

A Posteriori Error Estimates for Self-Similar Solutions to the Euler Equations

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Abstract

The main goal of this paper is to analyze a family of “simplest possible” initial data for which, as shown by numerical simulations, the incompressible Euler equations have multiple solutions. We take here a first step toward a rigorous validation of these numerical results. Namely, we consider the system of equations corresponding to a self-similar solution, restricted to a bounded domain with smooth boundary. Given an approximate solution obtained via a finite dimensional Galerkin method, we establish a posteriori error bounds on the distance between the numerical approximation and the exact solution having the same boundary data.

1 Introduction

The flow of a homogeneous, incompressible, non-viscous fluid in \mathbb{R}^2 is modeled by the Euler equations

$$\begin{cases} u_t + (u \cdot \nabla)u &= -\nabla p & \text{(balance of momentum),} \\ \operatorname{div} u &= 0 & \text{(incompressibility condition).} \end{cases} \quad (1.1)$$

Here $u = u(t, x)$ denotes the velocity of the fluid, while the scalar function p is a pressure. The condition $\operatorname{div} u = 0$ implies the existence of a stream function ψ such that

$$u = \nabla^\perp \psi, \quad (u_1, u_2) = (-\psi_{x_2}, \psi_{x_1}). \quad (1.2)$$

Denoting by $\omega = \operatorname{curl} u = (-u_{1,x_2} + u_{2,x_1})$ the vorticity of the fluid, it is well known that the Euler equations (1.1) can be reformulated in terms of the system

$$\begin{cases} \omega_t + \nabla^\perp \psi \cdot \nabla \omega &= 0, \\ \Delta \psi &= \omega. \end{cases} \quad (1.3)$$

The velocity u is then recovered from the vorticity by the Biot-Savart formula

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy. \quad (1.4)$$

Our eventual goal is to construct “simplest possible” initial data for which the equations (1.1) have multiple solutions. Numerical simulations, shown in Figures 2 and 3, indicate that two distinct solutions can be achieved for initial data where the vorticity

$$\bar{\omega}(x) = \omega(0, x) = \text{curl } u(0, x) \quad (1.5)$$

has the form

$$\bar{\omega}(x) = r^{-1/\mu} \bar{\Omega}(\theta), \quad x = (x_1, x_2) = (r \cos \theta, r \sin \theta). \quad (1.6)$$

Here $\frac{1}{2} < \mu < +\infty$, while $\bar{\Omega} \in \mathcal{C}^\infty(\mathbb{R})$ is a non-negative, smooth, periodic function which satisfies

$$\bar{\Omega}(\theta) = \bar{\Omega}(\pi + \theta), \quad \bar{\Omega}(\theta) = 0 \quad \text{if } \theta \in \left[\frac{\pi}{4}, \pi\right]. \quad (1.7)$$

As shown in Figure 1, left, the initial vorticity $\bar{\omega}$ is supported on two wedges, and becomes arbitrarily large as $|x| \rightarrow 0$.

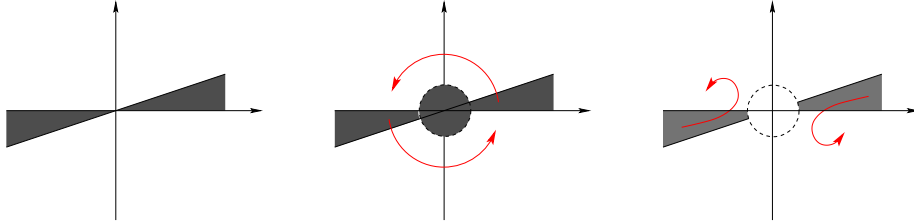


Figure 1: The supports of the initial vorticity considered in (1.6)-(1.8).

One can approximate $\bar{\omega}$ by two families of initial data $\bar{\omega}_\varepsilon, \bar{\omega}_\varepsilon^\dagger \in \mathbf{L}^\infty(\mathbb{R}^2)$, taking

$$\bar{\omega}_\varepsilon(x) = \begin{cases} \omega(x) & \text{if } |x| > \varepsilon, \\ \varepsilon^{-1/\mu} & \text{if } |x| \leq \varepsilon, \end{cases} \quad \bar{\omega}_\varepsilon^\dagger(x) = \begin{cases} \omega(x) & \text{if } |x| > \varepsilon, \\ 0 & \text{if } |x| \leq \varepsilon. \end{cases} \quad (1.8)$$

As $\varepsilon \rightarrow 0$, we have $\bar{\omega}_\varepsilon, \bar{\omega}_\varepsilon^\dagger \rightarrow \bar{\omega}$ in \mathbf{L}_{loc}^p , for a suitable p depending on the parameter μ in (1.6). By Yudovich’s theorem [25], for every $\varepsilon > 0$ these initial data yield a unique solution. However, the numerical simulations indicate that, as $\varepsilon \rightarrow 0$, two distinct limit solutions are obtained. In the first solution, shown in Figure 2, both wedges wind up together in a single spiral. In the second solution, shown in Figure 3, each wedge curls up on itself, and two distinct spirals are observed.

Remark 1.1 The fact that all approximate solutions $\omega_\varepsilon, \omega_\varepsilon^\dagger$ are uniquely determined by their initial data implies that the ill-posedness exhibited by this example is “uncurable”. Namely, there is no way to select one solution with initial datum as in (1.6)-(1.7), preserving the continuous dependence on initial data.

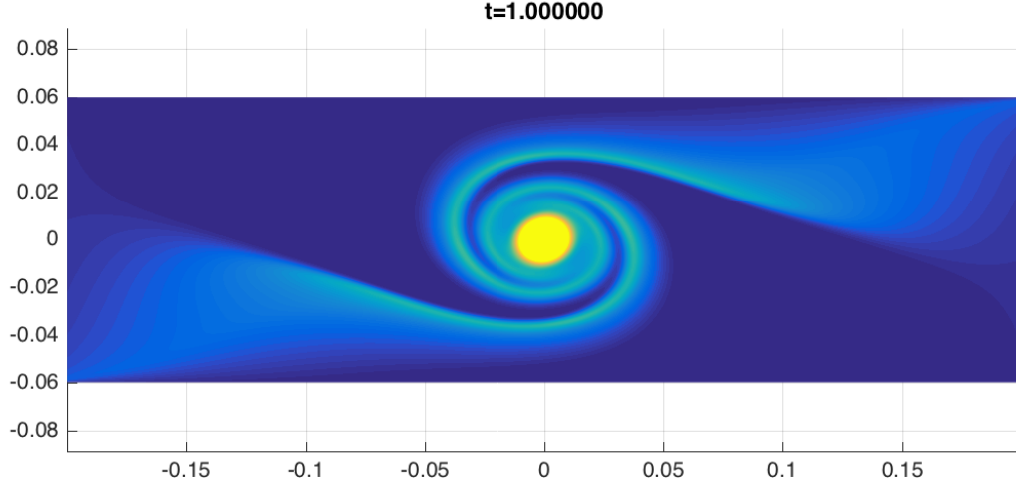


Figure 2: The vorticity distribution at time $t = 1$, for a solution to (1.3) with initial vorticity $\bar{\omega}_\varepsilon$.

We observe that both of these limit solutions are self-similar, i.e. they have the form

$$\begin{cases} u(t, x) = t^{\mu-1}U\left(\frac{x}{t^\mu}\right), \\ \omega(t, x) = t^{-1}\Omega\left(\frac{x}{t^\mu}\right), \\ \psi(t, x) = t^{2\mu-1}\Psi\left(\frac{x}{t^\mu}\right). \end{cases} \quad (1.9)$$

Notice that, by self-similarity, these solutions are completely determined as soon as we know their values at time $t = 1$. Indeed, these are given by U, Ω, Ψ . Inserting (1.9) in (1.3), one obtains the equations

$$\begin{cases} (\nabla^\perp \Psi - \mu y) \cdot \nabla \Omega = \Omega, \\ \Delta \Psi = \Omega, \end{cases} \quad (1.10)$$

while the velocity is recovered by

$$U = \nabla^\perp \Psi. \quad (1.11)$$

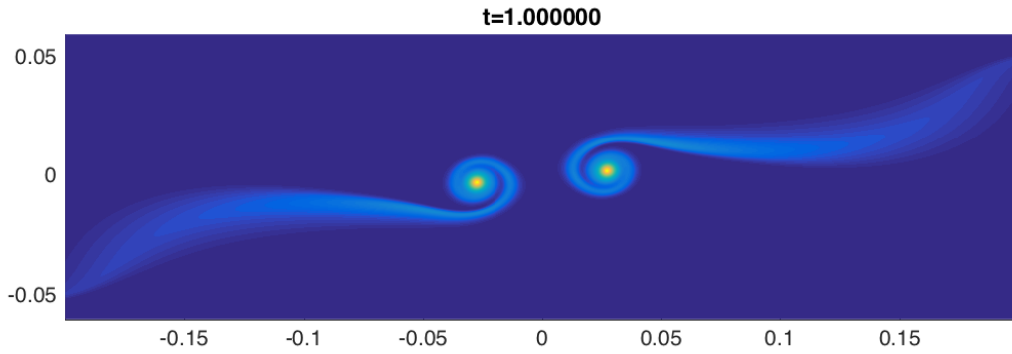


Figure 3: The vorticity distribution at time $t = 1$, for a solution to (1.3) with initial vorticity $\bar{\omega}_\varepsilon^\dagger$.

