On Self-similar Solutions to the Incompressible Euler Equations

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Abstract

Recent numerical simulations have shown the existence of multiple self-similar solutions to the Cauchy problem for the 2-dimensional incompressible Euler equation, with initial vorticity in $\mathbf{L}^p_{loc}(\mathbb{R}^2)$, $1 \leq p < +\infty$. Toward a rigorous validation of these computations, in this paper we analytically construct self-similar solutions (i) on an outer domain of the form $\{|x| > R\}$, and (ii) in a neighborhood of the points where the solution exhibits a spiraling vortex singularity. The outer solution is obtained as the fixed point of a contractive transformation, based on the Biot-Savart formula and integration along characteristics. The inner solution is constructed using a system of adapted coordinates, following the approach of V. Elling (*Comm. Math. Phys.*, 2016).

1 Introduction

The classical Euler equations, describing the motion of a non-viscous, homogeneous, incompressible fluid, take the form

$$\begin{cases} u_t + (u\cdot\nabla)u &= -\nabla p\,, \\ \operatorname{div} u &= 0\,. \end{cases} \tag{1.1}$$

Here $u = u(t,x)$ is the fluid velocity while p is the pressure. For a given initial condition

$$u(0,x) = \bar{u}(x)\,, \qquad x \in \mathbb{R}^d, \tag{1.2}$$

the well-posedness of the Cauchy problem, in various functional spaces, has been an outstanding mathematical problem [2, 3, 6, 11, 21, 24]. In recent years, a new approach based on convex integration [10, 13, 14, 15] has led to the construction of a large family of weak solutions, culminating with the proof of the famous Onsager's conjecture [5, 20]. This approach, relying on a Baire category argument, yields infinitely many solutions with the same initial data. Similar results have been obtained also for multi-dimensional inviscid compressible flow [8]. In these constructions, the usual admissibility criteria based on energy or entropy dissipation fail to select a unique solution. Yet, one may still hope that some other admissibility criterium can be found, selecting unique solutions which depend continuously on the initial data.

We believe that this is not the case. Indeed, as soon as the vorticity is only required be in $\mathbf{L}^p_{loc}$, for some $p < +\infty$, the ill-posedness of the Euler equations appears to be "incurable". To support this claim, in the present paper we study a simple example of 2-dimensional incompressible Euler flow where numerical simulations show the existence of two distinct solutions for the same initial data. In polar coordinates, $x = (x_1, x_2) = (r\cos\theta,\, r\sin\theta)$, the initial vorticity $\overline{\omega} = \text{curl}\ \bar{u}$ has the form

$$\overline{\omega}(x) \;=\; r^{-\frac{1}{\mu}}\,\overline{\Omega}(\theta), \tag{1.3}$$

where $\frac{1}{2} < \mu < +\infty$. As shown in Figure 1, left, we assume that $\overline{\Omega} \in \mathcal{C}^\infty(\mathbb{R})$ is a non-negative, smooth, periodic function which satisfies

$$\overline{\Omega}(\theta) \;=\; \overline{\Omega}(\pi + \theta), \qquad\qquad \overline{\Omega}(\theta) \;=\; 0 \quad \text{if} \;\; \theta \in \left[\frac{\pi}{4}\,,\,\pi\right]. \tag{1.4}$$

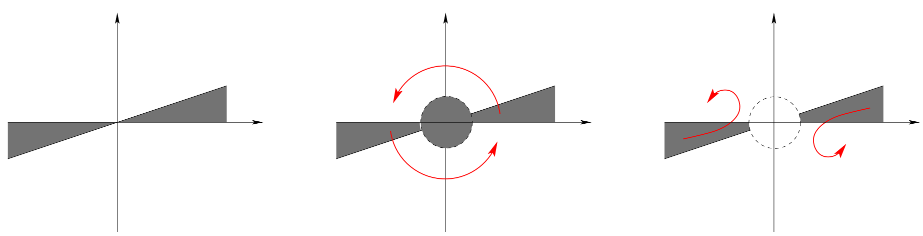

Figure 1: The supports of the initial vorticity $\overline{\omega}$, $\overline{\omega}_1^\varepsilon$, and $\overline{\omega}_2^\varepsilon$, considered at (1.3)-(1.6).

We then consider two sequences of initial data, with vorticity $\overline{\omega}_1^\varepsilon, \overline{\omega}_2^\varepsilon \in \mathbf{L}^\infty(\mathbb{R}^2)$, as shown in Figure 1, center and right. Namely

$$\overline{\omega}_1^\varepsilon(x) \;\doteq\; \begin{cases} \overline{\omega}(x) & \text{if} \quad |x| > \varepsilon, \\[2mm] \varepsilon^{-\frac{1}{\mu}} & \text{if} \quad |x| \leq \varepsilon, \end{cases} \tag{1.5}$$

$$\overline{\omega}_2^\varepsilon(x) \;\doteq\; \begin{cases} \overline{\omega}(x) & \text{if} \quad |x| > \varepsilon, \\[2mm] 0 & \text{if} \quad |x| \leq \varepsilon. \end{cases} \tag{1.6}$$

By Yudovich' theorem [24], for every $\varepsilon > 0$ the Cauchy problem for the incompressible Euler equation (1.1) with initial data (1.5) or (1.6) has a unique solution. Numerical simulations of these solutions, performed by Wen Shen [23], are shown in Figures 2 and 3, respectively. As $\varepsilon \to 0$, the initial data $\overline{\omega}_1^\varepsilon, \overline{\omega}_2^\varepsilon$ converge to the same limit $\overline{\omega}$ in $\mathbf{L}^p_{loc}$. However, at times $t > 0$ the corresponding solutions converge to two different limits. Both of these limits yield solutions to the Euler equations (1.1) with the same initial data (1.3).

At an intuitive level, this behavior is easy to explain. For the initial data $\overline{\omega}$ in (1.3)-(1.4), the vorticity is supported inside two wedges. In the approximations (1.5), the presence of a dense

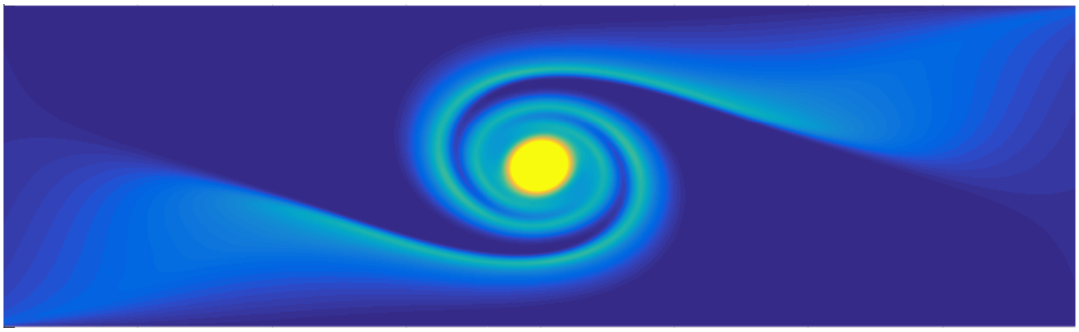

Figure 2: The vorticity distribution at time $t = 1$, for a solution to (1.1) with initial vorticity (1.5).

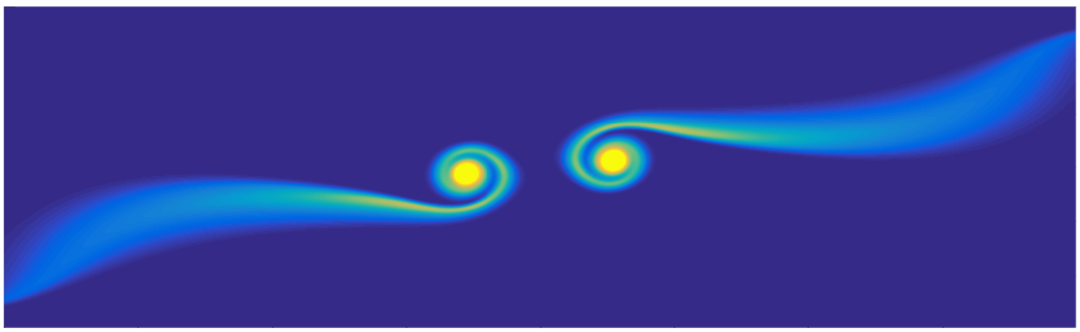

Figure 3: The vorticity distribution at time $t = 1$, for a solution to (1.1) with initial vorticity (1.6).

patch of vorticity around the origin stirs these two wedges together, forming a single spiraling vortex (Fig. 2). On the other hand, in the approximations (1.6) each of the two wedges curls up on itself, and two separate vortex spirals are generated (Fig. 3).

The above simulations provide numerical evidence of the existence of multiple self-similar solutions to the Cauchy problem (1.1)–(1.3). However, proving the existence of two distinct exact solutions, close to the computed ones, is a nontrivial task. Toward this goal, the main difficulties to overcome are:

(i) The exact solution is a function defined on the whole plane $\mathbb{R}^2$, while a numerical approximation is computed only on a bounded domain $\mathcal{D}$.

(ii) The exact solution is not smooth, but contains a singularity at the center of each spiral. In a neighborhood of such point, a-posteriori error estimates will break down.

We thus propose an approach based on a domain decomposition:

$$\mathbb{R}^2 \;=\; \mathcal{D}^o \cup \mathcal{D}^m \cup \mathcal{D}^i. \tag{1.7}$$

As shown in Fig. 4, here $\mathcal{D}^o$ is an outer domain, $\mathcal{D}^m$ is an intermediate, bounded domain where the solution remains smooth, and $\mathcal{D}^i$ is an inner domain consisting of one or two discs around the spirals' centers.

In the present paper we focus on the analytical construction of solutions on the two domains $\mathcal{D}^o$ and $\mathcal{D}^i$. Once this is accomplished, the existence of a globally defined solution is reduced to a problem that can be resolved by a numerical computation. Indeed, if a numerical solution is available on $\mathcal{D}^m$, which matches sufficiently well with the analytical solutions on the inner and outer domains, then a version of the Newton-Kantorovich theorem [1, 12] will yield the existence of an exact solution defined on the entire plane $\mathbb{R}^2$. We remark that our solutions are $\mathcal{C}^\infty$ on the intermediate domain $\mathcal{D}^m$. The analysis of a numerical algorithm, together with posteriori error estimates on an intermediate domain, has been carried out in [4]. Appropriate matching conditions will be studied in a future paper.

The remainder of this paper is organized as follows. Section 2 reviews the vorticity formulation of the Euler equations, deriving the system (2.9) satisfied by self-similar solutions. In Section 3 we construct self-similar solutions on an outer domain $\mathcal{D}^o = \{x \in \mathbb{R}^2\,;\ |x| > R\}$, with given asymptotic conditions as $|x| \to +\infty$. The existence and uniqueness of these solutions is established in Theorem 3.1.

The remaining sections carry out the construction of solutions in a small neighborhood of a spiral's center. Here the analysis heavily relies on the ground-breaking work of Volker Elling [16, 17, 18]. Self-similar solutions are obtained within an adapted system of coordinates which follow the pseudo-streamlines. As shown in Elling's papers, this approach is successful as long as our solutions remain sufficiently close to radially symmetric ones. Section 4 describes the construction of adapted coordinates, where the characteristic curves for the linear, first order PDE in (2.9) are taken as coordinate lines. Sections 5 and 6 analyze radially symmetric solutions and the linearized system of equations satisfied by a small perturbation of such solutions. In terms of a Fourier series, these take the form $Y(\beta,\phi) = \sum_k Y_k(\beta)e^{ik\phi}$.

We point out the major difference between our analysis and the results in [16, 17]. To construct a globally defined solution of (2.9) in terms adapted coordinates, one needs to assume that such solution is close to radially symmetric on the entire space $\mathbb{R}^2$. In [16, 17], this was guaranteed by the assumption that all Fourier components of the vorticity $Y_k(\beta)$ vanish, for $0 < |k| < N$, with N large enough.

On the other hand, our construction allows all components of the vorticity to be nonzero, including $k = \pm 1$ and $k = \pm 2$. In our case, however, the solution is constructed not on the entire space $\mathbb{R}^2$ but only in a neighborhood of the spiral's center. We remark that the analysis of the Fourier components $|k| = 2$ and $|k| = 1$ is essential in order to understand the behavior of the solutions shown respectively in Fig 2 and in Fig. 3, near the spirals' centers. Additionally, the boundary data for the stream function Ψ is allowed to be any function which is close enough to the radially symmetric one, with the restriction that one cannot assign the $k = \pm 1$ Fourier coefficients. This restriction reflects the fact that we need to construct solutions on a ball that is centered precisely at the singularity of the vortex, as being off center will yield non-zero coefficients for Ψ on the $k = \pm 1$ modes.

Section 7 contains the analysis of the nonlinear problem, in adapted coordinates. The main result, stated in Theorem 7.1, yields the existence of a solution operator, mapping the boundary data into the corresponding solution, on a suitably small neighborhood of the origin.

For solutions as in Figures 2, 3, we expect that the stream function will become close to radially symmetric, as a spiral's center is approached. On the other hand, the vorticity only satisfies a linear transport equation, which does not provide any smoothing. Hence, if it is not close to radially symmetric as $|x| \to \infty$, the vorticity will never approach radial symmetry. To address this issue, Proposition 7.3 gives a further refinement of Theorem 7.1. Here the existence of a solution is proved also for large vorticity, far from radially symmetric, but on a suitably small neighborhood of the spiral's center.

Finally, in Section 8 we analyze how the solution on the inner domain varies, depending on the boundary data. This is a basic step, toward the eventual goal of matching the solution on the inner domain $\mathcal{D}^i$ with a solution computed on the intermediate domain $\mathcal{D}^m$.

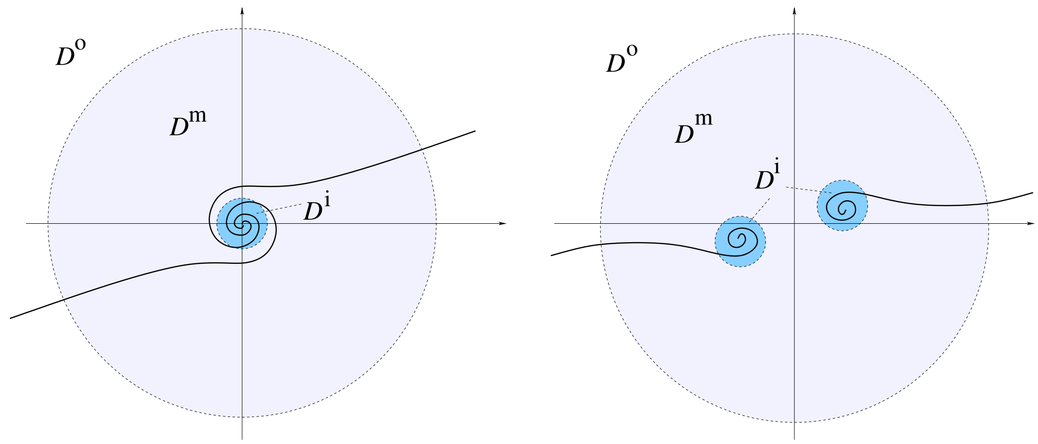

Figure 4: The decomposition (1.7) of the plane into an outer, a middle, and an inner domain. Left: the case of the single spiraling vortices, as in Fig. 2. Right: the case of two spiraling vortices, as in Fig. 3.

2 Self-similar solutions

Let $u = u(t, x)$ be a solution to the Euler equations (1.1). Calling $\omega = \text{curl } u = (-u_{1,x_2} + u_{2,x_1})$ and taking the curl of both sides of (1.1) one obtains the linear transport equation

$$\omega_t + u \cdot \nabla \omega \;=\; 0. \tag{2.1}$$

We recall that the velocity u can be recovered from the vorticity by the Biot-Savart formula

$$u(x) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(y)\, dy. \tag{2.2}$$

The assumption that div $u = 0$ implies the existence of a stream function ψ such that

$$u \;=\; \nabla^\perp \psi, \qquad\qquad (u_1, u_2) \;=\; (-\psi_{x_2},\, \psi_{x_1}). \tag{2.3}$$

The Euler equations (1.1) can be reformulated as follows: find two scalar functions ω, ψ defined for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$, such that

$$\begin{cases} \omega_t + \nabla^\perp \psi \cdot \nabla \omega \;=\; 0, \\[2ex] \qquad\qquad \Delta \psi \;=\; \omega. \end{cases} \tag{2.4}$$

We seek solutions which are self-similar, so that

$$u(t, x) \;=\; t^\alpha U\left(\frac{x}{t^\mu}\right), \qquad p(t, x) \;=\; t^\beta P\left(\frac{x}{t^\mu}\right), \tag{2.5}$$

for some exponents α, β, μ. Notice that, taking $t = 1$, one finds

$$U(x) \;=\; u(1,x), \qquad P(x) \;=\; p(1,x). \tag{2.6}$$

Therefore, the value of the solution at $t = 1$ determines its values at all positive times.

Inserting (2.5) into (1.1) one obtains

$$\alpha t^{\alpha-1} U - \mu t^{\alpha-1} \frac{x}{t^\mu} \cdot \nabla U + t^\alpha U \cdot t^{\alpha-\mu} \nabla U \;=\; -\, t^{\beta-\mu} \nabla P. \tag{2.7}$$

This forces $\alpha - 1 = 2\alpha - \mu = \beta - \mu$, hence $\alpha = \mu - 1$ and $\beta = 2\alpha$. Setting

$$y \;=\; \frac{x}{t^\mu}$$

for some constant μ, these self-similar solutions will be described using the notation

$$\begin{cases} u(t,x) \;=\; t^{\mu-1} U\left(\frac{x}{t^\mu}\right), \\ \omega(t,x) \;=\; t^{-1} \Omega\left(\frac{x}{t^\mu}\right), \\ \psi(t,x) \;=\; t^{2\mu-1} \Psi\left(\frac{x}{t^\mu}\right). \end{cases} \tag{2.8}$$

Inserting (2.8) in (2.3)-(2.4), one finds

$$t^{\mu-1} U(y) \;=\; (u_1, u_2)(t,x) \;=\; (-\psi_{x_2}, \psi_{x_1})(t,x) \;=\; t^{2\mu-1}(-t^{-\mu}\Psi_{y_2},\; t^{-\mu}\Psi_{y_1}) \;=\; t^{\mu-1} \nabla_y^\perp \Psi(y),$$

$$\Omega \;=\; t\,\omega \;=\; t\,\Delta_x \psi \;=\; \Delta_y \Psi,$$

$$0 \;=\; \omega_t + \nabla_x^\perp \psi \cdot \nabla_x \omega \;=\; -\,t^{-2}\Omega(y) - t^{-2}\mu \frac{x}{t^\mu} \cdot \nabla_y \Omega(y) + t^{-2}\left(-\Psi_{y_2}\Omega_{y_1} + \Psi_{y_1}\Omega_{y_2}\right).$$

This yields the equations

$$\begin{cases} \Big(\nabla^\perp \Psi - \mu y\Big) \cdot \nabla \Omega \;=\; \Omega\,, \\ \Delta \Psi \;=\; \Omega\,, \end{cases} \tag{2.9}$$

while the velocity is recovered by

$$U \;=\; \nabla^\perp \Psi. \tag{2.10}$$

The vector field

$$\mathbf{q}(y) \;\doteq\; U(y) - \mu y \tag{2.11}$$

is called the *pseudo-velocity*. Its integral curves are the *pseudo-streamlines*.

Notice that the system (2.9) is equivalent to the third order PDE for the stream function

$$\Big(\nabla^\perp \Psi - \mu y\Big) \cdot \nabla(\Delta \Psi) \;=\; \Delta \Psi\,. \tag{2.12}$$

Throughout this paper, we assume that the rescaling parameter μ satisfies

$$\mu \;>\; \frac{1}{2}, \qquad \mu, 2\mu \;\notin\; \mathbb{N}. \tag{2.13}$$

To construct solutions of (2.9), one can proceed as follows.

- Given Ψ, solve the first equation for Ω. This is a linear, first order equation that can be solved explicitly, integrating along characteristics, with suitable asymptotic conditions at $|y| \to \infty$.
- Insert Ω in the second equation, solve for Ψ.

This will yield a map $\Psi \mapsto \Omega \mapsto \Psi$. Working in a suitable function space, one needs to find a fixed point of this map, which will in turn provide a self-similar solution of (1.1).

Our eventual goal is to construct solutions of (2.9) on the entire plane $\mathbb{R}^2$ with prescribed asymptotic behavior as $|y| \to +\infty$. Our solutions will be everywhere smooth, except for one or two points where a spiraling singularity occurs. Solutions will be constructed separately on three domains:

- An outer domain, of the form $\mathcal{D}^o = \{x \in \mathbb{R}^2\,;\ \ |x| > R\}$, for some large radius R. Here the fluid velocity will be small, and the solution will be obtained as a small perturbation of the asymptotic data.
- A bounded intermediate domain $\mathcal{D}^m$, where the solution is still smooth and can be accurately computed by numerical approximations.
- An inner domain $\mathcal{D}^i$, consisting of one or two small discs centered at points where the spiraling singularities occur. On these domains, the solution will be constructed analytically, using the adapted coordinates introduced in [16, 17].

By suitably patching together these three solutions defined on $\mathcal{D}^o, \mathcal{D}^m, \mathcal{D}^i$, a globally defined solution of (2.9) can be obtained. The present paper will focus on the construction of solutions on the outer domain $\mathcal{D}^o$ and on the inner domain $\mathcal{D}^i$.

3 Solutions on an outer domain

In this section we construct solutions to the system

$$\left\{ \begin{array}{rcl} \Big(\nabla^\perp\Psi - \mu x\Big)\cdot\nabla\Omega &=& \Omega\,, \\ \Delta\Psi &=& \Omega\,, \end{array}\right. \tag{3.1}$$

on an outer domain of the form

$$\mathcal{D}_R \doteq \{x \in \mathbb{R}^2\,;\ \ |x| \geq R\}. \tag{3.2}$$

We recall that the velocity is recovered by $U(x) = \nabla^\perp\Psi(x)$. We seek a solution (Ω, Ψ) of (3.1) with suitable conditions given on the circumference $\{|x| = R\}$, and asymptotic values as $|x| \to +\infty$. It will be convenient to use the polar coordinate notation

$$x(r,\theta) \doteq (r\cos\theta,\, r\sin\theta) \in \mathbb{R}^2, \qquad r = |x|, \quad \theta = \measuredangle x, \tag{3.3}$$

$$\Omega(r,\theta) \doteq \Omega(r\cos\theta,\, r\sin\theta), \qquad \Psi(r,\theta) \doteq \Psi(r\cos\theta,\, r\sin\theta).$$

Moreover, we denote by $\mathbb{T}$ the unit circumference. Namely: $\mathbb{T} = [0, 2\pi]$ with endpoints identified. Any map $\phi : \mathbb{T} \mapsto \mathbb{R}$ can thus be identified with a 2π-periodic function of a real variable.

Given a function $\overline{\omega} : \mathbb{T} \mapsto \mathbb{R}$, we impose the asymptotic condition

$$\lim_{r\to+\infty} r^{\frac{1}{\mu}}\, \Omega(r,\theta) \;\doteq\; \overline{\omega}(\theta), \tag{3.4}$$

Together with $\overline{\omega}$ we also consider the 2π-periodic function $\overline{\psi} : \mathbb{T} \mapsto \mathbb{R}$, defined as the solution to the linear second order ODE

$$\overline{\psi}_{\theta\theta}(\theta) + \left(2 - \frac{1}{\mu}\right)^2 \overline{\psi}(\theta) \;=\; \overline{\omega}(\theta). \tag{3.5}$$

Notice that the homogeneous equation

$$\psi'' + \lambda^2\psi \;=\; 0$$

has nonzero 2π-periodic solutions only if λ is an integer. Since we always assume that μ satisfies (2.13), this guarantees that (3.5) has a unique 2π-periodic solution.

Remark 3.1 If λ is not an integer, the unique periodic solution to

$$\psi''(\theta) + \lambda^2\psi(\theta) \;=\; f(\theta)$$

is given by

$$\psi(\theta) \;=\; \frac{1}{2\lambda\sin\lambda\pi}\int_{-\pi}^{\pi} f(\theta+\pi-\alpha)\,\cos(\lambda\alpha)\,d\alpha. \tag{3.6}$$

Notice that for this periodic problem the Green's function has constant sign if $0 < \lambda < 1$, but changes sign if $\lambda > 1$.

As $r \to +\infty$, on the stream function Ψ we impose the asymptotic condition

$$\lim_{r\to+\infty} r^{\frac{1}{\mu}-2}\, \Psi(r,\theta) \;\doteq\; \overline{\psi}(\theta). \tag{3.7}$$

Remark 3.2 To motivate (3.7), consider two functions of the form

$$\Omega(r,\theta) \;=\; r^{-\frac{1}{\mu}}\,\overline{\omega}(\theta), \qquad\qquad \Psi(r,\theta) \;=\; r^{2-\frac{1}{\mu}}\,\overline{\psi}(\theta). \tag{3.8}$$

Assume that

$$\Omega \;=\; \Delta\Psi \;=\; \Psi_{rr} + \frac{\Psi_r}{r} + \frac{\Psi_{\theta\theta}}{r^2}\,. \tag{3.9}$$

Inserting (3.8) into (3.9) one obtains

$$r^{-\frac{1}{\mu}}\,\overline{\omega}(\theta) \;=\; \left(2-\frac{1}{\mu}\right)\left(1-\frac{1}{\mu}\right) r^{-\frac{1}{\mu}}\overline{\psi}(\theta) + \left(2-\frac{1}{\mu}\right) r^{-\frac{1}{\mu}}\overline{\psi}(\theta) + r^{-\frac{1}{\mu}}\overline{\psi}_{\theta\theta}(\theta). \tag{3.10}$$

Dividing both sides by $r^{-1/\mu}$ we obtain (3.5).

Finally, along the inner boundary $S_R \doteq \{x \in \mathbb{R}^2\,;\; |x| = R\}$, we impose a Dirichlet condition

$$\Psi(R,\theta) \;=\; \psi_0(\theta), \tag{3.11}$$

for some given function $\psi_0 : \mathbb{T} \mapsto \mathbb{R}$. Notice that no boundary value is imposed on $\Omega(R,\theta)$.

