Piecewise Smooth Solutions to the Burgers-Hilbert Equation

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Abstract

The paper is concerned with the Burgers-Hilbert equation $u_t + (u^2/2)_x = \mathbf{H}[u]$, where the right hand side is a Hilbert transform. Unique entropy admissible solutions are constructed, locally in time, having a single shock. In a neighborhood of the shock curve, a detailed description of the solution is provided.

1 Introduction

Consider the balance law obtained from Burgers' equation by adding the Hilbert transform as a source term:

$$u_t + \left(\frac{u^2}{2}\right)_x \;=\; \mathbf{H}[u]\,. \tag{1.1}$$

Here

$$\mathbf{H}[f](x) \;\doteq\; \lim_{\varepsilon\to 0+} \frac{1}{\pi}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy \tag{1.2}$$

denotes the Hilbert transform of a function $f \in \mathbf{L}^2(I\!R)$. The above equation was derived in [1] as a model for nonlinear waves with constant frequency. For initial data

$$u(0,x) \;=\; \bar u(x)\,, \tag{1.3}$$

in $H^2(I\!R)$, the local existence and uniqueness of the solution to (1.1) was proved in [7], together with a sharp estimate on the time interval where this solution remains regular. See also [8] for a shorter proof. For general initial data $\bar u \in \mathbf{L}^2(I\!R)$, the global existence of entropy weak solutions was recently proved in [4] together with a partial uniqueness result. We remark that, in this general setting, the well-posedness of the Cauchy problem remains a largely open question.

In the present paper we consider an intermediate situation. Namely, we construct solutions of (1.1) which are piecewise continuous, with a single shock. Our solutions have the form

$$u(t,x) \;=\; \varphi\big(x-y(t)\big) + w\big(t,\, x-y(t)\big),$$

where $t \mapsto y(t)$ denotes the location of the shock. Here $w \in H^2\big(]-\infty, 0[\cup]0, +\infty[\big)$, while $\varphi(x) = \frac{2}{\pi}|x|\ln|x|$, for x near the origin.

In Section 2 we write (1.1) in an equivalent form, and state an existence-uniqueness theorem, locally in time. The key a priori estimates on approximate solutions, and a proof of the main theorem, are then worked out in Sections 3 to 5.

The present results can be easily extended to the case of solutions with finitely many, non-interacting shocks. An interesting open problem is to describe the local behavior of a solution in a neighborhood of a point (t_0, x_0) where either (i) a new shock is formed, or (ii) two shocks merge into a single one. Motivated by the analysis in [12] we conjecture that, for generic initial data

$$\bar{u} \ \in \ H^2(I\!R) \cap \mathcal{C}^3(I\!R)\,,$$

the corresponding solution of (1.1) remains piecewise smooth with finitely many shock curves on any domain of the form $[0, T] \times I\!R$. We thus regard the present results as a first step toward a description of all generic singularities. For other examples of hyperbolic equations where generic singularities have been studied we refer to [2, 3, 5, 6, 9]. The possible emergence of singularities, for more general dispersive perturbations of Burgers' equation, has been recently studied in [10].

2 Statement of the main result

Consider a piecewise smooth solution of (1.1) with one single shock. Calling $y(t)$ the location of the shock at time t, by the Rankine-Hugoniot conditions we have

$$\dot{y}(t) \ = \ \frac{u^-(t) + u^+(t)}{2}\,. \tag{2.1}$$

where u^-, u^+ denote the left and right limits of $u(t, x)$ as $x \to y(t)$. Here and in the sequel, the upper dot denotes a derivative w.r.t. time. It is convenient to shift the space coordinate, replacing x with $x - y(t)$, so that in the new coordinate system the shock is always located at the origin. In these new coordinates, the equation (1.1) takes the equivalent form

$$u_t + \left(\frac{u^2}{2}\right)_x - \dot{y}\, u_x \ = \ \mathbf{H}[u]\,. \tag{2.2}$$

We shall construct solutions to (2.2) in a special form, providing a cancellation between leading order terms in the transport equation and the Hilbert transform.

Consider a smooth function with compact support $\eta \in \mathcal{C}_c^\infty(I\!R)$, with $\eta(x) = \eta(-x)$, and such that

$$\begin{cases} \eta(x) = 1 & \text{if} \quad |x| \leq 1\,, \\ \eta(x) = 0 & \text{if} \quad |x| \geq 2\,, \\ \eta'(x) \leq 0 & \text{if} \quad x \in [1, 2]\,. \end{cases} \tag{2.3}$$

Moreover, define

$$\varphi(x) \ \doteq \ \frac{2|x|\,\ln|x|}{\pi} \cdot \eta(x)\,. \tag{2.4}$$

Notice that φ has support contained in the interval $[-2, 2]$ and is smooth separately on the domains $\{x < 0\}$ and $\{x > 0\}$.

In addition, we consider the space of functions

$$\mathcal{H} \doteq H^2\big(\,]-\infty, 0[\,\cup\,]0, +\infty[\,\big). \tag{2.5}$$

Every function $w \in \mathcal{H}$ is continuously differentiable outside the origin. The distributional derivative of w_x is an $\mathbf{L}^2$ function restricted to the half lines $]-\infty, 0[$ and $]0, +\infty[$. However, both w and w_x can have a jump at the origin. It is clear that the traces

$$\begin{cases} u^- \doteq w(0-)\,, \\ u^+ \doteq w(0+)\,, \end{cases} \qquad \begin{cases} b^- \doteq w_x(0-)\,, \\ b^+ \doteq w_x(0+)\,, \end{cases} \tag{2.6}$$

are continuous linear functionals on $\mathcal{H}$.

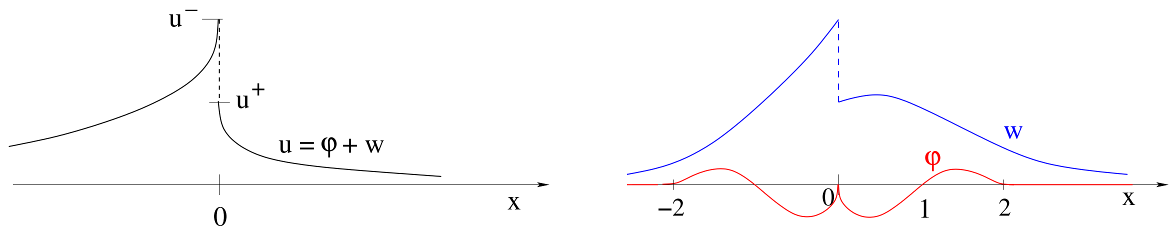

Figure 1: Decomposing a piecewise regular function $u = \varphi + w$ as a sum of the function φ defined at (2.4) and a function $w \in H^2(I\!R \setminus \{0\})$, continuously differentiable outside the origin.

Solutions of (2.2) will be constructed in the form

$$u(t, x) = \varphi(x) + w(t, x). \tag{2.7}$$

In order that the shock be entropy admissible, the function w should range in the open domain

$$\mathcal{D} \doteq \Big\{ w \in H^2\big(I\!R \setminus \{0\}\big)\,; \quad w(0-) > w(0+) \Big\}. \tag{2.8}$$

By (2.6)-(2.8), for $x \approx 0$ this solution has the asymptotic behavior

$$u(t, x) = \begin{cases} u^-(t) + b^-(t)\,x + \frac{2|x|\,\ln|x|}{\pi} + \mathcal{O}(1) \cdot |x|^{3/2} & \text{if } x < 0\,, \\ u^+(t) + b^+(t)\,x + \frac{2|x|\,\ln|x|}{\pi} + \mathcal{O}(1) \cdot |x|^{3/2} & \text{if } x > 0\,, \end{cases} \tag{2.9}$$

for suitable functions $u^\pm, b^\pm$. Here and throughout the sequel, the Landau symbol $\mathcal{O}(1)$ denotes a uniformly bounded quantity.