Constructing two distinct self-similar solutions of (1.3) with the same initial data (1.6) amounts to finding two distinct solutions (Ω, Ψ) , $(\Omega^\dagger, \Psi^\dagger)$ of (1.10) with the same asymptotic behavior as $|x| \rightarrow +\infty$. More precisely, writing the vorticity Ω in polar coordinates, this means

$$\lim_{r \rightarrow +\infty} r^{\frac{1}{\mu}} \Omega(r, \theta) = \lim_{r \rightarrow +\infty} r^{\frac{1}{\mu}} \Omega^\dagger(r, \theta) \doteq \bar{\Omega}(\theta), \quad (1.12)$$

for some smooth function $\bar{\Omega}$ as in (1.7).

Since the two solutions in Figures 2 and 3 are produced by numerical computations, a natural question is whether an exact self-similar solution of the Euler equations exists, close to each computed one. This requires suitable a posteriori error bounds. Toward this goal, two difficulties arise:

- (i) The self similar solution (Ω, Ψ) is defined on the entire plane \mathbb{R}^2 , while a numerical solution is computed only on some bounded domain.
- (ii) The solution is smooth, with the exception of one or two points corresponding to the spirals' centers. In a neighborhood of these points the standard error estimates break down.

To address these issues, we propose a domain decomposition method. As shown in Fig. 4, the plane can be decomposed into an outer domain $D^\# \supseteq \{x \in \mathbb{R}^2; |x| > R\}$, an inner domain \mathcal{D}^\flat containing a neighborhood of the spirals' centers where the solution has singularities, and a bounded intermediate domain \mathcal{D}^\natural where the solution is smooth. The solution is constructed analytically on $\mathcal{D}^\#$ and on \mathcal{D}^\flat , and numerically on \mathcal{D}^\natural . These three components are then patched together by suitable matching conditions.

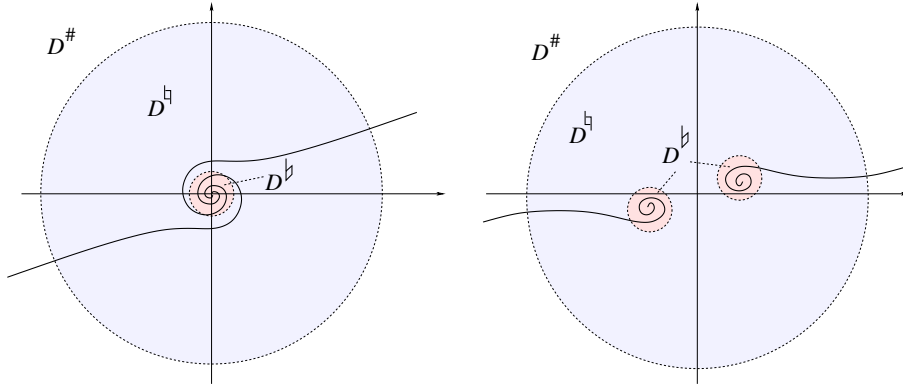


Figure 4: Decomposing the plane $\mathbb{R}^2 = \mathcal{D}^\# \cup \mathcal{D}^\natural \cup \mathcal{D}^\flat$ into an outer, a middle, and an inner domain. Left: the case of a single spiraling vortex, as in Fig. 2. Right: the case of two spiraling vortices, as in Fig. 3.

A detailed analysis of the solution to (1.10) in a neighborhood of infinity and near the spirals' centers will appear in the companion paper [5], relying on the approach developed in [15, 16, 17]. In the present paper we focus on the derivation of a posteriori error estimates for a numerically computed solution to (1.10), on a bounded domain $\mathcal{D} \subset \mathbb{R}^2$ with smooth boundary $\partial\mathcal{D}$. As shown in Fig. 5, we assume that this boundary can be decomposed as the union of two closed, disjoint components:

$$\partial\mathcal{D} = \Sigma_1 \cup \Sigma_2. \quad (1.13)$$

We seek a solution to (1.10), satisfying boundary conditions of the form

$$\begin{cases} \Psi(x) = g(x), & x \in \partial\mathcal{D}, \\ \Omega(x) = h(x), & x \in \Sigma_1, \end{cases} \quad (1.14)$$

where g, h are given smooth functions. Given an approximate solution $(\Omega_0, \Psi_0) \in \mathcal{C}^{0,\alpha}(\mathcal{D}) \times \mathcal{C}^{2,\alpha}(\mathcal{D})$, computed by a Galerkin finite dimensional approximation, we want to prove the existence of an exact solution close to the approximate one.

The remainder of the paper is organized as follows. Section 2 introduces the basic framework, specifying the main assumptions on the numerical scheme and on the approximate solution. Section 3 begins by analyzing the first equation in (2.5), regarded as a linear PDE for the vorticity function Ω . In this direction, Lemma 3.1 provides a detailed estimate on how the solution depends on the vector field $\nabla^\perp \Phi$. In addition, Lemma 3.2 yields a sharper regularity estimate on the solutions, deriving an a priori bound on their $\mathcal{C}^{0,\alpha}$ norm.

Finally, in Section 4 the exact solution Ω is constructed as the fixed point of a transformation which is contractive w.r.t. a norm equivalent to the \mathbf{L}^2 norm. We remark that, in order to achieve this contractivity, a bound on the norms $\|\Omega\|_{\mathbf{L}^2}$ and $\|\Phi\|_{H^2}$ is not good enough. Indeed, we need an a priori bound on $\|\Omega\|_{\mathcal{C}^{0,\alpha}}$ and on $\|\Phi\|_{\mathcal{C}^2}$. This is achieved by means of Lemma 3.2. At the end of Section 4 we collect all the various constants appearing in the estimates, and summarize our analysis by stating a theorem on the existence of an exact solution, close to the computed one.

For results on the uniqueness of solutions to the incompressible Euler equations we refer to [2, 21, 25]. Examples showing the non-uniqueness of solutions to the incompressible Euler equations were first constructed in [22, 23]. See also [24] for a different approach, yielding more regular solutions. Following the major breakthrough in [11] several examples of multiple solutions for Euler's equations have recently been provided [8, 9, 10, 12]. These solutions are obtained by means of convex integration and a Baire category argument. They have turbulent nature and their physical significance is unclear.

Our numerical simulations, on the other hand, suggest that “uncurable” non-uniqueness can arise from quite simple initial data. In our case, both solutions can be easily visualized; they remain everywhere smooth (with the exception of one or two points), and conserve energy at all times.