3.1 Rescaling of coordinates.

It will be convenient to rescale coordinates, introducing the variables

$$z \;=\; \frac{x}{R}\,,\qquad \widetilde{\Omega}(z) \;=\; \Omega(Rz),\qquad \widetilde{\Psi}(z) \;=\; R^{-2}\Psi(Rz). \tag{3.12}$$

By the relations

$$\nabla_z\widetilde{\Omega}(z) \;=\; R\,\nabla_x\Omega(x),\qquad \nabla_z\widetilde{\Psi}(z) \;=\; R^{-1}\,\nabla_x\Psi(x),\qquad \Delta_z\widetilde{\Psi}(z) \;=\; \Delta_x\Psi(x),$$

in these rescaled variables the equations (3.1) take exactly the same form, namely

$$\begin{cases} \Big(R\nabla_z^{\perp}\widetilde{\Psi} - \mu Rz\Big)\cdot R^{-1}\nabla_z\widetilde{\Omega} &= \widetilde{\Omega}\,,\\ \Delta_z\widetilde{\Psi} &= \widetilde{\Omega}\,. \end{cases} \tag{3.13}$$

However, we now have to solve (3.13) on the outer domain

$$\mathcal{D}_1 = \{z\in\mathbb{R}^2\,;\ \ |z|\geq 1\}\,. \tag{3.14}$$

Moreover, the asymptotic conditions (3.4), (3.7) and the boundary condition (3.11) now become

$$\begin{cases} \displaystyle\lim_{r\to+\infty} r^{\frac{1}{\mu}}\widetilde{\Omega}(y(r,\theta)) &= R^{-\frac{1}{\mu}}\overline{\omega}(\theta),\\ \displaystyle\lim_{r\to+\infty} r^{\frac{1}{\mu}-2}\,\widetilde{\Psi}(y(r,\theta)) &= R^{-\frac{1}{\mu}}\overline{\psi}(\theta),\\ \widetilde{\Psi}(y(1,\theta)) &= R^{-2}\psi_0(\theta). \end{cases} \tag{3.15}$$

In other words, solving the original system on the exterior domain $\mathcal{D}_R$ with R large is equivalent to solving it on $\mathcal{D}_1$, but with smaller boundary and asymptotic data.

Throughout the sequel, we shall thus seek a solution to the system (3.1) on the outer domain $\mathcal{D}_1$, with asymptotic and boundary conditions

$$\begin{cases} \displaystyle\lim_{r\to+\infty} r^{\frac{1}{\mu}}\,\Omega(r,\theta) &= \overline{\omega}(\theta),\\ \displaystyle\lim_{r\to+\infty} r^{\frac{1}{\mu}-2}\,\Psi(r,\theta) &= \overline{\psi}(\theta),\\ \Psi(1,\theta) &= \psi_0(\theta), \end{cases} \tag{3.16}$$

assuming that (3.5) holds and that the norms $\|\overline{\omega}\|_{\mathcal{C}^2(\mathbb{T})}$, $\|\psi_0\|_{\mathcal{C}^2(\mathbb{T})}$ are sufficiently small. Our analysis has two main goals:

(i) Prove the existence and uniqueness of a solution to the problem (3.1), (3.16), for all sufficiently small boundary and and asymptotic data.

(ii) Given a radius $r_1>1$, study the differential of the map

$$\Lambda_1 : \psi_0 \;\mapsto\; (\omega_1,\psi_1), \tag{3.17}$$

where $\omega_1,\psi_1:\mathbb{T}\mapsto\mathbb{R}$ denote the restrictions of the functions Ω,Ψ to the circumference $\partial B_{r_1} \doteq \{x\in\mathbb{R}^2\,;\ |x|=r_1\}$.

3.2 Statement of the main result.

We always assume that the rescaling parameter μ satisfies (2.13). Moreover, we let $N_0, N_1 \in \mathbb{N}$ be the smallest integers such that

$$\frac{N_0+3}{\mu} > 2, \qquad\qquad 1+\frac{N_1+3}{\mu} > 2\,, \tag{3.18}$$

and define $\epsilon > 0$ by the identity

$$2+\epsilon \doteq \min\left\{\frac{N_0+3}{\mu}\,,\ +\frac{N_1+3}{\mu}\right\}. \tag{3.19}$$

Theorem 3.1 *Assume that μ satisfies (2.13) and let the integers $N_0, N_1 \geq 1$ be as in (3.18). Consider any functions $\overline{\omega}, \overline{\psi}, \psi_0 : \mathbb{T} \mapsto \mathbb{R}$, with $\overline{\psi}$ satisfying (3.5). If the norms*

$$\|\overline{\omega}\|_{\mathcal{C}^{N_0+N_1}(\mathbb{T})}\,, \qquad\qquad \|\psi_0\|_{\mathcal{C}^2(\mathbb{T})}\,,$$

are sufficiently small, then the problem (3.1) on the exterior of the unit disc, with boundary and asymptotic conditions (3.16), has a unique solution (Ω, Ψ). Here $\Omega \in \mathcal{C}^1(\mathcal{D}_1)$ and $\Psi \in \mathcal{C}^2(\mathcal{D}_1)$.

A proof will be worked out in the remainder of this section.

3.3 Review of the Poisson equation on an outer domain.

A Green's function for the Laplace operator on the outer domain $\mathcal{D}_1 = \{x \in \mathbb{R}^2\,;\ |x| > 1\}$ is

$$G(x,y) = \frac{1}{2\pi}\ln\left(\frac{1}{|y|}\cdot\frac{|y-x|}{|y^*-x|}\right), \qquad\qquad y^* \doteq \frac{y}{|y|^2}\,. \tag{3.20}$$

This provides a solution to

$$\Delta_x G(x,y) = \delta_y\,, \qquad\qquad G(x,y) = 0 \quad \text{for} \quad |x| = 1,$$

where δ_y denotes the Dirac measure concentrating a unit mass at the point y. Assuming that g is continuous while f satisfies the bound

$$f(x) = \mathcal{O}(1)\cdot|x|^{-2-\frac{1}{\mu}}, \tag{3.21}$$

the solution to the boundary value problem

$$\begin{cases} \Delta u(x) = f(x) & |x| > 1, \\ u(x) = g(x) & |x| = 1, \end{cases} \tag{3.22}$$

can be written as

$$\begin{aligned} u(x) &= \int_{|y|>1} G(x,y)\,f(y)\,dy + \int_{|y|=1} \nabla_y G(x,y)\cdot\mathbf{n}(y)\,g(y)\,d\sigma \\ &= \frac{1}{2\pi}\int_{|y|>1}\ln\left(\frac{1}{|y|}\cdot\frac{|y-x|}{|y^*-x|}\right)f(y)\,dy + \frac{|x|^2-1}{2\pi}\int_{|y|=1}\frac{g(y)}{|y-x|^2}\,d\sigma. \end{aligned} \tag{3.23}$$

Here $\mathbf{n}(y) = -y$ is the unit outer normal to the domain $\mathcal{D}_1$ at the boundary point $y \in \partial\mathcal{D}_1$. Moreover, $d\sigma$ denotes integration w.r.t. arc-length, along the unit circumference.

Observe that, for every fixed x, the first integral on the right hand side of (3.23) is absolutely convergent provided that

$$f(y) \;=\; \mathcal{O}(1)\cdot|y|^{-\alpha} \qquad \text{for some } \alpha > 2. \tag{3.24}$$

Lemma 3.1 *Assume that $g \in \mathcal{C}^1(\mathbb{T})$ and (3.24) holds. Then the integrals in (3.23) are absolutely convergent. Moreover, there exist a constant C_0 independent of f, g such that*

$$|\nabla u(x)| \;=\; C_0\,|x|^{-1}\cdot\Big(\big\|\,|x|^\alpha f\big\|_{\mathbf{L}^\infty} + \|g\|_{\mathcal{C}^1}\Big). \tag{3.25}$$

Proof. 1. If (3.24) holds, differentiating w.r.t. x, one obtains

$$\begin{aligned}\nabla u(x) \;&=\; \frac{1}{2\pi}\int_{|y|>1}\left(\frac{x-y}{|x-y|^2} - \frac{x-y^*}{|x-y^*|^2}\right) f(y)\,dy \\ &\quad + \frac{2x}{2\pi}\int_{|y|=1}\frac{g(y)}{|x-y|^2}\,d\sigma - \frac{|x|^2-1}{2\pi}\int_{|y|=1}\frac{x-y}{|x-y|^4}\,g(y)\,d\sigma.\end{aligned} \tag{3.26}$$

We observe that

$$\begin{aligned}\int_{|y|>1}\frac{|y|^{-\alpha}}{|y-x|}\,dy \;&=\; \left(\int_{|y|>1,\,|y-x|<\frac{|x|}{2}} + \int_{|y|>1,\,|y-x|>\frac{|x|}{2}}\right)\frac{|y|^{-\alpha}}{|y-x|}\,dy \\ &\leq\; \left|\frac{x}{2}\right|^{-\alpha}\int_{|z|\leq\frac{|x|}{2}}\frac{1}{|z|}\,dz + \frac{2}{|x|}\int_{|y|>1}|y|^{-\alpha}\,dy \\ &=\; 2^\alpha\pi\cdot|x|^{1-\alpha} + \frac{4\pi}{\alpha-2}|x|^{-1}.\end{aligned}$$

Since we are assuming $\alpha > 2$, the first integral in (3.26) has size $\mathcal{O}(1)\cdot|x|^{-1}\big\|\,|x|^\alpha f\big\|_{\mathbf{L}^\infty}$.

2. We now consider the second and third integrals in (3.26). For $|x| \geq 2$, these integrals have size $\mathcal{O}(1)\cdot|x|^{-1}\|g\|_{\mathcal{C}^0}$. In the case $1 < |x| < 2$, using polar coordinates we write the second integral in (3.23) as

$$v(r,\theta) \;=\; \frac{r^2-1}{2\pi}\int_0^{2\pi}\frac{g(\theta+\alpha)}{|r-\cos\alpha|^2 + |\sin\alpha|^2}\,d\alpha.$$

Observing that $v_r(r,\theta)$ and $v_\theta(r,\theta)$ both have size $\mathcal{O}(1)\cdot\|g\|_{\mathcal{C}^1}$, we obtain the bound $\nabla v(x) = \mathcal{O}(1)\cdot|x|^{-1}\|g\|_{\mathcal{C}^1}$, also for $1 < |x| < 2$. □

3.4 A linear, first order PDE.

Motivated by (3.1), given a vector field $\mathbf{v} = (v_1, v_2)$, in this subsection we study the more general equation

$$(\mathbf{v}(x) - \mu x)\cdot\nabla u \;=\; u. \tag{3.27}$$

We seek a solution to (3.27) on the outer domain $\mathcal{D}_1 = \{x \in \mathbb{R}^2\,;\ |x| \geq 1\}$, with asymptotic behavior given by

$$\lim_{r\to+\infty} r^{1/\mu} u(r,\theta) \;=\; \overline{\omega}(\theta). \tag{3.28}$$

The linear, first order equation (3.27) can be solved by the classical method of characteristics [19, 22]. This yields the system

$$\begin{cases} \dot{x} &= \mathbf{v}(x) - \mu x\,, \\ \dot{U} &= U\,. \end{cases} \tag{3.29}$$

Throughout the following, we shall assume

$$|\mathbf{v}(x)| \;\leq\; \varepsilon_1 |x|^{1-\frac{1}{\mu}}, \qquad\qquad |\nabla \mathbf{v}(x)| \;\leq\; \varepsilon_1 |x|^{-\frac{1}{\mu}}, \tag{3.30}$$

for some $\varepsilon_1 > 0$ sufficiently small. It will be convenient to rewrite the above equations in polar coordinates $x = (r\cos\theta, r\sin\theta)$, using r as independent variable. At any point x, denote by $a(x), b(x)$ the components of $\mathbf{v}(x)$ in the radial and angular direction. More precisely introducing the unit vector $\mathbf{e} \doteq x/|x| = (\cos\theta, \sin\theta)$, let

$$\mathbf{v}(x) \;=\; a(x)\mathbf{e} + b(x)\mathbf{e}^{\perp}. \tag{3.31}$$

Using polar coordinates, from (3.29) one obtains

$$\begin{cases} \dot{r} &= a(r,\theta) - \mu r, \\ \dot{\theta} &= b(r,\theta)\, r^{-1}, \\ \dot{U} &= U, \end{cases} \qquad\qquad \begin{cases} \dfrac{d\theta}{dr} &= -\,\dfrac{b(r,\theta) r^{-1}}{\mu r - a(r,\theta)}, \\ \dfrac{dU}{dr} &= -\,\dfrac{U}{\mu r - a(r,\theta)}. \end{cases} \tag{3.32}$$

The last two equations in (3.32) can be written in the equivalent form

$$\begin{cases} \dfrac{d\theta}{dr} &= -\, r^{-1-\frac{1}{\mu}} A(r,\theta), \\ \dfrac{dU}{dr} &= -\,\dfrac{U}{\mu r}\Big(1 + r^{-1/\mu} B(r,\theta)\Big)\,, \end{cases} \tag{3.33}$$

where we defined

$$A(r,\theta) \;\doteq\; r^{-1+\frac{1}{\mu}} \frac{b(r,\theta)}{\mu - r^{-1} a(r,\theta)}\,, \qquad\qquad B(r,\theta) \;=\; r^{-1+\frac{1}{\mu}} \frac{a(r,\theta)}{\mu - r^{-1} a(r,\theta)}\,. \tag{3.34}$$

Notice that the bounds in (3.30) imply

$$\begin{cases} A(r,\theta) &= \mathcal{O}(1)\cdot\varepsilon_1\,, \\ B(r,\theta) &= \mathcal{O}(1)\cdot\varepsilon_1\,, \end{cases} \qquad \begin{cases} A_r(r,\theta) &= \mathcal{O}(1)\cdot\varepsilon_1\, r^{-1}, \\ B_r(r,\theta) &= \mathcal{O}(1)\cdot\varepsilon_1\, r^{-1}, \end{cases} \qquad \begin{cases} A_\theta(r,\theta) &= \mathcal{O}(1)\cdot\varepsilon_1, \\ B_\theta(r,\theta) &= \mathcal{O}(1)\cdot\varepsilon_1. \end{cases} \tag{3.35}$$

Fix an angle $\Theta \in [0, 2\pi]$. Given the asymptotic data

$$\lim_{r\to+\infty} \theta(r) \;=\; \Theta\,, \qquad \lim_{r\to+\infty} r^{\frac{1}{\mu}} U(r) \;=\; \overline{\omega}(\Theta), \tag{3.36}$$

the solution to (3.33) can be obtained as the fixed point of a contractive map. Indeed, given $\theta \in \mathcal{C}^0([1,+\infty[)$, consider the Picard operator

$$(\mathcal{P}\theta)(r) \doteq \Theta + \int_r^{+\infty} s^{-1-\frac{1}{\mu}} A(s,\theta(s))\,ds\,. \tag{3.37}$$

Consider any two functions $\theta, \tilde{\theta} \in \mathcal{C}^0([1,+\infty[)$ and set $\delta \doteq \|\theta - \tilde{\theta}\|_{\mathcal{C}^0}$. Using the bound on A_θ given in (3.35), for every $r \geq 1$ we obtain

$$\big|(\mathcal{P}\theta)(r) - (\mathcal{P}\tilde{\theta})(r)\big| = \mathcal{O}(1)\cdot\varepsilon_1 \int_r^{+\infty} s^{-1-\frac{1}{\mu}}\,\delta\,ds = \mathcal{O}(1)\cdot\varepsilon_1\mu\cdot\delta\,.$$

Therefore, for $\varepsilon_1 > 0$ small enough, the map (3.37) is a strict contraction in the space $\mathcal{C}^0$, with a unique fixed point $\theta(\cdot)$. By (3.37), this solution satisfies

$$|\theta(r) - \Theta| \leq \mu r^{-1/\mu}\,\|A\|_{\mathcal{C}^0}\,. \tag{3.38}$$

To solve the second equation in (3.33) we set

$$Z(r) \doteq r^{1/\mu}U(r).$$

This yields

$$\frac{dZ}{dr} = \frac{1}{\mu} r^{-1+\frac{1}{\mu}} U - r^{\frac{1}{\mu}} \frac{U}{\mu r}\Big(1 + r^{-1/\mu}B(r,\theta(r))\Big) = r^{-1-\frac{1}{\mu}} \frac{B(r,\theta(r))}{\mu}\cdot Z\,. \tag{3.39}$$

Integrating, one obtains

$$\int_{Z(r)}^{Z(+\infty)} \frac{dZ}{Z} = -\int_r^{+\infty} s^{-1-\frac{1}{\mu}} \frac{B(s,\theta(s))}{\mu}\,ds\,,$$

$$Z(r) = Z(+\infty)\cdot\exp\left\{\int_r^{+\infty} s^{-1-\frac{1}{\mu}} \frac{B(s,\theta(s))}{\mu}\,ds\right\}, \tag{3.40}$$

and finally

$$U(r) = \exp\left\{\int_r^{+\infty} s^{-1-\frac{1}{\mu}} \frac{B(s,\theta(s))}{\mu}\,ds\right\}\cdot r^{-1/\mu}\,\varpi(\Theta). \tag{3.41}$$

Next, we recall that the components of the gradient $p = (p_1,p_2) = (u_{x_1}, u_{x_2})$ satisfy the characteristic equations

$$\dot{p}_j = -\sum_i v_{i,x_j} p_i + (1+\mu)p_j\,.$$

Calling $\eta = |p|$, using r as independent variable, and integrating along a characteristic, we now obtain

$$-\frac{d\eta(r)}{dr} \leq \frac{|D\mathbf{v}(r)| + 1 + \mu}{\mu r - a(r,\theta)}\,\eta(r).$$

Hence, for any $r_2 > r_1 \geq 1$, one has

$$\begin{aligned}\frac{\eta(r_2)}{\eta(r_1)} &\geq \exp\left\{\int_{r_1}^{r_2} \frac{1+\mu+\mathcal{O}(1)\cdot\varepsilon_1 r^{-1/\mu}}{\mu r - \varepsilon_1 r^{1-\frac{1}{\mu}}}\,dr\right\}\\ &= \exp\left\{\frac{1+\mu}{\mu}\int_{r_1}^{r_2}\Big(r^{-1} + \mathcal{O}(1)\varepsilon_1 r^{-1-\frac{1}{\mu}}\Big)dr\right\}.\end{aligned} \tag{3.42}$$

$$\eta(r_1) \;\leq\; \left[\eta(r_2) r_2^{1+\frac{1}{\mu}}\right] \cdot r_1^{-1-\frac{1}{\mu}} \cdot \Big(1 + \mathcal{O}(1)\cdot \varepsilon_1 r_1^{-1/\mu}\Big).$$

Setting $r = r_1$ and letting $r_2 \to +\infty$, by the asymptotic condition (3.28) we obtain

$$|p(r)| \;=\; \eta(r) \;=\; \mathcal{O}(1)\cdot \|\overline{\omega}\|_{\mathcal{C}^1}\, r^{-1-\frac{1}{\mu}}. \tag{3.43}$$

The next lemma collects the main properties of the solution u of (3.27)-(3.28). Moreover, given a second vector field $\widetilde{\mathbf{v}}$, we estimate the difference between u and the solution $\tilde{u}$ of

$$(\widetilde{\mathbf{v}}(x) - \mu x)\cdot \nabla \tilde{u} \;=\; \tilde{u}, \tag{3.44}$$

with the same asymptotic condition (3.28)

Lemma 3.2 (asymptotic estimates). *Let $\mathbf{v}, \widetilde{\mathbf{v}} : \mathcal{D}_1 \mapsto \mathbb{R}^2$ be $\mathcal{C}^1$ vector fields satisfying the bounds*

$$|\mathbf{v}(x)|,\ |\widetilde{\mathbf{v}}(x)| \;\leq\; \varepsilon_1 |x|^{1-\frac{1}{\mu}}, \qquad\qquad |D\mathbf{v}(x)|,\ |D\widetilde{\mathbf{v}}(x)| \;\leq\; \varepsilon_1 |x|^{-\frac{1}{\mu}}, \tag{3.45}$$

$$|\mathbf{v}(x) - \widetilde{\mathbf{v}}(x)| \;\leq\; \varepsilon_2 |x|^{-1}, \tag{3.46}$$

for some constants $\mu > \frac{1}{2}$ and $\varepsilon_1, \varepsilon_2 > 0$, and for all $|x| \geq 1$. Moreover, assume that $\overline{\omega} \in \mathcal{C}^1(\mathbb{T})$.

(i) If $\varepsilon_1 > 0$ is sufficiently small, then the equation (3.27) has a unique solution with asymptotic behavior (3.28). This solution satisfies the bounds

$$|u(x)| \;\leq\; 2\,\|\overline{\omega}\|_{\mathcal{C}^0}\cdot |x|^{-\frac{1}{\mu}}, \tag{3.47}$$

$$|\nabla u(x)| \;\leq\; 3\,\|\overline{\omega}\|_{\mathcal{C}^1}\cdot |x|^{-1-\frac{1}{\mu}}. \tag{3.48}$$

(ii) If $\tilde{u}$ is a solution to (3.44) satisfying the same asymptotic conditions (3.28), then (3.46) implies

$$|u(x) - \tilde{u}(x)| \;\leq\; \mathcal{O}(1)\cdot \varepsilon_2\,\|\overline{\omega}\|_{\mathcal{C}^1}\cdot |x|^{-2-\frac{1}{\mu}}. \tag{3.49}$$

Proof. 1. For each $\Theta \in \mathbb{T}$, call

$$r \mapsto \theta(r, \Theta), \qquad r \mapsto U(r, \Theta)$$

the corresponding solution to the characteristic system of ODEs (3.33) with asymptotic conditions (3.36). By (3.37) we have

$$\frac{\partial}{\partial \Theta}\theta(r, \Theta) \;=\; 1 + \mathcal{O}(1)\cdot \varepsilon_1\, r^{-1/\mu}.$$

Hence, if $\varepsilon_1 > 0$ is small enough, for every $r \geq 1$ the map $\Theta \mapsto \theta(r, \Theta)$ is a bijection from $\mathbb{T}$ into itself. In particular, for every angle $\vartheta \in \mathbb{T}$ there exists a unique $\Theta = \Theta(r, \vartheta)$ such that $\theta(r, \Theta) = \vartheta$. The desired solution, obtained by the method of characteristics, is thus

$$u(r, \vartheta) \;=\; U(r, \Theta(r, \vartheta)).$$

From (3.41) and the bound on B in (3.35) it follows

$$|U(r,\Theta)| \;=\; \big(1+\mathcal{O}(1)\cdot\varepsilon_1\, r^{-1/\mu}\big)r^{-1/\mu}\overline{\omega}(\Theta). \tag{3.50}$$

This immediately implies (3.47). In turn, (3.48) is a consequence of (3.43).

2. To prove (3.49), we first estimate the difference between the corresponding solutions of the characteristic system of ODEs. The assumption (3.46) implies

$$|a(r,\theta)-\tilde a(r,\theta)| \;\leq\; \varepsilon_2 r^{-1}\,, \qquad\qquad |b(r,\theta)-\tilde b(r,\theta)| \;\leq\; \varepsilon_2 r^{-1},$$

hence

$$|A(r,\theta)-\widetilde A(r,\theta)|+|B(r,\theta)-\widetilde B(r,\theta)| \;=\; \mathcal{O}(1)\cdot\varepsilon_2 r^{-2+\frac{1}{\mu}}.$$

Using this bound in (3.37) and (3.41), we obtain

$$|\theta(r,\Theta)-\tilde\theta(r,\Theta)| \;=\; \mathcal{O}(1)\cdot\varepsilon_2\, r^{-2}, \tag{3.51}$$

$$|U(r,\Theta)-\widetilde U(r,\Theta)| \;=\; \mathcal{O}(1)\cdot\varepsilon_2\, r^{-2-\frac{1}{\mu}}\,\|\overline{\omega}\|_{\mathcal{C}^0}. \tag{3.52}$$

In addition to (3.52), we now use the bound (3.48) on the gradient ∇u together with (3.51), and obtain

$$|u(r,\vartheta)-\tilde u(r,\vartheta)| \;=\; \mathcal{O}(1)\cdot\varepsilon_2\, r^{-2-\frac{1}{\mu}}\,\|\overline{\omega}\|_{\mathcal{C}^1}.$$

This establishes the inequality (3.49). □

The following lemma provides an error estimate, bounding the difference between an approximation $u^\sharp$ and the exact solution u of (3.27)-(3.28).

Lemma 3.3 (error estimates). *Let $\mathbf{v}:\mathcal{D}_1\mapsto\mathbb{R}^2$ be a $\mathcal{C}^1$ vector field satisfying the assumptions in (3.45), for some $\varepsilon>0$ sufficiently small. Let $u^\sharp\in\mathcal{C}^0(\mathcal{D}_1)$ be a function which satisfies the asymptotic condition (3.28) and the approximate equation*

$$(\mathbf{v}-\mu x)\cdot\nabla u^\sharp \;=\; u^\sharp+e(x), \qquad\qquad |e(x)|\;\leq\;\varepsilon_4\,|x|^{-\alpha} \tag{3.53}$$

for some $\varepsilon_4,\alpha>0$. Then the difference between $u^\sharp$ and the exact solution u of (3.27) can be estimated as

$$|u(x)-u^\sharp(x)| \;=\; \mathcal{O}(1)\cdot\varepsilon_4\,|x|^{-\alpha-\frac{2}{\mu}}. \tag{3.54}$$

Proof. For a fixed $\Theta\in\mathbb{T}$, let $r\mapsto U(r)=u(t,\theta(r))$ be the functions considered in (3.41). To determine the corresponding function $U^\sharp(r)=u^\sharp(r,\theta(r))$, we define $Z^\sharp(r)\doteq r^{1/\mu}U^\sharp(r)$. The equations in (3.39)–(3.41) are now replaced by

$$\dot U^\sharp \;=\; U^\sharp+e, \qquad \frac{dU^\sharp}{dr} \;=\; -\frac{U^\sharp+e}{\mu r}\Big(1+r^{-1/\mu}B(r,\theta)\Big),$$

$$\frac{dZ^\sharp}{dr} \;=\; \frac{1}{\mu}r^{-1+\frac{1}{\mu}}U^\sharp - r^{\frac{1}{\mu}}\frac{U^\sharp+e}{\mu r}\Big(1+r^{-1/\mu}B(r,\theta(r))\Big) \;=\; r^{-1-\frac{1}{\mu}}\frac{B(r,\theta(r))}{\mu}\cdot Z^\sharp + e^\sharp(r), \tag{3.55}$$

where

$$e^\sharp(r) \;=\; r^{-1-\frac{1}{\mu}}\Big(1+r^{-1/\mu}B(r,\theta(r))\Big)\frac{e}{\mu} \;=\; \mathcal{O}(1)\cdot\varepsilon_4\, r^{-1-\alpha-\frac{2}{\mu}}.$$

Integrating, one obtains

$$Z^\sharp(r) \;=\; Z^\sharp(+\infty)\cdot \exp\left\{\int_r^{+\infty} s^{-1-\frac{1}{\mu}}\,\frac{B(s,\theta(s))}{\mu}\,ds\right\} + \mathcal{O}(1)\cdot \varepsilon_4\, r^{-\alpha-\frac{1}{\mu}}, \tag{3.56}$$

and finally

$$U^\sharp(r) \;=\; \exp\left\{\int_r^{+\infty} s^{-1-\frac{1}{\mu}}\,\frac{B(s,\theta(s))}{\mu}\,ds\right\}\cdot r^{-1/\mu}\,\overline{\omega}(\Theta) + \mathcal{O}(1)\cdot \varepsilon_4\, r^{-\alpha-\frac{2}{\mu}}. \tag{3.57}$$

Comparing (3.57) with (3.41) we obtain (3.54). □

3.5 An approximating sequence.

A solution to the system (3.1) on the domain $\mathcal{D}_1$, with asymptotic and boundary conditions (3.16), will be obtained as the limit of a sequence of approximations (Ω_n, Ψ_n). Assume that an initial guess (Ω_0, Ψ_0) is available, which satisfies the conditions (3.16) and provides an approximate solution to (3.1), namely

$$\Big|(\nabla^\perp\Psi_0 - \mu x)\cdot\nabla\Omega_0 - \Omega_0\Big| \;=\; \mathcal{O}(1)\cdot|x|^{-2-\frac{1}{\mu}}, \qquad\qquad \Delta\Psi_0 \;=\; \Omega_0\,. \tag{3.58}$$

We then set $\mathbf{v}_0 = \nabla^\perp\Psi_0$ and construct a sequence of approximations $(\Omega_n, \Psi_n, \mathbf{v}_n)_{n\geq 1}$ such that

$$\left\{\begin{array}{rcl} (\mathbf{v}_{n-1} - \mu x)\cdot\nabla\Omega_n &=& \Omega_n\,,\\ \Delta\Psi_n &=& \Omega_n\,,\\ \mathbf{v}_n &=& \nabla^\perp\Psi_n\,. \end{array}\right. \tag{3.59}$$

At each step, the above equations for Ω_n, Ψ_n are solved with the same asymptotic and boundary conditions as in (3.16).

