of solutions of the form $r = r^*(\varepsilon)$, with

$$r^*(0) \;=\; \bar{r}\,, \qquad\qquad \frac{\partial r^*}{\partial \varepsilon}(0) \;=\; -\,\frac{\partial^2 F}{\partial \varepsilon^2}(\bar{r},0)\cdot\left(\frac{\partial^2 F}{\partial r \partial \varepsilon}(\bar{r},0)\right)^{-1}.$$

Consider the formal asymptotic expansions

$$g_\varepsilon(\alpha,\rho) \;=\; \varepsilon g_1(\alpha,\rho) + \varepsilon^2 g_2(\alpha,\rho) + o(\varepsilon^2)\,, \tag{4.7}$$

$$\rho_\varepsilon(\alpha,r) \;=\; r + \varepsilon\rho_1(\alpha,r) + \varepsilon^2\rho_2(\alpha,r) + o(\varepsilon^2)\,, \tag{4.8}$$

$$r^*(\varepsilon) \;=\; \bar{r} + \varepsilon\, r_1 + \varepsilon^2 r_2 + o(\varepsilon^2)\,. \tag{4.9}$$

The conditions in (ii) for the existence of a nontrivial branch of solutions, bifurcating from the trivial branch at $(\bar{r},0)$, can be written as

$$\frac{\partial F}{\partial \varepsilon}(\bar{r},0) \;=\; \rho_1(2\pi,\bar{r}) \;=\; 0\,, \tag{4.10}$$

$$\frac{\partial^2 F}{\partial \varepsilon \partial r}(\bar{r},0) \;=\; \frac{\partial \rho_1}{\partial r}(2\pi,\bar{r}) \;\neq\; 0\,. \tag{4.11}$$

Inserting (4.7)-(4.8) in (4.4) and equating coefficients, to first order we obtain

$$\rho_1(2\pi,r) \;=\; \int_0^{2\pi} g_1(\beta,r)\,d\beta\,. \tag{4.12}$$

Hence (4.10) yields

$$\int_0^{2\pi} g_1(\beta,\bar{r})\,d\beta \;=\; \int_0^{2\pi} \frac{\partial g_\varepsilon}{\partial \varepsilon}(\beta,\bar{r})\,d\beta \;=\; 0\,. \tag{4.13}$$

Moreover, (4.11) yields

$$\int_0^{2\pi} \frac{\partial}{\partial r} g_1(\beta,\bar{r})\,d\beta \;=\; \int_0^{2\pi} \frac{\partial^2 g_\varepsilon}{\partial \varepsilon \partial r}(\beta,\bar{r})\,d\beta \;\neq\; 0\,. \tag{4.14}$$

Introducing the vector $x(\alpha,r) \doteq (r\cos\alpha,\, r\sin\alpha)$, from (4.5) and the definitions of the vector fields $\mathbf{w},\mathbf{v}$ at (4.1)-(4.2), it follows

$$g_1(\alpha,\rho) \;=\; c + \frac{1}{\rho}\Big\langle \mathbf{v}(x(\alpha,\rho))\,,\; x(\alpha,\rho)\Big\rangle\,.$$

Defining the function

$$\psi(r,R) \;\doteq\; \int_0^{2\pi} \Big\langle \mathbf{v}(x(\alpha,r))\,, \frac{x(\alpha,r)}{r}\Big\rangle\,d\alpha\,, \tag{4.15}$$

the conditions (4.13) and (4.14) can be written as

$$\psi(\bar{r},R) \;=\; -\,2\pi\,c\,. \tag{4.16}$$

$$\frac{\partial}{\partial r}\psi(r,R)\bigg|_{r=\bar{r}} \;\neq\; 0\,. \tag{4.17}$$

Theorem 5. *Assume that there exists a unique radius $\bar{r}$ for which (4.16) hold, and for such value $\bar{r}$ assume that the inequality (4.17) holds as well. Then, for every $\varepsilon > 0$ small enough, the following holds.*

(i) The ODE (4.4) has a unique periodic solution $r = r^*(\alpha)$.