2 Setting of the problem

Throughout the following, we consider the boundary value problem (1.10), (1.14) on a bounded, open domain $\mathcal{D} \subset \mathbb{R}^2$, with smooth boundary $\partial\mathcal{D}$ decomposed as in (1.13). The unit outer normal at the boundary point $y \in \partial\mathcal{D}$ will be denoted by $\mathbf{n}(y)$. We call Ψ^g the solution to the non-homogeneous boundary value problem

$$\begin{cases} \Delta\Psi(x) = 0 & x \in \mathcal{D}, \\ \Psi(x) = g(x) & x \in \partial\mathcal{D}, \end{cases} \quad (2.1)$$

and define the vector field

$$\mathbf{v}(x) \doteq \nabla^\perp \Psi^g(x) - \mu x. \quad (2.2)$$

Assuming that g is smooth, the same is true of \mathbf{v} . Given $\Omega \in \mathbf{L}^2(\mathcal{D})$, we define

$$\Phi = \Delta^{-1}\Omega \quad (2.3)$$

to be the solution of

$$\begin{cases} \Delta\Phi(x) = \Omega(x), & x \in \mathcal{D}, \\ \Phi(x) = 0, & x \in \partial\mathcal{D}. \end{cases} \quad (2.4)$$

Our problem thus amounts to finding a function Ω such that

$$\begin{cases} (\nabla^\perp\Phi(x) + \mathbf{v}(x)) \cdot \nabla\Omega(x) = \Omega(x), & x \in \mathcal{D}, \\ \Omega(x) = h(x), & x \in \Sigma_1, \end{cases} \quad (2.5)$$

where $\Phi = \Delta^{-1}\Omega$ and \mathbf{v} is the vector field at (2.2). In the following, for any given function $\Phi \in \mathcal{C}^2(\mathcal{D})$, we denote by $\Omega = \Gamma(\Phi)$ the solution to (2.5). We seek a fixed point of the composed map

$$\Omega \mapsto \Lambda(\Omega) \doteq \Gamma(\Delta^{-1}\Omega). \quad (2.6)$$

We shall consider approximate solutions of (2.5) which are obtained by finite dimensional Galerkin approximations. More precisely, given a finite set of linearly independent functions $\{\phi_j; 1 \leq j \leq N\} \subset \mathbf{L}^2(\mathcal{D})$, consider the orthogonal decomposition $\mathbf{L}^2(\mathcal{D}) = U \times V$, where

$$U = \text{span}\{\phi_1, \dots, \phi_N\}, \quad V = U^\perp, \quad (2.7)$$

with orthogonal projections

$$P : \mathbf{L}^2(\mathcal{D}) \mapsto U, \quad (I - P) : \mathbf{L}^2(\mathcal{D}) \mapsto V. \quad (2.8)$$

Example 1. We can choose the functions ϕ_j to be piecewise affine, obtained from a triangulation of the domain \mathcal{D} . This implies

$$\phi_j \in W^{1,\infty}(\mathcal{D}) \subset \mathcal{C}^{0,\alpha}(\mathcal{D}) \quad (2.9)$$

for every $0 < \alpha \leq 1$. In this case, the gradient $\nabla\phi_j \in \mathbf{L}^\infty \cap BV$ is piecewise constant, with jumps along finitely many segments.

Example 2. In alternative, one can choose $\phi_1, \dots, \phi_N \in \mathbf{L}^2(\mathcal{D})$ to be the first N normalized eigenfunctions of the Laplacian. More precisely, for $j = 1, \dots, N$ we require that the functions ϕ_j satisfy

$$\begin{cases} \Delta\phi_j + \lambda_j\phi_j = 0 & x \in \mathcal{D}, \\ \phi_j = 0 & x \in \Sigma_1, \\ \mathbf{n} \cdot \nabla\phi_j = 0 & x \in \Sigma_2, \end{cases} \quad (2.10)$$

with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$. Moreover, $\|\phi_j\|_{\mathbf{L}^2} = 1$.

Remark 2.1 In both of the above cases, there may be no linear combination $\sum_j c_j\phi_j$ of the basis functions that matches the boundary data h along Σ_1 . This issue can be addressed simply by adding to our basis an additional function $\phi_0 \in \mathcal{C}^\infty(\mathcal{D})$, chosen so that $\phi_0 = h$ on Σ_1 .

Our basic question can be formulated as follows.

(Q) Assume we can find a finite dimensional approximation

$$\Omega_0 = \sum_{j=1}^N c_j \phi_j, \quad \Phi_0 = \Delta^{-1} \Omega_0, \quad (2.11)$$

with error

$$\left\| \Omega_0 - P \Lambda(\Omega_0) \right\|_{\mathbf{L}^2(\mathcal{D})} = \delta_0. \quad (2.12)$$

How small should δ_0 be, to make sure that an exact solution (Ω, Φ) of (2.5), (2.3) exists, close to (Ω_0, Φ_0) ?

Given a function $\Phi \in \mathcal{C}^2(\mathcal{D})$, the linear, first order PDE (2.5) for Ω can be solved by the method of characteristics. Namely, consider the vector field

$$\mathbf{q}(x) = \nabla^\perp \Phi(x) + \mathbf{v}(x), \quad (2.13)$$

whose divergence is

$$\operatorname{div} \mathbf{q} = -2\mu. \quad (2.14)$$

We shall denote by $t \mapsto \exp(t\mathbf{q})(y)$ the solution to the ODE

$$\dot{x}(t) = \mathbf{q}(x(t)), \quad x(0) = y.$$

For convenience, we shall use the notation

$$t \mapsto x(t, y) \doteq \exp(-t\mathbf{q})(y) \quad (2.15)$$

for the solution to the ODE

$$\dot{x}(t) = -\mathbf{q}(x(t)), \quad x(0) = y.$$

Consider the set

$$\mathcal{D}^* \doteq \left\{ y \in \overline{\mathcal{D}}; \ x(\tau, y) \in \Sigma_1 \cap \operatorname{Supp}(h) \text{ for some } \tau \geq 0 \right\}. \quad (2.16)$$

In other words, $y \in \mathcal{D}^*$ if the characteristic through y reaches a boundary point in the support of h within finite time. Calling

$$\tau(y) \doteq \min \{ t \geq 0; \ x(t, y) \in \Sigma_1 \} \quad (2.17)$$

the first time when the characteristic starting at y reaches the boundary Σ_1 , the solution to (2.5) is computed by

$$\Omega(y) = \begin{cases} e^{\tau(y)} h(x(\tau(y), y)) & \text{if } x \in \mathcal{D}^*, \\ 0 & \text{if } x \notin \mathcal{D}^*. \end{cases} \quad (2.18)$$

The following transversality assumption will play a key role in the sequel (see Fig. 5).

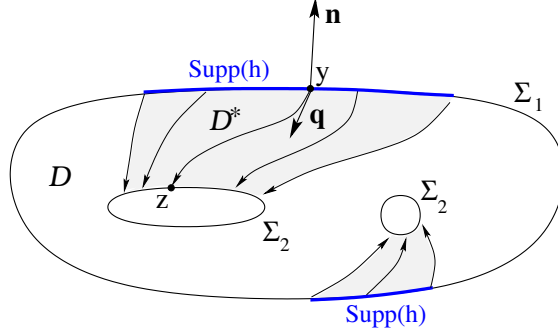


Figure 5: According to **(A1)**, every characteristic starting at a point $y \in \Sigma_1 \cap \text{Supp}(h)$ exits from the domain \mathcal{D} at some boundary point $z = z(y) \in \Sigma_2$, at a time $T(y) \leq T^*$. The shaded region represents the subdomain \mathcal{D}^* in (2.16).

(A1) *There exists constants $T^*, c_1 > 0$ such that, for every boundary point $y \in \Sigma_1 \cap \text{Supp}(h)$, the following holds.*

(i) *The vector \mathbf{q} is strictly inward pointing:*

$$\langle \mathbf{n}(y), \mathbf{q}(y) \rangle \leq -c_1. \quad (2.19)$$

(ii) *The characteristic $t \mapsto \exp(t\mathbf{q})(y)$ remains inside \mathcal{D} until it reaches a boundary point*

$$z(y) = \exp(T(y)\mathbf{q})(y) \in \Sigma_2 \quad (2.20)$$

within a finite time $T(y) \leq T^$, and exits transversally:*

$$\langle \mathbf{n}(z(y)), \mathbf{q}(z(y)) \rangle \geq c_1. \quad (2.21)$$

As in (2.7)-(2.8), we consider the decomposition $H \doteq \mathbf{L}^2(\mathcal{D}) = U \times V$, with perpendicular projections P and $I - P$, and write $\Omega = (u, v)$. The partial derivatives of the map $\Lambda = \Lambda(u, v)$ introduced at (2.6) will be denoted by $D_u\Lambda$, $D_v\Lambda$. Let a finite dimensional approximate solution $\Omega_0 = (u_0, 0) \in U$ of (2.5) be given, with $\Phi_0 = \Delta^{-1}\Omega_0$. Throughout the following, we denote by

$$A \doteq P \circ D_u\Lambda(\Omega_0). \quad (2.22)$$

the partial differential w.r.t. u of the map $(u, v) \mapsto P\Lambda(u, v)$, computed at the point Ω_0 . Notice that A is a linear map from the finite dimensional space U into itself.