Inductive estimates on the functions Ω_n will be obtained by applying Lemma 3.2 to the vector fields $\mathbf{v}_n$. In turn, the functions Ψ_n can be recovered by

$$\Psi_n(x) \;=\; \Psi_{n-1}(x) + \frac{1}{2\pi}\int_{|y|>1} \ln\left(\frac{1}{|y|}\cdot\frac{|y-x|}{|y^*-x|}\right)(\Omega_n(y) - \Omega_{n-1}(y))\,dy. \tag{3.60}$$

Assuming that the asymptotic and boundary data $\overline{\omega}, \overline{\psi}$, and ψ_0 are sufficiently small, we will show that the above sequence of approximations is convergent.

3.6 A leading order approximation.

To start the inductive procedure (3.59), it would be natural to choose

$$\Omega_0(r,\theta) \;=\; r^{-\frac{1}{\mu}}\,\overline{\omega}(\theta), \qquad\qquad \Psi_0 \;=\; r^{2-\frac{1}{\mu}}\,\overline{\psi}(\theta), \tag{3.61}$$

so that

$$-\mu x\cdot\nabla\Omega_0 \;=\; \Omega_0\,, \qquad \Delta\Psi_0 \;=\; \Omega_0\,, \qquad |\mathbf{v}_0| \;=\; |\nabla^\perp\Psi_0| \;=\; \mathcal{O}(1)\cdot r^{1-\frac{1}{\mu}}. \tag{3.62}$$

By (3.61)-(3.62) it follows

$$(\mathbf{v}_0 - \mu x) \cdot \nabla\Omega_0 - \Omega_0 \ \le\ |\nabla\Psi_0|\,|\nabla\Omega_0| \ =\ \mathcal{O}(1)\cdot|x|^{-\frac{2}{\mu}}\,. \tag{3.63}$$

As in (3.59), let Ω_1 be the solution to

$$(\mathbf{v}_0 - \mu x) \cdot \nabla\Omega_1 \ =\ \Omega_1\,,$$

with asymptotic condition given at (3.16). In view of (3.63), the error estimate in Lemma 3.3 with $\alpha = 2/\mu$ yields

$$\big|\Omega_1(x) - \Omega_0(x)\big| \ =\ \mathcal{O}(1)\cdot|x|^{-\frac{4}{\mu}}. \tag{3.64}$$

If $\mu < 2$, then for $n = 1$ the integral in (3.60) is absolutely convergent, and the inductive procedure can begin.

However, this is not the case if $\mu > 2$. To ensure that, when $n = 1$, the integral in (3.60) is absolutely convergent, we thus need to find a better approximation (Ω_0, Ψ_0). Namely, we shall require that

$$\begin{cases} (\nabla^\perp\Psi_0 - \mu x)\cdot\nabla\Omega_0 &=\ \Omega_0 + \mathcal{O}(1)\cdot|x|^{-\alpha}, \\[2mm] \Delta\Psi_0 &=\ \Omega_0\,. \end{cases} \tag{3.65}$$

for some $\alpha > 0$ such that $\alpha + \frac{2}{\mu} > 2$. To construct such an approximation, we use separation of variables and try with a finite sum of the form

$$\Omega_0(r,\theta) \ =\ \sum_{j=1}^{N_0} r^{-\frac{j}{\mu}}\omega_j(\theta) + \sum_{k=1}^{N_1} r^{-1-\frac{k}{\mu}}\widetilde{\omega}_k(\theta), \tag{3.66}$$

$$\widehat{\Psi}_0(r,\theta) \ =\ \sum_{j=1}^{N_0} r^{2-\frac{j}{\mu}}\psi_j(\theta) + \sum_{k=1}^{N_1} r^{1-\frac{k}{\mu}}\widetilde{\psi}_k(\theta), \tag{3.67}$$

choosing $N_0, N_1 \in \mathbb{N}$ to be the smallest integers such that (3.18) holds. Setting $\mathbf{e} \doteq (\cos\theta, \sin\theta)$, we compute

$$\begin{aligned} \nabla(r^\alpha\omega(\theta)) &=\ r^{\alpha-1}[\alpha\omega(\theta)\,\mathbf{e} + \omega'(\theta)\,\mathbf{e}^\perp]\,, \\ \nabla^\perp(r^\alpha\psi(\theta)) &=\ r^{\alpha-1}[-\psi'(\theta)\,\mathbf{e} + \alpha\psi(\theta)\,\mathbf{e}^\perp]\,, \\ \Delta(r^\alpha\psi(\theta)) &=\ r^{\alpha-2}[\alpha^2\psi(\theta) + \psi''(\theta)]\,. \end{aligned} \tag{3.68}$$

Using the identities (3.68), from (3.66)-(3.67) we thus obtain

$$\nabla\Omega_0(r,\theta) \ =\ \sum_{j\ge1} r^{-1-\frac{j}{\mu}}\left[-\frac{j}{\mu}\omega_j(\theta)\mathbf{e} + \omega_j'(\theta)\mathbf{e}^\perp\right] + \sum_{k\ge1} r^{-2-\frac{k}{\mu}}\left[\Big(-1-\frac{k}{\mu}\Big)\widetilde{\omega}_k(\theta)\mathbf{e} + \widetilde{\omega}_k'(\theta)\mathbf{e}^\perp\right], \tag{3.69}$$

$$-\mu x\cdot\nabla\Omega_0 \ =\ \sum_{j\ge1} r^{-\frac{j}{\mu}}\cdot j\,\omega_j(\theta) + \sum_{k\ge1} r^{-1-\frac{k}{\mu}}(\mu+k)\widetilde{\omega}_k(\theta), \tag{3.70}$$

$$\nabla^\perp\widehat{\Psi}_0(r,\theta) \ =\ \sum_{j\ge1} r^{1-\frac{j}{\mu}}\left[\psi_j'(\theta)\mathbf{e} + \Big(2-\frac{j}{\mu}\Big)\psi_j(\theta)\mathbf{e}^\perp\right] + \sum_{k\ge1} r^{-\frac{k}{\mu}}\left[\widetilde{\psi}_k'(\theta)\mathbf{e} + \Big(1-\frac{k}{\mu}\Big)\widetilde{\psi}_k(\theta)\mathbf{e}^\perp\right], \tag{3.71}$$

$$\Delta\widehat{\Psi}_0(r,\theta) \;=\; \sum_{j\geq 1} r^{-\frac{j}{\mu}}\left[\Big(2-\frac{j}{\mu}\Big)^2\psi_j(\theta)+\psi_j''(\theta)\right]+\sum_{k\geq 1} r^{-1-\frac{k}{\mu}}\left[\Big(1-\frac{j}{\mu}\Big)^2\widetilde{\psi}_j(\theta)+\widetilde{\psi}_j''(\theta)\right]. \tag{3.72}$$

As $r\to+\infty$, the asymptotic conditions (3.16) imply

$$\omega_1 \;=\; \overline{\omega}, \qquad\qquad \psi_1 \;=\; \overline{\psi}.$$

We need to determine all further terms in the expansions (3.66)-(3.67). Notice that, if $\omega_j,\widetilde{\omega}_k$ are known, then the equation $\Delta\Psi_0=\Omega_0$ uniquely determines the functions $\psi_j,\widetilde{\psi}_k$. Indeed,

$$\begin{cases} \psi_j''(\theta)+\Big(2-\frac{j}{\mu}\Big)^2\psi_j(\theta) &=& \omega_j(\theta),\\ \widetilde{\psi}_k''(\theta)+\Big(1-\frac{j}{\mu}\Big)^2\widetilde{\psi}_k(\theta) &=& \widetilde{\omega}_k(\theta). \end{cases} \tag{3.73}$$

We assume here that $\mu, 2\mu\notin\mathbb{N}$, hence all the above equations have a unique solution.

The functions $\omega_j,\widetilde{\omega}_k$ will be determined by an inductive procedure, based on the first identity in (3.65). Assume that $\omega_j,\widetilde{\omega}_k$ have already been determined for all integers $j<j^*$ and $k<k^*$, in such a way that (3.73) holds and moreover

$$(\nabla^\perp\widehat{\Psi}_0-\mu x)\cdot\nabla\Omega_0-\Omega_0 \;=\; o\big(|x|^{-\gamma}\big) \qquad \text{for all } \;\gamma \;<\; \min\left\{\frac{j^*}{\mu},\, 1+\frac{k^*}{\mu}\right\}.$$

Two cases must be considered.

CASE 1: $\frac{j^*}{\mu} \;<\; 1+\frac{k^*}{\mu}$.

We can then uniquely determine the function ω_{j^*} by imposing that

$$(\nabla^\perp\widehat{\Psi}_0-\mu x)\cdot\nabla\Omega_0-\Omega_0 \;=\; o(|x|^{-j^*/\mu}). \tag{3.74}$$

Indeed, inserting (3.66), (3.70),(3.71) in (3.74) and collecting terms of order $r^{-j^*/\mu}$, we obtain

$$(j^*-1)\,\omega_{j^*}(\theta) \;=\; F_{j^*}(\theta), \tag{3.75}$$

where the function $F_{j^*}(\theta)$ is computed in terms of the previous functions $\omega_j,\widetilde{\omega}_k$ and their first derivatives $\omega_j',\widetilde{\omega}_k'$, with $j<j^*$ and $k<k^*$.

CASE 2: $1+\frac{k^*}{\mu} \;<\; \frac{j^*}{\mu}$.

In this case we can uniquely determine the function $\widetilde{\omega}_{k^*}$ by imposing that

$$(\nabla^\perp\widehat{\Psi}_0-\mu x)\cdot\nabla\Omega_0-\Omega_0 \;=\; o(|x|^{1-(k^*/\mu)}). \tag{3.76}$$

Indeed, inserting (3.66), (3.70),(3.71) in (3.74) and collecting terms of order $r^{1-(k^*/\mu)}$, we obtain

$$(\mu+k^*-1)\,\widetilde{\omega}_{k^*}(\theta) \;=\; G_{k^*}(\theta), \tag{3.77}$$

where the function $G_{k^*}(\theta)$ is computed in terms of the previous functions $\omega_j,\widetilde{\omega}_k$, $\omega_j',\widetilde{\omega}_k'$, with $j<j^*$ and $k<k^*$.

By induction, all functions $\omega_j,\widetilde{\omega}_k$ can thus be uniquely determined in a finite number of steps.

We observe that, at each step, the formula for ω_{j^*} or $\widetilde{\omega}_{k^*}$ contains the first order derivatives of the previous functions ω_j or $\widetilde{\omega}_k$ with $j < j^*$ and $k < k^*$. We thus have an estimate of the form

$$\max_{1\leq j\leq N_0} \|\omega_j\|_{\mathcal{C}^1(\mathbb{T})} + \max_{1\leq k\leq N_1} \|\widetilde{\omega}_k\|_{\mathcal{C}^1(\mathbb{T})} \;\leq\; C\cdot \|\overline{\omega}\|_{\mathcal{C}^N(\mathbb{T})}, \qquad\qquad N \doteq N_0+N_1. \tag{3.78}$$

Example 3.1 Assume $2<\mu<\frac{5}{2}$. By (3.18) we can take $N_0=2$, $N_1=0$. The approximation this takes the form

$$\Omega_0(r,\theta) \;=\; r^{-\frac{1}{\mu}}\omega_1(\theta)+r^{-\frac{2}{\mu}}\omega_2(\theta), \qquad\qquad \widehat{\Psi}_0(r,\theta) \;=\; r^{2-\frac{1}{\mu}}\psi_1(\theta)+r^{2-\frac{2}{\mu}}\psi_2(\theta).$$

Given the function $\overline{\omega}:\mathbb{T}\mapsto\mathbb{R}$, we begin by taking $\omega_1=\overline{\omega}$, $\psi_1=\overline{\psi}$, where $\overline{\psi}$ is uniquely determined by (3.5). In turn, ω_2 is obtained by imposing the condition

$$(\nabla^\perp\widehat{\Psi}_0-\mu x)\cdot\nabla\Omega_0-\Omega_0 \;=\; o(|x|^{-2/\mu}).$$

In view of (3.69)–(3.71), this yields

$$\Big(r^{1-\frac{1}{\mu}}\psi_1'-\mu r\Big)\left[-\frac{1}{\mu}r^{-1-\frac{1}{\mu}}\omega_1-\frac{2}{\mu}r^{-1-\frac{2}{\mu}}\omega_2\right]+r^{1-\frac{1}{\mu}}\left(2-\frac{1}{\mu}\right)\psi_1\left[r^{-1-\frac{1}{\mu}}\omega_1'+r^{-1-\frac{2}{\mu}}\omega_2'\right]$$
$$-\Big(r^{-\frac{1}{\mu}}\omega_1+r^{-\frac{2}{\mu}}\omega_2\Big) \;=\; o(|x|^{-2/\mu}).$$

Notice that terms of order $\mathcal{O}(1)\cdot r^{-1/\mu}$ already cancel. Collecting terms of order $\mathcal{O}(1)\cdot r^{-2/\mu}$ we obtain

$$\psi_1'\omega_1+2\omega_2+\left(2-\frac{1}{\mu}\right)\psi_1\omega_1'-\omega_2 \;=\; 0,$$

$$\omega_2 \;=\; -\,\psi_1'\omega_1-\left(2-\frac{1}{\mu}\right)\psi_1\omega_1'\,.$$

Finally, ψ_2 is determined by the first equation in (3.73), with $j=2$.

Having determined all coefficients in the finite sums (3.66)-(3.67), it now remains to adjust the boundary condition (3.16) at $r=1$. Since in general $\widehat{\Psi}_0(1,\theta)\neq\psi_0(\theta)$, a correction term must be added. Recalling (3.23), we thus consider the function

$$\Psi_0(x) \;=\; \widehat{\Psi}_0(x)+\Psi_0^\sharp(x), \qquad\qquad \Psi_0^\sharp(x) \;\doteq\; \frac{|x|^2-1}{2\pi}\int_0^{2\pi}\frac{\psi_0(\theta)-\widehat{\Psi}_0(1,\theta)}{|x_1-\cos\theta|^2+|x_2-\sin\theta|^2}\,d\theta. \tag{3.79}$$

Notice that the above definition implies $\nabla^\perp\Psi_0^\sharp(x) \;=\; \mathcal{O}(1)\cdot|x|^{-1}$. Hence

$$\nabla^\perp\Psi_0^\sharp(x)\cdot\nabla\Omega_0(x) \;=\; \mathcal{O}(1)\cdot|x|^{-2-\frac{1}{\mu}}. \tag{3.80}$$

3.7 Convergence of the approximating sequence.

Having determined the functions $\Omega_0,\Psi_0,\widehat{\Psi}_0$ as in (3.66)-(3.67) and (3.79), we now define

$$\mathbf{v}_0(x) \;\doteq\; \nabla^\perp\Psi_0(x). \tag{3.81}$$

We then construct the sequence $(\Omega_n,\Psi_n,\mathbf{v}_n)_{n\geq1}$ according to (3.59), always with asymptotic and boundary conditions (3.16). The following analysis will establish

- a priori bounds in a strong norm,
- convergence estimates in a weaker norm.

Consider the spaces

$$X \doteq \Big\{\Omega \in \mathcal{C}^0(\mathcal{D}_1)\,;\quad \|\Omega\|_X \doteq \sup_x |x|^{2+\frac{1}{\mu}}|\Omega(x)| < +\infty\Big\},$$

$$Y \doteq \Big\{\mathbf{v} \in \mathcal{C}^0(\mathcal{D}_1;\mathbb{R}^2)\,;\quad \|\mathbf{v}\|_Y \doteq \sup_x |x|\,|\mathbf{v}(x)| < +\infty\Big\}.$$

Recalling (3.78), by induction we will establish the bounds

$$|\Omega_n(x)| = \mathcal{O}(1)\cdot|x|^{-1/\mu}\big(\|\psi_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\big), \tag{3.82}$$

$$|\nabla\Omega_n(x)| = \mathcal{O}(1)\cdot|x|^{-1-\frac{1}{\mu}}\big(\|\psi_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\big), \tag{3.83}$$

$$\|\Omega_{n+1}-\Omega_n\|_X = \mathcal{O}(1)\cdot\|\mathbf{v}_n-\mathbf{v}_{n-1}\|_Y\cdot\|\overline{\omega}\|_{\mathcal{C}^N}\,, \tag{3.84}$$

$$|\mathbf{v}_n(x)| = \mathcal{O}(1)\cdot|x|^{1-\frac{1}{\mu}}\big(\|\psi_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\big), \tag{3.85}$$

$$|D\mathbf{v}_n(x)| = \mathcal{O}(1)\cdot|x|^{-1/\mu}\big(\|\psi_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\big), \tag{3.86}$$

$$\|\mathbf{v}_n-\mathbf{v}_{n-1}\|_Y = \mathcal{O}(1)\cdot\|\Omega_n-\Omega_{n-1}\|_X\,. \tag{3.87}$$

1. We begin by observing that, by the construction of $\Omega_0, \Psi_0, \mathbf{v}_0$ at (3.66)-(3.67), (3.79), and (3.81), the above estimates (3.82), (3.83), (3.85), and (3.86) hold true when $n = 0$.

2. To estimate the difference $\Omega_1 - \Omega_0$, we observe that Ω_0 satisfies the approximate equation

$$(\mathbf{v}_0-\mu x)\cdot\nabla\Omega_0 = \Omega_0+\mathcal{O}(1)\cdot|x|^{-2-\epsilon}\Big(\|\overline{\psi}_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\Big),$$

where ϵ is the constant in (3.19). We compare Ω_0 with the exact solution Ω_1 of

$$(\mathbf{v}_0-\mu x)\cdot\nabla\Omega_1 = \Omega_1\,,$$

always with the same asymptotic condition (3.16). The error estimate proved in Lemma 3.3 yields

$$\Omega_1(x)-\Omega_0(x) = \mathcal{O}(1)\cdot|x|^{-2-\epsilon}\Big(\|\overline{\psi}_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\Big), \tag{3.88}$$

for all $x \in \mathcal{D}_1$.

In turn, the estimate (3.25) in Lemma 3.1 yields

$$\begin{aligned}\big|\mathbf{v}_1(x)-\mathbf{v}_0(x)\big| &= \big|\nabla\Psi_1(x)-\nabla\Psi_0(x)\big|\\ &= \mathcal{O}(1)\cdot|x|^{-1}\cdot\big\|\,|x|^{2+\varepsilon}(\Omega_1-\Omega_0)\big\|_{\mathbf{L}^\infty} = \mathcal{O}(1)\cdot|x|^{-1}\Big(\|\overline{\psi}_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\Big)\,.\end{aligned} \tag{3.89}$$

3. The remainder of the proof is achieved by induction. By (3.89), the estimate (3.49) in Lemma 3.2 yields

$$\big|\Omega_2(x)-\Omega_1(x)\big| = \mathcal{O}(1)\cdot|x|^{-2-\frac{1}{\mu}}\cdot\big\|\,|x|(\mathbf{v}_2-\mathbf{v}_1)\big\|_{\mathbf{L}^\infty} \le C_1\,|x|^{-2-\frac{1}{\mu}}\Big(\|\overline{\psi}_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\Big), \tag{3.90}$$

for some constant C_1. In turn, the bound (3.25) in Lemma 3.1 yields

$$\big|\nabla\Psi_2(x)-\nabla\Psi_1(x)\big| \;=\; \mathcal{O}(1)\cdot|x|^{-1}\cdot\Big\|\,|x|^{2+\varepsilon}(\Omega_1-\Omega_0)\Big\|_{\mathbf{L}^\infty} \;=\; \mathcal{O}(1)\cdot|x|^{-1}\Big(\|\overline{\psi}_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\Big)\,. \tag{3.91}$$

Assuming that Ω_n, Ψ_n satisfy uniform bounds for all $n\geq 2$, the bound (3.84) follows from (3.49), while (3.87) is a consequence of (3.25). Notice that, by taking $\|\overline{\omega}\|_{\mathcal{C}^N}$ small enough, this implies

$$\|\Omega_{n+1}-\Omega_n\|_X \;\leq\; \frac{1}{2}\|\Omega_n-\Omega_{n-1}\|_X\,. \tag{3.92}$$

Hence the the sequence $(\Omega_n)_{n\geq 1}$ is absolutely convergent.

4. Assuming that the vector field $\mathbf{v}_{n-1}$ satisfies the uniform bounds as in (3.85)-(3.86), the estimates (3.82)-(3.83) on the solution Ω_n of

$$(\mathbf{v}_{n-1}-\mu x)\cdot\nabla\Omega_n \;=\; \Omega_n\,, \qquad\qquad \lim_{r\to\infty} r^{1/\mu}\Omega_n(r,\theta) \;=\; \overline{\omega}(\theta),$$

follow from (3.47)-(3.48) in Lemma 3.2. It thus remains to give a proof of (3.85)-(3.86).

5. The bound (3.85) follows from Lemma 3.1 and the formula

$$|\mathbf{v}_n(x)| \;\leq\; |\mathbf{v}_1(x)|+\frac{1}{2\pi}\int_{|y|>1}\left\{\frac{1}{|x-y|}+\frac{1}{|x-y^*|}\right\}\big|\Omega_n(y)-\Omega_0(y)\big|\,dy. \tag{3.93}$$

Indeed, by (3.90) and (3.92) one has

$$\big|\Omega_n(x)-\Omega_1(x)\big| \;\leq\; 2C_1\,|x|^{-2-\frac{1}{\mu}}\Big(\|\overline{\psi}_0\|_{\mathcal{C}^1}+\|\overline{\omega}\|_{\mathcal{C}^N}\Big)\,. \tag{3.94}$$

6. To prove (3.86), we write $\mathbf{v}_n=\mathbf{v}_1+\mathbf{w}_n$, where

$$\mathbf{w}_n(x) \;\doteq\; \nabla^\perp\left(\frac{1}{2\pi}\int_{|y|>1}\ln\left(\frac{1}{|y|}\cdot\frac{|y-x|}{|y^*-x|}\right)\big(\Omega_n(y)-\Omega_0(y)\big)\,dy\right). \tag{3.95}$$

The Jacobian matrix $D\mathbf{w}_n(x)$ can be estimated in terms of the Hessian matrix of second derivatives $D^2\phi$, where

$$\phi(x) \;=\; \Psi_n-\Psi_0 \;=\; \frac{1}{2\pi}\int_{|y|>1}\ln\left(\frac{1}{|y|}\cdot\frac{|y-x|}{|y^*-x|}\right)\big(\Omega_n(y)-\Omega_0(y)\big)\;dy \tag{3.96}$$

is the solution to

$$\left\{\begin{array}{rcll}\Delta\phi &=& \Omega_n-\Omega_0\,, & |x|>1,\\[2mm] \phi(x) &=& 0, & |x|=1.\end{array}\right. \tag{3.97}$$

A bound on $D\mathbf{w}_n(x)$ will be obtained using Schauder's regularity estimates on ϕ.

For any $\rho\geq 3$ consider the annular region

$$\Gamma_\rho \;\doteq\; \Big\{x\in\mathbb{R}^2\,;\;\;\frac{\rho}{3}\leq|x|\leq 3\rho\Big\}\,.$$

Using the rescaled variable $y = \rho x$, and setting $\widehat{\phi}(y) = \phi(\rho y)$, one obtains an equation on the fixed domain Γ_1, namely

$$\Delta\widehat{\phi}(y) \;=\; \rho^2\big(\Omega_n(\rho y) - \Omega_0(\rho y)\big) \;\doteq\; f(y). \tag{3.98}$$

We observe that, by (3.66) and (3.83), the right hand side of (3.98) satisfies the estimates

$$|f(y)| \;=\; \mathcal{O}(1)\cdot\rho^2\big|\Omega_n(\rho y) - \Omega_0(\rho y)\big| \;=\; \mathcal{O}(1)\cdot\rho^2\cdot\rho^{-2-\frac{1}{\mu}} \;=\; \mathcal{O}(1)\cdot\rho^{-1/\mu}. \tag{3.99}$$

In addition, by (3.83), the gradient of $\Omega_n - \Omega_0$ over Γ_ρ can be bounded as

$$\sup_{x\in\Gamma_\rho} |\nabla\Omega_n(x) - \nabla\Omega_0(x)| \;=\; \mathcal{O}(1)\cdot\rho^{-1-\frac{1}{\mu}}.$$

In turn, this implies

$$|\nabla_y f(y)| \;=\; \mathcal{O}(1)\cdot\rho^3\cdot\rho^{-1-\frac{1}{\mu}}. \tag{3.100}$$

Together, (3.99) and (3.100) imply that, for any $y, y' \in \Gamma_1$,

$$|f(y) - f(y')| \;\leq\; \mathcal{O}(1)\cdot\min\Big\{|y-y'|\cdot\rho^{2-\frac{1}{\mu}}\,,\;\rho^{-1/\mu}\Big\} \;=\; \mathcal{O}(1)\cdot|y-y'|^{\frac{1}{2\mu}}. \tag{3.101}$$

By (3.99) and (3.100), the right hand side of (3.98) is bounded and Hölder continuous, uniformly w.r.t. ρ. Moreover, the function $\widehat{\phi}$ satisfies uniform bounds for $|y| = 1/3$ and $|y| = 3$. Schauder's interior regularity estimates yield

$$|D_y^2\widehat{\phi}(y)| \;\leq\; \kappa_4\,, \qquad\qquad \frac{1}{2} \;<\; |y| \;<\; 2, \tag{3.102}$$

for some constant κ_4 independent of ρ. Returning to the original coordinates, for any $\rho \geq 3$ we obtain

$$|D^2\phi(x)| \;\leq\; \kappa_4|x|^{-2}\,, \qquad\qquad \frac{\rho}{2} \;<\; |x| \;<\; 2\rho. \tag{3.103}$$

To complete the proof, it now remains to prove that (3.103) holds also for $1 < |x| < 2/3$, possibly with a larger constant κ_4. But this is an immediate consequence of Schauder's boundary regularity estimates, because $\phi(x) = 0$ for $|x| = 1$.