Inserting (2.7) in the equation (2.2) and recalling (2.6), one obtains

$$w_t + \left(\varphi + w - \frac{u^- + u^+}{2} \right) (\varphi_x + w_x) = \mathbf{H}[\varphi] + \mathbf{H}[w]\,. \tag{2.10}$$

To derive estimates on the Hilbert transform, the following observation is useful. Consider a function f with compact support, continuously differentiable for $x < 0$ and for $x > 0$, with a jump at the origin. Then, for any $x \neq 0$, an integration by parts yields[1]

$$\mathbf{H}[f](x) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} f'(y)\, \ln|x-y|\, dy + \frac{1}{\pi}\big[f(0+) - f(0-)\big]\, \ln|x|\,. \tag{2.11}$$

A similar computation shows that, to leading order, the Hilbert transform of w near the origin is given by

$$\mathbf{H}[w](x) \;=\; \frac{u^+ - u^-}{\pi}\, \ln|x| + \mathcal{O}(1)\,, \tag{2.12}$$

with u^-, u^+ as in (2.6). On the other hand, for $x \approx 0$ one has

$$\begin{aligned}
&\left(\varphi(x) + w(x) - \frac{w(0-)+w(0+)}{2}\right)\varphi_x(x)\\
&\quad= \left(\mathrm{sign}\,(x)\cdot\frac{u^+ - u^-}{2} + \mathcal{O}(1)\cdot |x|\ln|x|\right)\cdot\frac{2\,\mathrm{sign}\,(x)\cdot\big(1+\ln|x|\big)}{\pi}\\
&\quad= \frac{u^+ - u^-}{\pi}\, \ln|x| + \mathcal{O}(1)\,.
\end{aligned} \tag{2.13}$$

The identity between the leading terms in (2.12) and (2.13) achieves a crucial cancellation between the two sides of (2.10). It is thus convenient to write this equation in the equivalent form

$$w_t + \left(\varphi + w - \frac{u^- + u^+}{2}\right) w_x \;=\; \mathbf{H}[\varphi] - \varphi\varphi_x + \left(\mathbf{H}[w] - \Big(w - \frac{u^- + u^+}{2}\Big)\varphi_x\right)\,. \tag{2.14}$$

Definition. *By an* **entropic solution** *to the Cauchy problem (2.10) with initial data*

$$w(0,\cdot) \;=\; \bar w \;\in\; \mathcal{D}\,, \tag{2.15}$$

we mean a function $w : [0,T]\times I\!R \mapsto I\!R$ *such that*

(i) For every $t \in [0,T]$*, the norm* $\|w(t,\cdot)\|_{H^2(I\!R\setminus\{0\})}$ *remains uniformly bounded. As* $x \to 0$*, the limits satisfy*

$$u^-(t) \;\doteq\; u(t,0-) \;>\; u(t,0+) \;\doteq\; u^+(t). \tag{2.16}$$

[1]Indeed, if $f \in \mathcal{C}^\infty_c(I\!R)$, then for a suitably large constant M we have

$$\begin{aligned}
\pi\cdot\mathbf{H}[f](x) \;&=\; \lim_{\varepsilon\to 0+}\int_{|y-x|>\varepsilon} \frac{f(x-y)}{y}\, dy \;=\; -\lim_{\varepsilon\to 0+}\int_{|y-x|>\varepsilon}\frac{f(x+y)}{y}\, dy\\
&=\; -\lim_{\varepsilon\to 0+}\left(\int_{-M}^{-\varepsilon} + \int_{\varepsilon}^{M}\right)\frac{f(x+y)-f(x)}{y}\, dy\\
&=\; \lim_{\varepsilon\to 0+}\left(\int_{-M}^{-\varepsilon} + \int_{\varepsilon}^{M}\right) f'(x+y)\, \ln|y|\, dy - \lim_{\varepsilon\to 0+}[f(x-\varepsilon)-f(x)]\ln\varepsilon\\
&\qquad + \lim_{\varepsilon\to 0+}[f(x+\varepsilon)-f(x)]\ln\varepsilon + [f(x-M)-f(x)]\ln M - [f(x+M)-f(x)]\ln M\\
&=\; \int_{-\infty}^{\infty} f'(x+y)\, \ln|y|\, dy \;=\; \int_{-\infty}^{\infty} f'(y)\, \ln|x-y|\, dy\,.
\end{aligned}$$

By approximating f with a sequence of smooth functions with compact support we obtain (2.11).

(ii) *The equation (2.14) is satisfied in integral sense. Namely, for every* $t_0 \geq 0$ *and* $x_0 \neq 0$, *calling* $t \mapsto x(t; t_0, x_0)$ *the solution to the Cauchy problem*

$$\dot{x} \doteq \varphi(x) + w(t,x) - \frac{u^-(t) + u^+(t)}{2}, \qquad x(t_0) = x_0, \tag{2.17}$$

one has

$$w(t_0, x_0) = \bar{w}(x(0; t_0, x_0)) + \int_0^{t_0} F(t, x(t;\, t_0, x_0))\, dt, \tag{2.18}$$

with

$$F \doteq \mathbf{H}[\varphi] - \varphi\varphi_x + \left(\mathbf{H}[w] - \left(w - \frac{u^- + u^+}{2}\right)\varphi_x\right). \tag{2.19}$$

A few remarks are in order:

(i) The bound on the norm $\|w(t,\cdot)\|_{H^2}$ implies that the limits in (2.16) are well defined. By requiring that the inequality in (2.16) holds we make sure that the shock is entropy admissible.

(ii) Since $w(t,\cdot) \in H^2(\mathbb{R} \setminus \{0\})$, the right hand side of the ODE in (2.17) is continuously differentiable w.r.t. x. Combined with the inequalities in (2.16), this implies that the backward characteristic $t \mapsto x(t; t_0, x_0)$ is well defined for all $t \in [0, t_0]$.

(iii) In [11], a function satisfying the integral equations (2.18) was called a **broad solution**. The regularity assumption on $w(t,\cdot)$ and the fact that the source term F in (2.19) is continuous outside the origin imply that $w = w(t,x)$ is continuously differentiable w.r.t. both variables t, x, for $x \neq 0$. Therefore, the identity in (2.14) is satisfied at every point (t,x), with $x \neq 0$.

The main result of this paper provides the existence and uniqueness of an entropic solution, locally in time.

Theorem 1. *For every* $\bar{w} \in \mathcal{D}$ *there exists* $T > 0$ *such that the Cauchy problem (2.2), (2.15) admits a unique entropic solution, defined for* $t \in [0, T]$.

In turn, Theorem 1 yields the existence of a piecewise regular solution to the Burgers-Hilbert equation (1.1), locally in time, for initial data of the form

$$u(0,x) = \varphi(x) + \bar{w}(x),$$

with $\bar{w} \in \mathcal{D}$.

The solution $w = w(t,x)$ of (2.14) will be obtained as a limit of a sequence of approximations. More precisely, for $n = 1$, we define

$$w_1(t,\cdot) = \bar{w} \qquad \text{for all } t \geq 0. \tag{2.20}$$

Next, let the n-th approximation $w_n(t,x)$ be constructed. By induction, we then define $w_{n+1}(t,x)$ to be the solution of the linear, non-homogeneous Cauchy problem

$$w_t + \left(\varphi + w_n - \frac{u_n^- + u_n^+}{2}\right) w_x \;=\; \mathbf{H}[\varphi] - \varphi\varphi_x + \left(\mathbf{H}[w] - \Big(w - \frac{u^- + u^+}{2}\Big)\varphi_x\right). \tag{2.21}$$

with initial data (2.15).

The induction argument requires three steps:

(i) Existence and uniqueness of solutions to the linear problem (2.21) with initial data (2.15).

(ii) A priori bounds on the strong norm $\|w_n(t)\|_{H^2(I\!R\setminus\{0\})}$, uniformly valid for $t \in [0,T]$ and all $n \geq 1$.

(iii) Convergence in a weak norm. This will follow from the bound

$$\sum_{n\geq 1} \|w_{n+1}(t) - w_n(t)\|_{H^1(I\!R\setminus\{0\})} \;<\; \infty\,.$$

In the following sections we shall provide estimates on each term on the right hand side of (2.21), and complete the above steps (i)–(iii).

3 Estimates on the source terms

To estimate the right hand side of (2.21), we consider again the cutoff function η in (2.3) and split an arbitrary function $w \in H^2(I\!R\setminus\{0\})$ as a sum:

$$w \;=\; v_1 + v_2 + v_3\,, \tag{3.1}$$

where

$$v_1(x) \;\doteq\; \begin{cases} w(0-)\cdot\eta(x) & \text{if } x<0,\\ w(0+)\cdot\eta(x) & \text{if } x>0,\end{cases} \qquad v_2(x) \;\doteq\; \begin{cases} w_x(0-)\cdot x\,\eta(x) & \text{if } x<0,\\ w_x(0+)\cdot x\,\eta(x) & \text{if } x>0,\end{cases} \tag{3.2}$$

$$v_3 \;=\; w - v_1 - v_2\,. \tag{3.3}$$

The right hand side of (2.21) can be expressed as the sum of the following terms:

$$A \doteq \mathbf{H}[\varphi]\,, \qquad B \doteq \varphi\varphi_x\,, \qquad C \doteq \mathbf{H}[v_2+v_3]\,, \qquad D \doteq \mathbf{H}[v_1] - \Big(w - \frac{u^- + u^+}{2}\Big)\varphi_x\,. \tag{3.4}$$

The goal of this section is to provide a priori bounds of the size of these source terms and on their first and second derivatives.