Since we are seeking a fixed point, in the same spirit of the Newton-Kantorovich theorem [1, 6, 14] we shall assume the invertibility of the Jacobian map, restricted to the finite dimensional subspace U .

(A2) *The operator $I - A$ from U into itself has a bounded inverse, with*

$$\|(I - A)^{-1}\|_{\mathcal{L}(U)} \leq \gamma < +\infty \quad (2.23)$$

for some $\gamma \geq 1$.

Here the left hand side refers to the operator norm, in the space of linear operators on $U \subset \mathbf{L}^2(\mathcal{D})$. Notice that (2.23) implies that the operator $I - AP : H \mapsto H$ is also invertible, with

$$\|(I - AP)^{-1}\|_{\mathcal{L}(H)} \leq \gamma. \quad (2.24)$$

Concerning the V -component, a key assumption used in our analysis will be

(A3) *The orthogonal spaces U, V in (2.7) are chosen so that*

$$\|\Delta^{-1} \circ (I - P)\| < \varepsilon_0 \ll 1. \quad (2.25)$$

Intuitively this means that, in the decomposition $\Omega = (u, v) \in U \times V$, the component $v \in V$ captures the high frequency modes, which are heavily damped by the inverse Laplace operator [3, 7, 20].

From an abstract point of view, proving the existence of a fixed point of the map $\Omega \mapsto \Lambda(\Omega)$ in (2.6) is a simple matter. Using the decomposition $\Omega = (u, v) \in U \times V$ with orthogonal projections as in (2.8), we start with an initial guess $\Omega_0 = (u_0, 0)$. Assuming (2.23), we write (2.5) in the form

$$\Omega - AP\Omega = \Lambda(\Omega) - AP\Omega.$$

Equivalently,

$$\Omega = \Upsilon(\Omega) \doteq (I - AP)^{-1}(\Lambda(\Omega) - AP\Omega). \quad (2.26)$$

The heart of the matter is to show that, on a neighborhood \mathcal{N} of Ω_0 , the map $\Omega \mapsto \Upsilon(\Omega)$ is a strict contraction. If the initial error $\|\Upsilon(\Omega_0) - \Omega_0\|$ is sufficiently small, the iterates $\Omega_n = \Upsilon^n(\Omega_0)$ will thus remain inside \mathcal{N} and converge to a fixed point. The contraction property is proved as follows. By (2.24) one has $\|(I - AP)^{-1}\| \leq \gamma$. On the other hand, computing the differential of the map $\Omega \mapsto \Lambda(\Omega) - AP\Omega$ w.r.t. the components $(u, v) \in U \times V$, at the point $\Omega = \Omega_0 = (u_0, 0)$ one obtains

$$D(\Lambda(u, v) - Au) = \begin{pmatrix} 0 & PD_v\Lambda \\ (I - P)D_u\Lambda & (I - P)D_v\Lambda \end{pmatrix}. \quad (2.27)$$

Because of (2.25), we expect

$$\|D_v\Lambda\| \leq \|D\Gamma\| \cdot \|\Delta^{-1} \circ (I - P)\| \ll 1. \quad (2.28)$$

By possibly using an equivalent norm on the product space $U \times V$ (see (4.3) for details), we thus achieve the strict contractivity of the map Υ in (2.26).

In the next section, more careful estimates will be derived on the differentials of Γ and Λ in (2.6). In this direction we remark that, while the operator Δ^{-1} is well defined on $\mathbf{L}^2(\mathcal{D})$, a difficulty arises in connection with the differential of Γ . Indeed, to compute this differential, we need to perturb the function Φ in (2.5) and estimate the change in the corresponding solution Ω . This can be done assuming that $\Phi \in \mathcal{C}^2$, hence $\nabla^\perp \Phi \in \mathcal{C}^1$. Unfortunately, the assumption $\Omega \in \mathbf{L}^2$ only implies $\Phi = \Delta^{-1}\Omega \in H^2$, which does not yield any bound on $\|\Phi\|_{\mathcal{C}^2}$. In order to establish the desired a posteriori error bound, an additional argument will thus be needed, showing that our approximate solutions actually enjoy some additional regularity.

3 Preliminary lemmas

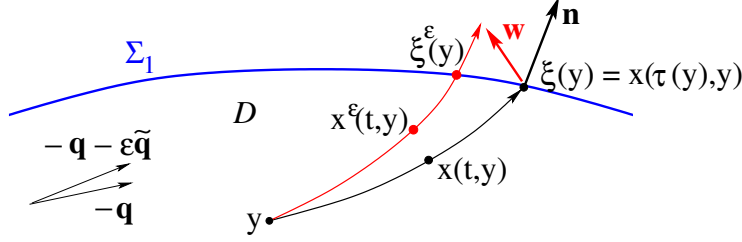


Figure 6: Computing the perturbed solution $\Omega^\varepsilon(y)$ in (3.4), by estimating the change in the characteristic through y .

3.1 Continuous dependence of solutions to a first order linear PDE.

As remarked in Section 2, solutions to the linear PDE (2.5) can be found by the method of characteristics (2.18). Our present goal is to understand how the solution changes, depending on the vector field \mathbf{q} in (2.13). To fix the ideas, consider the boundary value problem

$$\begin{cases} \mathbf{q}(x) \cdot \nabla \Omega = \Omega, & x \in \mathcal{D}, \\ \Omega = h, & x \in \Sigma_1, \end{cases} \quad (3.1)$$

assuming that $\mathbf{q} : \mathcal{D} \mapsto \mathbb{R}^2$ is a \mathcal{C}^1 vector field satisfying **(A1)** together with

$$\|\mathbf{q}\|_{\mathcal{C}^1(\mathcal{D})} \leq M, \quad \operatorname{div} \mathbf{q}(x) = -2\mu \quad \text{for all } x \in \mathcal{D}. \quad (3.2)$$

Given a second vector field $\tilde{\mathbf{q}} \in \mathcal{C}^1$, consider the family of perturbations

$$\mathbf{q}^\varepsilon(x) = \mathbf{q}(x) + \varepsilon \tilde{\mathbf{q}}(x), \quad (3.3)$$

and let

$$\Omega^\varepsilon(x) = \Omega(x) + \varepsilon \tilde{\Omega}(x) + o(\varepsilon) \quad (3.4)$$

be the corresponding solutions of (3.1). Here and in the sequel, the notation $o(\varepsilon)$ indicates a higher order infinitesimal, so that $\varepsilon^{-1}o(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The next lemma provides an \mathbf{L}^2 estimate on size of the first order perturbation $\tilde{\Omega}$. Setting $\Sigma_1^* \doteq \Sigma_1 \cap \operatorname{Supp}(h)$, we introduce the constant

$$\tilde{C} = \sup_{x \in \Sigma_1^*} \frac{1}{|\langle \mathbf{n}(x), \mathbf{q}(x) \rangle|} \cdot \|h\|_{\mathcal{C}^0} + \left(1 + \sup_{x \in \Sigma_1^*} \frac{|\mathbf{q}(x)|}{|\langle \mathbf{n}(x), \mathbf{q}(x) \rangle|} \right) \|Dh\|_{\mathcal{C}^0}, \quad (3.5)$$

and define

$$K(M, t) = e^t \left(\frac{e^{(2M+1)t} - e^{-2\mu t}}{4M + 4\mu + 2} \right)^{1/2}. \quad (3.6)$$

Lemma 3.1 *Recalling (2.16), assume that $\mathbf{q} \in \mathcal{C}^1(\mathcal{D})$ satisfies (3.2) together with **(A1)**. In the setting considered at (3.3)-(3.4), the first order perturbation $\tilde{\Omega}$ satisfies*

$$\|\tilde{\Omega}\|_{\mathbf{L}^2(\mathcal{D})} \leq \tilde{C} \cdot K\left(\|D\mathbf{q}\|_{\mathcal{C}^0(\mathcal{D}^*)}, T^*\right) \cdot \|\tilde{\mathbf{q}}\|_{\mathbf{L}^2(\mathcal{D}^*)}. \quad (3.7)$$

Proof. 1. In analogy with (2.15), we denote by

$$t \mapsto x^\varepsilon(t, y) = \exp(-t\mathbf{q}^\varepsilon)(y)$$

the solution to

$$\dot{x}^\varepsilon = -\mathbf{q}(x) - \varepsilon\tilde{\mathbf{q}}(x), \quad x(0) = y. \quad (3.8)$$