7. By the previous estimates, we have the convergence $\|\Omega_n - \Omega\|_X \to 0$ and $\|\mathbf{v}_n - \mathbf{v}\|_Y \to 0$, as $n \to \infty$. In particular, this implies the uniform convergence of the functions Ω_n and $\mathbf{v}_n$ on the outer domain $\mathcal{D}_1$. Since Ω_n is recovered from $\mathbf{v}_{n-1}$ by solving the PDE in (3.59), while $\Psi_n - \Psi_0$ is recovered by (3.96), we conclude that (Ω, Ψ) provide a solution to the problem (3.1), (3.16). This completes the proof of Theorem 3.1. □

Remark 3.3 (Dependence on the boundary data). It is of interest to see how the solution (Ω, Ψ) depends on the data ψ_0 in (3.16). In the special case where the asymptotic data $\overline{\omega} \equiv 0$, according to (3.23) the explicit solution is

$$\Psi(x) \;=\; \frac{|x|^2 - 1}{2\pi}\int_{|y|=1}\frac{\psi_0(y)}{|y-x|^2}\,d\sigma, \qquad\qquad \Omega(x) \;=\; 0. \tag{3.104}$$

In general, for $\overline{\omega}$ small, a good approximation to the differential of the solution map $\psi_0 \;\mapsto\; (\Omega, \Psi)$ is provided by differential of the first iteration of our scheme (3.59).

4 Solutions on an inner domain

In the remainder of this paper we consider the problem of constructing solutions to (2.9) on a small disc

$$B_R \doteq \{x \in \mathbb{R}^2\,;\quad |x| < R\},$$

with boundary data

$$\begin{cases} \Omega(R,\theta) &= \overline{\omega}(\theta), \\ \Psi(R,\theta) &= \overline{\psi}(\theta). \end{cases} \tag{4.1}$$

Remark 4.1 Throughout the following, we assume that the integral curves of the *pseudo-velocity* $\mathbf{q}(y) = \nabla^\perp \Psi(y) - \mu y$ are spirals winding around the origin. We notice that, by adding an appropriate linear function to Ψ in (2.9), one may shift the centers of these spirals to an arbitrary point in the plane. More generally, this shift can be achieved by adding a function which is linear on a large disc, and satisfies a suitable decay estimate as $|x| \to +\infty$. In the sequel, we will thus restrict our attention to the case of spirals centered at zero. Solutions on more general inner domains, as shown in Fig. 4, right, can be constructed in a similar way.

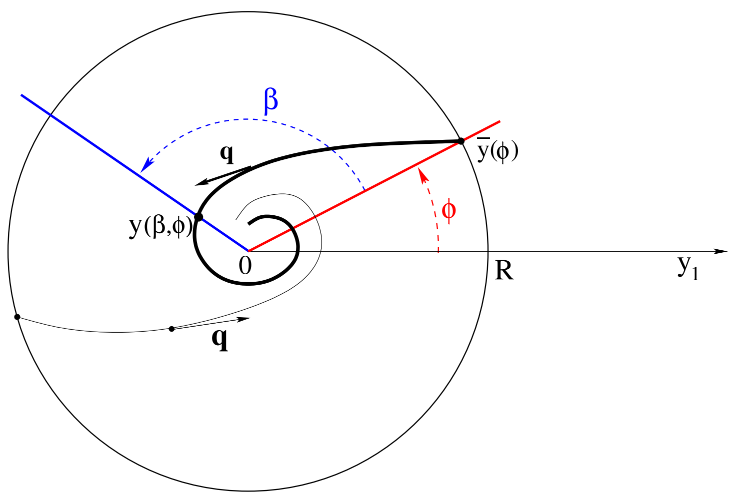

Figure 5: Construction of the adapted coordinate system. When $\beta = 0$, one starts by defining $y(0,\phi) = \bar{y}(\phi) \doteq (R\cos\phi, R\sin\phi)$. For $\beta > 0$, the point $y(\beta,\phi)$ lies at the intersection of the pseudo-streamline through $\bar{y}(\phi)$ with the ray making an angle $\phi + \beta$ with the y_1-axis.

4.1 Adapted coordinates.

We regard the first equation in (2.9) as a linear, first order equation for Ω. The characteristic curves, i.e. the pseudo-streamlines, are the integral curves of the vector field $\mathbf{q} = \nabla^\perp \Psi - \mu y$. Following [16, 17], it will be convenient to make a change of variables $(y_1, y_2) \mapsto (\beta, \phi)$, such that (see Fig. 5)

- Pseudo-streamlines have equation $\phi = \text{constant}$.
- $\beta = \theta - \phi$, where $r = |y|$, $\theta = \measuredangle\, y$ are the standard polar coordinates on $\mathbb{R}^2$.

- For fixed ϕ we have

$$\lim_{\beta\to 0+} r(\beta,\phi) \;=\; R\,, \qquad\qquad \lim_{\beta\to 0+} \theta(\beta,\phi) \;=\; \phi\,. \tag{4.2}$$

We calculate the left hand side of (2.9) in terms of the new variables (β, ϕ).

$$\begin{aligned}
(\nabla^\perp \Psi(y) - \mu y)\cdot\nabla\Omega \;&=\; (-\Psi_{y_2} - \mu y_1)\Omega_{y_1} + (\Psi_{y_1} - \mu y_2)\Omega_{y_2} \\
=\; &\Big[- (\Psi_\beta \beta_{y_2} + \Psi_\phi \phi_{y_2}) - \mu y_1)\Big](\Omega_\beta\beta_{y_1} + \Omega_\phi\phi_{y_1}) \\
&+\Big[(\Psi_\beta\beta_{y_1} + \Psi_\phi\phi_{y_1}) - \mu y_2\Big](\Omega_\beta\beta_{y_2} + \Omega_\phi\phi_{y_2}) \\
=\; &\Big[(\beta_{y_1}\phi_{y_2} - \beta_{y_2}\phi_{y_1})\Psi_\beta - \mu(y_1\phi_{y_1} + y_2\phi_{y_2})\Big]\Omega_\phi \\
&+\Big[(\beta_{y_2}\phi_{y_1} - \beta_{y_1}\phi_{y_2})\Psi_\phi - \mu(y_1\beta_{y_1} + y_2\beta_{y_2})\Big]\Omega_\beta\,.
\end{aligned} \tag{4.3}$$

If $\{\phi = \text{constant}\}$ is the equation of a pseudo-streamline, then in the above expression the coefficient of Ω_ϕ should vanish. Namely

$$0 \;=\; (\beta_{y_1}\phi_{y_2} - \beta_{y_2}\phi_{y_1})\Psi_\beta - \mu(y_1\phi_{y_1} + y_2\phi_{y_2}). \tag{4.4}$$

We now observe that the 2×2 Jacobian matrix of the variable transformation satisfies

$$\begin{bmatrix} y_{1,\beta} & y_{1,\phi} \\ y_{2,\beta} & y_{2,\phi} \end{bmatrix} \;=\; \begin{bmatrix} \beta_{y_1} & \beta_{y_2} \\ \phi_{y_1} & \phi_{y_2} \end{bmatrix}^{-1} \;=\; \frac{1}{\beta_{y_1}\phi_{y_2} - \beta_{y_2}\phi_{y_1}} \cdot \begin{bmatrix} \phi_{y_2} & -\beta_{y_2} \\ -\phi_{y_1} & \beta_{y_1} \end{bmatrix}. \tag{4.5}$$

In view of (4.5), from (4.4) it follows

$$\Psi_\beta + \mu\, y \times y_\beta \;=\; \Psi_\beta + \mu(y_1 y_{2,\beta} - y_2 y_{1,\beta}) \;=\; 0. \tag{4.6}$$

Using the variables

$$\theta(\beta,\phi) \;=\; \beta + \phi, \qquad\qquad y \;=\; y(r,\theta), \tag{4.7}$$

we can write

$$\begin{aligned}
y_\beta \;&=\; y_r r_\beta + y_\theta\theta_\beta\,, \\
y \times y_\beta \;&=\; (y\times y_r) r_\beta + (y \times y_\theta)\theta_\beta \;=\; |y|^2.
\end{aligned} \tag{4.8}$$

Together with (4.6), this yields

$$\Psi_\beta \;=\; -\,\mu\,|y|^2. \tag{4.9}$$

Using polar coordinates, the map $(\beta,\phi)\mapsto(r,\theta)$ thus takes the form

$$\begin{cases} r \;=\; (-\Psi_\beta/\mu)^{1/2}, \\ \theta \;=\; \beta + \phi. \end{cases} \tag{4.10}$$

It will be convenient to use the auxiliary variable $\ell = \ln r$, so that

$$\begin{cases} \ell \;=\; \frac{1}{2}\ln(-\Psi_\beta/\mu)\,, \\ \theta \;=\; \beta + \phi. \end{cases} \tag{4.11}$$

For future use, we also introduce the differential operator

$$\partial_\varphi \;=\; \partial_\phi - \partial_\beta, \tag{4.12}$$

describing a directional derivative in the radial direction, i.e. with $\theta =$ constant.

We now compute the Jacobian matrix of the transformation (4.11) and its inverse.

$$J \;\doteq\; \begin{pmatrix} \ell_\beta & \ell_\phi \\ \theta_\beta & \theta_\phi \end{pmatrix} \;=\; \begin{pmatrix} \frac{\Psi_{\beta\beta}}{2\Psi_\beta} & \frac{\Psi_{\beta\phi}}{2\Psi_\beta} \\ 1 & 1 \end{pmatrix}, \tag{4.13}$$

$$J^{-1} \;=\; \begin{pmatrix} \beta_\ell & \beta_\theta \\ \phi_\ell & \phi_\theta \end{pmatrix} \;=\; \frac{1}{\ell_\varphi}\begin{pmatrix} -1 & \ell_\phi \\ 1 & -\ell_\beta \end{pmatrix}. \tag{4.14}$$

Notice that the Jacobian determinant is

$$\det J \;=\; \ell_\beta - \ell_\phi \;=\; \frac{\Psi_{\beta\beta}-\Psi_{\beta\phi}}{2\Psi_\beta} \;\doteq\; -\frac{\Psi_{\beta\varphi}}{2\Psi_\beta} \;=\; -\,\ell_\varphi\,. \tag{4.15}$$

Moreover, by (4.10)-(4.11), the determinant of the coordinate transformation $(\beta,\phi)\mapsto(y_1,y_2)$ is computed by

$$\begin{aligned} \det\begin{pmatrix} y_{1,\beta} & y_{1,\phi} \\ y_{2,\beta} & y_{2,\phi} \end{pmatrix} \;&=\; r\,\det\begin{pmatrix} r_\beta & r_\phi \\ \theta_\beta & \theta_\phi \end{pmatrix} \;=\; r\,r_\ell\det\begin{pmatrix} \ell_\beta & \ell_\phi \\ \theta_\beta & \theta_\phi \end{pmatrix} \\ &=\; r^2\,\frac{\Psi_{\beta\phi}-\Psi_{\beta\beta}}{-2\Psi_\beta} \;=\; \frac{\Psi_{\beta\varphi}}{2\mu}\,. \end{aligned} \tag{4.16}$$

For future use we observe that, for any function f, one has

$$\ell_\varphi\, f_\theta \;=\; \ell_\varphi f_\phi - \ell_\phi f_\varphi \;=\; (\ell_\varphi f)_\phi - (\ell_\phi f)_\varphi\,. \tag{4.17}$$

We also observe that

$$\partial_\theta \;=\; \beta_\theta\,\partial_\beta + \phi_\theta\,\partial_\phi \;=\; \frac{1}{\ell_\varphi}\big(\ell_\phi\partial_\beta - \ell_\beta\partial_\phi\big) \;=\; \frac{1}{\ell_\varphi}\big(\ell_\phi\partial_\phi - \ell_\phi\partial_\varphi - \ell_\beta\partial_\phi\big) \;=\; \partial_\phi - \frac{\ell_\phi}{\ell_\varphi}\partial_\varphi\,. \tag{4.18}$$

Therefore, partial derivatives w.r.t. ℓ,θ can be expressed as

$$\begin{cases} \partial_\ell \;=\; \dfrac{1}{\ell_\varphi}\,\partial_\varphi\,, \\[2mm] \partial_\theta \;=\; \partial_\phi - \dfrac{\ell_\phi}{\ell_\varphi}\partial_\varphi\,. \end{cases} \tag{4.19}$$

4.2 The equation in the new coordinates.

Adopting a notation from differential geometry, we denote by $t\mapsto \exp(t\mathbf{q})(y_0)$ the solution to

$$\dot y \;=\; \mathbf{q}(y), \qquad\qquad y(0) \;=\; y_0\,. \tag{4.20}$$

Here $\mathbf{q}(y) = \nabla^\perp \Psi(y) - \mu y$ is the pseudo-velocity, introduced at (2.11). By construction, the point $y(\beta, \phi)$ lies on the integral curve of $\mathbf{q}$ starting from $\bar y(\phi) = (R\cos\phi, R\sin\phi)$. Hence

$$y(\beta,\phi) \;=\; \exp(\tau\mathbf{q})(\bar y(\phi)) \tag{4.21}$$

for some $\tau = \tau(\beta)$. The first equation in (2.9) thus leads to

$$\Omega(\beta,\phi) \;=\; e^{\tau(\beta)}\,\Omega(0,\phi). \tag{4.22}$$

The time $\tau(\beta)$ can now be estimated by comparing the area elements at the two points $\bar y$ and y. Namely, with reference to Fig. 6, we claim that

$$e^{-2\mu\tau(\beta)} \;=\; \frac{\Big(y_\phi\times(\nabla^\perp\Psi-\mu y)\Big)(y(\beta,\phi))}{\Big(y_\phi\times(\nabla^\perp\Psi-\mu y)\Big)(\bar y(\phi))}\,. \tag{4.23}$$

To prove (4.23), we rely on the identities

$$\dot y \;=\; y_\beta\,\dot\beta \;=\; \nabla^\perp\Psi - \mu y, \tag{4.24}$$

$$\mathrm{div}\,(\nabla^\perp\Psi - \mu y) \;\equiv\; -\,2\mu\,, \tag{4.25}$$

$$\frac{d}{dt}\big(y_\phi\times y_\beta\dot\beta\big) \;=\; -\,2\mu\big(y_\phi\times y_\beta\dot\beta\big). \tag{4.26}$$

Integrating (4.26) over the time interval $[0,\tau(\beta)]$ and using (4.24) and (4.21), we obtain (4.23).

Observing that

$$y_\phi\times\nabla^\perp\Psi \;=\; y_{1,\phi}\Psi_{y_1} + y_{2,\phi}\Psi_{y_2} \;=\; \Psi_\phi$$

and using (4.22)-(4.23), one obtains

$$\begin{aligned}\Omega(\beta,\phi) \;&=\; e^{\tau(\beta)}\,\Omega(0,\phi) \;=\; \big(e^{-2\mu\tau(\beta)}\big)^{-\frac{1}{2\mu}}\,\overline{\omega}(\phi)\\ &=\; \left(\frac{\Big(\Psi_\phi+\mu\,y\times y_\phi\Big)(\beta,\phi)}{\Big(\Psi_\phi+\mu\,y\times y_\phi\Big)(0,\phi)}\right)^{-\frac{1}{2\mu}}\overline{\omega}(\phi)\,,\end{aligned} \tag{4.27}$$

where $\overline{\omega}$ is the boundary value prescribed at (4.1). When $\beta = 0$, by (4.1) one has

$$\Big(\Psi_\phi+\mu\,y\times y_\phi\Big)(0,\phi) \;=\; \overline{\psi}'(\phi)+\mu R^2. \tag{4.28}$$

Using the relations

$$y\times y_\phi \;=\; r^2\,\theta_\phi \;=\; -\,\frac{\Psi_\beta}{\mu}\,,$$

we can write (4.27) in the form

$$\Omega(\beta,\phi) \;=\; (\Psi_\phi-\Psi_\beta)^{-\frac{1}{2\mu}}\,\omega^*(\phi) \;=\; \Psi_\varphi^{-\frac{1}{2\mu}}\,\omega^*(\phi). \tag{4.29}$$

Here, in view of (4.28), we defined

$$\omega^*(\phi) \;\doteq\; \Big(\overline{\psi}'(\phi)+\mu R^2\Big)^{\frac{1}{2\mu}}\overline{\omega}(\phi). \tag{4.30}$$

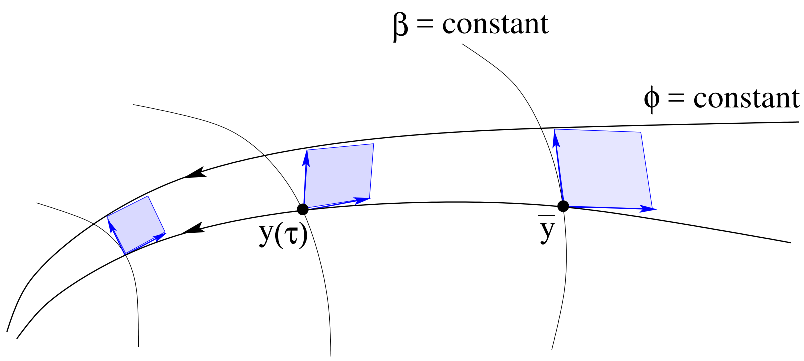

Figure 6: By (4.25), along any trajectory $\dot{y} = \nabla^\perp \Psi(y) - \mu y$, the area element decreases at an exponential rate. Namely, $dA(y(\tau)) = e^{-2\mu\tau} dA(\bar{y})$. This motivates the formula (4.27).

It remains to write the second equation in (2.9) in terms of the coordinates β, ϕ. Writing the Laplace operator in polar coordinates, one obtains

$$\Psi_{rr} + \frac{\Psi_r}{r} + \frac{\Psi_{\theta\theta}}{r^2} \;=\; \Omega. \tag{4.31}$$

Recalling that $\ell = \log r$, this can be written as

$$\ell_\varphi(\Psi_{\ell\ell} + \Psi_{\theta\theta}) \;=\; r^2\, \ell_\varphi\, \Omega. \tag{4.32}$$

Using (4.19)-(4.17) we compute

$$\Psi_{\ell\ell} + \Psi_{\theta\theta} \;=\; \partial_\ell \left[\frac{1}{\ell_\varphi}\,\Psi_\varphi\right] + \partial_\theta \left[\Psi_\phi - \frac{\ell_\phi}{\ell_\varphi}\Psi_\varphi\right], \tag{4.33}$$

$$\begin{aligned} \ell_\varphi \cdot (\Psi_{\ell\ell} + \Psi_{\theta\theta}) \;&=\; \left(\frac{\Psi_\varphi}{\ell_\varphi}\right)_\varphi + \left[\ell_\varphi\left(\Psi_\phi - \frac{\ell_\phi}{\ell_\varphi}\Psi_\varphi\right)\right]_\phi - \left[\ell_\phi\left(\Psi_\phi - \frac{\ell_\phi}{\ell_\varphi}\Psi_\varphi\right)\right]_\varphi \\ &=\; \left[\big(1+(\ell_\phi)^2\big)\frac{\Psi_\varphi}{\ell_\varphi} - \ell_\phi\Psi_\phi\right]_\varphi + \Big[\ell_\varphi\Psi_\phi - \ell_\phi\Psi_\varphi\Big]_\phi . \end{aligned} \tag{4.34}$$

Using (4.34), (4.29), and the first identity in (4.10), from (4.32) one obtains

$$\left[\big(1+(\ell_\phi)^2\big)\frac{\Psi_\varphi}{\ell_\varphi} - \ell_\phi\Psi_\phi\right]_\varphi + \Big[\ell_\varphi\Psi_\phi - \ell_\phi\Psi_\varphi\Big]_\phi \;=\; -\frac{\Psi_\beta}{\mu}\,\ell_\varphi\Psi_\varphi^{-\frac{1}{2\mu}}\,\omega^*(\phi)\,. \tag{4.35}$$

Replacing the expressions ℓ_ϕ, ℓ_φ by their values in terms of the partial derivatives of Ψ, given at (4.13), as in [16, 17] one eventually obtains the non-linear PDE

$$\left[\left(1+\Big(\frac{\Psi_{\beta\phi}}{2\Psi_\beta}\Big)^2\right)\frac{2\Psi_\beta\Psi_\varphi}{\Psi_{\beta\varphi}} - \frac{\Psi_{\beta\phi}\Psi_\phi}{2\Psi_\beta}\right]_\varphi + \left[\frac{\Psi_{\beta\varphi}\Psi_\phi - \Psi_{\beta\phi}\Psi_\varphi}{2\Psi_\beta}\right]_\phi \;=\; -\frac{\Psi_{\beta\varphi}}{2\mu}\,\Psi_\varphi^{-\frac{1}{2\mu}}\,\omega^*(\phi)\,. \tag{4.36}$$

Since the variable ϕ represents an angle, a solution of (4.36) will be defined for $\beta > 0$, $\phi \in \mathbb{R}$. It will be 2π-periodic w.r.t. the variable ϕ.

In view of (4.1) and (4.9), the third order equation (4.36) should be supplemented by the two boundary conditions

$$\begin{cases} \Psi(0,\phi) \;=\; \overline{\psi}(\phi), \\ \Psi_\beta(0,\phi) \;=\; -\,\mu R^2. \end{cases} \tag{4.37}$$

Notice that the first boundary condition in (4.1) does not show up in (4.37). Instead, it is used in (4.30) to obtain the explicit formula (4.29) for the vorticity in the adapted coordinates.

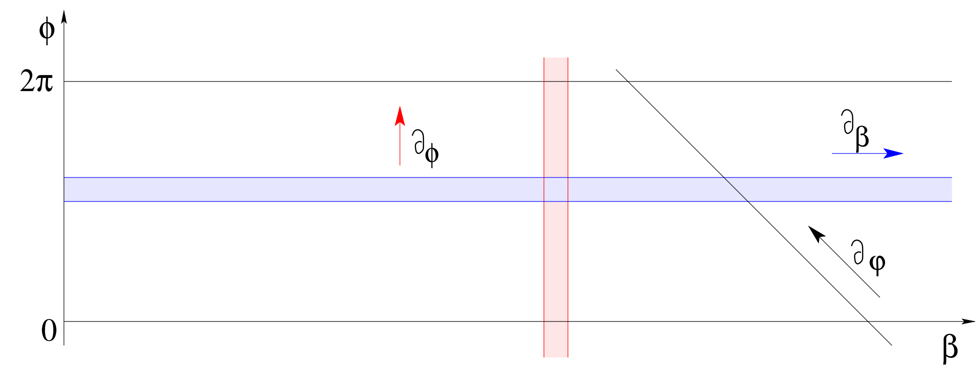

Figure 7: The coordinates β, ϕ and the vector field $\partial_\varphi \doteq \partial_\phi - \partial_\beta$.

4.3 Recovering the solution in the original coordinates.

As soon as the stream function $\Psi = \Psi(\beta, \phi)$ of (4.36) has been constructed as a function of the auxiliary variables β, ϕ, one can recover Ψ in terms of the polar coordinates r, θ by inverting (4.10). Namely,

$$\Psi\big(r(\beta, \phi),\, \theta(\beta, \phi)\big) \;=\; \Psi(\beta, \phi). \tag{4.38}$$

Here one has to check that (4.2) holds and that the determinant of the coordinate transformation at (4.16) does not vanish. In our context, this will be justified by the conclusion of Theorem 7.1.

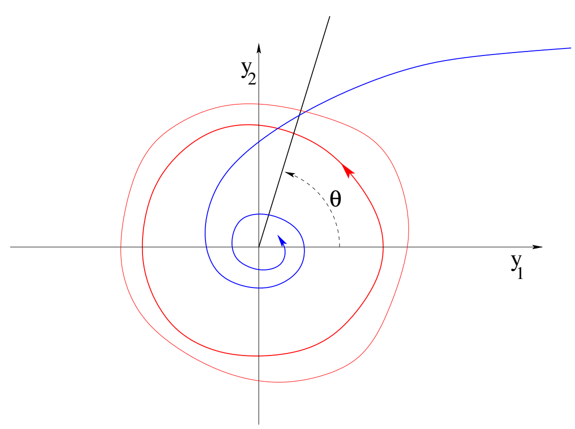

Figure 8: The images of the coordinate lines. The blue spiral corresponds to $\{\phi = \text{constant}\}$, while closed red curves correspond to $\{\beta = \text{constant}\}$. Finally, the ray $\theta = \text{constant}$ corresponds to the line $\{\varphi \doteq \phi - \beta = \text{constant}\}$.

Remark 4.2 At first sight, the equation (4.36) looks much harder to solve than the third order equation (2.12) or the equivalent system (2.9). The key advantage offered by (4.36) is that, at least in some cases, it allows us to resolve the singularity near the spiral's center. In other words, while the solution $\Psi(y)$ loses regularity near the center of the vortex spiral, in terms of the coordinates (β, ϕ) we expect that the same function Ψ will remain uniformly smooth.

Due to its complexity, the equation (4.36) will be solved by a perturbation argument. Starting from a radially symmetric solution, we first solve the linearized equation describing a first order perturbation.

In a second step, using the implicit function theorem, we obtain a solution to the original nonlinear problem. As shown in [16, 17], this approach can be successful as long as the solutions remain sufficiently close to a radially symmetric one.

5 Radially symmetric solutions

Radially symmetric solutions of (2.9), defined for all $r \geq 0$, can be easily constructed. As in [16, 17], such solutions play a fundamental role in our analysis. Indeed, the solutions we construct on the inner domain are obtained by suitably small perturbations of a radially symmetric solution.