Lemma 1. *There exist constants K_0, K_1 such that the following holds. For any $\delta \in]0,\,1/2]$ and any $w \in H^2(I\!R\setminus\{0\})$, the source terms in (3.4) satisfy*

$$\|A\|_{H^2(I\!R\setminus[-\delta,\delta])} + \|B\|_{H^2(I\!R\setminus[-\delta,\delta])} \;\leq\; K_0\cdot\delta^{-2/3}\,, \tag{3.5}$$

$$\|C\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} + \|D\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\leq\; K_1\delta^{-2/3}\cdot\|w\|_{H^2(\mathbb{R}\setminus\{0\})}\,. \tag{3.6}$$

Proof. 1. We begin by observing that the function φ is continuous with compact support, smooth outside the origin. Therefore, the Hilbert transform $A = \mathbf{H}[\varphi]$ is smooth outside the origin. As $|x| \to \infty$ one clearly has

$$A(x) \;=\; \mathcal{O}(1)\cdot x^{-1}\,, \qquad A_x(x) \;=\; \mathcal{O}(1)\cdot x^{-2}\,, \qquad A_{xx}(x) \;=\; \mathcal{O}(1)\cdot x^{-3}. \tag{3.7}$$

In addition, as $x \to 0$, we claim that

$$A(x) \;=\; \mathcal{O}(1)\cdot x\,\ln^2|x|\,, \qquad A_x(x) \;=\; \mathcal{O}(1)\cdot \ln^2|x|\,, \qquad A_{xx}(x) \;=\; \mathcal{O}(1)\cdot \frac{\ln|x|}{x}. \tag{3.8}$$

Indeed, to fix the ideas, let $0 < x < 1/2$. By (2.11) we have

$$\pi\cdot\mathbf{H}[\varphi](x) \;=\; \int_{-2}^{2}\varphi'(y)\,\ln|x-y|\,dy \;=\; I_1 + I_2 + I_3, \tag{3.9}$$

where:

$$I_1 \;\doteq\; \left(\int_{-2}^{-1} + \int_1^2\right)\varphi'(y)\,\ln|x-y|\,dy \;=\; \mathcal{O}(1)\cdot x\,, \tag{3.10}$$

$$\frac{\pi}{2}\,I_2 \;\doteq\; \int_{-1}^{0} -\ln|x-y|\,dy + \int_0^1 \ln|x-y|\,dy \;=\; \left(\int_{-x}^{x} - \int_{1-x}^{1+x}\right)\ln|y|\,dy \;=\; O(1)\cdot x\ln x\,, \tag{3.11}$$

and moreover,

$$\begin{aligned}
\frac{\pi}{2}\,I_3 \;&\doteq\; \int_0^1 \ln|y|\,\ln|x-y|\,dy + \int_{-1}^{0} -\ln|y|\,\ln|x-y|dy \\
&=\; \left(\int_0^{x/2} + \int_{x/2}^{x} + \int_{x-1}^{0} - \int_{-1}^{0}\right)\ln|y|\ln|x-y|\,dy \\
&=\; \left(\int_0^{x/2} + \int_{x/2}^{x}\right)\ln|y|\ln|x-y|\,dy - \int_0^x \ln|y-1|\ln|x-y+1|\,dy \\
&\doteq\; I_{31} + I_{32} + I_{33}\,.
\end{aligned} \tag{3.12}$$

We now have

$$\begin{aligned}
|I_{31}| \;&\leq\; \ln\left|\frac{x}{2}\right|\cdot\int_0^{x/2}\ln|y|\,dy \;=\; O(1)\cdot x\ln^2|x|, \\
|I_{32}| \;&\leq\; \ln\left|\frac{x}{2}\right|\cdot\int_{x/2}^{x}\ln|x-y|\,dy \;=\; O(1)\cdot x\ln^2|x|, \\
|I_{33}| \;&\leq\; \int_0^x \ln|1-x|\,\ln|1+x||\,dy \;=\; O(1)\cdot x^3.
\end{aligned} \tag{3.13}$$

Hence $\mathbf{H}[\varphi] \;=\; O(1)\cdot x\ln^2|x|$. This yields the first estimate in (3.8).

Next, we estimate the derivative $\pi\partial_x\mathbf{H}[\varphi] = \partial_x I_1 + \partial_x I_2 + \partial_x I_3$. The term $|\partial_x I_1|$ is uniformly bounded, while

$$\frac{\pi}{2}\,\partial_x I_2 \;=\; \int_0^{2x}\frac{1}{x-y}\,dy + \int_{2x}^{1}\frac{1}{x-y}\,dy - \int_{-1}^{0}\frac{1}{x-y}\,dy \;=\; O(1)\cdot\ln|x|. \tag{3.14}$$

Differentiating I_3 w.r.t. x we obtain

$$\begin{aligned}\frac{\pi}{2}\,\partial_x I_3 \;=\;& \left(\int_{-1}^{-x/2} + \int_{-x/2}^{0}\right)\frac{-\ln|y|}{x-y}\,dy + \left(\int_0^{x/2} + \int_{3x/2}^{1}\right)\frac{\ln|y|}{x-y}\,dy \\ & + \lim_{\epsilon\to 0}\left(\int_{x/2}^{x-\epsilon} + \int_{x+\epsilon}^{3x/2}\right)\frac{\ln y}{x-y}\,dy.\end{aligned} \tag{3.15}$$

Assuming $0 < x < 1/2$, we obtain

$$\int_{-1}^{-x/2}\frac{-\ln|y|}{x-y}\,dy \;\le\; \int_{-1}^{x/2}\frac{-\ln|y|}{|y|}\,dy \;=\; O(1)\cdot\ln^2|x|,$$

$$\int_{-x/2}^{0}\frac{-\ln|y|}{x-y}\,dy \;\le\; \int_{-x/2}^{0}\frac{-\ln|y|}{x}\,dy \;=\; O(1)\cdot\ln|x|,$$

$$\int_0^{x/2}\frac{\ln|y|}{x-y}\,dy \;\le\; \int_0^{x/2}\frac{\ln|y|}{x/2}\,dy \;=\; O(1)\cdot\ln|x|,$$

$$\int_{3x/2}^{1}\frac{\ln|y|}{x-y}\,dy \;\le\; \ln\left|\frac{3x}{2}\right|\int_{3x/2}^{1}\frac{1}{x-y}\,dy \;=\; O(1)\cdot\ln^2|x|.$$

The remaining term is estimated as

$$\left(\int_{x/2}^{x-\epsilon} + \int_{x-\epsilon}^{3x/2}\right)\frac{\ln y}{x-y}\,dy \;=\; \left(\int_{x/2}^{x-\epsilon} + \int_{x-\epsilon}^{3x/2}\right)\frac{\ln y - \ln x}{x-y}\,dy \;\le\; \frac{2}{x}(x-2\epsilon) \;\le\; 2.$$

Combining the previous estimates we obtain $\partial_x\mathbf{H}[\varphi](x) = O(1)\cdot\ln^2|x|$. This gives the second estimate in (3.8).

Finally, we estimate the second derivative of the Hilbert transform $\partial_{xx}\mathbf{H}[\varphi] = \sum_{i=1}^{3}\partial_{xx}(I_i)$. By (3.10) and (3.14) we obtain

$$\begin{aligned}\partial_{xx} I_1 \;&=\; \mathcal{O}(1), \\ \frac{\pi}{2}\,\partial_{xx} I_2 \;&=\; -\int_{2x}^{1}\frac{1}{(x-y)^2}\,dy + \int_{-1}^{0}\frac{1}{(x-y)^2}\,dy \;=\; O(1)\cdot\frac{\ln|x|}{x}.\end{aligned} \tag{3.16}$$

$$\begin{aligned}\frac{\pi}{2}\,\partial_{xx} I_3 \;=\;& \left(\int_{-1}^{-x/2} + \int_{-x/2}^{0}\right)\frac{\ln|y|}{(x-y)^2}\,dy - \left(\int_0^{x/2} + \int_{3x/2}^{1}\right)\frac{\ln|y|}{(x-y)^2}\,dy \\ & + \frac{\ln|x/2|}{x} + \frac{3\ln|3x/2|}{x} + \partial_x\left(\int_{x/2}^{3x/2}\frac{\ln|y|}{x-y}\,dy\right).\end{aligned} \tag{3.17}$$

Assuming $0 < x < 1/2$, we obtain

$$
\begin{aligned}
\Big|\int_{-1}^{-x/2} \frac{\ln|y|}{(x-y)^2}\,dy\Big| \;&\le\; \ln\Big|\frac{x}{2}\Big|\int_{-1}^{x/2} \frac{1}{(x-y)^2}\,dy \;=\; O(1)\cdot\frac{\ln|x|}{x}\,,\\
\Big|\int_{-x/2}^{0} \frac{\ln|y|}{(x-y)^2}\,dy\Big| \;&\le\; \int_{-x/2}^{0} \frac{-\ln|y|}{x^2}\,dy \;=\; O(1)\cdot\frac{\ln|x|}{x}\,,\\
\Big|\int_{0}^{x/2} \frac{\ln|y|}{(x-y)^2}\,dy\Big| \;&\le\; \int_{0}^{x/2} \frac{\ln|y|}{(x/2)^2}\,dy \;=\; O(1)\cdot\frac{\ln|x|}{x}\,,\\
\Big|\int_{3x/2}^{1} \frac{\ln|y|}{(x-y)^2}\,dy\Big| \;&\le\; \ln\Big|\frac{3x}{2}\Big|\int_{3x/2}^{1} \frac{1}{(x-y)^2}\,dy \;=\; O(1)\cdot\frac{\ln|x|}{x}\,.
\end{aligned}
\tag{3.18}
$$