(ii) The region $S^* \doteq \Big\{(r\cos\alpha,\, r\sin\alpha)\,;\quad 0 \le r \le r^*(\alpha), \quad \alpha \in [0, 2\pi]\Big\}$ *is positively invariant for the differential inclusion (4.3).*

(iii) If, in addition, $\psi(r, R) < 2\pi c$ *for all* $0 < r < \bar{r}$, *then for any initial set* $S_0 \subseteq S^*$, *the corresponding reachable set* $S(t)$ *satisfies*

$$\lim_{t\to\infty} d_H(S(t), S^*) \;=\; 0\,. \tag{4.18}$$

Moreover, for all t *sufficiently large the set* $S(t)$ *admits a polar coordinate representation*

$$S(t) \;=\; \Big\{r(\cos\theta,\, \sin\theta)\,;\quad 0 \le r \le r(t,\theta)\Big\} \tag{4.19}$$

with $\lim_{t\to\infty} r(t,\theta) = r^*(\theta)$ *uniformly for* $\theta \in [0, 2\pi]$.

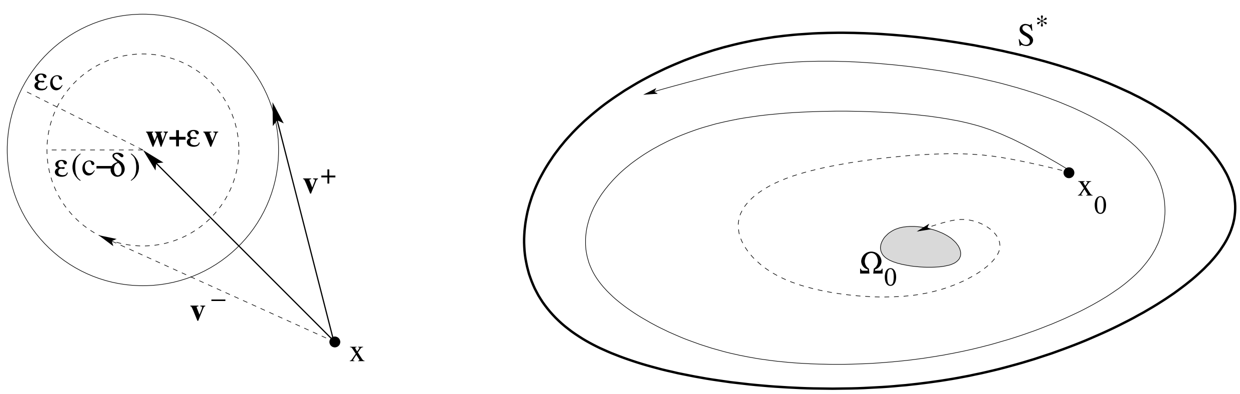

Figure 1: Left: construction of the vector fields $\mathbf{v}^+$, $\mathbf{v}^-$. Right: any trajectory of $\dot{x} = \mathbf{v}^+(x)$ (solid curve) approaches the periodic orbit providing the boundary of S^*. Any trajectory of $\dot{x} = \mathbf{v}^-(x)$ (dashed curve) enters the set Ω_0.

Proof. 1. By standard results of bifurcation theory [11], the existence and uniqueness of the periodic solution are an immediate consequence of the assumptions.

2. The positive invariance of the set S^* follows from the definition (4.5). Indeed, calling $T_{S^*}(x)$ the tangent cone to the set S^* at any boundary point x, (4.5) implies

$$B\Big(\mathbf{w}(x) + \varepsilon\mathbf{v}(x)\;,\; c\varepsilon\Big) \;\subseteq\; T_{S^*}(x)\,. \tag{4.20}$$

Moreover, for every $x \in S^*$,

$$B\Big(-\mathbf{w}(x) - \varepsilon\mathbf{v}(x)\;,\; c\varepsilon\Big) \cap T_{S^*}(x) \;\neq\; \emptyset\,. \tag{4.21}$$

Since the right hand side of (4.3) is a Lipschitz continuous multifunction, by (4.20) every trajectory of (4.3) starting at a point $x_0 \in S^*$ remains inside S^* for all times $t \in [0, \infty[$. On the other hand, by (4.21), for every point $x_0 \in S^*$ there exists a trajectory $t \mapsto x(t) \in S^*$ defined for $t \in]-\infty, 0]$ with $x(0) = x_0$. Together, these two properties yield the positive invariance of S^*.