In the radially symmetric case where $\Omega = \Omega(r)$, the first equation in (2.9) reduces to

$$-\mu r \, \Omega_r \;=\; \Omega. \tag{5.1}$$

Hence the vorticity has the form

$$\Omega(r) \;=\; c_0 \, r^{-\frac{1}{\mu}} \tag{5.2}$$

for some constant c_0. In turn the second equation yields

$$\Psi_{rr} + \frac{\Psi_r}{r} \;=\; c_0 r^{-\frac{1}{\mu}}, \tag{5.3}$$

$$(r\Psi_r)_r \;=\; c_0 r^{1-\frac{1}{\mu}},$$

$$r\Psi_r \;=\; c_0 \left(2 - \frac{1}{\mu}\right)^{-1} r^{2-\frac{1}{\mu}}.$$

Assuming $\mu > 1/2$, the stream function is thus computed as

$$\Psi(r) \;=\; c_0 \left(2 - \frac{1}{\mu}\right)^{-2} r^{2-\frac{1}{\mu}}. \tag{5.4}$$

Finally, the identity $U = \nabla^\perp \Psi$ implies that the velocity satisfies

$$|U| \;=\; |\Psi_r| \;=\; c_0 \left(2 - \frac{1}{\mu}\right)^{-1} r^{1-\frac{1}{\mu}}, \tag{5.5}$$

hence

$$U(x) \;=\; c_0 \left(2 - \frac{1}{\mu}\right)^{-1} |x|^{-\frac{1}{\mu}} \, x^\perp. \tag{5.6}$$

We rewrite this solution in terms of the adapted coordinates β, ϕ. By radial symmetry it follows $\beta = \beta(r)$. From the identity (4.9) we obtain

$$(\beta_r)^{-1} \Psi_r \;=\; -\mu r^2,$$

hence

$$\beta_r \;=\; -\frac{\Psi_r}{\mu r^2} \;=\; -\frac{1}{\mu r^2} \, c_0 \left(2 - \frac{1}{\mu}\right)^{-1} r^{1-\frac{1}{\mu}} \;=\; -\frac{c_0}{2\mu - 1} \, r^{-1-\frac{1}{\mu}}.$$

This yields

$$\beta \;=\; \frac{c_0\,\mu}{2\mu-1}\, r^{-\frac{1}{\mu}}. \tag{5.7}$$

Next, since the angular variable is $\theta=\beta+\phi$, the curves $\{\phi=\text{constant}\}$ correspond to algebraic spirals (see Fig. 9):

$$\theta \;=\; \frac{c_0\,\mu}{2\mu-1} r^{-\frac{1}{\mu}} + \phi\,. \tag{5.8}$$

Remark 5.1 The present analysis refers to radially symmetric solutions defined over the entire plane $\mathbb{R}^2$. For solutions defined on the disc $B_R=\{x\in\mathbb{R}^2\,;\;\; |x|<R\}$, the adapted coordinates β,ϕ defined in Section 4.1 were chosen so that

$$r\,=\,R \qquad \Longrightarrow \qquad \beta\,=\,0, \qquad \phi\,=\,\theta.$$

This only produces a shift in these coordinates, replacing (5.7)-(5.8) with

$$\beta \;=\; \frac{c_0\,\mu}{2\mu-1}\left(r^{-\frac{1}{\mu}}-R^{-\frac{1}{\mu}}\right), \qquad\qquad \theta \;=\; \frac{c_0\,\mu}{2\mu-1}\left(r^{-\frac{1}{\mu}}-R^{-\frac{1}{\mu}}\right)+\phi\,. \tag{5.9}$$

For convenience, throughout the following we shall use the coordinates (5.7)-(5.8). In particular, on the inner domain where $|y|\le R$, we have

$$\beta\,\in\,[b,+\infty[\,, \qquad\qquad b\,=\,\frac{c_0\mu}{2\mu-1}\,R^{-1/\mu}. \tag{5.10}$$

The boundary conditions (4.37) become

$$\left\{\begin{array}{rcl} \Psi(b,\phi) &=& \overline{\psi}(\phi),\\[2mm] \Psi_\beta(b,\phi) &=& -\,\mu R^2. \end{array}\right. \tag{5.11}$$

It is of interest to compute the various quantities appearing in the equations (4.36), in this radially symmetric case. Setting

$$\kappa \;\doteq\; \left(\frac{c_0\,\mu}{2\mu-1}\right)^{\mu},$$

one finds

$$\left\{\begin{array}{rcl} r &=& \kappa\,\beta^{-\mu},\\[2mm] r_\varphi &=& -\,r_\beta \;=\; \mu\kappa\,\beta^{-1-\mu},\\[2mm] \Psi_\phi &=& \Omega_\phi \;=\; r_\phi \;=\; 0,\\[2mm] \Omega &=& \Omega_0\,\beta,\\[2mm] \Psi &=& \frac{\mu\kappa^2}{2\mu-1}\,\beta^{1-2\mu},\\[2mm] \Psi_\beta &=& -\,\mu r^2 \;=\; -\,\mu\kappa^2\,\beta^{-2\mu},\\[2mm] \Psi_{\beta\beta} &=& 2\mu^2\kappa^2\,\beta^{-1-2\mu}. \end{array}\right. \tag{5.12}$$

Remark 5.2 Inserting

$$\Psi_\beta \;=\; -\,\mu\kappa^2\beta^{-2\mu}, \qquad \Psi_{\beta\beta} \;=\; 2\mu^2\kappa^2\beta^{-1-2\mu}, \qquad \Psi_\phi\;=\;0, \qquad \Psi_\varphi\;=\;-\Psi_\beta\,,$$

in (4.36), one obtains

$$
\begin{aligned}
-\left[\frac{2(\Psi_\beta)^2}{\Psi_{\beta\beta}}\right]_\beta + \left[\frac{1}{2\mu-1}(-\Psi_\beta)^{1-\frac{1}{2\mu}}\overline{\Omega}\right]_\beta &= 0,\\
-\left[\frac{2\mu^2\kappa^4\beta^{-4\mu}}{2\mu^2\kappa^2\beta^{-1-2\mu}}\right]_\beta + \left[\frac{1}{2\mu-1}(\mu\kappa^2\beta^{-2\mu})^{1-\frac{1}{2\mu}}\right]_\beta \Omega_0 &= 0,\\
-\kappa\cdot\left[\beta^{1-2\mu}\right]_\beta + \frac{(\mu\kappa^2)^{1-\frac{1}{2\mu}}}{2\mu-1}\left[\beta^{1-2\mu}\right]_\beta \Omega_0 &= 0.
\end{aligned}
$$

This yields

$$
\Omega_0 \;=\; (2\mu-1)\mu^{\frac{1}{2\mu}-1}\kappa^{\frac{1}{\mu}-1}. \tag{5.13}
$$

Remark 5.3 Consider a fluid in steady motion, with velocity $U = \nabla^\perp\Psi$ independent of time, as in (5.6). Consider all fluid particles that at time $t = 0$ lie on the positive x_1-axis, which In radial coordinates is

$$
\{(r,\theta)\,;\quad \theta = 0, \quad r > 0\}.
$$

At time $t = 1$, these moving particles are located along the spiral

$$
\Big\{(r,\theta)\,;\quad \theta \;=\; |U(r)|/r\,, \quad r > 0\Big\}.
$$

By (5.5) and (5.8), one checks that this spiral coincides with the spiral

$$
\Big\{\big(r(\beta,\phi),\theta(\beta,\phi)\big)\,;\quad \beta > 0, \quad \phi = 0\Big\}.
$$

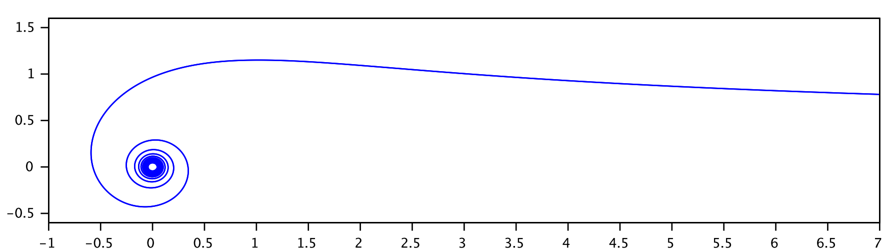

Figure 9: The algebraic spiral $\theta = \frac{3}{2}r^{-4/3}$ obtained by taking $\kappa = 1$, $\mu = 3/4$ in (5.8).

5.1 Rescaled equations

Motivated by (5.12), we expect that our solutions will have size $\Psi = \mathcal{O}(1)\cdot\beta^{1-2\mu}$. In order to construct solutions on a domain where $\beta >> 1$, again following [16, 17] it is thus convenient to introduce the rescaled variables

$$
\widetilde{\Psi} \;=\; \beta^{2\mu-1}\Psi\,. \tag{5.14}
$$

Defining the operators

$$
A \;\doteq\; -\beta\partial_\varphi\,, \qquad\qquad B_0 \;\doteq\; \beta\partial_\beta\,, \tag{5.15}
$$

$$
\begin{aligned}
\overline{\partial_\varphi} \;&\doteq\; \beta\partial_\varphi + 2\mu - 1 \;=\; -A + 2\mu - 1,\\
\overline{\partial_\beta} \;&\doteq\; \beta\partial_\beta + 1 - 2\mu \;=\; B_0 + 1 - 2\mu\,,
\end{aligned} \tag{5.16}
$$

from (5.14) one obtains

$$
\begin{cases}
\Psi_\beta &= \beta^{1-2\mu}\widetilde{\Psi}_\beta + (1-2\mu)\beta^{-2\mu}\widetilde{\Psi} \doteq \beta^{-2\mu}\overline{\partial_\beta}\widetilde{\Psi}\,, \\
\Psi_\phi &= \beta^{1-2\mu}\widetilde{\Psi}_\phi\,, \\
\Psi_\varphi &= \beta^{1-2\mu}\widetilde{\Psi}_\varphi - (1-2\mu)\beta^{-2\mu}\widetilde{\Psi} \doteq \beta^{-2\mu}\overline{\partial_\varphi}\widetilde{\Psi}\,, \\
\Psi_{\beta\varphi} &= \beta^{-2\mu}\Big(\beta\widetilde{\Psi}_{\beta\varphi} + (1-2\mu)(\widetilde{\Psi}_\varphi - \widetilde{\Psi}_\beta) + (1-2\mu)2\mu\beta^{-1}\widetilde{\Psi}\Big)\,.
\end{cases}
\tag{5.17}
$$

Using (5.17), we can write (4.36) in the equivalent form

$$
0 = \overline{\partial_\varphi}T^{(\varphi)} + \partial_\phi T^{(\phi)} + T^{(0)}\omega^*, \tag{5.18}
$$

where

$$
T^{(\varphi)} \doteq 2\left(1 + \left(\frac{\overline{\partial_\beta}\partial_\phi\widetilde{\Psi}}{2\overline{\partial_\beta}\widetilde{\Psi}}\right)^2\right) \times \frac{\overline{\partial_\beta}\widetilde{\Psi}\,\overline{\partial_\varphi}\widetilde{\Psi}}{(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\widetilde{\Psi}} - \frac{\partial_\phi\overline{\partial_\beta}\widetilde{\Psi}\cdot\partial_\phi\widetilde{\Psi}}{2\overline{\partial_\beta}\widetilde{\Psi}}\,, \tag{5.19}
$$

$$
T^{(\phi)} \doteq \frac{(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\widetilde{\Psi}\cdot\partial_\phi\widetilde{\Psi} - \partial_\phi\overline{\partial_\beta}\widetilde{\Psi}\cdot\overline{\partial_\varphi}\widetilde{\Psi}}{2\overline{\partial_\beta}\widetilde{\Psi}}\,, \tag{5.20}
$$

$$
T^{(0)} \doteq \frac{1}{2\mu}(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\widetilde{\Psi}\cdot(\overline{\partial_\varphi}\widetilde{\Psi})^{-1/2\mu}\,. \tag{5.21}
$$

More explicitly, substituting (5.17) in the third order equation (4.36) and dividing both sides by $\beta^{-2\mu}$, we eventually obtain

$$
\begin{aligned}
0 \;=\; & \frac{\omega^*}{2\mu}\Big((1-2\mu)\beta\widetilde{\Psi}_\varphi + \beta^2\widetilde{\Psi}_{\beta\varphi} - (1-2\mu)\beta\widetilde{\Psi}_\beta + (1-2\mu)2\mu\widetilde{\Psi}\Big)\big(\beta\widetilde{\Psi}_\varphi - (1-2\mu)\widetilde{\Psi}\big)^{-1/2\mu} \\
& + \left(\frac{\big((1-2\mu)\beta\widetilde{\Psi}_\varphi + \beta^2\widetilde{\Psi}_{\beta\varphi} - (1-2\mu)\beta\widetilde{\Psi}_\beta + (1-2\mu)2\mu\widetilde{\Psi}\big)\widetilde{\Psi}_\phi - \big((1-2\mu)\widetilde{\Psi}_\phi + \beta\widetilde{\Psi}_{\beta\phi}\big)\big(\beta\widetilde{\Psi}_\varphi - (1-2\mu)\widetilde{\Psi}\big)}{2((1-2\mu)\widetilde{\Psi} + \beta\widetilde{\Psi}_\beta)}\right)_\phi \\
& + \big(\beta\partial_\varphi - (1-2\mu)\big)\Bigg[2\big((1-2\mu)\widetilde{\Psi} + \beta\widetilde{\Psi}_\beta\big)\left(1 + \left(\frac{(1-2\mu)\widetilde{\Psi}_\phi + \beta\widetilde{\Psi}_{\beta\phi}}{2((1-2\mu)\widetilde{\Psi} + \beta\widetilde{\Psi}_\beta)}\right)^2\right) \\
& \qquad \times \frac{\beta\widetilde{\Psi}_\varphi - (1-2\mu)\widetilde{\Psi}}{(1-2\mu)\beta\widetilde{\Psi}_\varphi + \beta^2\widetilde{\Psi}_{\beta\varphi} - (1-2\mu)\beta\widetilde{\Psi}_\beta + (1-2\mu)2\mu\widetilde{\Psi}} - \frac{\widetilde{\Psi}_\phi((1-2\mu)\widetilde{\Psi}_\phi + \beta\widetilde{\Psi}_{\phi\beta})}{2((1-2\mu)\widetilde{\Psi} + \beta\widetilde{\Psi}_\beta)}\Bigg].
\end{aligned}
\tag{5.22}
$$

It remains to identify the appropriate boundary conditions for the rescaled equations. Recalling (5.10) and (5.14), from (5.11) one obtains

$$
\begin{cases}
\widetilde{\Psi}(b,\phi) &= b^{2\mu-1}\Psi(b,\phi) = b^{2\mu-1}\overline{\psi}(\phi) \doteq \widetilde{\psi}(\phi)\,, \\
-\mu\kappa^2 b^{-2\mu} &= -\mu R^2 = \Psi_\beta(b,\phi) = (1-2\mu)b^{-2\mu}\widetilde{\Psi}(b,\phi) + b^{1-2\mu}\widetilde{\Psi}_\beta(b,\phi).
\end{cases}
\tag{5.23}
$$

In these boundary conditions, we must guarantee that the second equation is satisfied, as this is intimately related to the change of variables for the adapted coordinates. The validity of the second equation will also be a major concern later on, when we construct our function spaces. As a consequence, this will require us to give up some flexibility in the choice of $\widetilde{\psi}$.

We remark that, after the rescaling, all the expressions (5.19)–(5.21) involve only the differential operators ∂_ϕ, $\overline{\partial_\beta}$ and $\overline{\partial_\varphi}$. A crucial part of the analysis in Section 6 is that the last two operators have bounded inverses within the space of bounded continuous functions (see Propositions 6.1 and Lemma 7.5). This is because the unbounded part of these operators, namely $\beta\partial_\beta$ and $\beta\partial_\varphi$, couple differentiation with multiplication by β. This will help facilitate the construction of appropriate function spaces in which solutions will exist.

6 Linearizing around the radially symmetric solution

Always following [16, 17], we now linearize the rescaled equation (5.22) around the constant solution $\widetilde{\Psi} \equiv 1$. Consider a family of perturbed solutions of the form

$$\begin{cases} \widetilde{\Psi}_\epsilon &= 1 + \epsilon\, Y(\phi,\beta) + o(\epsilon), \\ \Omega_\epsilon &= \dfrac{(2\mu-1)^{1+\frac{1}{2\mu}}}{\mu} + \epsilon\,\omega(\phi) + o(\epsilon). \end{cases} \tag{6.1}$$

Inserting (6.1) into (5.22) and matching terms of order ϵ, we obtain the linearized equation

$$\begin{aligned} 0 \;=\; & \frac{\omega}{2\mu}\big((1-2\mu)2\mu\big)(2\mu-1)^{-1/2\mu} \\ & + \frac{(2\mu-1)^{1+1/2\mu}}{2\mu^2}\Bigg(\Big((1-2\mu)(\beta\partial_\varphi - \beta\partial_\beta + 2\mu)Y + \beta^2 Y_{\beta\varphi}\Big)(2\mu-1)^{-1/2\mu} \\ & \qquad + \big(2\mu(1-2\mu)\big)\frac{-1}{2\mu}(2\mu-1)^{-1-1/2\mu}\big(\beta Y_\varphi + (2\mu-1)Y\big)\Bigg) \\ & + \frac{(1-2\mu)2\mu Y_{\phi\phi}}{2(1-2\mu)} + \frac{(1-2\mu)\big((1-2\mu)Y_{\phi\phi} + \beta Y_{\beta\phi\phi}\big)}{2(1-2\mu)} \\ & + 2\big(\beta\partial_\varphi - (1-2\mu)\big)\Bigg[\frac{2\mu-1}{(1-2\mu)2\mu}\big(\beta\partial_\beta + (1-2\mu)\big)Y + \frac{(1-2\mu)}{2\mu(1-2\mu)}\big(\beta\partial_\beta - (1-2\mu)\big)Y \\ & + (1-2\mu)^2\frac{1}{\big((1-2\mu)2\mu\big)^2}\Big((1-2\mu)2\mu + (1-2\mu)\beta\partial_\varphi + \beta^2\partial_{\beta\varphi} - (1-2\mu)\beta\partial_\beta\Big)Y\Bigg]. \end{aligned} \tag{6.2}$$

Using the fact that $\beta^2 Y_{\beta\varphi} = (\beta\partial_\varphi + 1)\beta\partial_\beta Y$, setting

$$\widetilde{\omega} \;=\; 2\mu^2(2\mu-1)^{1-1/2\mu}\,\omega\,,$$

and performing some simple cancellations, we obtain

$$\begin{aligned} \widetilde{\omega} \;=\; & (2\mu-1)\Big(\beta\partial_\varphi + (1-2\mu)\beta\partial_\varphi + (\beta\partial_\varphi+1)\beta\partial_\beta - (1-2\mu)\beta\partial_\beta - (2\mu-1)^2\Big)Y \\ & + \mu^2(\beta\partial_\beta + 1)Y_{\phi\phi} \\ & + \Big(\beta\partial_\varphi - (1-2\mu)\Big)\Bigg(2\mu\big((\beta\partial_\varphi - (1-2\mu)\big) - \big(\beta\partial_\beta + (1-2\mu)\big) \\ & \qquad + \Big((1-2\mu)2\mu + (1-2\mu)\beta\partial_\varphi + (\beta\partial_\varphi+1)\beta\partial_\beta - (1-2\mu)\beta\partial_\beta\Big)\Bigg)Y. \end{aligned} \tag{6.3}$$

We can factor the above operators as

$$\begin{aligned}\widetilde{\omega} \;&=\; (2\mu-1)\Big(\beta Y_\varphi + \beta Y_\beta + (\beta\partial_\varphi - (1-2\mu))(\beta\partial_\beta + (1-2\mu))\Big)Y \\ &\quad +\mu^2(\beta\partial_\beta+1)Y_{\phi\phi} \\ &\quad +\big(\beta\partial_\varphi - (1-2\mu)\big)\Big(\big(\beta\partial_\varphi - (1-2\mu)\big)(\beta\partial_\beta+1) - (2\mu-1)\big(\beta\partial_\beta + (1-2\mu)\big)\Big)Y.\end{aligned} \tag{6.4}$$

Combining terms, one obtains

$$\widetilde{\omega} \;=\; (2\mu-1)(\beta\partial_\varphi + \beta\partial_\beta)Y + \mu^2\partial_\phi^2(\beta\partial_\beta+1)Y + (\beta\partial_\varphi + (2\mu-1))^2(\beta\partial_\beta+1)Y. \tag{6.5}$$

We thus recover the linearized equation in the same form derived in [16, 17], namely

$$\widetilde{\omega} \;=\; (2\mu-1)\beta Y_\phi + \Big(\beta\partial_\varphi + (2\mu-1) + i\mu\partial_\phi\Big)\Big(\beta\partial_\varphi + (2\mu-1) - i\mu\partial_\phi\Big)(\beta\partial_\beta+1)Y. \tag{6.6}$$

Remark 6.1 The equation for the rescaled perturbation is independent of the constant κ in (5.12). For convenience, in the above we chose $\kappa = \left(\frac{2\mu-1}{\mu}\right)^{1/2}$.

6.1 Fourier transform in ϕ

We shall compute the solution of (6.6), and later also of the original equation (5.18), in terms of a Fourier series in ϕ. Writing

$$Y(\beta,\phi) \;=\; \sum_{k\in\mathbb{Z}} Y_k(\beta)\, e^{ik\phi}, \qquad\qquad \widetilde{\omega}(\phi) \;=\; \sum_{k\in\mathbb{Z}} \omega_k\, e^{ik\phi}, \tag{6.7}$$

from (6.6) we obtain a countable family of decoupled ODEs for the coefficients Y_k, namely

$$A_k^- A_k^+ B\, Y_k + (2\mu-1)ik\beta\, Y_k \;=\; \omega_k\,. \tag{6.8}$$

Here A_k^+, A_k^-, and B denote first order differential operators:

$$\begin{cases} A_k^- &=\; \beta(\partial_\beta - ik) - (2\mu-1) - k\mu, \\ A_k^+ &=\; \beta(\partial_\beta - ik) - (2\mu-1) + k\mu, \\ B &=\; \beta\partial_\beta + 1. \end{cases} \tag{6.9}$$

We remark that, when $k = 0$, the equation (6.8) has the simple solution

$$Y_0(\beta) \;=\; (2\mu-1)^{-2}\omega_0 + c_0\beta^{-1}\,, \tag{6.10}$$

Notice that here we neglect the other two homogeneous solutions, namely $\beta^{2\mu-1}$ and $\log(\beta)\beta^{2\mu-1}$, because they are unbounded as $\beta \to \infty$. We observe that, when we invert the rescaling (5.14) and go back to the original variable Ψ, adding these two homogeneous solutions to $\widetilde{\Psi}$ amounts to adding constants to Ψ and to the Green's function of the Laplace operator in $\mathbb{R}^2$. For convenience, we choose to re-introduce the first of these homogeneous solutions only at the very end of our analysis.

Toward the analysis of the nonlinear problem, we shall need to consider the equation with $k = 0$ also in the presence of a non constant right hand side. This will be studied together with the cases $k \neq 0$.

When $k \neq 0$, a more careful analysis is needed. In the following, we focus on the case $k \geq 1$. The analysis for $k \leq -1$ is entirely analogous.

Looking at (6.8), it would seem natural to treat the term $(2\mu - 1)ik\beta$ as a relatively bounded perturbation of the third order operator $A_k^- A_k^+ B$. As shown in [17, 18], this approach works well for large values of k. However, when k is small, this strategy is not effective. For this reason, in the following we write the left hand side of (6.8) in a different way, which is effective for all μ, k on the domain $\beta \in [b, +\infty[$, with b large. More precisely, for $k \neq 0$, we consider the equivalent linear differential equation

$$\widehat{A}_k^+ \widehat{A}_k^- BY_k + (2\mu - 1)(ik\beta - B)Y_k \;=\; \omega_k\,, \tag{6.11}$$

where we let

$$\widehat{A}_k^+ \doteq \beta(\partial_\beta - ik) - a_k^+, \tag{6.12}$$

$$\widehat{A}_k^- \doteq \beta(\partial_\beta - ik) - a_k^-, \tag{6.13}$$

$$a_k^+ \doteq (2\mu - 1) + \sqrt{k^2\mu^2 - (2\mu - 1)}, \qquad a_k^- \doteq (2\mu - 1) - \sqrt{k^2\mu^2 - (2\mu - 1)}. \tag{6.14}$$

Since we are always assuming $\mu > 1/2$, a direct computation yields

$$0 < a_1^- < a_1^+\,, \qquad\qquad -1 < a_2^- < 0 < a_2^+\,, \tag{6.15}$$

$$a_{k+1}^- < a_k^- < -1 < 0 < a_k^+ < a_{k+1}^+ \qquad \text{for} \;\; k \geq 3. \tag{6.16}$$

The advantage of this factorization will become apparent in the proof of Proposition 6.2, where it will achieve a cancellation of certain terms after integration by parts. In the special case $k = 0$, instead of (6.12)-(6.13) we define

$$\widehat{A}_0^+ \;\doteq\; \widehat{A}_0^- \;\doteq\; \beta\,\partial_\beta - (2\mu - 1).$$

Notice that, by (6.9), these operators coincide with A_0^- and A_0^+.

6.2 Solutions to the linearized equation.

The main goal of this section is to establish the solvability of (6.11), within an appropriate function space, on the half line $\beta \in [b, +\infty[$, for some $b > 0$ suitably large and with suitable boundary conditions. We begin by introducing some appropriate function spaces in which the three operators $\widehat{A}_k^+$, $\widehat{A}_k^-$, B, possess a right inverse.

Definition 6.1 *For a fixed $\delta > 0$, we define the spaces of continuous functions $g : [b, +\infty[\mapsto \mathbb{R}$ by setting*

$$\mathcal{H}_\delta \doteq \Big\{ g \in \mathcal{C}^0([b, +\infty[\;; \quad \|g\|_\delta \doteq \sup_{\beta \in [b,\infty]} \big|\beta^\delta g(\beta)\big| \;<\; \infty \Big\}. \tag{6.17}$$

$$\begin{aligned} \mathcal{H}_\delta^+ \;\doteq\; \Big\{ g \in \mathcal{C}^0([b, +\infty[\;; &\quad \textit{there exists the limit } g_\infty \;=\; \lim_{\beta \to \infty} g(\beta), \\ &\textit{and moreover} \quad \|g\|_\delta^+ \;\doteq\; |g_\infty| + \|g - g_\infty\|_\delta \;<\; \infty \Big\}. \end{aligned} \tag{6.18}$$

In other words, $\mathcal{H}_\delta$ is the space of all continuous functions which converge to zero at rate $g(\beta) = \mathcal{O}(1)\cdot\beta^{-\delta}$, while $\mathcal{H}_\delta^+$ contains all functions converging to some limit g_∞ at rate $g(\beta) - g_\infty = \mathcal{O}(1)\cdot\beta^{-\delta}$. The form of (6.10) requires that we use $\mathcal{H}_\delta^+$ in constructing solutions for the $k = 0$ mode. On the other hand, we shall see that for $k \neq 0$ we may restrict our attention to $\mathcal{H}_\delta$.

In the same spirit as Proposition 11 in [17], we now establish the right invertibility of the above operators.