The remaining term is estimated by

$$
\begin{aligned}
\partial_x\left(\int_{x/2}^{3x/2} \frac{\ln|y|}{x-y}\right)dy \;&=\; \partial_x\left(\int_{-x/2}^{x/2} \frac{\ln|x-y|}{y}\,dy\right)\\
&=\; \int_{-x/2}^{x/2} \frac{1}{y(x-y)}\,dy + \frac{\ln|x/2|}{x} - \frac{\ln|3x/2|}{x}\,,
\end{aligned}
\tag{3.19}
$$

where

$$
\left|\int_{-x/2}^{x/2} \frac{1}{y(x-y)}\,dy\right| \;=\; \left|\int_{-x/2}^{x/2} \frac{1}{y}\Big(\frac{1}{x-y}-\frac{1}{x}\Big)\,dy\right| \;=\; \left|\int_{-x/2}^{x/2} \frac{1}{x(x-y)}\,dy\right| \;\le\; \frac{2}{x}\,. \tag{3.20}
$$

Therefore, by (3.16) and (3.18) – (3.20), we have $\partial_{xx}\mathbf{H}[\varphi](x) = O(1)\cdot\dfrac{\ln|x|}{x}$.

2. The function $B = \varphi\varphi_x$ is smooth outside the origin and vanishes for $|x| \ge 2$. As $x \to 0$, the following estimates are straightforward:

$$
B(x) \;=\; \mathcal{O}(1)\cdot|x|\,\ln^2|x|\,,\qquad B_x(x) \;=\; \mathcal{O}(1)\cdot\ln^2|x|\,,\qquad B_{xx}(x) \;=\; \mathcal{O}(1)\cdot\frac{\ln|x|}{|x|}. \tag{3.21}
$$

3. Next, we observe that $v_3 \in H^2(I\!R)$. Moreover, there exists a constant C_η such that

$$
\|v_3\|_{H^2(I\!R)} \;\le\; C_\eta\cdot\|w\|_{H^2(I\!R\setminus\{0\})}\,.
$$

Clearly, the Hilbert transform $\mathbf{H}[v_3]$ satisfies the same bounds. Hence

$$
\big\|\mathbf{H}[v_3]\big\|_{H^2(I\!R)} \;=\; \mathcal{O}(1)\cdot\|w\|_{H^2(I\!R\setminus\{0\})}\,. \tag{3.22}
$$

We observe that v_2 is Lipschitz continuous, has compact support and is continuously differentiable outside the origin. Since v_2 has better regularity properties than φ, the same arguments used to estimate the Hilbert transform of φ also apply to $\mathbf{H}[v_2]$. More precisely, as in (3.7) for $|x| \to \infty$ we have

$$
\mathbf{H}[v_2](x) \;=\; \mathcal{O}(1)\cdot x^{-1}\,,\qquad \mathbf{H}[v_2]_x(x) \;=\; \mathcal{O}(1)\cdot x^{-2}\,,\qquad \mathbf{H}[v_2]_{xx}(x) \;=\; \mathcal{O}(1)\cdot x^{-3}. \tag{3.23}
$$

As in (3.8), for $x \to 0$ we have

$$\mathbf{H}[v_2](x) \;=\; \mathcal{O}(1)\cdot x\,\ln^2|x|\,, \qquad \mathbf{H}[v_2]_x(x) \;=\; \mathcal{O}(1)\cdot \ln^2|x|\,, \qquad \mathbf{H}[v_2]_{xx}(x) \;=\; \mathcal{O}(1)\cdot \frac{\ln|x|}{x}. \tag{3.24}$$

The only difference is that in (3.23)-(3.24) by $\mathcal{O}(1)$ we now denote a quantity such that

$$|\mathcal{O}(1)| \;\leq\; C\cdot \|w\|_{H^2(\mathbb{R}\setminus\{0\})}\,, \tag{3.25}$$

for some constant C independent of w.

4. Finally, observing that the the function v_1 in (3.2) has compact support, for $|x|\to\infty$ we have the bounds

$$D(x) \;=\; \mathbf{H}[v_1](x) \;=\; \mathcal{O}(1)\cdot x^{-1} \qquad D_x(x) \;=\; \mathcal{O}(1)\cdot x^{-2}\,, \qquad D_{xx}(x) \;=\; \mathcal{O}(1)\cdot x^{-3}. \tag{3.26}$$

On the other hand, for $x\to 0$ we claim that

$$D(x) \;=\; \mathcal{O}(1)\,, \qquad D_x(x) \;=\; \mathcal{O}(1)\cdot\ln|x|\,, \qquad D_{xx}(x) \;=\; \mathcal{O}(1)\cdot|x|^{-1}, \tag{3.27}$$

where $\mathcal{O}(1)$ is a quantity satisfying (3.25). Indeed, without loss of generality we can assume $0<x<1/2$. Recalling the construction of w and φ, we have

$$\Big(w-\frac{u^-+u^+}{2}\Big)\varphi_x \;=\; \frac{(u^+-u^-)\ln|x|}{\pi}+\mathcal{O}(1). \tag{3.28}$$

The Hilbert transform of v_1 is computed by

$$\begin{aligned}\pi\mathbf{H}[v_1] \;&=\; \int_{-\infty}^{+\infty}\frac{v_1(y)}{x-y}\,dy\\ &=\; \Big(\int_{-2}^{-1}+\int_1^2\Big)\frac{v_1(y)}{x-y}\,dy+\int_{-1}^0\frac{u^-}{x-y}\,dy+\Big(\int_0^{x/2}+\int_{3x/2}^1\Big)\frac{u^+}{x-y}\,dy+\int_{x/2}^{3x/2}\frac{u^+}{x-y}\,dy\end{aligned}$$

The first term on the right hand side is bounded and the last term vanishes, in the principal value sense. The second term is computed by

$$\int_{-1}^0\frac{u^-}{x-y}\,dy \;=\; u^-(-\ln|x|+\ln|x+1|) \;=\; -\,u^-\ln|x|+\mathcal{O}(1)\cdot|x|\,,$$

while the remaining integrals are estimated by

$$\Big(\int_0^{x/2}+\int_{3x/2}^1\Big)\frac{u^+}{x-y}\,dy \;=\; u^+(\ln|x|-\ln|x-1|) \;=\; u^+\ln|x|+\mathcal{O}(1)\cdot|x|\,.$$

Combining the previous estimates we obtain

$$\mathbf{H}[v_1] \;=\; \frac{(u^+-u^-)\ln|x|}{\pi}+\mathcal{O}(1). \tag{3.29}$$

Next, we estimate the derivative $D_x(x)$. We have

$$\partial_x\Big(w-\frac{u^++u^-}{2}\Big)\cdot\varphi_x \;=\; \mathcal{O}(1)\cdot\ln|x|, \qquad \Big(w-\frac{u^++u^-}{2}\Big)\varphi_{xx} \;=\; \frac{u^+-u^-}{\pi x}+\mathcal{O}(1). \tag{3.30}$$

To estimate the derivative of $\mathbf{H}[v_1]$ we write

$$\begin{aligned}\pi \cdot \partial_x \mathbf{H}[v_1] &= \left(\int_{-2}^{-1} + \int_1^2\right) \frac{-v_1(y)}{(x-y)^2}\,dy - \int_{-1}^0 \frac{u^-}{(x-y)^2}\,dy \\ &\quad + \partial_x \left(\int_0^{x/2} + \int_{3x/2}^1\right) \frac{v_1(y)}{x-y}\,dy + \partial_x \int_{x/2}^{3x/2} \frac{v_1(y)}{x-y}\,dy\,.\end{aligned} \tag{3.31}$$

The first term on the right hand side of (3.31) is uniformly bounded. The second term is estimated by

$$-\int_{-1}^0 \frac{u^-}{(x-y)^2}\,dy = -\frac{u^-}{x} + \mathcal{O}(1).$$

Furthermore, we have

$$\begin{aligned}\partial_x \left(\int_0^{x/2} + \int_{3x/2}^1\right) \frac{v_1(y)}{x-y}\,dy &= \left(\int_0^{x/2} + \int_{3x/2}^1\right) \frac{-v_1(y)}{(x-y)^2}\,dy + \frac{4u^+}{x} \\ &= \frac{-3u^+}{x} + \mathcal{O}(1) + \frac{4u^+}{x} = \frac{u^+}{x} + \mathcal{O}(1).\end{aligned} \tag{3.32}$$