3. In the remainder of the proof we assume that $\psi(r, R) < 2\pi c$ for all $0 < r < \bar{r}$. Since $\psi(r, R) \to 0$ as $r \to 0+$ and $\psi(\bar{r}, R) = -2\pi c$, by continuity we can find $\delta > 0$ such that $\psi(r, R) < 2\pi(c - \delta)$ for all $r \in [0, \bar{r}]$.

It is convenient to introduce the vector fields $\mathbf{v}^+$, $\mathbf{v}^-$, as in Fig. 1. At a given point x, these are defined as the tangents to the circumferences centered at $\mathbf{w}(x) + \varepsilon \mathbf{v}(x)$ and with radii εc, $\varepsilon(c - \delta)$, respectively. By Pythagoras' theorem,

$$|\mathbf{v}^+(x)| \; = \; \sqrt{|\mathbf{w}(x) + \varepsilon \mathbf{v}(x)|^2 - \varepsilon^2 c^2}, \qquad\qquad |\mathbf{v}^-(x)| \; = \; \sqrt{|\mathbf{w}(x) + \varepsilon \mathbf{v}(x)|^2 - \varepsilon^2 (c-\delta)^2}, \tag{4.22}$$

and the vectors $\mathbf{v}^+(x)$, $\mathbf{v}^-(x)$ are well defined as soon as the right hand sides of (4.22) are ≥ 0.

Observe that the above assumptions on the function ψ in (4.15) imply that, for every $\varepsilon > 0$ sufficiently small, the following holds.

- The vector field $\mathbf{v}^+$ has a unique periodic solution. This is precisely the boundary of the domain S^*. In polar coordinates, it corresponds to a periodic solution of (4.4).
- The vector field $\mathbf{v}^-$ has no periodic solution inside S^*.

Consider any point $x_0 \in S^*$, and denote by $t \mapsto x(t, x_0)$ the solution of

$$\dot{x} \; = \; \mathbf{v}^-(x) \,, \qquad\qquad x(0) \; = \; x_0 \,.$$

Notice that this trajectory is well defined, as long as it does not touch the set

$$\Omega_0^- \; \doteq \; \Big\{ x \,; \;\; |\mathbf{w}(x) + \varepsilon \mathbf{v}(x)| \leq \varepsilon(c - \delta) \Big\} \; \subset \; \Omega_0 \; \doteq \; \Big\{ x \,; \;\; |\mathbf{w}(x) + \varepsilon \mathbf{v}(x)| \leq \varepsilon c \Big\}.$$

Since S^* is positively invariant, we have $x(t) \in S^*$. By the Poincaré-Bendixson theorem, $x(\cdot)$ must either approach a periodic orbit, or a point where $\mathbf{v}^- = 0$. By assumption, there are no periodic orbits inside S^*. We conclude that there exists a time $\tau \geq 0$ such that $\Big| \mathbf{w}(x(\tau)) + \varepsilon \, \mathbf{v}(x(\tau)) \Big| \; \leq \; \varepsilon(c - \frac{\delta}{2})$. Hence, $x(\tau)$ lies in the interior of Ω_0.

By the above arguments, for every initial set $S_0 \subseteq S^*$ there exists $\tau \geq 0$ such that $S(\tau, S_0) \cap int\Omega_0 \neq \emptyset$.