Proposition 6.1 *Let $0 < \delta < -a_2^-$. Then the operators defined by*

$$\begin{aligned}
\big[(\widehat{A}_k^+)^\dagger g\big](\beta) &= e^{ik\beta}\beta^{a_k^+}\int_\beta^\infty s^{-a_k^+-1}e^{-iks}g(s)\,ds \qquad \textit{for}\quad k\geq 1,\\
\big[(\widehat{A}_k^-)^\dagger g\big](\beta) &= \begin{cases} e^{ik\beta}\beta^{a_k^-}\displaystyle\int_b^\beta s^{-a_k^--1}e^{-iks}g(s)\,ds & \textit{if}\quad k\geq 2,\\ e^{ik\beta}\beta^{a_k^-}\displaystyle\int_\beta^\infty s^{-a_k^--1}e^{-iks}g(s)\,ds & \textit{if}\quad k=1,\end{cases}\\
\big[B^\dagger g\big](\beta) &= \beta^{-1}\int_b^\beta g(s)\,ds,
\end{aligned}$$

are injective operators on H_δ, providing the right inverses of $\widehat{A}_k^+$, $\widehat{A}_k^-$, and B, respectively. Moreover, they are bounded linear operators on H_δ, with norms

$$\|(\widehat{A}_k^+)^\dagger\|_{L(H_\delta)} \;\leq\; \frac{1}{a_k^+ + \delta}\,, \tag{6.19}$$

$$\|(\widehat{A}_k^-)^\dagger\|_{L(H_\delta)} \;\leq\; \begin{cases}\dfrac{1}{-a_k^- - \delta} & \textit{if}\quad k\geq 2,\\[2mm] \dfrac{1}{a_k^- + \delta} & \textit{if}\quad k=1,\end{cases} \tag{6.20}$$

$$\|B^\dagger\|_{L(H_\delta)} \;\leq\; \frac{1}{1-\delta}\,. \tag{6.21}$$

Furthermore, for any $\epsilon > 0$ and $k \neq 0$, one has that

$$\|(\widehat{A}_k^+)^\dagger\|_{L(H_\epsilon)} \;\leq\; \frac{1}{a_k^+ + \epsilon}, \qquad\qquad \|(\widehat{A}_k^+)^\dagger\mathbf{1}\|_{H_\epsilon} \;\leq\; \frac{C}{k}b^{-1+\epsilon}, \tag{6.22}$$

where C does not depend on k, and $\mathbf{1}$ denotes the constant function.

Proof: We first show boundedness of the operators. In the case of $B^\dagger$, it is straightforward to check that

$$\|B^\dagger g\|_\delta \;\leq\; \sup_\beta \beta^{\delta-1}\int_b^\beta |g(s)|\,ds \;\leq\; \|g\|_\delta\cdot\sup_\beta \beta^{\delta-1}\int_b^\beta s^{-\delta}\,ds \;\leq\; \frac{\|g\|_\delta}{1-\delta}\,,$$

where in the second inequality we used the fact that $0 < \delta < 1$.

Concerning $(\widehat{A}_k^-)^\dagger$, for $k \geq 2$, recalling that $0 < \delta < -a_2^- < -a_k^-$ we estimate

$$\begin{aligned}
\|(\widehat{A}_k^-)^\dagger g\|_\delta &\leq \sup_{\beta\in[b,\infty)} \beta^{\delta+a_k^-}\int_b^\beta s^{-a_k^--1}|g(s)|\,ds\\
&\leq \|g\|_\delta \sup_{\beta\in[b,\infty)} \beta^{\delta+a_k^-}\int_b^\beta s^{-a_k^--1-\delta}\,ds \;\leq\; \frac{\|g\|_\delta}{-a_k^- - \delta}.
\end{aligned}$$

The same estimate also works for $(A_k^+)^\dagger$ and $(A_k^-)^\dagger$ (when $k=1$), with the additional observation that in this case any choice of $0<\delta$ is possible. On the other hand, a direct computation shows that these are right inverses, and hence must be invertible.

The last estimate in (6.22), concerning the action of $(\widehat{A}_k^+)^\dagger$ on constant functions, is achieved by an integration by parts. This concludes the proof. $\square$

Remark 6.2 The previous estimates are valid on any domain $[b,+\infty[$. Under the above assumptions on δ, in the cases where b shows up in the expressions (namely in $(\widehat{A}_k^-)^\dagger$ and in $B^\dagger$), we use the fact that, after integrating, we obtain a positive power $s>0$. Therefore, we can bound $|\beta^s - b^s|$ with β^s.

We notice that the previous proposition, along with the equation (6.10), directly establishes the solvability of (6.11) within H_δ^+.

Next, we define the functions

$$\begin{cases} E_k^0(\beta) & \doteq \ \beta^{-1} & \text{for } k\in\mathbb{Z}, \\ E_k^-(\beta) & \doteq \ b^{-a_k^-}\cdot\beta^{-1}\int_b^\beta e^{iks}s^{a_k^-}\,ds & \text{for } k\in\mathbb{Z},\ k\neq 0, \\ E_k^+(\beta) & \doteq \ b^{-a_k^+}\cdot\beta^{-1}\int_b^\beta e^{iks}s^{a_k^+}\,ds & \text{for } k\in\mathbb{Z},\ k\neq 0. \end{cases} \tag{6.23}$$

Remark 6.3 It is straightforward to verify that each of these functions is in the kernel of $\widehat{A}_k^+\widehat{A}_k^-B$. Moreover, E_k^+ is unbounded as long as $a_k^+>1$. Since this is the case for many choices of μ and k, we will not include E_k^+ as a part of our function space. It is also easy to check that E_k^- decays faster than β^{-1}, for all $|k|\geq 2$. Later on, we shall identify situations where we can include E_k^- in our function spaces, in order to satisfy the boundary conditions. Here the factors $b^{-a_k^-}$ and $b^{-a_k^+}$ provide a natural normalization, see Proposition 6.3.

With these definitions at hand, we are prepared to define the appropriate function spaces on which we will construct solutions to these linearized equations. It is important to observe that the choice of these function spaces, which leads to an intermediate Banach algebra, later in Section 7, will allow us to handle non-linear terms as well. For convenience, we use the notation

$$\langle k\rangle \ \doteq \ (1+k^2)^{1/2}$$

Definition 6.2 *Recalling (6.23), we introduce the spaces*

$$\mathcal{E}_k \ \doteq \ \begin{cases} B^\dagger(\widehat{A}_k^-)^\dagger H_\delta\oplus\mathbb{C}E_k^0\oplus\mathbb{C}E_k^- & \textit{for } |k|\geq 2, \\ B^\dagger(\widehat{A}_k^-)^\dagger H_\delta\oplus\mathbb{C}E_k^0 & \textit{for } |k|=1, \\ B^\dagger(\widehat{A}_k^-)^\dagger H_\delta^+\oplus\mathbb{C}E_k^0 & \textit{for } k=0, \end{cases} \tag{6.24}$$

$$\mathcal{F}_k \ \doteq \ \begin{cases} \widehat{A}_k^+H_\delta & \textit{if } k\neq 0, \\ \widehat{A}_k^+H_\delta^+ & \textit{if } k=0, \end{cases} \tag{6.25}$$

with norms

$$\begin{cases} \|g_k + a_1 E_k^0 + a_2 E_k^-\|_{\mathcal{E}_k} = \|A_k^- B g_k\|_\delta + \langle k\rangle |a_1| + \langle k\rangle |a_2| & \text{for } |k| \geq 2, \\ \|g_k + a_1 E_k^0\|_{\mathcal{E}_k} = \|A_k^- B g_k\|_\delta + \langle k\rangle |a_1| & \text{for } |k| = 1, \\ \|g_k + a_1 E_k^0\|_{\mathcal{E}_k} = \|A_k^- B g_k\|_\delta^+ + \langle k\rangle |a_1| & \text{for } k = 0, \end{cases} \tag{6.26}$$

$$\begin{cases} \|\widehat{A}_k^+ g_k\|_{\mathcal{F}_k} = \|g_k\|_\delta, & \text{for } k \neq 0, \\ \|\widehat{A}_k^+ g_k\|_{\mathcal{F}_k} = \|g_k\|_\delta^+, & \text{for } k = 0. \end{cases} \tag{6.27}$$

Here the elements of $\mathcal{F}_k$ are understood in the sense of distributions. We notice that, if $g_k \in \mathcal{F}_k$ is a bounded continuous function, then (for $k \neq 0$) we have $\|g_k\|_{\mathcal{F}_k} = \|(\widehat{A}_k^+)^\dagger g_k\|_\delta$.

Notice that we have not included E_k^- as part of our function spaces when $|k| = 1$. This is because, in these cases, the term AE_k^- (which shows up in the expression for $T^{(\varphi)}$) may not be in H_δ^+. See Lemma 7.4 for more details.

Our first main goal is to prove the solvability of the linear equation (6.11) within $\mathcal{E}_k$, for a right hand side in $\mathcal{F}_k$. For this purpose, we show that the perturbation term in (6.11) does not spoil the invertibility of the equation within these spaces, provided we work on a domain $\beta \in [b, +\infty[$ with b large enough. Notice that here we do not yet prescribe boundary conditions at $\beta = b$. This will be done at a later stage.

Proposition 6.2 *Let $\mu > 1/2$ and $0 < \delta < -a_2^-$ be given. Then there exists a constant $C > 0$ so that for all $k \in \mathbb{Z}$, $k \neq 0$,*

$$\|(2\mu - 1)(ik\beta - B)\|_{L(B^\dagger (\widehat{A}_k^-)^\dagger H_\delta; \mathcal{F}_k)} < Cb^{-1}. \tag{6.28}$$

By choosing b large enough, one can construct a linear, bounded (uniformly in k) solution operator for (6.11)*, mapping a right hand side $\omega_k \in \mathcal{F}_k$ to a solution $Y_k \in B^\dagger (\widehat{A}_k^-)^\dagger H_\delta$.*

This solution operator can be written as

$$Y_k = \Big[B^\dagger (\widehat{A}_k^-)^\dagger (\widehat{A}_k^+)^\dagger + Q_k\Big]\omega_k\,,$$

where Q_k satisfies

$$\|Q_k\|_{L(\mathcal{F}_k; B^\dagger (\widehat{A}_k^-)^\dagger H_\delta)} \leq Cb^{-1}.$$

Proof. We need to bound the operator norm

$$\big\|(\widehat{A}_k^+)^\dagger ((2\mu - 1)ik\beta - (2\mu - 1)B) B^\dagger (\widehat{A}_k^-)^\dagger\big\|_{L(H_\delta)}.$$

Recalling the formula for $(\widehat{A}_k^-)^\dagger$, writing $(\widehat{A}_k^-)^\dagger g(s) = e^{iks} h(s)$ and integrating by parts, one obtains

$$
\begin{aligned}
&((2\mu-1)ik\beta-(2\mu-1)B)B^\dagger(\widehat{A}_k^-)^\dagger g\\
&\quad= \ (2\mu-1)\left(ik\int_b^\beta e^{iks}h(s)\,ds-e^{ik\beta}h(\beta)\right)\\
&\quad= \ -(2\mu-1)\left(-e^{ikb}h(b)-\int_b^\beta e^{iks}h'(s)\,ds\right)\\
&\quad= \ -(2\mu-1)((\widehat{A}_k^-)^\dagger g)(b)\\
&\qquad+\begin{cases}-(2\mu-1)\displaystyle\int_b^\beta\left(s^{-1}g(s)+a_k^-e^{iks}s^{a_k^--1}\int_b^s e^{-ikt}t^{-a_k^--1}g(t)\,dt\right)ds & \text{for } k>2,\\ -(2\mu-1)\displaystyle\int_b^\beta\left(s^{-1}g(s)+a_k^-e^{iks}s^{a_k^--1}\int_s^\infty e^{-ikt}t^{-a_k^--1}g(t)\,dt\right)ds & \text{for } k=1,2.\end{cases}
\end{aligned}
$$

By changing the order of integration and integrating by parts, the double integral in each of the above expressions yields only a term of order $\beta^{-1-\delta}+b^{-1-\delta}$. Applying $(\widehat{A}_k^+)^\dagger$ and using the bound (6.22), we thus obtain terms which are bounded in the H_δ norm by $C(b^{-2}+b^{-1})$, with a constant C independent of k. Notice that the first term in this expression, namely $(2\mu-1)((\widehat{A}_k^-)^\dagger g)(b)$ (which does not depend on β), after applying $(\widehat{A}_k^+)^\dagger$ for the same reason is bounded by Cb^{-1} in the H_δ norm.

Concerning the single integral, by applying $(\widehat{A}_k^-)^\dagger$ one obtains

$$
e^{ik\beta}\beta^{a_k^+}\int_\beta^\infty s^{-a_k^+-1}e^{-iks}\int_b^s t^{-1}g(t)\,dt\,ds \ = \ e^{ik\beta}\beta^{a_k^+}\int_b^\infty t^{-1}g(t)\int_{t\vee\beta}^\infty s^{-a_k^+-1}e^{-iks}\,ds\,dt.
$$

Integrating by parts, we find that this expression is bounded by b^{-1} in the H_δ norm. This establishes the estimate (6.28). In turn, for b large enough, we obtain the solvability of (6.11). The bounds on the solution operator follow by a standard perturbation argument. □

Remark 6.4 *By setting $\beta=b$ and computing the composed operator $B^\dagger(\widehat{A}_k^-)^\dagger(\widehat{A}_k^+)^\dagger$, one checks that the solutions constructed in Proposition 6.2 satisfy $Y_k(b)=0$.*

Next, we will show that, by including the functions E_k^0, E_k^- in the space $\mathcal{E}_k$, we can solve (6.11) with some prescribed boundary conditions. As a first step, we introduce an operator corresponding to our rescaled boundary condition:

$$
\mathbb{H}(Y) \ \doteq \ (1-2\mu)b^{-2\mu}Y+b^{1-2\mu}\partial_\beta Y. \tag{6.29}
$$

Proposition 6.3 *Fix $\mu>1/2$ and $0<\delta<-a_2^-$. Let b be sufficiently large, so that the conclusion of Proposition 6.2 holds. Then the functions E_k^0 and (when $|k|\geq 2$) E_k^- are not elements of $B^\dagger(\widehat{A}_k^-)^\dagger H_\delta$ (respectively, of $B^\dagger(\widehat{A}_k^-)^\dagger H_\delta^+$ when $k=0$).*

However, there exist functions $\widetilde{E}_k^0=E_k^0+Z_k^0$ and, for $|k|\geq 2$, $\widetilde{E}_k^-=E_k^-+Z_k^-$, such that

$$
A_k^-A_k^+B\,E_k^0+(2\mu-1)ik\beta\,\widetilde{E}_k^0 \ = \ 0 \qquad A_k^-A_k^+B\,E_k^-+(2\mu-1)ik\beta\,\widetilde{E}_k^- \ = \ 0. \tag{6.30}
$$

Here the additional terms Z_k^0, Z_k^- satisfy

$$Z_k^0 \;=\; B^\dagger(\widehat{A}_k^-)^\dagger(\widehat{A}_k^+)^\dagger(2\mu-1)ik + W_k^0, \qquad \textit{with} \qquad \|W_k^0\|_{\mathcal{E}_k} \;\leq\; Cb^{-2+\delta},$$

$$Z_k^- \;= B^\dagger(\widehat{A}_k^-)^\dagger(\widehat{A}_k^+)^\dagger\left(-(2\mu-1)e^{ikb} - (2\mu-1)b^{-a_k^-}\int_b^\beta e^{iks}a_k^- s^{a_k^- -1}\,ds\right) + W_k^-\,,$$

with $\|W_k^-\|_{\mathcal{E}_k} \;\leq\; Cb^{-2+\delta}$*, where $C>0$ is a constant independent of k, not necessarily the same as in Proposition 6.2.*

Furthermore, there exists a unique solution (in $\mathcal{E}_k$) to the homogeneous problem with boundary conditions

$$A_k^- A_k^+ B\, Y_k + (2\mu-1)ik\beta\, Y_k \;=\; 0, \qquad \begin{cases} \mathbb{H}(Y_k)(b) = b_k, \quad Y_k(b) = d_k & \textit{for} \;\; k\geq 3,\\ \mathbb{H}(Y_k)(b) = b_k & \textit{for} \;\; k = 0,1,2. \end{cases}$$

Such solution can be written in the form

$$Y_k \;=\; \begin{cases} c_k^0\widetilde{E}_k^0 + c_k^-\widetilde{E}_k^- & \textit{for} \;\; k\geq 3,\\ c_k^0\widetilde{E}_k^0 & \textit{for} \;\; k=0,1,2, \end{cases}$$

with

$$c_k^0 \;=\; \mathcal{O}(1)\cdot bd_k\,, \qquad c_k^- \;=\; \mathcal{O}(1)\cdot\left(\frac{b^{-2\mu}d_k + b_k}{b^{-2\mu}}\right) \qquad \textit{when} \;\; k\geq 3,$$

$$c_k^0 \;=\; \mathcal{O}(1)\cdot b^{2\mu+1}b_k \qquad \textit{when} \;\; k=0,1,2.$$

Proof. 1. The case $k=0$ is resolved by a direct computation, as no perturbation term is present.

In the case $k\geq 1$, since by construction E_k^0 and E_k^- lie in the kernel of $\widehat{A}_k^- B$, they are not included in the space $B^\dagger(\widehat{A}_k^-)^\dagger H_\delta$. Moreover, the definition (6.23) implies $E_k^-(b)=0$.

2. We now construct Z_k^0 and Z_k^- so that the equation (6.30) is satisfied. We observe that

$$\big(\widehat{A}_k^+\widehat{A}_k^- B - (2\mu-1)B + (2\mu-1)ik\beta\big)E_k^0 \;=\; (2\mu-1)ik. \tag{6.31}$$

Since

$$\widehat{A}_k^\dagger(2\mu-1)ik \;=\; (2\mu-1)\beta^{-1} + O(1)\cdot\beta^{-2}\,,$$

the H_δ norm of the above quantity has magnitude $\mathcal{O}(1)\cdot b^{\delta-1}$. Thus by taking the constant function $\omega_k(\beta) = (2\mu-1)ik$ for all $\beta\in[b,+\infty[$, in (6.11), by applying Proposition 6.2 we obtain the existence of Z_k^0 with the desired properties.

On the other hand, to study E_k^- for $|k|\geq 2$, integrating by parts we obtain

$$\Big(\widehat{A}_k^+\widehat{A}_k^- B - (2\mu-1)B + (2\mu-1)ik\beta\Big)E_k^- \;=\; (2\mu-1)b^{-a_k^-}\left(-e^{ik\beta}\beta^{a_k^-} + ik\int_b^\beta e^{iks}s^{a_k^-}\,ds\right)$$

$$=\; -(2\mu-1)e^{ikb} - (2\mu-1)b^{-a_k^-}\int_b^\beta e^{iks}a_k^- s^{a_k^- -1}\,ds.$$

We recall that, for $|k| \geq 2$, one has $a_k^- < 0$. Applying $(\widehat{A}_k^+)^\dagger$ to the previous expression one obtains a function in H_δ whose norm has size $\mathcal{O}(1)\cdot b^{-1+\delta}$. Once again, applying Proposition 6.2 we obtain the existence of Z_k^- satisfying the desired bounds.

3. Having constructed $\widetilde{E}_k^0$ and $\widetilde{E}_k^-$, we now show that they allow us to solve the ODE (6.11) with the prescribed boundary conditions, satisfying the required bounds.

In the case $|k| = 1$, it suffices to take

$$b_k \;=\; \mathbb{H}(c_k^0 \widetilde{E}_k^0)(b) \;=\; c_k^0(1-2\mu)b^{-2\mu}(E_k^0(b) + Z_k^0(b)) + c_k^0 b^{1-2\mu}((E_k^0)'(b) + (Z_k^0)'(b)).$$

By a direct computation one checks that $E_k^0(b) = b^{-1}$ and $(E_k^0)'(b) = -b^{-2}$. Hence

$$\mathbb{H}(E_k^0)(b) \;=\; -\,2\mu b^{-2\mu-1}.$$

Furthermore, since $Z_k^0 \in \mathcal{E}_k$, we have that $Z_k^0(b) = 0$. Another direct computation yields $(Z_k^0 - W_k^0)'(b) = \mathcal{O}(1)\cdot b^{-3}$.

On the other hand, by using the fact that $\|W_k^0\|_{\mathcal{E}_k} \leq b^{-2+\delta}$, then taking a derivative of the expression for B and recalling that $\widehat{A}_k^-$ is bounded on H_δ, we obtain

$$(W_k^0)'(b) \;=\; \mathcal{O}(1)\cdot b^{-3+\delta}.$$

Combining these facts, always in the case $k = 1$, we conclude

$$c_k^0 \;=\; \mathcal{O}(1)\cdot b^{2\mu-1} b_k\,.$$

On the other hand, when $|k| \geq 2$, we have $\widetilde{E}_k^-(b) = 0$, because $E_k^-(b) = 0$ and Z_k^- is an element of $\mathcal{E}_k$. The form of E_k^0, along with the construction of Z_k^0 and Remark 6.4, implies that, for $|k| \geq 2$, one has $c_k^0 = b d_k$. Hence

$$b_k \;=\; \mathbb{H}(c_k^0 \widetilde{E}_k^0 + c_k^- \widetilde{E}_k^-)(b) \;=\; \mathcal{O}(1)\cdot d_k b^{-2\mu} + c_k^- \mathbb{H}(\widetilde{E}_k^-).$$

We now compute

$$\mathbb{H}(E_k^-)(b) \;=\; e^{ikb} b^{-2\mu}, \qquad \begin{cases} \mathbb{H}(Z_k^-)(b) \;=\; \mathcal{O}(1)\cdot b^{-2\mu-2+\delta}, \\ \mathbb{H}(Z_k^- - W_k^-)(b) \;=\; \mathcal{O}(1)\cdot b^{-2\mu-2}. \end{cases}$$

In turn this implies

$$c_k^- \;=\; \mathcal{O}(1)\cdot\left(\frac{b^{-2\mu} d_k + b_k}{b^{-2\mu}}\right).$$

□

Define the spaces of admissible boundary values

$$\mathcal{B}_k = \begin{cases} \mathbb{C}^2 & \text{for } |k| \geq 2, \\ \mathbb{C} & \text{for } |k| = 0,1, \end{cases} \qquad \begin{cases} \big\|(b_k, d_k)\big\|_{\mathcal{B}_k} \doteq \langle k\rangle(|b_k| + |d_k|) & \text{for } |k| \geq 2, \\ \|b_k\|_{\mathcal{B}_k} \doteq \langle k\rangle |b_k| & \text{for } |k| = 0,1. \end{cases} \tag{6.32}$$

Notice that we assign two boundary values for $|k| \geq 2$, but only one for $|k| = 0, 1$. The following proposition summarizes the results of this subsection.

Proposition 6.4 *Consider the linear boundary value problem for* (6.8), *namely*

$$A_k^- A_k^+ B\, Y_k + (2\mu - 1) ik\beta\, Y_k \;=\; \omega_k, \qquad \begin{cases} \mathbb{H}(Y_k)(b) = b_k, \quad Y_k(b) = d_k\,, & \textit{for } |k| \geq 2, \\ \mathbb{H}(Y_k)(b) = b_k\,, & \textit{for } |k| = 0, 1. \end{cases}$$

Under the same assumptions as in Proposition 6.2, this equation admits a linear, bounded solution operator, mapping the source term and initial data $(\omega_k, b_k, d_k) \in \mathcal{F}_k \times \mathcal{B}_k$ *to the solution* $Y_k \in \mathcal{E}_k$ *(or, in the case* $|k| = 0, 1$, *mapping* (ω_k, b_k) *to* Y_k*). The norm of this linear solution operator is bounded uniformly w.r.t.* k.

7 Solution of the non-linear problem

In this section we construct a solution of the original non-linear problem (5.18). This will be achieved using an implicit function theorem, again following the analysis in [17].

For convenience, we recall the form of the equation:

$$0 \;=\; \overline{\partial_\varphi} T^{(\varphi)} + \partial_\phi T^{(\phi)} + T^{(0)} \omega^*,$$

where

$$T^{(\varphi)} \doteq\; 2\left(1 + \left(\frac{\overline{\partial_\beta}\partial_\phi \widetilde{\Psi}}{2\overline{\partial_\beta}\widetilde{\Psi}}\right)^2\right) \times \frac{\overline{\partial_\beta}\widetilde{\Psi}\, \overline{\partial_\varphi}\widetilde{\Psi}}{(\overline{\partial_\varphi} + 1)\overline{\partial_\beta}\widetilde{\Psi}} - \frac{\partial_\phi \overline{\partial_\beta}\widetilde{\Psi} \cdot \partial_\phi \widetilde{\Psi}}{2\overline{\partial_\beta}\widetilde{\Psi}}\,, \tag{7.33}$$

$$T^{(\phi)} \doteq\; \frac{(\overline{\partial_\varphi} + 1)\overline{\partial_\beta}\widetilde{\Psi} \cdot \partial_\phi \widetilde{\Psi} - \partial_\phi \overline{\partial_\beta}\widetilde{\Psi} \cdot \overline{\partial_\varphi}\widetilde{\Psi}}{2\overline{\partial_\beta}\widetilde{\Psi}}\,, \tag{7.34}$$

$$T^{(0)} \doteq\; \frac{1}{2\mu}(\overline{\partial_\varphi} + 1)\overline{\partial_\beta}\widetilde{\Psi} \cdot (\overline{\partial_\varphi}\widetilde{\Psi})^{-1/2\mu}\,. \tag{7.35}$$

To help the reader, we outline here the main steps of the analysis.

1. In Section 7.1 we begin by defining an intermediate function space $\mathcal{G}_-$, which is a Banach algebra obtained by summing functions which are periodic in ϕ and which (in the $k \neq 0$ modes) decay rapidly enough as $\beta \to +\infty$. In a similar way, we also define an output space $\mathcal{F}$, obtained by multiplying elements of $\mathcal{F}_k$ by $e^{ik\phi}$, and a space $\mathcal{E}$, obtained by multiplying elements of $\mathcal{E}_k$ by $e^{ik\phi}$.

2. In Section 7.2 we show that the outside derivatives appearing in (5.18) (namely $\overline{\partial_\varphi}, \partial_\phi$) are bounded as operators from $\mathcal{G}_-$ into $\mathcal{F}$. This is indeed the primary reason for defining the space $\mathcal{F}_k$ in (6.25) in terms of the operators $\widehat{A}_k^-$. This will allow us to more easily handle the derivatives that are on the outside of the non-linearities.

3. In Section 7.3 we prove that all of the derivative terms in the expressions for $T^{(\varphi)}, T^{(\phi)}$, and $T^{(0)}$ are bounded operators from $\mathcal{E}$ into $\mathcal{G}_-$ (up to some mild modifications). This step will also make it clear why we impose additional decay on the boundary conditions, including the weight $\langle k \rangle$ in the norm of the boundary terms in (6.32).

4. In Section 7.4, using Banach algebras and power series expansions, we show that the various non-linear terms in $T^{(\varphi)}$, $T^{(\phi)}$, and $T^{(0)}$, are bounded, $\mathcal{C}^1$ operators on $\mathcal{G}_-$ (up to some mild modifications).