Lastly, since $v_1(x) = u^+$ for $x \in]0,1]$, we have

$$\partial_x \int_{x/2}^{3x/2} \frac{v_1(y)}{x-y}\,dy = \partial_x \int_{-x/2}^{x/2} \frac{u^+}{y}\,dy = 0\,. \tag{3.33}$$

Combining the previous estimates we thus obtain

$$\partial_x \mathbf{H}[v_1](x) = \frac{u^+ - u^-}{\pi x} + \mathcal{O}(1).$$

Together with (3.30), as $x \to 0$ this yields the asymptotic estimate

$$D_x(x) = \mathbf{H}[v_1]_x - \left[\left(w - \frac{u^- + u^+}{2}\right)\varphi_x\right]_x = \mathcal{O}(1) \cdot \ln|x|. \tag{3.34}$$

The second derivative D_{xx} is estimated in a similar way. Indeed, by (3.1)–(3.3) and (3.30), we have

$$\begin{aligned}\partial_{xx}\left(w - \frac{u^- + u^+}{2}\varphi_x\right) &= \partial_{xx}\left(w - \frac{u^+ + u^-}{2}\right)\varphi_x + \partial_x\left(w - \frac{u^+ + u^-}{2}\right)\varphi_{xx} \\ &\quad + \partial_x\left(w - \frac{u^- + u^+}{2}\varphi_x\right)\varphi_{xx} + \left(w - \frac{u^- + u^+}{2}\varphi_x\right)\varphi_{xxx} \\ &= -\frac{u^+ - u^-}{\pi x^2} + \mathcal{O}(1) \cdot \frac{1}{x}.\end{aligned} \tag{3.35}$$

On the other hand, differentiating (3.31) and recalling (3.32) and (3.33) we have

$$\begin{aligned}\pi \cdot \partial_{xx} \mathbf{H}[v_1] &= \left(\int_{-2}^{-1} + \int_1^2\right) \frac{2v_1(y)}{(x-y)^3}\,dy + \int_{-1}^0 \frac{2u^-}{(x-y)^3}\,dy \\ &\quad + \partial_x \left(\int_0^{x/2} + \int_{3x/2}^1\right) \frac{-v_1(y)}{(x-y)^2}\,dy - \frac{4u^+}{x^2} + \partial_{xx} \int_{-x/2}^{x/2} \frac{u^+}{y}\,dy\,.\end{aligned} \tag{3.36}$$

As before, the first term is uniformly bounded while the last term is zero. The second term is computed by

$$\int_{-1}^{0} \frac{2u^-}{(x-y)^3}\,dy \;=\; \frac{u^-}{x^2} + \mathcal{O}(1). \tag{3.37}$$

The third term is estimated by

$$\begin{aligned}
\partial_x \left(\int_0^{x/2} + \int_{3x/2}^{1}\right) \frac{-v_1(y)}{(x-y)^2}\,dy \;&=\; \left(\int_0^{x/2} + \int_{3x/2}^{1}\right) \frac{2v_1(y)}{(x-y)^3}\,dy - \frac{2u^+}{x^2} + \frac{6u^+}{x^2} \\
&=\; \frac{3u^+}{x^2} + \mathcal{O}(1).
\end{aligned} \tag{3.38}$$

Combining above estimates (3.35)–(3.38) we obtain

$$\begin{aligned}
D_{xx} \;&=\; \mathbf{H}[v_1]_{xx} - \left[\Big(w - \frac{u^- + u^+}{2}\Big)\varphi_x\right]_{xx} \\
&=\; \frac{1}{\pi}\left(\frac{u^-}{x^2} + \frac{3u^+}{x^2} - \frac{4u^+}{x^2}\right) + \frac{u^+ - u^-}{\pi x^2} + \mathcal{O}(1)\cdot\frac{1}{x} \;=\; \mathcal{O}(1)\cdot\frac{1}{x}.
\end{aligned} \tag{3.39}$$

5. By the estimates (3.8), (3.21) it follows

$$\begin{aligned}
\|A+B\|_{H^2(I\!\!R\setminus[-\delta,\delta])} \;&=\; \mathcal{O}(1)\cdot\left(\int_\delta^1 \frac{\ln^2|x|}{x^2}\,dx\right)^{1/2} \;=\; \mathcal{O}(1)\cdot\left(\int_\delta^1 \frac{dx}{x^{7/3}}\right)^{1/2} \\
&=\; \mathcal{O}(1)\cdot(\delta^{-4/3})^{1/2} \;=\; \mathcal{O}(1)\cdot\delta^{-2/3}.
\end{aligned} \tag{3.40}$$

Similarly, the estimates (3.6) follow from (3.22), and (3.26)-(3.27). □

4 Construction of approximate solutions

In this section, given an initial datum $\bar{w} \in \mathcal{D}$, we prove that all the approximate solutions w_n at (2.20)-(2.21) are well defined, on a suitably small time interval $[0, T]$.

As in (2.6), we define

$$\begin{cases} \bar{u}^- \;\doteq\; \bar{w}(0-)\,, \\ \bar{u}^+ \;\doteq\; \bar{w}(0+), \end{cases} \qquad\qquad \begin{cases} u_n^-(t) \;\doteq\; w_n(t, 0-)\,, \\ u_n^+(t) \;\doteq\; w_n(t, 0+). \end{cases}$$

To fix the ideas, assume that the initial data $\bar{w} \in H^2(I\!\!R \setminus \{0\})$ satisfies

$$\bar{u}^- - \bar{u}^+ \;=\; 6\delta_0\,, \qquad\qquad \|\bar{w}\|_{H^2(I\!\!R\setminus\{0\})} \;=\; \frac{M_0}{2}\,, \tag{4.1}$$

for some (possibly large) constants $\delta_0, M_0 > 0$.

Choosing a time interval $[0, T]$ sufficiently small, we claim that for each $n \geq 1$ the approximate solution w_n satisfies the a priori bounds

$$\begin{cases} |u_n^-(t) - \bar{u}^-| \;\leq\; \delta_0\,, \\ |u_n^+(t) - \bar{u}^+| \;\leq\; \delta_0\,, \end{cases} \qquad \|w_n(t)\|_{H^2(I\!\!R\setminus\{0\})} \;\leq\; M_0\,, \qquad \text{for all } t \in [0, T]. \tag{4.2}$$

This will be proved by induction. For $n = 1$ these bounds are a trivial consequence of the definition (2.20). In the following, we assume that the function $w_n = w_n(t,x)$ satisfies (4.2), and show that the same bounds are satisfied by w_{n+1}. We recall that w_{n+1} is defined as the solution to the linear equation (2.21), with initial data (2.15).

A sequence of approximate solutions $w^{(k)}$ to the linear equation (2.21) will be constructed by induction on $k = 1, 2, \ldots$ For notational convenience we introduce the function

$$a(t,x) \doteq \varphi(x) + w_n(t,x) - \frac{u_n^-(t) + u_n^+(t)}{2}. \tag{4.3}$$

As in (2.17), call $t \mapsto x(t; t_0, x_0)$ the solution to the Cauchy problem

$$\dot{x} \doteq a(t, x(t)), \qquad x(t_0) = x_0. \tag{4.4}$$

We begin by defining

$$w^{(1)}(t,x) \doteq \bar{w}(x). \tag{4.5}$$

By induction, if $w^{(k)}$ has been constructed, we then set

$$w^{(k+1)}(t_0, x_0) = \bar{w}(x(0; t_0, x_0)) + \int_0^{t_0} F^{(k)}(t, x(t;\, t_0, x_0))\, dt, \tag{4.6}$$

where $F^{(k)}$ is defined as in (2.19), with w replaced by $w^{(k)}$ and $u^\pm(t) = w(t, 0\pm)$ replaced by $w^{(k)}(t, 0\pm)$, respectively.

Assuming that w_n satisfies (4.2), we will show that every approximation $w^{(k)}$ to the linear Cauchy problem (2.21), (2.15) satisfies the same bounds, on a sufficiently small time interval $[0, T]$. Our first result deals with solution to the linear transport equation (4.7). We show that, within a sufficiently short time interval, the H^2 norm of the solution can be amplified at most by a factor of $3/2$.