For $0 \leq t < \tau(y)$, the tangent vector

$$\mathbf{w}(t, y) \doteq \lim_{\varepsilon \rightarrow 0} \frac{x^\varepsilon(t, y) - x(t, y)}{\varepsilon} \quad (3.9)$$

provides a solution to the linearized equation

$$\dot{\mathbf{w}}(t, y) = -D\mathbf{q}(x(t, y)) \cdot \mathbf{w}(t, y) - \tilde{\mathbf{q}}(x(t, y)), \quad \mathbf{w}(0, y) = 0. \quad (3.10)$$

For notational convenience, we extend the definition of the vector $\mathbf{w}(t, y)$ by setting

$$\mathbf{w}(t, y) = \mathbf{w}(\tau(y), y) \quad \text{if} \quad t \in [\tau(y), T^*]. \quad (3.11)$$

We denote by

$$\begin{aligned} \xi(y) &= x(\tau(y), y) = \exp(-\tau(y)\mathbf{q})(y), \\ \xi^\varepsilon(y) &= x^\varepsilon(\tau^\varepsilon(y), y) = \exp(-\tau^\varepsilon(y)\mathbf{q}^\varepsilon)(y), \end{aligned} \quad (3.12)$$

the points where the characteristic through y crosses the boundary Σ_1 , and consider the expansions

$$\xi^\varepsilon(y) = \xi(y) + \varepsilon\tilde{\xi}(y) + o(\varepsilon), \quad \tau^\varepsilon(y) = \tau(y) + \varepsilon\tilde{\tau}(y) + o(\varepsilon). \quad (3.13)$$

Observing that

$$\left\langle \mathbf{n}(x(\tau(y), y)), \mathbf{w}(\tau(y), y) - \tilde{\tau}(y) \cdot \mathbf{q}(x(\tau(y), y)) \right\rangle = 0, \quad (3.14)$$

we obtain

$$\tilde{\tau}(y) = \frac{\left\langle \mathbf{n}(\xi(y)), \mathbf{w}(\tau(y), y) \right\rangle}{\left\langle \mathbf{n}(\xi(y)), \mathbf{q}(\xi(y)) \right\rangle}, \quad (3.15)$$

$$\tilde{\xi}(y) = \mathbf{w}(\tau(y), y) - \tilde{\tau}(y) \mathbf{q}(\xi(y)), \quad (3.16)$$

$$|\tilde{\xi}(y)| \leq |\mathbf{w}(\tau(y), y)| \cdot \left(1 + \left| \frac{\mathbf{q}(\xi(y))}{\langle \mathbf{n}(\xi(y)), \mathbf{q}(\xi(y)) \rangle} \right| \right). \quad (3.17)$$

Finally, for $y \in \mathcal{D}^*$ we have

$$\tilde{\Omega}(y) = \tilde{\tau}(y) e^{\tau(y)} h(\xi(y)) + e^{\tau(y)} \nabla h(\xi(y)) \cdot \tilde{\xi}(y). \quad (3.18)$$

In view of (3.15)–(3.17) and the definition of \tilde{C} at (3.5), this yields

$$\begin{aligned} |\tilde{\Omega}(y)| &\leq e^{\tau(y)} \|h\|_{\mathbf{L}^\infty} \cdot \frac{|\mathbf{w}(\tau(y), y)|}{|\langle \mathbf{n}(\xi(y)), \mathbf{q}(\xi(y)) \rangle|} \\ &\quad + e^{\tau(y)} \|\nabla h\|_{\mathbf{L}^\infty} \cdot \left(1 + \frac{|\mathbf{q}(\xi(y))|}{|\langle \mathbf{n}(\xi(y)), \mathbf{q}(\xi(y)) \rangle|} \right) |\mathbf{w}(\tau(y), y)| \\ &\leq e^{\tau(y)} \tilde{C} |\mathbf{w}(\tau(y), y)| \leq e^{T^*} \tilde{C} |\mathbf{w}(\tau(y), y)|. \end{aligned} \quad (3.19)$$

2. It now remains to derive a bound on the \mathbf{L}^2 norm of \mathbf{w} . Keeping (3.11) in mind, define

$$Z(t) \doteq \int_{\mathcal{D}} |\mathbf{w}(t, y)|^2 dy.$$

For any $t \in [0, T^*]$, by (3.2) the Jacobian determinant of the map $y \mapsto x(t, y)$ satisfies

$$\det \left(\frac{\partial x(t, y)}{\partial y} \right) = e^{2\mu t}. \quad (3.20)$$

Using the above identity to change variables of integration, setting

$$\mathcal{D}_t^* \doteq \{y \in \mathcal{D}^*; \tau(y) < t\},$$

we obtain

$$\int_{\mathcal{D}_t^*} |\tilde{\mathbf{q}}(x(t, y))|^2 dy \leq e^{-2\mu t} \int_{\mathcal{D}^*} |\tilde{\mathbf{q}}(x)|^2 dx = e^{-2\mu t} \|\tilde{\mathbf{q}}\|_{\mathbf{L}^2(\mathcal{D}^*)}^2. \quad (3.21)$$

In turn, by the elementary inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, this yields

$$\begin{aligned} \frac{d}{dt} Z(t) &\leq 2 \int_{\mathcal{D}_t^*} \left| \langle \mathbf{w}(t, y), \dot{\mathbf{w}}(t, y) \rangle \right| dy \\ &\leq 2 \int_{\mathcal{D}_t^*} \left\{ |D\mathbf{q}(x(t, y))| |\mathbf{w}(t, y)|^2 + |\mathbf{w}(t, y)| |\tilde{\mathbf{q}}(x(t, y))| \right\} dy \\ &\leq 2 \left\{ \|D\mathbf{q}\|_{\mathcal{C}^0} \cdot \|\mathbf{w}(t, \cdot)\|_{\mathbf{L}^2(\mathcal{D}_t^*)}^2 + \|\mathbf{w}(t, \cdot)\|_{\mathbf{L}^2(\mathcal{D}_t^*)} e^{-\mu t} \|\tilde{\mathbf{q}}\|_{\mathbf{L}^2(\mathcal{D}^*)} \right\} \\ &\leq 2 \left\{ C Z(t) + \frac{Z(t)}{2} + \frac{e^{-2\mu t}}{2} \|\tilde{\mathbf{q}}\|_{\mathbf{L}^2(\mathcal{D}^*)}^2 \right\}. \end{aligned}$$

Hence

$$\|\mathbf{w}(t, \cdot)\|_{\mathbf{L}^2(\mathcal{D})}^2 = Z(t) \leq \kappa(t) \cdot \|\tilde{\mathbf{q}}\|_{\mathbf{L}^2(\mathcal{D})}^2, \quad (3.22)$$

where we set

$$\kappa(t) \doteq \frac{1}{2} \int_0^t e^{(2M+1)(t-s)} e^{-2\mu s} ds = \frac{e^{(2M+1)t} - e^{-2\mu t}}{4M + 4\mu + 2}. \quad (3.23)$$

Taking $t = T^*$ we conclude

$$\int_{\mathcal{D}} |\mathbf{w}(\tau(y), y)|^2 dy \leq \frac{e^{(2M+1)T^*} - e^{-2\mu T^*}}{4M + 4\mu + 2} \cdot \|\tilde{\mathbf{q}}\|_{\mathbf{L}^2(\mathcal{D})}^2. \quad (3.24)$$

Using this bound, from (3.19) one obtains (3.7). \square

3.2 A regularity estimate.

Given a Hölder continuous function $f : \mathcal{D} \mapsto \mathbb{R}$, for $0 < \alpha, \delta < 1$, we introduce the notation

$$\|f\|_{\alpha, \delta} \doteq \sup_{x, y \in \mathcal{D}, 0 < |x-y| < \delta} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \quad (3.25)$$

Notice that, compared with the standard definition of the norm in the Hölder space $\mathcal{C}^{0, \alpha}$ (see for example [4, 18]), in (3.25) the supremum is taken only over couples with $|x - y| < \delta$. From the above definition it immediately follows

$$\|f\|_{\alpha, \delta} \leq \|f\|_{\mathcal{C}^{0, \alpha}}, \quad \|f\|_{\alpha, \delta} \leq \delta^{1-\alpha} \|\nabla f\|_{\mathbf{L}^\infty}. \quad (3.26)$$

The next lemma shows that the $\mathcal{C}^{0,\alpha}$ norm can be controlled in terms of the seminorm $\|\cdot\|_{\alpha,\delta}$ together with the \mathbf{L}^2 norm.