4. For $\varepsilon \ll 1$, $\Omega_0 = \Big\{ x \,; \;\; |\mathbf{w}(x) + \varepsilon \mathbf{v}(x)| \leq \varepsilon c \Big\}$ is convex and diffeomorphic to disk $B(o, \varepsilon c)$. Consider the one-to-one map $x \mapsto H(x) \doteq \mathbf{w}(x) + \varepsilon \mathbf{v}(x)$. Notice that $\mathbf{v}(x)$ is smooth and $\lim_{|x| \to \infty} |\mathbf{v}(x)| = 0$,

$$\mathcal{J}_H(x) \; = \; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \varepsilon \mathcal{J}_{\mathbf{v}}(x) \,, \tag{4.23}$$

hence, $\mathcal{J}_H(x)$ is invertible and continuous when ε is very small. $H(x)$ is the diffeomorphism we need, provided $\varepsilon \ll 1$. There exist a point $x_0 \in \Omega_0$ which is the preimage of the origin.

$$H(x) \; = \; \mathbf{w}(x) + \varepsilon \mathbf{v}(x) \; \approx \; \mathcal{J}_H(x_0)(x - x_0) + o(\varepsilon) \; \approx \; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (x - x_0) + o(\varepsilon) \,.$$

5. By a straightforward comparison argument, if $S(\tau, S_0) \supseteq S_1$, then $S(\tau + t, S_0) \supseteq S(t, S_1)$. To prove the convergence (4.18), by the previous step it is thus not restrictive to assume that

$S_0 \subseteq \Omega_0$. By definition, Ω_0 is the set of stationary points for the differential inclusion (4.3). In other words, if $\bar{x} \in \Omega_0$, then $x(t) \equiv \bar{x}$ is a solution to (4.3). The assumption $S_0 \subseteq \Omega_0$ thus implies

$$S(t, S_0) \supseteq S(s, S_0) \supset S_0 \qquad \text{for all } t \geq s \geq 0 . \tag{4.24}$$

Therefore, the closure of the union

$$S_\infty \doteq \overline{\bigcup_{t \geq 0} S(t, S_0)} \subseteq S^*$$

must be a positively invariant set, consisting of all the trajectories of $\mathbf{v}^+$ starting inside Ω_0. Hence the boundary ∂S_∞ should be a periodic orbit of $\mathbf{v}^+$. By uniqueness, we conclude that $S_\infty = S^*$.

6. Finally, we show that the stronger convergence (4.19) holds.

Denote by $\Gamma^* \doteq \partial S^*$, $\Gamma(t) \doteq \partial S(t)$ the boundaries of S^* and $S(t)$, respectively. By the previous analysis we know that Γ^* is a smooth curve and that

$$\lim_{t \to +\infty} d_H(\Gamma(t), \Gamma^*) = 0 . \tag{4.25}$$

To prove the Lipschitz regularity of the curve $\Gamma(t)$ we use the fact that each $S(t)$ satisfies an interior ball condition [10]:

There exists a constant $\rho > 0$ such that, for all $t \geq 1$, every point $P \in S(t)$ is contained in some closed disc $D \subseteq S(t)$ of radius ρ.

For convenience, we consider the transformation mapping the point $P = r(\cos\theta, \sin\theta)$ to the point $P' = \frac{r}{r^*(\theta)}(\cos\theta, \sin\theta)$, where $r^*(\cdot)$ yields the polar representation of the curve Γ^*. Relying on this change of coordinates, it is not restrictive to assume that the set S^* is the closed unit disc, so that $\Gamma^* = \{x \in I\!R^2 ;\ \ |x| = 1\}$. The images of the sets $S(t)$ in these new coordinates will satisfy an interior ball condition, possibly with a different radius $\rho > 0$.