Lemma 2. *Let $w_n = w_n(t,x)$ be a function that satisfies the bounds (4.2) for all $t > 0$, and define $a = a(t,x)$ as in (4.3). Then there exists $T > 0$ small enough, depending only on δ_0, M_0, so that the following holds. For any $\tau \in [0, T]$ and any solution w of the linear equation*

$$w_t + a(t,x) w_x = 0 \tag{4.7}$$

with initial datum

$$w(0) = \bar{w} \in H^2\big(I\!R \setminus [-\delta_0\tau,\, \delta_0\tau]\big),$$

one has

$$\|w(\tau)\|_{H^2(I\!R\setminus\{0\})} \leq \frac{3}{2}\, \|\bar{w}\|_{H^2\left(I\!R\setminus[-\delta_0\tau,\, \delta_0\tau]\right)}. \tag{4.8}$$

Proof. 1. The equation (4.7) can be solved by the method of characteristics, separately on the regions where $x < 0$ and $x > 0$. We observe that characteristics move toward the origin from both sides. In this first step we prove that all characteristics starting at time $t = 0$ inside the interval $[-\delta_0\tau\,,\ \delta_0\tau]$ hit the origin before time τ (see Fig. 2). Hence the profile $w(\tau, \cdot)$ does not depend on the values of $\bar{w}$ on this interval.

We claim that there exists $\delta_1 > 0$ such that

$$\begin{cases} a(t,x) \;\leq\; -\,\delta_0 & \text{for all } x \in\,]0,\delta_1]\,, \\ a(t,x) \;\geq\; \delta_0 & \text{for all } x \in [-\delta_1,\, 0[\,. \end{cases} \tag{4.9}$$

Indeed, (4.1) and (4.2) imply

$$a(t,0+) \;=\; \frac{u_n^+(t) - u_n^-(t)}{2} \;\leq\; -\,2\delta_0\,. \tag{4.10}$$

Moreover, for $x > 0$ we have

$$\big|a(t,x) - a(t,0+)\big| \;\leq\; \frac{2}{\pi}\big|x\ln x\big| + \int_0^x |w_{n,x}(t,y)|\,dy \;\leq\; C_0\,|x|^{1/2}\,, \tag{4.11}$$

for some constant C_0 depending only on the norm $\|w_n(t,\cdot)\|_{H^2}$, hence only on M_0 in (4.2). Choosing $\delta_1 > 0$ small enough so that $C_0\delta_1^{1/2} < \delta_0$, from (4.10)-(4.11) we obtain the first inequality in (4.9). The second inequality is proved in the same way. In addition, by choosing the time interval $[0,T]$ small enough, we can also assume

$$\delta_0\,T \;\leq\; \delta_1\,. \tag{4.12}$$

2. Multiplying (4.7) by $2w$ one finds

$$(w^2)_t + (aw^2)_x \;=\; a_x w^2. \tag{4.13}$$

Integrating (4.13) over the domain

$$\Omega \;\doteq\; \Big\{(t,x)\,;\;\; |x| > \delta_0(\tau - t)\,,\;\; t\in[0,\tau]\Big\} \tag{4.14}$$

shown in Fig. 2, we obtain

$$\int_{-\infty}^{\infty} w^2(\tau,x)\,dx \;\leq\; \int_{|x|>\delta_0\tau} \bar w^2\,dx + \int_0^\tau \int_{|x|>\delta_0(\tau-t)} a_x w^2\,dx\,dt\,. \tag{4.15}$$

Indeed, by (4.9) and (4.12), for every $\tau \in]0,T[$ the flow points outward along the boundary of the domain Ω. By (4.3) the derivative a_x satisfies a bound of the form

$$|a_x(t,x)| \;\leq\; C_a\big(1 + |\ln|x||\big), \tag{4.16}$$

where C_a is a constant depending only on the norm $\|w_n\|_{H^2}$ in (4.2). Taking the supremum of $|a_x(t,x)|$ over the set

$$\Omega_t \;\doteq\; \{x\,;\;\; |x| > \delta_0(\tau - t)\}, \tag{4.17}$$

from (4.15) we thus obtain

$$\|w(\tau)\|^2_{\mathbf{L}^2(I\!R)} \;\leq\; \|\bar w\|^2_{\mathbf{L}^2(\Omega_0)} + \int_0^\tau C_a\Big(1 + |\ln(\delta_0(\tau-t))|\Big)\,\|w(t)\|^2_{\mathbf{L}^2(\Omega_t)}\,dt\,. \tag{4.18}$$

By Gronwall's lemma, this yields a bound on $\|w(\tau)\|^2_{\mathbf{L}^2}$.

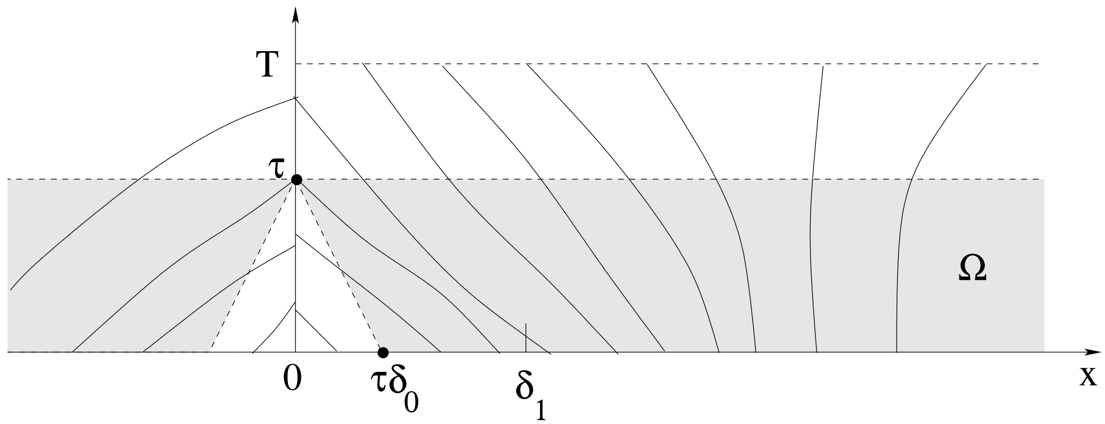

Figure 2: The norm $\|w(\tau)\|_{H^2(I\!R\setminus\{0\})}$ is estimated by using the balance laws for w^2, w_x^2, w_{xx}^2 on the shaded domain Ω. By (4.9), along the boundary where $|x| = \delta_0(\tau - t)$ all characteristics move outward. Hence no inward flux is present.

3. Next, differentiating (4.7) w.r.t. x and multiplying by $2w_x$ we obtain

$$w_{xt} + aw_{xx} = -a_x w_x , \qquad w_x(0,\cdot) = \bar{w}_x . \tag{4.19}$$

$$(w_x^2)_t + (aw_x^2)_x = -a_x w_x^2 . \tag{4.20}$$

Integrating (4.20) over the domain Ω in (4.14) and using the bound (4.16), by similar computations as before we now obtain

$$\|w_x(\tau)\|^2_{\mathbf{L}^2(I\!R)} \leq \|\bar{w}_x\|^2_{\mathbf{L}^2(\Omega_0)} + \int_0^\tau C_a\Big(1 + |\ln(\delta_0(\tau - t))|\Big)\,\|w_x(t)\|^2_{\mathbf{L}^2(\Omega_t)}\,dt . \tag{4.21}$$

By Gronwall's lemma, this yields a bound on $\|w_x(\tau)\|^2_{\mathbf{L}^2}$.

4. Differentiating (4.19) once again and multiplying all terms by $2w_{xx}$ we find

$$w_{xxt} + aw_{xxx} = -2a_x w_{xx} - a_{xx}w_x , \qquad w_{xx}(0,\cdot) = \bar{w}_{xx} , \tag{4.22}$$

$$\big(w_{xx}^2\big)_t + \big(a\,w_{xx}^2\big)_x = -3\,a_x w_{xx}^2 - 2a_{xx}w_x w_{xx} . \tag{4.23}$$

Integrating (4.23) over the domain Ω in (4.14), we obtain

$$\int_{-\infty}^{\infty} w_{xx}^2(\tau,x)\,dx \leq \int_{|x|>\delta\tau} \bar{w}_{xx}^2(y)\,dy + \int_0^\tau \int_{|x|>\delta_0(\tau-t)} \Big(-3\,a_x w_{xx}^2 - 2a_{xx}w_x w_{xx}\Big)\,dx\,dt . \tag{4.24}$$

To estimate the right hand side of (4.24) we observe that, for $|x|$ small,

$$|a_x| = |\varphi_x + w_{n,x}| = \mathcal{O}(1)\cdot\Big(|\ln|x|| + \|w_n\|_{H^2}\Big), \qquad |a_{xx}| = |\varphi_{xx} + w_{n,xx}| = \mathcal{O}(1)\cdot\frac{1}{|x|} + |w_{n,xx}| . \tag{4.25}$$