As a preliminary we observe that, since \mathcal{D} is an open set with smooth boundary, it has a positive inner radius $\rho > 0$. Namely, every point $y \in \mathcal{D}$ lies in an open ball of radius ρ , entirely contained inside \mathcal{D} .

Lemma 3.2 *Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded open set with inner radius $\rho > 0$, and let $\alpha, \delta \in]0, 1[$ and $c > 0$ be given. Then there exists $\delta_c > 0$ such that the following holds. If*

$$\|f\|_{\alpha,\delta} \leq c, \quad \|f\|_{\mathbf{L}^2(\mathcal{D})} \leq \delta_c, \quad (3.27)$$

then

$$\|f\|_{\mathcal{C}^0(\mathcal{D})} \leq \frac{c}{2} \delta^\alpha. \quad (3.28)$$

In turn, this implies

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{D})} \leq c. \quad (3.29)$$

Proof. 1. We claim that, by choosing $\delta_c > 0$ small enough, the inequalities (3.27) imply (3.28). Indeed, consider the function

$$\phi(x) \doteq \max \left\{ 0, \frac{c}{2} \delta^\alpha - c|x|^\alpha \right\}.$$

and the disc

$$B = \{(x_1, x_2); (x_1 - \rho)^2 + x_2^2 < \rho^2\}.$$

Setting

$$\delta_c \doteq \|\phi\|_{\mathbf{L}^2(B)} = \left(\int_B \phi^2(x) dx \right)^{1/2}, \quad (3.30)$$

we claim that the assumptions (3.27) imply (3.28). Indeed, assume that $f(y_0) > c\delta^\alpha/2$ at some point $y_0 \in \mathcal{D}$. Let $B_0 \subset \mathcal{D}$ be a disc of radius ρ which contains y_0 . A comparison argument now yields

$$\begin{aligned} \|f\|_{\mathbf{L}^2(\mathcal{D})}^2 &\geq \int_{B_0} f^2(x) dx \geq \int_{B_0} \left(\max \left\{ 0, |f(y_0)| - c|x - y_0|^\alpha \right\} \right)^2 dx \\ &> \int_B \phi^2(x) dx = \delta_c^2, \end{aligned}$$

reaching a contradiction. Hence (3.28) holds.

2. By the assumptions, we already know that

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq c \quad (3.31)$$

when $0 < |x - y| < \delta$. It remains to prove that the same holds when $|x - y| \geq \delta$. But in this case by (3.28) one trivially has

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq \frac{|f(x)| + |f(y)|}{\delta^\alpha} \leq c.$$

Together with (3.28), this yields

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathcal{D})} \doteq \max \left\{ \sup_x |f(x)|, \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right\} \leq c,$$

proving (3.29). \square

4 Construction of an exact solution

As in (2.7)-(2.8), we consider the decomposition $H \doteq \mathbf{L}^2(\mathcal{D}) = U \times V$, with perpendicular projections P and $I - P$, and write $\Omega = (u, v)$. We recall that $D_u \Lambda$, $D_v \Lambda$ denote the partial derivatives of the map $\Lambda = \Lambda(u, v)$ introduced at (2.6).

As anticipated in Section 2, we write the fixed point problem $\Omega = \Lambda(\Omega)$ in the equivalent form

$$\Omega = \Upsilon(\Omega) \doteq (I - AP)^{-1}(\Lambda(\Omega) - AP\Omega). \quad (4.1)$$

In terms of the components (u, v) , at $\Omega = \Omega_0 = (u_0, 0)$, the differential of the map $\Omega \mapsto \Lambda(\Omega) - AP\Omega$ has the form (2.27).

To achieve the contraction property, on the product space $H = U \times V$ we consider the equivalent inner product

$$\langle (u, v), (u', v') \rangle_* = \langle u, u' \rangle + \eta_0 \langle v, v' \rangle, \quad (4.2)$$

for a suitable constant $0 < \eta_0 \leq 1$. The corresponding norm is

$$\|(u, v)\|_* = (\|u\|^2 + \eta_0 \|v\|^2)^{1/2}. \quad (4.3)$$

Based on (2.28), we assume that a constant η_0 can be chosen so that, at the point Ω_0 , the corresponding norm of the linear operator (2.27) is $\leq 1/4\gamma$. By (2.24) and the definition of Υ at (4.1), this implies

$$\|D\Upsilon(\Omega_0)\|_* \leq \frac{1}{4}.$$

Here and in the sequel, we also denote by $\|\cdot\|_*$ the norm of a linear operator, corresponding to the norm (4.3) on the product space $H = U \times V$.

By continuity, we can determine a radius $r_0 > 0$ such that, denoting by $B_*(\Omega_0, r_0)$ a ball centered at Ω_0 with radius r_0 w.r.t. the equivalent norm $\|\cdot\|_*$, one has the implication

$$\Omega, \Omega' \in B_*(\Omega_0, r_0) \implies \|\Upsilon(\Omega) - \Upsilon(\Omega')\|_* \leq \frac{1}{2} \|\Omega - \Omega'\|_*. \quad (4.4)$$

If the approximate solution Ω_0 satisfies

$$\|\Upsilon(\Omega_0) - \Omega_0\|_* \leq \frac{r_0}{2}, \quad (4.5)$$

we can then define the iterates

$$\Omega_n \doteq \Upsilon(\Omega_{n-1}), \quad \Phi_n = \Delta^{-1} \Omega_n. \quad (4.6)$$

Taking the limit

$$\bar{\Omega} = \lim_{n \rightarrow \infty} \Omega_n, \quad (4.7)$$

we thus obtain a fixed point $\bar{\Omega} = \Upsilon(\bar{\Omega})$, with

$$\|\bar{\Omega} - \Omega_0\|_{\mathbf{L}^2(\mathcal{D})} \leq \|\bar{\Omega} - \Omega_0\|_* \leq r_0. \quad (4.8)$$

While this approach is entirely straightforward, our main concern here is to derive more precise estimates on the various constants, which guarantee that an exact solution actually exists.

Notice that the contraction property, in a norm equivalent to \mathbf{L}^2 , implies that $\|\Upsilon^n(\Omega_0) - \Omega_0\|_{\mathbf{L}^2}$ will be small, for every $n \geq 1$. In turn, recalling (4.6), we conclude that $\|\Phi_n - \Phi_0\|_{H^2}$ also remains small. However, this estimate is not enough to provide an a priori bound on $\|\nabla^\perp \Phi_n\|_{C^1}$, which is needed to estimate the differential $D\Lambda$. For this reason, an additional regularity estimate for the iterates $\Omega_n = \Upsilon^n(\Omega_0)$ will be derived.

4.1 Regularity estimates.

Assuming that the C^1 vector field $\mathbf{q}_0 = \nabla^\perp \Phi_0 + \mathbf{v}$ satisfies the transversality assumptions **(A1)**, we can find $0 < \delta_1 \leq 1$ with the following property.

(P1) *If $\|\Phi - \Phi_0\|_{C^2} \leq \delta_1$, then the vector field $\mathbf{q} = \nabla^\perp \Phi + \mathbf{v}$ still satisfies (2.19)–(2.21), possibly with a smaller constant $c_1 > 0$ and with T^* replaced by $T^* + 1$.*

In particular, all characteristics starting from a point $y \in \Sigma_1$ in the support of h still exit from the domain \mathcal{D} within time $T^* + 1$. Moreover, the right hand side of (3.5) remains uniformly bounded by some constant, which we still denote by \tilde{C} .

Our next goal is to ensure that all our approximations satisfy

$$\|\Phi_n - \Phi_0\|_{C^2} \leq \delta_1 \quad \text{for all } n \geq 1. \quad (4.9)$$

This bound will be achieved by an inductive argument, in several steps.