By (4.25), for any $\varepsilon > 0$ we can find t_ε sufficiently large such that

$$\Gamma(t) \subseteq \Big\{ x \in I\!R^2 ;\ \ 1 - \varepsilon \leq |x| \leq 1 \Big\} \qquad \text{for all } t \geq t_\varepsilon .$$

Choosing $\varepsilon = \rho/2$, we claim that, for $t \geq t_\varepsilon$, the curve $\Gamma(t)$ can be written in polar coordinates as

$$\Gamma(t) = \Big\{ r^{(t)}(\theta)(\cos\theta, \sin\theta) ;\ \ \theta \in [0, 2\pi] \Big\} \tag{4.26}$$

for some Lipschitz continuous function $r^{(t)}$. Indeed, fix an angle α, and define

$$r^{(t)}(\alpha) \doteq \max \Big\{ r \geq 0 ;\ \ \lambda(\cos\alpha, \sin\alpha) \in \mathcal{S}(t) \ \text{ for all } \lambda \in [0, r] \Big\}.$$

Referring to Fig. 2, consider the point

$$P = r^{(t)}(\alpha)(\cos\alpha, \sin\alpha) \in \Gamma(t).$$

Let B_1, B_2 be the two closed discs with radius ρ, tangent to Γ^* and containing P as a boundary point. Consider the open region Σ, bounded between Γ^* and the two discs. By construction, if $Q \in \Sigma$, then $Q \notin \Gamma(t)$, because there exists no disc of radius ρ containing Q and contained in $S(t)$. The above argument shows that, for every $t \geq t_\varepsilon$, the set $\Gamma(t)$ admits the polar coordinate representation (4.26), where the function $r^{(t)}$ satisfies an estimate of the form

$$1 - \varepsilon \ \leq \ r^{(t)}(\beta) \ \leq \ r^{(t)}(\alpha) + C\,|\beta - \alpha|\,.$$

for some uniform constant C, valid for all times $t \geq t_\varepsilon$ and all angles α, β. Hence $r^{(t)}$ is Lipschitz continuous with Lipschitz constant C. □

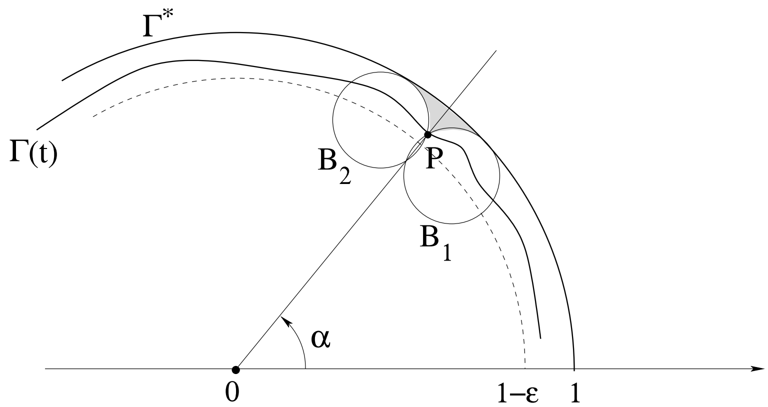

Figure 2: No point of $\Gamma(t)$ can lie in the shaded area, because the interior ball condition would otherwise be violated.

Remark. Comparing the definitions of ψ in (4.15) and of ϕ at (3.8)-(3.9), by (3.11), we obtain

$$\psi(r,R) \ = \ 2\pi\,\phi(r,R) \ = \ \frac{1}{r}\int_{R-r}^{R+r} 2s\left(\frac{\varphi(s)}{s} + \varphi'(s)\right)\arccos\left(\frac{s^2+R^2-r^2}{2Rs}\right)\,ds\,.$$

Assume that there exists a radius $\rho > 0$ such that

$$\psi(\rho,R) \leq -2\pi c\,, \qquad \frac{\partial}{\partial r}\,\psi(r,R) < 0 \qquad \text{for all } r \in\,]0,\rho]\,. \tag{4.27}$$

Since $\psi(r,R) \to 0$ as $r \to 0$, if the conditions in (4.27) hold then there exists a unique $\bar{r} \in\,]0,\rho]$ for which all the assumptions of Theorem 5 hold.

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