Recalling that $\varphi(x) = 0$ for $|x| \geq 2$, we have the bounds

$$\begin{aligned} E &\doteq |3\,a_x\,w_{xx}^2 + 2a_{xx}w_x w_{xx}| \\ &\leq \mathcal{O}(1)\cdot\big(1 + |\ln|x||\big)w_{xx}^2 + \mathcal{O}(1)\cdot\left(\frac{1}{|x|} + |w_{n,xx}|\right)\|w\|_{H^2}\,w_{xx} , \end{aligned} \tag{4.26}$$

$$\int_{\delta_0(\tau-t)}^{2} \frac{|w_{xx}(t,x)|}{x}\,dx \;\leq\; \left(\int_{\delta_0(t-s)}^{2} \frac{1}{x^2}\right)^{1/2} \|w_{xx}\|_{\mathbf{L}^2(\Omega_t)} \;\leq\; \left(\frac{1}{\delta_0(t-s)}\right)^{1/2} \|w_{xx}\|_{\mathbf{L}^2(\Omega_t)}\,, \tag{4.27}$$

$$\int_0^\tau \int_{|x|>\delta_0(\tau-t)} E(t,x)\,dxdt \;\leq\; \mathcal{O}(1)\cdot \int_0^\tau \big(1+|\ln\delta_0(\tau-t)|\big)\cdot \|w(t)\|^2_{H^2(\Omega_t)}\,dt$$
$$+\mathcal{O}(1)\cdot\int_0^\tau [\delta_0(\tau-t)]^{-1/2}\cdot\|w(t)\|^2_{H^2(\Omega_t)}\,dt + \mathcal{O}(1)\cdot\int_0^\tau \|w_n(t)\|_{H^2}\cdot\|w(t)\|^2_{H^2(\Omega_t)}\,dt. \tag{4.28}$$

5. Calling $Z(t) \doteq \|w(t)\|_{H^2(\Omega_t)}$, by the estimates (4.18), (4.21), and (4.28) we obtain an integral inequality of the form

$$Z^2(\tau) \;\leq\; Z^2(0) + C_1\cdot\int_0^\tau \Big(1+|\ln\delta_0(\tau-t)| + [\delta_0(\tau-t)]^{-1/2} + M_0\Big) Z^2(t)\,dt\,. \tag{4.29}$$

By Gronwall's lemma, if $\tau > 0$ is sufficiently small this yields $Z(\tau) \;\leq\; \frac{3}{2}Z(0)$, proving (4.8). □

The above estimate can be easily extended to the linear, non-homogeneous problem

$$w_t + a(t,x)w_x \;=\; F(t,x), \qquad\qquad w(0,x) = \bar w(x). \tag{4.30}$$

Indeed, in the same setting as Lemma 2, using (4.8) and Duhamel's formula, for $\tau\in[0,T]$ we obtain

$$\|w(\tau,\cdot)\|_{H^2(I\!R\setminus\{0\})} \;\leq\; \frac{3}{2}\,\|\bar w\|_{H^2\left(I\!R\setminus[-\delta_0\tau,\,\delta_0\tau]\right)} + \frac{3}{2}\int_0^\tau \|F(t,\cdot)\|_{H^2\left(I\!R\setminus[-\delta_0(\tau-t),\,\delta_0(\tau-t)]\right)}\,dt\,. \tag{4.31}$$

Relying on Lemma 1 we now prove uniform H^2 bounds on all approximations $w^{(k)}$, on a suitably small time interval $[0,T]$.

Lemma 3. *Let $w_n = w_n(t,x)$ be a function that satisfies the bounds (4.2) for all $t>0$, and define $a = a(t,x)$ as in (4.3). Then there exists $T>0$ small enough, depending only on δ_0, M_0 in (4.1), so that the following holds. For every $k\geq 1$ and every $\tau\in[0,T]$, one has*

$$\|w^{(k)}(\tau)\|_{H^2(I\!R\setminus\{0\})} \;\leq\; M_0\,, \tag{4.32}$$

$$|w^{(k)}(\tau,0-) - \bar u^-| \;\leq\; \delta_0\,, \qquad\qquad |w^{(k)}(\tau,0+) - \bar u^+| \;\leq\; \delta_0\,. \tag{4.33}$$

Proof. 1. Recalling the constants K_0, K_1 in Lemma 1, choose $T>0$ small enough so that

$$\int_0^T (\delta_0 s)^{-2/3}\,ds \;<\; \frac{M_0}{6(K_0+K_1M_0)}\,. \tag{4.34}$$

2. The estimate (4.32) trivially holds for $w^{(1)}(\tau) \doteq \bar w$. Assuming that it holds for $w^{(k)}(t)$, $t \in [0,T]$, by (4.31) for any $\tau \in [0,T]$ we have the estimate

$$\begin{aligned}
\|w^{(k+1)}(\tau)\|_{H^2(I\!R\setminus\{0\})} &\leq \frac{3}{2}\|\bar w\|_{H^2(I\!R\setminus\{0\})} + \frac{3}{2}\int_0^\tau \|A+B+C+D\|_{H^2(I\!R\setminus[-\delta_0(\tau-t)\,,\,\delta_0(\tau-t)])}\,ds \\
&\leq \frac{3}{4}M_0 + \frac{3}{2}\int_0^\tau K_0[\delta_0(\tau-t)]^{-2/3}\,dt + \frac{3}{2}\int_0^\tau K_1[\delta_0(\tau-t)]^{-2/3}\|w^{(k)}(t)\|_{H^2(I\!R\setminus\{0\})}\,dt \\
&\leq \frac{3}{4}M_0 + \frac{3}{2}(K_0+K_1M_0)\int_0^\tau (\delta_0 s)^{-2/3}\,ds \\
&< \frac{3}{4}M_0 + \frac{3}{2}(K_0+K_1M_0)\cdot\frac{M_0}{6(K_0+K_1M_0)} = M_0\,.
\end{aligned} \tag{4.35}$$

By induction, this proves the bound (4.32).

3. To prove the two estimates in (4.33), we write

$$\big|w^{(k+1)}(\tau,0+) - \bar u^+\big| \leq |\bar w(x(0;\tau,0+)) - \bar u^+| + \tau\cdot\sup_{t\in[0,\tau]}\big\|(A+B+C+D)(t)\big\|_{\mathbf{L}^\infty}. \tag{4.36}$$

The a priori bound on $\|w^{(k)}(t,\cdot)\|_{H^2(I\!R\setminus\{0\})}$ implies that the $\mathbf{L}^\infty$ norm in (4.36) is uniformly bounded. By possibly choosing a smaller $T>0$, both terms on the right hand side of (4.36) will be $<\delta_0/2$. This yields the second inequality in (4.33). The first inequality is proved in the same way. □

The next lemma shows that the sequence of approximations $w^{(k)}$ defined at (4.5)–(4.6) converges to a solution to (2.21).

Lemma 4. *For some $T>0$ sufficiently small, the sequence of approximations $w^{(k)}(t,\cdot)$ converges in $H^2(I\!R\setminus\{0\})$ to a function $w = w(t,\cdot)$. The convergence is uniform for $t\in[0,T]$. This limit function provides a solution to the initial value problem (2.21) with initial data (2.15).*

Proof. 1. By the previous bounds, the difference between two approximations can be estimated by

$$\begin{aligned}
&\|w^{(k+1)}(\tau) - w^{(k)}(\tau)\|_{H^2(I\!R\setminus\{0\})} \\
&\qquad \leq \frac{3}{2}\int_0^\tau [\delta_0(\tau-t)]^{-2/3}K_1\,\|w^{(k)}(t) - w^{(k-1)}(t)\|_{H^2(I\!R\setminus[-\delta_0(\tau-t),\,\delta_0(\tau-t)]}\,dt\,.
\end{aligned} \tag{4.37}$$

If $T>0$ is small enough, so that

$$\frac{3}{2}\int_0^T (\delta_0 s)^{-2/3}K_1\,ds \leq \frac{1}{2},$$

then for every $\tau\in[0,T]$ the sequence $w^{(k)}(\tau,\cdot)$ is Cauchy in $H^2(I\!R\setminus\{0\})$, hence it converges to a unique limit function $w(\tau,\cdot)$.

2. It remains to prove that that w provides a solution to (2.21) with initial data (2.15), in the sense that the integral identities (2.18) are satisfied for all $t_0\in[0,T]$ and $x_0\neq 0$.

This is clear, because for every $\epsilon > 0$ as $k \to \infty$ the source terms on the right hand side of (2.21) converge uniformly on the set $\{(t,x)\,;\ \ t \in [0,T]\,,\ \ |x| \geq \epsilon\}$. □

5 Convergence of the approximate solutions

By the analysis in the previous section, the sequence of approximate solutions w_n of (2.21), (2.15) is well defined, on a suitably small time interval $[0,T]$. Moreover, the uniform bounds (4.2) hold.

To complete the proof of Theorem 1, it remains to show that the w_n converge to a limit function w, providing an entropic solution to the Cauchy problem (2.10), (2.15). Toward this goal we prove that on a suitably small time interval $[0,T]$ the sequence $(w_n)_{n\geq 1}$ constructed at (2.21) is Cauchy w.r.t. the norm of $H^1(I\!R\setminus\{0\})$, hence it converges to a unique limit. This will be achieved in several steps.