1. If (4.9) holds, then

$$\|\mathbf{q}_n\|_{C^1} = \|\nabla^\perp \Phi_n + \mathbf{v}\|_{C^1} \leq \|\nabla^\perp \Phi_0 + \mathbf{v}\|_{C^1} + \|\nabla^\perp \Phi_n - \nabla^\perp \Phi_0\|_{C^1} \leq M + 1. \quad (4.10)$$

Assuming (4.9), the solution $\Omega = \Gamma(\Phi_n)$ of (2.5) satisfies

$$\|\Omega\|_{C^0} \leq e^{T^*+1} \|h\|_{C^0}. \quad (4.11)$$

To provide a bound on the gradient $\nabla \Omega$, recalling the notation introduced at (2.15)–(2.17), fix any $x_0 \in \mathcal{D}$ such that $x_0 = \exp(t\mathbf{q}_n)(y)$ for some $y \in \Sigma_1 \cap \text{Supp}(h)$ and $t = \tau(x_0) > 0$. Using the notation $x(t, x_0) \doteq \exp(-t\mathbf{q}_n)(x_0)$, consider the vector

$$\mathbf{w}(t) \doteq \lim_{\varepsilon \rightarrow 0} \frac{x(t, x_0 + \varepsilon \mathbf{e}) - x(t, x_0)}{\varepsilon},$$

where $\mathbf{e} \in \mathbb{R}^2$ is any unit vector. Then

$$\dot{\mathbf{w}}(t) = D\mathbf{q}_n(x(t, x_0)) \cdot \mathbf{w}(t), \quad \mathbf{w}(0) = \mathbf{e}.$$

Hence

$$|\mathbf{w}(t)| \leq \exp\{t\|D\mathbf{q}_n\|_{C^0}\} \leq e^{(T^*+1)(M+1)}. \quad (4.12)$$

By (2.18), the same computations performed at (3.14)–(3.19) now yield

$$|\nabla\Omega(x_0) \cdot \mathbf{e}| \leq e^{T^*+1} \tilde{C} |\mathbf{w}(\tau(x_0))| \leq \tilde{C} e^{(T^*+1)(M+2)}.$$

Since the unit vector \mathbf{e} was arbitrary, this implies that the gradient of $\Omega = \Gamma(\Phi_n)$ satisfies

$$\|\nabla\Omega\|_{C^0} \leq \tilde{C} e^{(T^*+1)(M+2)}. \quad (4.13)$$

2. Recalling (4.1) and the definition of Γ at (2.5)–(2.6), by (4.11) and (4.13) we now obtain

$$\|\Gamma(\Delta^{-1}\Omega_n)\|_{C^1} \leq e^{T^*+1}\|h\|_{C^0} + \tilde{C} e^{(T^*+1)(M+2)}, \quad (4.14)$$

as long as (4.9) holds. In turn, this yields a bound of the form

$$\|\Omega_{n+1}\|_{C^1} = \|(I - AP)^{-1}(\Gamma(\Delta^{-1}\Omega_n) - AP\Omega_n)\|_{C^1} \leq C_1, \quad (4.15)$$

where the constant C_1 can be estimated in terms of (4.14) and the properties of the linear operator A .

3. Since the domain \mathcal{D} has smooth boundary, Schauder's regularity estimates [18, 19] with $\alpha = 1/2$ yield a bound of the form

$$\|\Phi_n - \Phi_0\|_{C^2} = \|\Delta^{-1}(\Omega_n - \Omega_0)\|_{C^2} \leq C_2 \|\Omega_n - \Omega_0\|_{C^{0,1/2}}, \quad (4.16)$$

for some constant C_2 depending only on \mathcal{D} .

Assume we have the inductive estimate

$$\|\Omega_n - \Omega_0\|_{C^1} \leq 2C_1. \quad (4.17)$$

Choosing $0 < \delta < 1$ so that $\delta \leq (2C_1C_2)^{\frac{1}{\alpha-1}} = (2C_1C_2)^{-2}$, we obtain

$$\|\Omega_n - \Omega_0\|_{\alpha,\delta} \leq \delta^{1/2} \|\nabla\Omega_n - \nabla\Omega_0\|_{C^0} \leq 2C_1\delta^{1/2} \leq \frac{1}{C_2}. \quad (4.18)$$

4. Let $\delta_1 > 0$ be the constant introduced in **(P1)**. We now use Lemma 3.2, with $\delta > 0$ as in (4.18) and $c = \delta_1/C_2$. This yields a constant $\delta_c \in]0, \delta_1]$ such that (3.27) implies (3.28)–(3.29). In the present setting, this means that the two inequalities

$$\|\Omega_n - \Omega_0\|_{C^1} \leq 2C_1, \quad \|\Omega_n - \Omega_0\|_{\mathbf{L}^2} \leq \delta_c, \quad (4.19)$$

together imply

$$\|\Omega_n - \Omega_0\|_{C^{0,1/2}} \leq 2C_1\delta^{1/2} \leq \frac{\delta_1}{C_2}, \quad \|\Phi_n - \Phi_0\|_{C^2} \leq \delta_1. \quad (4.20)$$

In other words, if the \mathbf{L}^2 distance between Ω_0 and every Ω_n remains small, then all these approximate solutions have uniformly bounded $C^{0,1/2}$ norm. In turn, property **(P1)** applies.

4.2 Convergence of the approximations.

As long as $\|\Omega - \Omega_0\|_{\mathbf{L}^2} \leq \delta_c$ we have $\|\Phi\|_{\mathcal{C}^2} \leq \delta_1$, and hence

$$\|D\Gamma(\Omega)\|_{\mathbf{L}^2} \leq \tilde{C} \cdot K(M + \delta_1, T^* + 1) \doteq \kappa_0. \quad (4.21)$$

$$\|D\Lambda(\Omega)\|_{\mathbf{L}^2} \leq \|D\Gamma(\Omega)\|_{\mathbf{L}^2} \cdot \|\Delta^{-1}\|_{\mathbf{L}^2} \leq \kappa_0/\lambda_1, \quad (4.22)$$

where $\lambda_1 > 0$ denotes the first eigenvalue of the Laplace operator on \mathcal{D} .

On the other hand, choosing a sufficiently large number N of functions in the orthogonal basis at (2.7), we achieve (2.25). When $(u, v) = (u_0, 0) \in U \times V$, the four blocks in the matrix of partial derivatives (2.27) have norms which can be dominated respectively by the entries of the matrix $\begin{pmatrix} 0 & \kappa_0\varepsilon_0 \\ \kappa_0/\lambda_1 & \kappa_0\varepsilon_0 \end{pmatrix}$. More generally, we can determine $\delta_2 \in]0, \delta_c]$ such that

$$\|\Omega - \Omega_0\|_{\mathbf{L}^2} \leq \delta_2 \implies \|P D_u \Lambda(\Omega) - P D_u \Lambda(\Omega_0)\|_{\mathbf{L}^2} \leq \kappa_0\varepsilon_0. \quad (4.23)$$

Computing the matrix of partial derivatives at (2.27) at such a point $\Omega = (u, v)$, and recalling that $A \doteq P D_u \Lambda(\Omega_0)$, we obtain the relation

$$D(\Lambda(u, v) - Au) = \begin{pmatrix} P D_u \Lambda(\Omega) - P D_u \Lambda(\Omega_0) & P D_v \Lambda \\ (I - P)D_u \Lambda & (I - P)D_v \Lambda \end{pmatrix} \prec \begin{pmatrix} \kappa_0\varepsilon_0 & \kappa_0\varepsilon_0 \\ \kappa_0/\lambda_1 & \kappa_0\varepsilon_0 \end{pmatrix}. \quad (4.24)$$

Here we have used the notation $A \prec B$, meaning that every entry in the 2×2 matrix A has norm bounded by the corresponding entry in the matrix B . Notice that the constant δ_2 in (4.23) involves only the behavior of $\Gamma \circ \Delta^{-1}$ on a neighborhood of Ω_0 in the finite dimensional subspace U , and can be directly estimated.