5. In Section 7.5, relying on the previous analysis, we can use the implicit function theorem on Banach spaces and deduce the solvability of the non-linear PDE. We also analyze the dependence of the solution on the boundary data, and explain what kind of solution is obtained by this procedure.

7.1 Banach algebras

In this section we construct appropriate spaces for boundary data and for solutions of (5.18).

Definition 7.1 (spaces for the boundary data). *The Banach space Q^s is the space of 2π-periodic functions defined as*

$$Q^s = \left\{ f = \sum_{k\in\mathbb{Z}} a_k e^{ik\theta}\,; \qquad \|f\|_{Q^s} \doteq \sum_{k\in\mathbb{Z}} |a_k|\langle k\rangle^s, < +\infty \right\}, \tag{7.36}$$

where $\langle k\rangle = (1+k^2)^{1/2}$.

Here the exponent s establishes a degree of regularity in terms of the decay of the Fourier modes. Following [17], we will be considering vorticity which, at $\beta = b$, lies in $Q^{-1/2}$. This allows rather ill-behaved data (non-smooth, with mixed sign). Further details motivating this choice of the function space are given in Lemma 7.9.

Next, we introduce some spaces where solutions will be constructed.

Definition 7.2 (spaces for the solutions). *Consider a function $f = f(\beta,\phi)$, given by*

$$f(\beta,\phi) = \sum_{k\in\mathbb{Z}} e^{ik\phi} g_k(\beta). \tag{7.37}$$

Recalling the norms introduced at (6.17)-(6.18) and (6.26)-(6.27), we define the following norms:

$$\begin{aligned}
\|f\|_{\mathcal{E}} &\doteq \sum_{k\in\mathbb{Z}} \langle k\rangle^{1/2} \|g_k\|_{\mathcal{E}_k}\,,\\
\|f\|_{\mathcal{F}} &\doteq \sum_{k\in\mathbb{Z}} \langle k\rangle^{1/2} \|g_k\|_{\mathcal{F}_k}\,,\\
\|f\|_{\mathcal{G}} &\doteq \sum_{k\in\mathbb{Z}} \langle k\rangle^{1/2} \|g_k\|_{\delta}^{+}\,,\\
\|f\|_{\mathcal{G}_-} &\doteq \|g_0\|_{\delta}^{+} + \sum_{k\in\mathbb{Z},k\neq 0} \langle k\rangle^{1/2} \|g_k\|_{\delta}\,,\\
\|f\|_{\mathcal{G}_0} &\doteq \sum_{k\in\mathbb{Z}} \langle k\rangle^{1/2} \|g_k\|_{\delta}\,.
\end{aligned} \tag{7.38}$$

It will also be convenient to write

$$\mathcal{G}_-^k \doteq \begin{cases} \mathcal{H}_\delta & \text{for } k \neq 0,\\ \mathcal{H}_\delta^+ & \text{for } k = 0. \end{cases}$$

We remind the reader that our solutions are built around the space $\mathcal{G}_-$, which requires decay in β for all the Fourier modes *except* for the $k = 0$ mode.

The following properties of the spaces $\mathcal{G}, \mathcal{G}_-$ are important in order to conduct the non-linear analysis.

Lemma 7.1 *Suppose that $f_1, f_2 \in \mathcal{G}$. Then*

$$\|f_1 f_2\|_{\mathcal{G}} \;\leq\; \sqrt{2}\,\|f_1\|_{\mathcal{G}}\,\|f_2\|_{\mathcal{G}}\,. \tag{7.39}$$

The previous inequality remains valid if one replaces $\mathcal{G}$ with $\mathcal{G}_-$ throughout.

Proof. Let $f_1 = \sum_k e^{ik\phi} g_k^1(\beta)$, $f_2 = \sum_k e^{ik\phi} g_k^2(\beta)$. We estimate

$$\begin{aligned}
\|f_1 f_2\|_{\mathcal{G}} &= \sum_{n\in\mathbb{Z}} \langle n\rangle^{1/2} \Big\| \sum_{k\in\mathbb{Z}} g_{n-k}^1 g_k^2 \Big\|_\delta^+ && (7.40)\\
&\leq \sum_{n\in\mathbb{Z}} \langle n\rangle^{1/2} \sum_{k\in\mathbb{Z}} \|g_{n-k}^1 g_k^2\|_\delta^+ && (7.41)\\
&\leq \sum_{n\in\mathbb{Z}} \langle n\rangle^{1/2} \sum_{k\in\mathbb{Z}} \|g_{n-k}^1\|_\delta^+ \|g_k^2\|_\delta^+\,, && (7.42)
\end{aligned}$$

Here we have used the fact that, for $b > 1$, the norm (6.17) of the space $\mathcal{H}_\delta$ satisfies

$$\|h_1 h_2\|_\delta^+ \;\leq\; \|h_1\|_\delta^+ \|h_2\|_\delta^+\,.$$

Next, one can verify by a straightforward computation that $\langle n\rangle \leq 2\langle k\rangle\langle n-k\rangle$ for all $n, k \in \mathbb{Z}$. This yields

$$\|f_1 f_2\|_{\mathcal{G}} \;\leq\; \sqrt{2} \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle n-k\rangle^{1/2} \langle k\rangle^{1/2} \|g_{n-k}^1\|_\delta^+ \|g_k^2\|_\delta^+ \;\leq\; \sqrt{2}\|f_1\|_{\mathcal{G}}\|f_2\|_{\mathcal{G}}. \tag{7.43}$$

This concludes the proof in the case $f_1, f_2 \in \mathcal{G}$. The proof for the other case is analogous. □

Extending this idea, the following lemma (which we will repeatedly apply to obtain our estimates) is straightforward. We denote here by $B_{\mathcal{E}}(\psi_1, r)$ the ball of radius r in $\mathcal{E}$ centered at ψ_1.

Lemma 7.2 *Suppose that F, G are in $\mathcal{C}^1(B_{\mathcal{E}}(\psi_1, r); \mathcal{G})$, i.e., Fréchet differentiable from $B_{\mathcal{E}}(\psi_1, r)$ into $\mathcal{G}$ with bounded, continuous Fréchet derivative. Consider the product $H(\psi) = F(\psi)\cdot G(\psi)$. Then $H \in \mathcal{C}^1(B_{\mathcal{E}}(\psi_1, r); \mathcal{G})$ as well, and the following bounds hold:*

$$\begin{aligned}
\|H(\psi)\|_{\mathcal{G}} &\leq \sqrt{2}\|F(\psi)\|_{\mathcal{G}}\|G(\psi)\|_{\mathcal{G}}\,,\\
\|DH\| &\leq \sqrt{2}(\|DF\|\|G\| + \|F\|\|DG\|).
\end{aligned} \tag{7.44}$$

Analogous bounds hold for mappings that are in $\mathcal{C}^1(\mathcal{E}; \mathcal{G}_-)$ or in $\mathcal{C}^1(\mathcal{G}; \mathcal{F})$.

7.2 Outer divergence

We now establish some estimates relating our intermediate Banach algebra $\mathcal{G}_-$ and the output space $\mathcal{F}$. An important point must here be highlighted: the operator $\overline{\partial_\varphi}$ is bounded from $\mathcal{G}_-$ into $\mathcal{F}$, but not from $\mathcal{G}$. This fact is responsible for some of the technicalities in the proofs.

Lemma 7.3 *For a constant C independent of k, the following bounds hold:*

$$\|\partial_\phi\|_{L(\mathcal{G},\mathcal{F})} \;\leq\; \sup_{k\in\mathbb{N}} \frac{k}{|a_k^+ - \delta|} \;\leq\; C, \tag{7.45}$$

$$\|\overline{\partial_\varphi}\|_{L(\mathcal{G}_-,\mathcal{F})} \;\leq\; 1 + \sup_{k\in\mathbb{N}} \frac{a_k^+ + (2\mu-1)}{|a_k^+ - \delta|} \;\leq\; C. \tag{7.46}$$

Proof: We first notice that by (7.37) and the definition of our norms (7.38), we can bound the overall norm of ∂_ϕ in the following way

$$\|\partial_\phi f\|_{\mathcal{F}} \;\leq\; \sum_{k\in\mathbb{Z}} \langle k\rangle^{1/2} \sup_\phi |\partial_\phi e^{ik\phi}| \|g_k\|_{\mathcal{F}_k} \;\leq\; \sup_{k\in\mathbb{Z}} |k| \|Id\|_{L(H_\delta^+,\mathcal{F}^k)}$$

Similarly, using the fact that $\widehat{A}_k^+ = \beta(\partial_\beta - ik) - a_k^+$ and $\overline{\partial_\varphi} = A_k^+ + a_k^+ + (2\mu - 1)$, we deduce

$$\|\overline{\partial_\varphi}\|_{L(\mathcal{G}_-,\mathcal{F})} \;\leq\; 1 + \sup_{k\in\mathbb{Z}} (a_k^+ + (2\mu-1)) \|Id\|_{L(\mathcal{G}_-^k,\,\mathcal{F}^k)}.$$

Hence proving the desired inequalities can be reduced to bounding the operator norm of $\|Id\|_{L(H_\delta^+,\mathcal{F}^k)} = \|(\widehat{A}_k^+)^\dagger\|_{L(H_\delta^+;\mathcal{G}_-^k)}$. We observe that the additional constant term allowed in functions in H_δ^+ does not affect the estimates (as long as $\delta < 1$), because

$$(A_k^+)^\dagger \mathbf{1} \;=\; \frac{1}{k}\left(\frac{1}{i\beta} + O(\beta^{-2})\right).$$

Equation (6.19) then provides the desired operator bound, which concludes the proof. □

7.3 Linear expressions

In order to show that the non-linear terms in (7.33) are $\mathcal{C}^1$, we will first show that all of the derivatives used in the construction of $T^{(\varphi)}, T^{(\phi)}$ and $T^{(0)}$ are bounded operators between $\mathcal{E}$ and $\mathcal{G}_-$. We begin by estimating some differential operators that will be used as building blocks.

Lemma 7.4 *The following bounds hold:*

$$\|Id\|_{L(\mathcal{E}_k,\mathcal{G}_-^k)} \;\leq\; \frac{C}{k}\,, \tag{7.47}$$

$$\|B_0\|_{L(\mathcal{E}_k,H_\delta)} \;\leq\; \frac{C}{k}\,, \tag{7.48}$$

$$\|AB\|_{L(\mathcal{E}_k,H_\delta)} \;\leq\; C\,, \tag{7.49}$$

$$\|A\|_{L(\mathcal{E}_k,H_\delta^+)} \;\leq\; C\,, \tag{7.50}$$

In particular, one has

$$\begin{aligned}
\|Id\|_{L(\mathcal{E},\mathcal{G}_-)} &\le \frac{C}{k}\,, && (7.51)\\
\|B_0\|_{L(\mathcal{E},\mathcal{G}_0)} &\le \frac{C}{k}\,, && (7.52)\\
\|AB\|_{L(\mathcal{E},\mathcal{G}_0)} &\le C\,, && (7.53)\\
\|A\|_{L(\mathcal{E},\mathcal{G})} &\le C\,. && (7.54)
\end{aligned}$$

Proof: We write the formulas in the case $|k| \ge 2$, since the other cases are only simpler (due to the omission of E_k^-). For the first item, we simply estimate using (6.21) and (6.20)

$$\|Id\|_{L(\mathcal{E}_k,\mathcal{G}_-^k)} \le \|B^\dagger(\widehat{A}_k^-)^\dagger\|_{L(\mathcal{G}_-^k)} + \frac{1}{\langle k\rangle}(\|E_k^0\|_{\mathcal{G}_-^k} + \|E_k^-\|_{\mathcal{G}_-^k}) \le \frac{1}{|a_k^- - \delta|(1-\delta)} + \frac{b^{\delta-1} + b^{\delta+a_k^- -1}}{k} \le \frac{C}{k}\,. \tag{7.55}$$

We recall that $B_0 = B - 1$. Thus we can estimate

$$\begin{aligned}
\|B_0\|_{L(\mathcal{E}_k,\mathcal{G}_-^k)} &\le \frac{C}{k} + \|BB^\dagger(\widehat{A}_k^-)^\dagger\|_{L(\mathcal{G}_-^k)} + \frac{1}{k}(\|BE_k^0\|_{\mathcal{G}_-^k} + \|BE_k^-\|_{\mathcal{G}_-^k})\\
&\le \frac{C}{k} + \frac{1}{|a_k^- - \delta|} + \frac{b^{a_k^- +\delta}}{k} \le \frac{C}{k}.
\end{aligned}$$

Similarly, as $A = \widehat{A}_k^- - a_k^-$ on $\mathcal{E}_k$, and as $BE_k^0 = 0, \widehat{A}_k^- BE_k^- = 0$, we have

$$\begin{aligned}
\|AB\|_{L(\mathcal{E}_k,\mathcal{G}_-^k)} &\le |a_k^-|\|B\|_{L(\mathcal{E}_k,\mathcal{G}_k^-)} + \|\widehat{A}_k^- B\|_{L(\mathcal{E}_k,\mathcal{G}_k^-)}\\
&\le \frac{|a_k^-|}{|a_k^- - \delta|} + \frac{|a_k^-|b^{a_k^- +\delta}}{k} + 1 \le C.
\end{aligned}$$

The last step, namely estimating the norm $\|A\|$, is much more delicate. Here it is important to note that the operator A is bounded as a mapping into H_δ^+, but not as a mapping into $\mathcal{G}_-^k$. This is because commuting A with B in general creates a constant term (in β).

To begin, we note that $A = B - ik\beta$. Therefore, it suffices to estimate the norm of $ik\beta$ as an operator from $\mathcal{E}_k$ into H_δ^+. We check that this operator takes E_k^0 and E_k^- into H_δ^+. Indeed, this is true because we have included the factor $\langle k\rangle$ in the definition of the $\mathcal{E}_k$ norm. Moreover, in the definition (6.24) we included E_k^- only when $a_k^- < 0$. Notice that, in the case of E_k^0, this already requires that we use H_δ^+ instead of H_δ.

Thus it only remains to estimate the action of the operator $ik\beta B^\dagger(\widehat{A}_k^-)^\dagger$ on elements of $\mathcal{G}_k^-$. Here we can directly compute, taking $\|g\|_{\mathcal{G}_k^-} \le 1$ and changing the order of integration:

$$ik\beta B^\dagger(\widehat{A}_k^-)^\dagger g \;=\; \int_b^\beta t^{-a_k^- -1} e^{-ikt} g(t) \left(\int_t^\beta s^{a_k^-} e^{iks}\, ds\right) dt.$$

Integrating the inner integral by parts, and then using the bound on g, we obtain a constant term of order $b^{-\delta}$, plus a term in H_δ with norm $(-a_k^- - \delta)^{-1}$. This achieves the desired estimate. □

With these building blocks in hand, we can now provide bounds for each of the scaled differential operators which make up the terms in $T^{(\varphi)}, T^{(\phi)}$, and $T^{(0)}$.

Lemma 7.5 *The following bounds hold:*

$$\|\overline{\partial_\varphi}\|_{L(\mathcal{E},\mathcal{G})} \leq C, \tag{7.56}$$

$$\|\overline{\partial_\beta}\|_{L(\mathcal{E},\mathcal{G}_-)} \leq C, \tag{7.57}$$

$$\|\partial_\phi\|_{L(\mathcal{E},\mathcal{G}_-)} \leq C, \tag{7.58}$$

$$\|(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\|_{L(\mathcal{E},\mathcal{G})} \leq C, \tag{7.59}$$

$$\|\partial_\phi\overline{\partial_\beta}\|_{L(\mathcal{E},\mathcal{G}_-)} \leq C. \tag{7.60}$$

Proof: We estimate the above terms one by one, using the building blocks from the previous lemma.

$$\begin{aligned}\|\overline{\partial_\varphi}\|_{L(\mathcal{E},\mathcal{G})} &\leq \|A\|_{L(\mathcal{E},\mathcal{G})} + \|2\mu-1\|_{L(\mathcal{E},\mathcal{G})} \leq C,\\ \|\overline{\partial_\beta}\|_{L(\mathcal{E},\mathcal{G}_-)} &\leq \|B_0\|_{L(\mathcal{E},\mathcal{G}_-)} + \|2\mu-1\|_{L(\mathcal{E},\mathcal{G}_-)} \leq C.\end{aligned}$$

We note that ∂_ϕ acts as a multiplication operator in our Fourier spaces, and that a_k^- grows like a constant times k. Hence, using (7.55), we obtain

$$\|\partial_\phi\|_{L(\mathcal{E},\mathcal{G}_-)} \leq \sup_{k\in\mathbb{N}} k\|Id\|_{L(\mathcal{E}_k,\mathcal{G}_k^-)} \leq \sup_{k\in\mathbb{N}} \frac{k}{|a_k^- - \delta|(1-\delta)} + (b^{\delta-1} + b^{\delta+a_k^- -1}) \leq C.$$

The fourth estimate is a straightforward application of the previous lemma:

$$\begin{aligned}\|(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\|_{L(\mathcal{E},\mathcal{G}_-)} &= \|(A+2\mu)(B-2\mu)\|_{L(\mathcal{E},\mathcal{G}_-)}\\ &\leq \|AB\|_{L(\mathcal{E},\mathcal{G}_-)} + 2\mu\big(\|B\|_{L(\mathcal{E},\mathcal{G}_-)} + \|A\|_{L(\mathcal{E},\mathcal{G}_-)} + 2\mu\|Id\|_{L(\mathcal{E},\mathcal{G}_-)}\big) \leq C.\end{aligned}$$

The last estimate is achieved by

$$\begin{aligned}\|\partial_\phi\overline{\partial_\beta}\|_{L(\mathcal{E},\mathcal{G}_-)} &\leq 2\mu\|\partial_\phi\|_{L(\mathcal{E},\mathcal{G}_-)} + \|\partial_\phi B\|_{L(\mathcal{E},\mathcal{G}_-)}\\ &\leq C + \sup_{k\in\mathbb{Z}} k\|(\widehat{A}_k^-)^\dagger\|_{L(\mathcal{G}_k^-)} + \frac{k}{\langle k\rangle}\|e^{ik\beta}\beta^{a_k^-}\|_{\mathcal{G}_k^-} \leq C,\end{aligned}$$

where we have used the fact that $BE_k^0 = 0$ and $BE_k^- = e^{ik\beta}\beta^{a_k^-}$, together with (6.20). □

Remark 7.1 We highlight the fact that, in the previous two lemmas, one needs the inequality $a_k^- < -\delta < 0$. For this reason, in the definition (6.24), for $|k| = 1$ the functions E_k^- are not included in the space $\mathcal{E}_k$. Put in another way, the linear operators which make up the terms of $T^{(\varphi)}, T^{(\phi)}$ and $T^{(0)}$ induce a reduction in the decay rate of E_k^-. As a consequence, we cannot construct an intermediate algebra (such as $\mathcal{G}$) which will allow the inclusion of the E_k^- when $|k| = 1$.

This exclusion has a clear physical explanation in the $k = \pm 1$ case, because these Fourier components induce a net velocity of the vortex center. As a consequence, there is no self-similar solution of the original equations (3.1) with a vortex centered exactly at the origin. By a suitable positioning of the vortex center (in self-similar coordinates), one can achieve appropriate values of the Fourier components for $k = \pm 1$.

At this stage, to streamline the presentation, we do not permit assigning Dirichlet boundary data for the $k = 0$ mode of Ψ. This restriction will be removed at the very end of our analysis.

7.4 Non-linear terms

The goal of this section will be to show that all the non-linear terms in $T^{(\varphi)}, T^{(\phi)}$, and $T^{(0)}$ induce $\mathcal{C}^1$ mappings from $\mathcal{E}$ into $\mathcal{G}$. Here the main idea is to write a Taylor expansion of each of these terms, and then use the Banach algebra construction of $\mathcal{G}$. Throughout this section we will be expanding about the constant solution $\psi_0 = \frac{1}{2\mu-1}$, which corresponds to a radially symmetric solution as discussed in Section 5.

In order to illustrate this approach, for the sake of clarity we begin with one of the simplest of the non-linear terms of interest, working out all details. Throughout the following, we denote by $\mathbb{1}$ the constant function identically equal to 1.

Lemma 7.6 *The mapping from $\psi \mapsto \frac{1}{\overline{\partial_\varphi}\psi}$ is $\mathcal{C}^1(\mathcal{E},\mathcal{G})$ in a neighborhood of the function ψ_0 (which satisfies $\overline{\partial_\varphi}\psi_0 = \mathbb{1}$). In particular*

$$\left\| D\left(\frac{1}{\overline{\partial_\varphi}\psi}\right)\right\|_{L(\mathcal{E},\mathcal{G})} \;\leq\; C\|\overline{\partial_\varphi}\|_{L(\mathcal{E},\mathcal{G})},$$

as long as

$$\|\psi - \psi_0\|_{\mathcal{E}} \;\leq\; \frac{1}{2\|\overline{\partial_\varphi}\|_{L(\mathcal{E},\mathcal{G})}}\,. \tag{7.61}$$

Proof: First, we recall that $\|\overline{\partial_\varphi}\|_{L(\mathcal{E},\mathcal{G})} < \infty$. This makes all of the estimates in what follows well-defined.

By (7.61), we have that $\|\overline{\partial_\varphi}\psi - 1\|_{\mathcal{G}} \leq \frac{1}{2}$. Then, a Taylor expansion about 1 yields

$$\frac{1}{\overline{\partial_\varphi}\psi} \;=\; \sum_{n=0}^{\infty}(-1)^n((\overline{\partial_\varphi}\psi - 1))^n. \tag{7.62}$$

Using the bound (7.44) n times, one obtains

$$\|D(\overline{\partial_\varphi}\psi - 1)^n\|_{L(\mathcal{E},\mathcal{G})} \;\leq\; n(\sqrt{2})^n\|\overline{\partial_\varphi}\|_{L(\mathcal{E},\mathcal{G})}\|\overline{\partial_\varphi}\psi - 1\|_{\mathcal{G}}^{n-1}.$$

Since $\|\overline{\partial_\varphi}\psi - 1\|_{\mathcal{G}} \leq \frac{1}{2}$, using the triangle inequality we now obtain

$$\left\| D\left(\frac{1}{\overline{\partial_\varphi}\psi}\right)\right\|_{L(\mathcal{E},\mathcal{G})} \;\leq\; 2\sum_{n=0}^{\infty} n\left(\frac{1}{\sqrt{2}}\right)^n \|\overline{\partial_\varphi}\|_{L(\mathcal{E},\mathcal{G})} \;\leq\; C\|\overline{\partial_\varphi}\|_{L(\mathcal{E},\mathcal{G})}.$$

Having given a bound on the Fréchet derivative of our mapping, it only remains to show that the Fréchet derivative varies smoothly in ψ. This can be proven using equation (7.62) along with standard Calculus bounds. This concludes the proof. □

Using the same technique, we obtain the following estimates on the remaining non-linear terms.

Lemma 7.7 *For $\psi \in \mathcal{E}$, define*

$$
\begin{aligned}
H_1(\psi) &\doteq \frac{1}{\overline{\partial_\beta}\psi}\,, && (7.63)\\
H_2(\psi) &\doteq \frac{1}{(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\psi}\,, && (7.64)\\
H_3(\psi) &\doteq (\overline{\partial_\varphi}\psi)^{-1/2\mu}\,, && (7.65)\\
H_4(\psi) &\doteq \frac{\overline{\partial_\varphi}\psi}{(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\psi}\,. && (7.66)
\end{aligned}
$$

Then, on a suitable neighborhood of ψ_0 in the space $\mathcal{E}$, we have

$$
H_1,\, H_4 \in \mathcal{C}^1(\mathcal{E},\mathcal{G}_-), \qquad H_2,\, H_3 \in \mathcal{C}^1(\mathcal{E},\mathcal{G}). \tag{7.67}
$$

Remark 7.2 We emphasize here that H_4 takes values inside $\mathcal{G}_-$ and not in $\mathcal{G}$. This fact will be crucial in closing all of our estimates. We also note that the size of the $\mathcal{E}$-neighborhood, as well as the $\mathcal{C}^1$ norm of the above expressions, depends only on the norms of the linear operators, and can be explicitly determined.

Proof. We first note that

$$
\begin{aligned}
H_1(\psi_0) &= -\mathbf{1}, && (7.68)\\
H_2(\psi_0) &= -2\mu\cdot\mathbf{1}, && (7.69)\\
H_3(\psi_0) &= \mathbf{1}, && (7.70)\\
H_4(\psi_0) &= \frac{-\mathbf{1}}{2\mu}\,. && (7.71)
\end{aligned}
$$

A Taylor expansion yields

$$
\begin{aligned}
H_1(\psi) &= -\sum_{n=0}^{\infty}(-1)^n(\overline{\partial_\beta}\psi+1)^n && \text{for } \|\overline{\partial_\beta}\psi+1\|_{\mathcal{G}_-} < \tfrac{1}{2}\,,\\
H_2(\psi) &= -2\mu\sum_{n=0}^{\infty}(-1)^n\left(2\mu(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\psi+1\right)^n && \text{for } \|2\mu(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\psi+1\|_{\mathcal{G}} < \tfrac{1}{2}\,,\\
H_3(\psi) &= \sum_{n=0}^{\infty}\prod_{k=0}^{n-1}\left(-\frac{1}{2\mu}-k\right)\frac{(\overline{\partial_\varphi}\psi-1)^n}{n!} && \text{for } \|\overline{\partial_\varphi}\psi-1\|_{\mathcal{G}} < \tfrac{1}{2}\,.
\end{aligned}
\tag{7.72}
$$

Thanks to the bounds proved in Lemma 7.5, all of these series are well-defined in $\mathcal{G}$ or in $\mathcal{G}_-$. For H_4, we utilize a convenient cancellation, namely:

$$
(H_4(\psi))^{-1} = \frac{(\overline{\partial_\varphi}+1)\overline{\partial_\beta}\psi}{\overline{\partial_\varphi}\psi} = \frac{(\overline{\partial_\varphi}+1)(B-2\mu)\psi}{\overline{\partial_\varphi}\psi} = -2\mu+\frac{(2\mu B_0-AB)\psi}{\overline{\partial_\varphi}\psi}\,. \tag{7.73}
$$

We note here that $-2\mu \in \mathcal{G}_-$ (regarded as a function which is constant w.r.t. both ϕ and β). Furthermore, by Lemma 7.4 both $AB\psi$ and $B_0\psi$ lie in $\mathcal{G}_0$. Finally, we note that the product of a function in $\mathcal{G}_0$ and a function in $\mathcal{G}_-$ will be in $\mathcal{G}_-$. This implies that as long as the denominator is bounded away from zero (which is true on a neighborhood of ψ_0 in $\mathcal{E}$), we have the inclusion $(H_4)^{-1} \in \mathcal{G}_-$. More precisely, we can express H_4 via the series (in the space $\mathcal{G}_-$)

$$\begin{aligned} H_4(\psi) \;&=\; -\frac{1}{2\mu}\sum_{n=0}^{\infty}(-1)^n\left(\frac{(B_0-(2\mu)^{-1}AB)\psi}{\overline{\partial_\varphi}\psi}\right)^n, \\ \|(B_0+(2\mu)^{-1}A(B+1))\psi\|_{\mathcal{G}_0} \;&<\; \epsilon \qquad \text{for } \|\overline{\partial_\varphi}\psi-1\|_{\mathcal{G}} \;<\; \frac{1}{2}, \end{aligned} \tag{7.74}$$

where ϵ is chosen small enough so that the previous expression is well-defined, as a series in $\mathcal{G}_-$.