1. For a fixed n, consider the differences

$$\begin{cases} W & \doteq\ w_{n+1} - w_n\,, \\ W_n & \doteq\ w_n - w_{n-1}\,, \end{cases} \qquad \begin{cases} U^- & \doteq\ u^-_{n+1} - u^-_n\,, \\ U^-_n & \doteq\ u^-_n - u^-_{n-1}\,, \end{cases} \qquad \begin{cases} U^+ & \doteq\ u^-_{n+1} - u^+_n\,, \\ U^+_n & \doteq\ u^+_n - u^+_{n-1}\,. \end{cases}$$

From (2.21) we deduce

$$W_t + \left(\varphi + w_n - \frac{u^-_n - u^+_n}{2}\right) W_x + \left(W_n - \frac{U^-_n + U^+_n}{2}\right) w_{n,x} \;=\; \mathbf{H}[W] - \left(W - \frac{U^- + U^+}{2}\right)\varphi_x\,. \tag{5.1}$$

Multiplying both sides by $2W$ we obtain the balance law

$$\begin{aligned} (W^2)_t + \left[\left(\varphi + w_n - \frac{u^-_n - u^+_n}{2}\right) W^2\right]_x \;&=\; (\varphi + w_n)_x\, W^2 \\ &- \left(W_n - \frac{U^-_n + U^+_n}{2}\right) 2W w_{n,x} + 2\mathbf{H}[W]\cdot W - \left(W - \frac{U^- + U^+}{2}\right) 2W\varphi_x\,. \end{aligned} \tag{5.2}$$

Integrating over the domain Ω in (4.14) and observing that $\varphi_x(x) = \mathcal{O}(1)\big(1 + |\ln|x||\big)$, we obtain

$$
\begin{aligned}
\frac{1}{2}\int W^2(\tau,x)\,dx \;&\leq\; -\int_0^\tau \int_{|x|>\delta_0(\tau-t)} \Big\{ (\varphi+w_n)_x\cdot W^2 \\
&- \left(W_n - \frac{U_n^-+U_n^+}{2}\right) 2W w_{n,x} + 2\mathbf{H}[W]\cdot W - \left(W - \frac{U^-+U^+}{2}\right) 2W\varphi_x \Big\}\, dxdt \\
&= \mathcal{O}(1)\cdot \int_0^\tau \Big\{ \big|\ln(\tau-t)\big|\cdot \|W(s)\|_{\mathbf{L}^2}^2 + \|W_n(t)\|_{H^1}\|W(t)\|_{\mathbf{L}^2} + \|W(t)\|_{\mathbf{L}^2}^2 \\
&\qquad + \big|\ln(\tau-t)\big|\cdot \|W(t)\|_{H^1}\|W(t)\|_{\mathbf{L}^2} \Big\}\, dt \\
&\leq C_3\cdot \int_0^\tau \|W(t)\|_{\mathbf{L}^2}\cdot \Big(\|W_n(t)\|_{H^1} + \big|\ln(\tau-t)\big|\, \|W(t)\|_{H^1}\Big)\, dt\,,
\end{aligned}
\tag{5.3}
$$

for some constant C_3.

2. Next, differentiating (5.1) w.r.t. x we obtain

$$
\begin{aligned}
W_{xt} &+ \left(\varphi + w_n - \frac{u_n^- - u_n^+}{2}\right) W_{xx} + (\varphi_x + w_{n,x})\, W_x + \left(W_n - \frac{U_n^-+U_n^+}{2}\right) w_{n,xx} + W_{n,x}\, w_{n,x} \\
&= \mathbf{H}[W_x] - \left(W - \frac{U^-+U^+}{2}\right)\varphi_{xx} - \varphi_x W_x\,.
\end{aligned}
\tag{5.4}
$$

Multiplying both sides by $2W_x$ we obtain the balance law

$$
\begin{aligned}
(W_x^2)_t + \left[\left(\varphi + w_n - \frac{u_n^- - u_n^+}{2}\right) W_x^2\right]_x \;&=\; -\,(\varphi_x + w_{n,x})\; W_x^2 - \left(W_n - \frac{U_n^-+U_n^+}{2}\right) 2W_x w_{n,xx} \\
&- 2w_{n,x} W_{n,x} W_x + 2\mathbf{H}[W_x]\, W_x - \left(W - \frac{U^-+U^+}{2}\right) 2W_x\varphi_{xx} - 2\varphi_x W_x^2\,.
\end{aligned}
\tag{5.5}
$$

By the definition (2.4) one has

$$
\|\varphi_{xx}\|_{\mathbf{L}^2(I\!R\setminus[-\delta_0(\tau-t),\delta_0(\tau-t)])} \;=\; \mathcal{O}(1)\cdot(\tau-t)^{-1/2}. \tag{5.6}
$$

Integrating (5.5) over the domain Ω in (4.14) we obtain

$$
\begin{aligned}
\int_0^\infty W_x^2(t,x)\,dx \;&=\; \mathcal{O}(1)\cdot \int_0^\tau \Big\{ \big|\ln(\tau-t)\big|\, \|W_x(t)\|_{\mathbf{L}^2}^2 + \|W_n(t)\|_{H^1}\, \|W_x(t)\|_{\mathbf{L}^2} \\
&+ \|W(t)\|_{H^1}\|W_x(t)\|_{\mathbf{L}^2}\cdot(\tau-t)^{-1/2} \Big\}\, dt\,.
\end{aligned}
\tag{5.7}
$$

3. Calling $Z(t) \doteq \|W(t)\|_{H^1(I\!R\setminus\{0\})}$, from (5.3) and (5.7) we obtain an integral inequality of the form

$$
Z^2(\tau) \;\leq\; C_4 \int_0^\tau Z(t)\cdot\Big(\|W_n(t)\|_{H^1} + Z(t)\Big)\cdot(\tau-t)^{-1/2}\, dt, \tag{5.8}
$$

for some constant C_4.

We now set

$$\varepsilon_0 \doteq \sup_{t\in[0,T]} \|W_n(t)\|_{H^1(\mathbb{R}\setminus\{0\})}.$$

Since $Z(0) = 0$, calling τ^* the first time where $Z \geq \varepsilon_0/2$ one has

$$\frac{\varepsilon_0}{2} \leq C_4 \int_0^{\tau^*} \frac{\varepsilon_0}{2} \cdot \left(\epsilon_0 + \frac{\varepsilon_0}{2}\right) (\tau^* - t)^{-1/2}\, dt = \frac{3}{2} C_4 \varepsilon_0^2 \tau^*.$$

Hence $\tau^* \geq (3C_4)^{-1}$. Choosing $0 < T < (3C_4)^{-1}$, we thus obtain

$$Z(t) \leq \frac{\epsilon_0}{2} \qquad \text{for all} \quad t \in [0,T].$$

This establishes the desired contraction property:

$$\sup_{t\in[0,T]} \|w_{n+1}(t) - w_n(t)\|_{H^1(\mathbb{R}\setminus\{0\})} \leq \frac{1}{2} \cdot \sup_{t\in[0,T]} \|w_n(t) - w_{n-1}(t)\|_{H^1(\mathbb{R}\setminus\{0\})}. \tag{5.9}$$

4. By (5.9), for every $t \in [0,T]$ the sequence of approximations $w_n(t,\cdot)$ is Cauchy in the space $H^1(\mathbb{R} \setminus \{0\})$, hence it converges to a unique limit $w(t,\cdot)$.

It remains to check that this limit function w is an entropic solution, i.e. it satisfies the integral equation (2.18). But this is clear, because for every $\epsilon > 0$ the sequence of functions

$$F_n \doteq \mathbf{H}[\varphi] - \varphi\varphi_x + \left(\mathbf{H}[w_n] - \left(w_n - \frac{u_n^- + u_n^+}{2}\right)\varphi_x\right) \tag{5.10}$$

converges to the corresponding function F in (2.19), uniformly for $t \in [0,T]$ and $|x| \geq \epsilon$.

5. Finally, to prove uniqueness, assume that $w, \tilde{w}$ are two entropic solutions. Consider the differences

$$W \doteq w - \tilde{w}, \qquad \begin{cases} U^- \doteq u^- - \tilde{u}^-, \\ U^+ \doteq u^+ - \tilde{u}^+, \end{cases}$$

and call $Z(t) \doteq \|W(t)\|_{H^1(\mathbb{R}\setminus\{0\})}$. Since $Z(0) = 0$, the same arguments used to prove (5.8) now yield

$$Z^2(\tau) \leq C_4 \int_0^{\tau} Z(t) \cdot \big[Z(t) + Z(t)\big] \cdot (\tau - t)^{-1/2}\, dt.$$

For $\tau \in [0,T]$ sufficiently small, we thus obtain $Z(\tau) = 0$. This completes the proof of Theorem 1. □

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