According to **(A3)**, we now make the key assumption that the constant $\varepsilon_0 > 0$ in (2.25) is small enough so that

$$\sqrt{2} \kappa_0 \left(\varepsilon_0^2 + \frac{\varepsilon_0}{\lambda_1} \right)^{1/2} \leq \frac{1}{2\gamma}. \quad (4.25)$$

This allows us to introduce on $U \times V$ the equivalent norm (4.3), choosing

$$\eta_0 \doteq \lambda_1 \varepsilon_0.$$

Without loss of generality, we can assume that $0 < \eta_0 \leq 1$, so that

$$\|\cdot\|_* \leq \|\cdot\|_{\mathbf{L}^2} \leq \frac{1}{\sqrt{\eta_0}} \|\cdot\|_*. \quad (4.26)$$

In term of this new norm, a direct computation shows, at any point (u, v) where (4.24) holds, the corresponding operator norm satisfies¹

¹Indeed, assume $u, v \in \mathbb{R}$, $u^2 + \eta_0 v^2 \leq 1$. Set $z = \sqrt{\eta_0}v$, so that $u^2 + z^2 \leq 1$. This implies

$$\begin{aligned} (\kappa_0\varepsilon_0 u + \kappa_0\varepsilon_0 v)^2 + \eta_0 \left(\frac{\kappa_0}{\lambda_1} u + \kappa_0\varepsilon_0 v \right)^2 &= \left(\kappa_0\varepsilon_0 u + \frac{\kappa_0\varepsilon_0}{\sqrt{\lambda_1\varepsilon_0}} z \right)^2 + \lambda_1\varepsilon_0 \left(\frac{\kappa_0}{\lambda_1} u + \frac{\kappa_0\varepsilon_0}{\sqrt{\lambda_1\varepsilon_0}} z \right)^2 \\ &= \kappa_0^2 \left[\left(\varepsilon_0 u + \sqrt{\frac{\varepsilon_0}{\lambda_1}} z \right)^2 + \left(\sqrt{\frac{\varepsilon_0}{\lambda_1}} u + \varepsilon_0 z \right)^2 \right] = \kappa_0^2 \left[\varepsilon_0^2 (u^2 + z^2) + \frac{\varepsilon_0}{\lambda_1} (u^2 + z^2) + 4\varepsilon_0 \sqrt{\frac{\varepsilon_0}{\lambda_1}} uz \right] \\ &\leq 2\kappa_0^2 \left(\varepsilon_0^2 + \frac{\varepsilon_0}{\lambda_1} \right) (u^2 + z^2). \end{aligned}$$

$$\left\| D(\Lambda(u, v) - Au) \right\|_* \leq \frac{1}{2\gamma}. \quad (4.27)$$

Hence

$$\|D\Upsilon\|_* \leq \|(I - PA)^{-1}\| \cdot \left\| D(\Lambda(u, v) - Au) \right\|_* \leq \frac{1}{2}. \quad (4.28)$$

By the previous analysis, restricted to the set

$$\mathcal{S} \doteq \left\{ \Omega; \|\Omega - \Omega_0\|_{\mathbf{L}^2} \leq \delta_2, \quad \|\Omega - \Omega_0\|_{C^{0,\alpha}} \leq \frac{\delta_1}{C_2} \right\},$$

the map $\Omega \mapsto \Upsilon(\Omega)$ is a strict contraction, w.r.t. the equivalent norm $\|\cdot\|_*$ introduced at (4.3). Indeed, by (4.28),

$$\Omega, \Omega' \in \mathcal{S} \quad \Longrightarrow \quad \|\Upsilon(\Omega) - \Upsilon(\Omega')\|_* \leq \frac{1}{2} \|\Omega - \Omega'\|_*. \quad (4.29)$$

Recalling that δ_1 is the constant in **(P1)**, we now have the chain of implications

$$\begin{aligned} \|\Omega - \Omega_0\|_* \leq \frac{\delta_2}{\sqrt{\eta_0}} &\quad \Longrightarrow \quad \|\Omega - \Omega_0\|_{\mathbf{L}^2} \leq \delta_2 \\ \Longrightarrow \quad \|\Omega - \Omega_0\|_{C^{0,\alpha}} \leq \frac{\delta_1}{C_2} &\quad \Longrightarrow \quad \|\Delta^{-1}(\Omega - \Omega_0)\|_{C^2} \leq \delta_1. \end{aligned}$$

If the initial guess Ω_0 satisfies

$$\|\Upsilon(\Omega_0) - \Omega_0\|_{\mathbf{L}^2} \leq \frac{\delta_2 \sqrt{\eta_0}}{2}, \quad (4.30)$$

then by (4.26) and (4.29) all the iterates Ω_n in (4.6) will remain inside \mathcal{S} . Namely,

$$\|\Omega_n - \Omega_0\|_{\mathbf{L}^2} \leq \frac{1}{\sqrt{\eta_0}} \|\Omega_n - \Omega_0\|_* \leq \frac{1}{\sqrt{\eta_0}} \cdot 2 \|\Upsilon(\Omega_0) - \Omega_0\|_* \leq \delta_2.$$

Letting $n \rightarrow \infty$ we have the convergence $\Omega_n \rightarrow \bar{\Omega} \in \mathcal{S}$. This limit function $\bar{\Omega}$ provides a solution to the boundary value problem (2.5), (2.3), with

$$\|\bar{\Omega} - \Omega_0\|_{\mathbf{L}^2} \leq \delta_2. \quad (4.31)$$

4.3 An existence theorem.

For readers' convenience, we summarize the previous analysis, recalling the various constants introduced along the way. All these constants can be estimated in terms of the domain \mathcal{D} , the boundary data g, h in (1.14), and the finite dimensional approximation Ω_0 produced by the numerical algorithm.

An entirely similar computation, with the numbers u, v replaced by $\|u\|$ and $\|v\|$ respectively, shows that the corresponding norm of the linear operator (4.24) is bounded by $\sqrt{2} \kappa_0 \left(\varepsilon_0^2 + \frac{\varepsilon_0}{\lambda_1} \right)^{1/2}$.

- \tilde{C} is the constant at (3.5), related to the stability of the ODEs for the characteristics of the linear PDE (3.1). This applies to any vector field $\mathbf{q} = \nabla^\perp \Phi + \mathbf{v}$ with $\|\Phi - \Phi_0\|_{C^2} \leq \delta_1$.
- $K(M, t)$ is the function in (3.6). Together with \tilde{C} , it provides an estimate (3.7) on how the solution to the linear PDE (3.1) varies in the \mathbf{L}^2 distance, depending on the vector field $\nabla^\perp \Phi$.
- C_2 is the Schauder regularity constant for the smooth domain \mathcal{D} , introduced at (4.16).
- $\lambda_1 > 0$ is the lowest eigenvalue of the Laplace operator on \mathcal{D} .
- ε_0 is the small constant in (2.25), determining the rate at which the components in the orthogonal space V are damped by the inverse Laplace operator Δ^{-1} .
- κ_0 is the constant in (4.21), estimating how the solution of the linear PDE (2.5) varies, depending on the vector field $\nabla^\perp \Phi_1$, w.r.t. the \mathbf{L}^2 norm.
- δ_1 is the constant in **(P1)**, measuring by how much we can perturb the vector field $\mathbf{q} = \nabla^\perp \Phi + \mathbf{v}$, and still retain the transversality conditions at the boundary, and a finite exit time.
- δ_c is the constant in the (4.19), providing the regularity estimate (4.20).
- γ is a bound on the norm $\|(I - PD_u \Lambda(\Omega_0))^{-1}\|$ of the inverse Jacobian matrix in (2.24). It gives a measure of stability for the finite dimensional fixed point problem $\Omega = P \circ \Lambda(\Omega)$.
- δ_2 is the constant in (4.23), determining a neighborhood of the initial approximation Ω_0 , where the differential $D\Lambda(\Omega)$ remains close to $D\Lambda(\Omega_0)$.

Recalling the definition of the finite dimensional linear operator A at (2.22) and the equivalent formulation of the boundary value problem (2.4)-(2.5) as a fixed point problem (4.1), we can now summarize all of the previous analysis as follows.

Theorem 4.1 *Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded open domain with smooth boundary, decomposed as $\partial\mathcal{D} = \Sigma_1 \cup \Sigma_2$. Given smooth boundary data g, h , consider the boundary value problem (2.4)-(2.5), where \mathbf{v} is the vector field in (2.2). Assume that the properties **(A1)**-**(A2)** hold, together with **(P1)**.*

Consider the orthogonal decomposition $\mathbf{L}^2(\mathcal{D}) = U \times V$ as in (2.7)-(2.8), and let $\Omega_0 = (u_0, 0) \in U \times V$ be an approximate solution such that

$$\|\Upsilon(\Omega_0) - \Omega_0\|_{\mathbf{L}^2(\mathcal{D})} \leq \frac{\delta_2 \sqrt{\lambda_1 \varepsilon_0}}{2}. \quad (4.32)$$

Then an exact solution $\bar{\Omega}$ exists, with

$$\|\bar{\Omega} - \Omega_0\|_{\mathbf{L}^2(\mathcal{D})} \leq \delta_2. \quad (4.33)$$

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