Having written these power series, using the same techniques as in the proof of Lemma 7.6, it is then straightforward to establish $\mathcal{C}^1$ bounds for these expressions, completing the proof.
□

With these estimates at hand, we now seek to estimate the $\mathcal{C}^1$ norm of the non-linear terms in (5.18)–(5.21).

Lemma 7.8 *Within a sufficiently small neighborhood of ψ_0 in $\mathcal{E}$, we have*

$$T^{(\phi)} \in \mathcal{C}^1(\mathcal{E};\mathcal{G}), \qquad T^{(0)} \in \mathcal{C}^1(\mathcal{E};\mathcal{G}), \qquad T^{(\varphi)} \in \mathcal{C}^1(\mathcal{E};\mathcal{G}_-). \tag{7.75}$$

Proof: 1. The denominator in $T^{(\phi)}$ is exactly H_1. The numerator consists of products of the scaled linear operators from Lemma 7.5. A repeated application of Lemma 7.2 now shows that $T^{(\phi)} \in \mathcal{C}^1(\mathcal{E};\mathcal{G})$, on the region where H_1 is well-defined, namely on a neighborhood of ψ_0 in $\mathcal{E}$.

2. Similarly, $T^{(0)}$ is a product of H_3 and one of the operators in Lemma 7.5. Therefore, Lemma 7.2 establishes that $T^{(0)} \in \mathcal{C}^1(\mathcal{E};\mathcal{G})$, on the region where H_3 is well-defined, which includes a neighborhood of ψ_0 in $\mathcal{E}$.

3. Concerning $T^{(\varphi)}$, we write

$$T^{(\varphi)} \;=\; H_5(\psi)\cdot H_4(\psi) + H_6(\psi)$$

where

$$H_5(\psi) \;\doteq\; 2\left(1+\left(\frac{\overline{\partial_\beta}\partial_\phi\widetilde{\Psi}}{2\overline{\partial_\beta}\widetilde{\Psi}}\right)^2\right), \qquad H_6(\psi) \;\doteq\; -\frac{\partial_\phi\overline{\partial_\beta}\widetilde{\Psi}\cdot\partial_\phi\widetilde{\Psi}}{2\overline{\partial_\beta}\widetilde{\Psi}}.$$

In terms of our previously defined maps, we can rewrite this as

$$H_5 = 2\big(1 + H_1\cdot H_1\cdot\overline{\partial_\beta}\partial_\phi\psi\cdot\overline{\partial_\beta}\partial_\phi\psi\big), \qquad H_6 \;=\; -\frac{1}{2}H_1\cdot\partial_\phi\psi\cdot\partial_\phi\overline{\partial_\beta}\psi$$

Again, by Lemmas 7.5 and 7.2 it follows that H_5 and H_6 are both in $\mathcal{C}^1(\mathcal{E};\mathcal{G}_-)$, on the region where H_1 is well-defined. Lemma 7.7 applied to H_4 then establishes the desired $\mathcal{C}^1$ bounds, on the region where the series for both H_1 and H_4 are well-defined. This completes the proof.
□

Relying on the previous lemma, we can finally provide an estimate on the right hand side of the rescaled equation (5.18). We recall that Q^s is the space of boundary data introduced at (7.36).

Proposition 7.1 *As (ψ, ω) range on a neighborhood of (ψ_0, ω_0), the map*

$$F(\psi,\omega) \doteq \overline{\partial_\varphi} T^{(\varphi)} + \partial_\phi T^{(\phi)} + T^{(0)}\omega \tag{7.76}$$

is in $\mathcal{C}^1(\mathcal{E} \times Q^{-1/2}; \mathcal{F})$.

Proof: By Lemma 7.8 it follows that the maps $T^{(\phi)}, T^{(0)}$ are $\mathcal{C}^1$ with values in $\mathcal{G}$, while $T^{(\varphi)}$ is $\mathcal{C}^1$ with values in $\mathcal{G}_-$. Lemma 7.3 implies that $\partial_\phi \in L(\mathcal{G};\mathcal{F})$, while $\overline{\partial_\varphi} \in L(\mathcal{G}_-;\mathcal{F})$. This immediately establishes that the first two terms on the right hand side of (7.76) are in $\mathcal{C}^1(\mathcal{E};\mathcal{F})$.

Concerning the last term, we recall that $\omega = \sum e^{ik\phi}\omega_k$, where $\omega_k \in \mathbb{C}$. Recalling that $\langle k\rangle^{-s} \leq \langle k-m\rangle^{s}\langle m\rangle^{-s}$, and the fact that the operator norm of $(\widehat{A}_k^+)^\dagger$ in H_δ^+ decays like k^{-1}, we obtain

$$\begin{aligned}
\|T^{(0)}\omega\|_{\mathcal{F}} &= \sum_k \langle k\rangle^{1/2} \|(T^{(0)}\omega)_k\|_{\mathcal{F}_k} \leq \sum_k \langle k\rangle^{-1/2}\|(T^{(0)}\omega)_k\|_{H_\delta^+} \\
&\leq \sum_k \sum_m \langle k-m\rangle^{1/2}\langle m\rangle^{-1/2}\|(T^{(0)})_{k-m}\|_{H_\delta^+}|\omega_m| \leq \|T^{(0)}\|_{\mathcal{G}}\|\omega\|_{Q^{-1/2}},
\end{aligned}$$

where here we've let $(\cdot)_k$ denote the k-th mode (in ϕ) of the function.

Next, we need to show boundedness of the Fréchet derivative of the third term. First, we notice that linearity in ω along with the last inequality gives that $D_\omega(T^{(0)}\omega)$ is a bounded operator between the spaces. Next, letting $\check{\psi}$ be a variation of ψ, we may bound

$$\begin{aligned}
\|D_\psi(T^{(0)}\omega)[\check{\psi}]\|_{\mathcal{F}} &= \sum_k \langle k\rangle^{1/2}\|(D_\psi(T^{(0)}\omega)[\check{\psi}])_k\|_{\mathcal{F}_k} \\
&\leq \sum_k \langle k\rangle^{-1/2}\|(\omega D_\psi(T^{(0)})[\check{\psi}])_k\|_{H_\delta^+} \\
&\leq \sum_k \sum_m \langle k-m\rangle^{1/2}\langle m\rangle^{-1/2}|\omega_m|\|(D_\psi(T^{(0)})[\check{\psi}])_{k-m}\|_{H_\delta^+} \\
&\leq \|\omega\|_{Q^{-1/2}}\|D_\psi(T^{(0)})[\check{\psi}]\|_{\mathcal{G}}
\end{aligned}$$

This implies the desired $\mathcal{C}^1$ bounds. □

The following proposition provides an additional estimate on the function F in (7.76), that will be useful later.

Proposition 7.2 *For any $\epsilon > 0$, there exists $r > 0$ such that, if $\psi_1, \psi_2 \in B\Big(\psi_0, \dfrac{r}{\max\{1, \|\omega\|_{Q^{-1/2}}\}}\Big)$, then*

$$\|D_\psi F(\psi_1,\omega) - D_\psi F(\psi_2,\omega)\| \leq \epsilon.$$

Proof. This follows from the fact that $T^{(0)}, T^{(\varphi)}, T^{(\phi)}$ are all $\mathcal{C}^1$, while the dependence of F on ω is linear (see Proposition 7.1 for details). □

Lemma 7.9 *The mapping $f \mapsto f(b)$ lies in $\mathcal{C}^1(\mathcal{E}; Q^{3/2})$. The same is true of the map $f \mapsto f'(b)$.*

Proof. These statements follow because

(i) by Lemma 7.4, Id and B_0 are bounded operators from $\mathcal{E}_k$ into $\mathcal{G}_-$ with norm $\leq \frac{C}{k}$,

(ii) pointwise evaluations are continuous in $\mathcal{G}_-$, and

(iii) the norm (7.38) on $\mathcal{G}_-$ weights the Fourier modes in ϕ by $\langle k \rangle^{1/2}$.

□

7.5 Existence of a solution on the inner domain.

We are now prepared to state the main theorem of this section. Roughly speaking, it shows that we can solve the non-linear PDE (5.18) for data which are sufficiently close to the explicit solution $(\bar{\psi}_0, \omega_0)$, i.e., to the radially symmetric solution (before the coordinate transformations). Moreover, it provides a space in which we can measure the stability of such solutions w.r.t. changes in the boundary conditions.

We remind the reader of some of the notation used here. The operator $\mathbb{H}$ introduced at (6.29) corresponds to a mixed boundary condition, which was chosen to ensure that $\Psi_\beta(0,\phi) = -\mu R^2$ in equation (4.37). This choice is crucial in the construction of our coordinate changes. The space Q^s is a Wiener-type space, which measures the regularity of a function in terms of the decay of its Fourier coefficients. Here we are always considering Fourier coefficients in the ϕ variables.

In the following, recalling (7.36) we consider the subspace

$$\widehat{Q}^{3/2} \doteq \left\{ f = \sum_{|k|\geq 2} a_k e^{ik\theta}; \qquad \|f\|_{Q^{3/2}} \doteq \sum_{|k|\geq 2} |a_k| \langle k \rangle^{3/2}, < +\infty \right\}. \tag{7.77}$$

Notice that here we are removing the components with $|k| \leq 1$.

Theorem 7.1 *There exists a constant $C_0 > 0$ small enough so that the following holds. On the set of all triples $(\bar{\psi}, \widehat{\psi}, \omega)$ such that*

$$\|\bar{\psi} - \bar{\psi}_0\|_{\widehat{Q}^{3/2}} \leq C_0, \qquad \|\widehat{\psi} - \mathbb{H}(\psi_0)(b)\|_{Q^{3/2}} \leq C_0, \qquad \|\omega - \omega_0\|_{Q^{-1/2}} \leq C_0\,, \tag{7.78}$$

there is a $\mathcal{C}^1$ solution operator $H : \widehat{Q}^{3/2} \times Q^{3/2} \times Q^{-1/2} \to \mathcal{E}$ so that $\psi = H(\bar{\psi}, \widehat{\psi}, \omega)$ solves the equation $F(\psi, \omega) = 0$. Moreover, $\psi(\theta, b)$ matches the Fourier modes of $\bar{\psi}$ for $|k| \geq 2$, and $\mathbb{H}(\psi)(b, \theta) = \widehat{\psi}$.

Remark: We note that the function spaces used in this construction do not allow us to choose the $k = 0, \pm 1$ terms in the Fourier expansion of $\bar{\psi}$ at $\beta = b$ (i.e. the Dirichlet condition); these are determined by the required boundary condition involving $\mathbb{H}$.

Proof. Indeed, the result follows from the implicit function theorem on Banach spaces [1, 12], using Proposition 6.4, Proposition 7.1, and Lemma 7.9. □

Corollary 7.1 *The equations (5.22) and (5.23) have a solution, for all values of ω and $\bar{\psi}$ which are sufficiently close to ω_0 and $\bar{\psi}_0$ in $Q^{-1/2}$ and in $\widehat{Q}^{3/2}$, respectively.*

We remark that the spaces Q^s, which are defined in terms of Fourier coefficients, are well-controlled in terms of Hölder norms. More specifically, for $0 < s < 1$, one has $\|f\|_{Q^s} \leq \|f\|_{\mathcal{C}^s}$. Hence in the next lemma we measure the distances in terms of these Hölder norms. We will also denote by $P_{|k|\neq 1}$ the projection of a function, defined on the unit circle, to the space of all Fourier modes with $|k| \neq 1$.

Having established the existence of solutions, we now seek to better understand their properties, by inverting the coordinate transformations.

Setting $\Psi \doteq \beta^{1-2\mu} H(\bar{\psi}, \widehat{\psi}, \omega)$, we obtain a solution to the boundary value problem (4.36) and (4.37). In turn, using (4.16) and recalling that $\Psi_{\beta\varphi} = \beta^{-1-2\mu}(\overline{\partial_\varphi} + 1)\overline{\partial_\beta}\widetilde{\Psi}$ satisfies the bound (7.59), we can convert our solution from the adapted coordinates to the standard cartesian coordinates. In the case where $\mu > 2/3$, this solution will be a weak solution to the original equation, see Proposition 40 in [17], while when $1/2 < \mu \leq 2/3$ this solution is a weak solution outside the vortex center. Finally, by adding a constant to Ψ, which does not effect whether the differential equation is satisfied, we can match the $k = 0$ coefficient of the boundary data for Ψ. We summarize these remarks in the following lemma. As before, the unit circumference is here denoted by $\mathbb{T} = [0, 2\pi]$, with endpoints identified.

Lemma 7.10 *Let $\mu > 2/3$. There exists $\delta > 0$ such that the following holds:*

Given a radius $r_0 \leq \delta$, consider boundary data $(\bar{\psi}, \omega)$ on $\partial B(0, r_0)$ sufficiently close to the radially symmetric solution $(\bar{\psi}_0, \omega_0)$, in the sense that

$$\left\| P_{|k|\neq 1}\bar{\psi} \right\|_{\mathcal{C}^{1,1/2}(\mathbb{T})} \leq \delta, \qquad \|\breve{\omega}\|_{\mathcal{C}^{1/2}(\mathbb{T})} \leq \delta, \qquad \breve{\omega}(\theta) = \int_0^\theta \omega(\theta') - \omega_0 \, d\theta'.$$

Then there exists a weak solution to the self-similar Euler equations on the disc $B(x, r_0)$ corresponding to the given boundary data, in the sense that the vorticity and stream function of the solution will match ω and the $|k| \neq 1$ Fourier components of $\bar{\psi}$ at the boundary.

In the previous theorems we do not permit one to set arbitrary boundary data for $\bar{\psi}$ in the first few Fourier coefficients. This is related to the fact that the $k = \pm 1$ modes correspond to a non-zero net velocity of the vortex center: this is addressed by properly centering the vortex center as described in Remark 4.1.

7.6 An alternative approach to constructing the non-linear solutions

The previous approach provides the existence of solutions, but it requires that all the boundary data be close to radially symmetric. We would like to relax this assumption, particularly on the vorticity ω, which we do not expect to gain any regularity near the center of the vortex spiral. In this section, this will be achieved by a more careful analysis of our constructive procedure. Throughout the following, to simplify some computations, we will assume that the $k = 0$ mode of ω matches that of ω_0. In other words, the average values of ω and ω_0 on $\mathbb{T}$ coincide.

Our goal is to construct a solution operator $\psi = H(\bar{\psi}, \hat{\psi}, \omega)$, as in Theorem 7.1, in this more general situation. For this purpose, we consider the map

$$G(\psi) \doteq \psi - (D_\psi F)^{-1}(\psi_0, \omega)[F(\psi, \omega)].$$

Here the partial differential $(D_\psi F)^{-1}(\psi_0, \omega)$ denotes the linear solution operator given in Proposition 6.4, which solves the infinite system of ordinary differential equations and matches the boundary terms given by $\bar{\psi}$ and $\hat{\psi}$. We remark that this operator does not vary with ω when $\psi = \psi_0$. Our goal is to prove that, for fixed ω (not necessarily small), and for b sufficiently large, the iterates of the map G converge to a fixed point.

Following a standard approach, for $\psi_1, \psi_2 \in B(\psi_0, r)$ we can write

$$\begin{aligned}
\|G(\psi_1) - G(\psi_2)\|_{\mathcal{E}} &= \|\psi_1 - \psi_2 - (D_\psi F)^{-1}(\psi_0, \omega)(F(\psi_1, \omega) - F(\psi_2, \omega))\|_{\mathcal{E}} \\
&\leq \|(D_\psi F)^{-1}(\psi_0, \omega)\|_{L(\mathcal{F};\mathcal{E})} \|D_\psi F(\psi_0, \omega)(\psi_1 - \psi_2) - (F(\psi_1, \omega) - F(\psi_2, \omega))\|_{\mathcal{F}} \\
&\leq \left\|(D_\psi F)^{-1}(\psi_0, \omega)\right\|_{L(\mathcal{F};\mathcal{E})} \|\psi_1 - \psi_2\|_{\mathcal{E}} \cdot \sup_{\psi \in B(\psi_0, r)} \|D_\psi F(\psi_0, \omega) - D_\psi F(\psi, \omega)\|_{L(\mathcal{E}, \mathcal{F})}
\end{aligned}$$

Thanks to Proposition 6.4, we have an a priori bound on $(D_\psi F)^{-1}(\psi_0, \omega)$. By Proposition 7.2, we can make the last factor on the right hand side as small as we like, as long as $r \leq C\|\omega\|^{-1}$. Hence

$$\|G(\psi_1) - G(\psi_2)\|_{\mathcal{E}} \leq \frac{1}{2} \|\psi_1 - \psi_2\|_{\mathcal{E}},$$

showing that G is a strictly non-expansive, restricted to the ball $B(\psi_0, r)$.

Since the function $F(\psi_0, \omega) = c(\omega - \omega_0)$ is independent of β and is zero for the $k = 0$ mode, we can apply (6.22) to deduce

$$\|G(\psi_0) - \psi_0\|_{\mathcal{E}} \leq C\|\omega\|_{Q^{-1/2}} b^{-1+\delta}. \tag{7.79}$$

These facts together imply that G is a $\mathcal{C}^1$ uniform contraction as long as $\|\omega\|_{Q^{-1/2}} b^{-1+\delta} \leq r \leq C\|\omega\|^{-1}_{Q^{-1/2}}$. The uniform contraction theorem (see, e.g. Theorem 1.244 in [7]) then implies that, on the set where $\|\omega\|_{Q^{-1/2}} \leq Cb^{(1-\delta)/2}$ and for $\hat{\psi}$ and $\bar{\psi}$ sufficiently small, there is a $\mathcal{C}^1$ solution operator $H : \widehat{Q}^{3/2} \times Q^{3/2} \times Q^{-1/2} \to \mathcal{E}$ solving the boundary value problem (5.22)-(5.23).

Summarizing the previous analysis, we have:

Proposition 7.3 *Assuming that the $k = 0$ mode of ω matches ω_0. Then there exists $\epsilon > 0$ such that, if $\|\omega\|^2_{Q^{-1/2}} \leq \epsilon b^{1-\delta}$ and $\|\bar{\psi} - \bar{\psi}_0\|_{\widehat{Q}^{3/2}} < \epsilon$, then there exists a solution to (5.22) with boundary data (5.23), with $\mathcal{C}^1$ dependence on ω and $\bar{\psi}$.*

In the same way as before, this can be restated in terms of Hölder norms and the original equations, but with the additional gain that, as the radius r_0 becomes smaller, one can allow the vorticity ω to be farther away from radially symmetric.

8 Dependence on boundary data

We now provide estimates on how changes on the boundary data will affect the solution. In particular, given two solutions, Ψ_1, Ψ_2, with corresponding boundary data (ψ_1, ω_1) and (ψ_2, ω_2), our goal is to establish estimates of the type

$$|\Psi_1(r_0,\theta) - \Psi_2(r_0,\theta)| \;\le\; C\big(\|\psi_1 - \psi_2\|_{Q^{3/2}} + \|\omega_1 - \omega_2\|_{Q^{-1/2}}\big).$$

We assume that r_0 is chosen sufficiently small, so that the ball $B(0, r_0)$ is contained in the domain where the the solutions are defined.

The first task is to convert from the adapted coordinates (β, ϕ) to the polar coordinates (r_0, θ). To this end, we recall that $-\Psi_\beta = \mu r^2$, and hence we seek to express the region $\{(r_0, \theta) : \theta \in [0, 2\pi)\}$ as a graph of a function in adapted coordinates, namely $\{(\beta(\phi), \phi) : \phi \in [0, 2\pi)\}$. This is the content of the following lemma.

Lemma 8.1 *Let Ψ be a solution of* (5.22) *and* (5.23) *as constructed in Theorem 7.1. Then there exists a $\mathcal{C}^1$ function $\beta(\phi)$ satisfying $\Psi_\beta(\beta(\phi), \phi) = -\mu r_0^2$. Furthermore, for any two solutions $\Psi^{(1)}, \Psi^{(2)}$ we have*

$$|\beta_1(\phi) - \beta_2(\phi)| \;\le\; C\big(\|\psi_1 - \psi_2\|_{\widehat{Q}^{3/2}} + \|\omega_1 - \omega_2\|_{Q^{-1/2}}\big). \tag{8.1}$$

Proof. 1. We first observe that

$$\Psi_{\beta\varphi} = \beta^{-1-2\mu}(\overline{\partial_\varphi} + 1)\overline{\partial_\beta}\widetilde{\Psi}, \qquad\qquad \Psi_{\beta\phi} = \beta^{-2\mu}\partial_\phi\overline{\partial_\beta}\widetilde{\Psi}.$$

The smoothness of our map from boundary data to solution, considered in Theorem 7.1), along with the bounds (7.59)-(7.60), then implies that for any of the solutions that we are considering,

$$|\Psi^{(1)}_{\beta\beta} - \Psi^{(2)}_{\beta\beta}| \;\le\; C\beta^{-1-2\mu}\big(\|\psi_1 - \psi_2\|_{\widehat{Q}^{3/2}} + \|\omega_1 - \omega_2\|_{Q^{-1/2}}\big). \tag{8.2}$$

In particular, since the radially symmetric solution satisfies $\Psi_{\beta\beta} = 2\mu^2\kappa^2\beta^{-1-2\mu} > 0$, both of our solutions will satisfy bounds of the form

$$C_1\beta^{-1-2\mu} \;>\; \Psi_{\beta\beta} \;>\; C_2\beta^{-1-2\mu}. \tag{8.3}$$

Since Ψ_β is locally $\mathcal{C}^1$ w.r.t. the (β, ϕ) variables, we can use the implicit function theorem to construct the corresponding functions $\beta_1(\phi)$ and $\beta_2(\phi)$ in (8.1).

2. To establish the bounds (8.1) we shall assume, without loss of generality, that $\beta_1(\phi) \le \beta_2(\phi)$. Note that

$$\int_b^{\beta_1(\phi)} \Psi^{(1)}_{\beta\beta}(z,\phi)\,dz \;=\; \int_b^{\beta_2(\phi)} \Psi^{(2)}_{\beta\beta}(z,\phi)\,dz.$$

Using equations (8.3) and (8.2), we then have, for some $C_3, C_4 > 0$,

$$\begin{aligned} 0 \le C_3\big(\beta_1(\phi)^{-2\mu} - \beta_2(\phi)^{-2\mu}\big) \;&\le\; \int_{\beta_1(\phi)}^{\beta_2(\phi)} \Psi^{(2)}_{\beta\beta}\,dz \;=\; \int_b^{\beta_1(\phi)} \Psi^{(1)}_{\beta\beta} - \Psi^{(2)}_{\beta\beta}\,dz \\ &\le\; \int_b^{\beta_2} \big|\Psi^{(1)}_{\beta\beta} - \Psi^{(2)}_{\beta\beta}\big|\,dz \le C_4\big(\|\psi_1 - \psi_2\|_{\widehat{Q}^{3/2}} + \|\omega_1 - \omega_2\|_{Q^{-1/2}}\big), \end{aligned}$$

In turn, this implies equation (8.1). $\square$

The next lemma provides bounds on the difference between coordinate changes (mapping polar coordinates to adapted coordinates) for two different solutions.

Lemma 8.2 *Let $\Psi^{(1)}$ and $\Psi^{(2)}$ be two solutions on the inner domain, in adapted coordinates. Given a point with polar coordinates (r_0, θ), the corresponding points (β_1, ϕ_1) and (β_2, ϕ_2) in adapted coordinates satisfy*

$$|\phi_1 - \phi_2| \ \leq \ C\big(\|\psi_1 - \psi_2\|_{\widehat{Q}^{3/2}} + \|\omega_1 - \omega_2\|_{Q^{-1/2}}\big). \tag{8.4}$$

Proof. The definition of adapted coordinates implies

$$\theta \ = \ \phi_1 + \beta_1(\phi_1) \ = \ \phi_2 + \beta_2(\phi_2).$$

In turn, $\phi_1 - \phi_2 = \beta_1(\phi_1) - \beta_2(\phi_2)$. By then using the triangle inequality

$$|\phi_1 - \phi_2| \ \leq \ \|\partial_\phi \beta_1\|_\infty |\phi_1 - \phi_2| + |\beta_1(\phi_2) - \beta_2(\phi_2)|\,.$$

Since the operators $\partial_\phi \partial_\beta$ and $\partial_\beta \partial_\beta$ are bounded in our space, and as the $\partial_\phi \beta = 0$ for the radially symmetric solution, we may conclude that, for all solutions sufficiently close to the symmetric one

$$|\phi_1 - \phi_2| \ \leq \ 2|\beta_1(\phi_2) - \beta_2(\phi_2)|.$$

Lemma 8.1 then gives the desired result. □

Finally, we provide bounds on the mapping from the boundary data to the values of the solution, in polar coordinates.

Proposition 8.1 *Let $\Psi^{(1)}$ and $\Psi^{(2)}$ be two solutions on the inner domain. Then*

$$|\Psi^{(1)}(r_0, \theta) - \Psi^{(2)}(r_0, \theta)| \ \leq \ C(\|\psi_1 - \psi_2\|_{\widehat{Q}^{3/2}} + \|\omega_1 - \omega_2\|_{Q^{-1/2}}).$$

Proof. By the triangle inequality,

$$|\Psi^{(1)}(r_0, \theta) - \Psi^{(2)}(r_0, \theta)| \ \leq \ |\Psi^{(1)}(\beta_1, \phi_1) - \Psi^{(1)}(\beta_2, \phi_2)| + |\Psi^{(1)}(\beta_2, \phi_2) - \Psi^{(2)}(\beta_2, \phi_2)|.$$

Since $\Psi^{(1)}$ is locally $\mathcal{C}^1$ in the adapted coordinates (with uniform bounds on the derivatives), for the first term we simply use the Lipschitz bounds on the change of coordinates from the previous two lemmas. The second term can be bounded immediately in terms of the boundary data, using Theorem 7.1. This concludes the proof. □

Remark 8.1 The previous result is stated as an $\mathbf{L}^\infty$ bound. This could be immediately upgraded to a $\mathcal{C}^0$ bound, since the solutions $\Psi^{(1)}$ and $\Psi^{(2)}$ are uniformly $\mathcal{C}^1$ in the adapted coordinates. Hence the difference will be a smooth function of θ.

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