

A MULTI-DIMENSIONAL OPTIMAL HARVESTING PROBLEM WITH MEASURE VALUED SOLUTIONS

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ABSTRACT. The paper is concerned with the optimal harvesting of a marine resource, described by an elliptic equation with Neumann boundary conditions and a nonlinear source term. Since the cost function has linear growth, an optimal solution is found within the class of measure-valued control strategies. The analysis also provides results on the existence and uniqueness of strictly positive solutions to an elliptic equation with a measure-valued source and Neumann boundary data.

1. THE BASIC MODEL

In this paper we study an optimal harvesting problem in a multi-dimensional domain. Consider a bounded connected open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary. Denote by $\varphi = \varphi(t, x)$ the density of fish at time t at the point $x \in \Omega$. In absence of fishing activity, we assume that the fish population evolves according to the parabolic equation with source term

$$\varphi_t = \Delta\varphi + g(x, \varphi) \quad x \in \Omega,$$

with Neumann boundary conditions

$$(1.1) \quad \nabla\varphi \cdot \mathbf{n} = 0 \quad x \in \partial\Omega.$$

Here $\mathbf{n} = \mathbf{n}(x)$ denotes the unit outer normal to the set Ω at the point $x \in \partial\Omega$. A typical choice for the source term is

$$g(x, \varphi) = \alpha(x)(h(x) - \varphi)\varphi.$$

Here $h(x)$ denotes the maximum fish population that can be supported by the habitat at x , while α is a reproduction speed.

We denote by $u = u(t, x)$ the intensity of harvesting conducted by a fishing company. In the presence of this harvesting activity, the population evolves according to

$$\varphi_t = \Delta\varphi + g(x, \varphi) - \varphi u.$$

Assuming that the harvesting rate remains constant in time, the fish population will reach an equilibrium described by

$$(1.2) \quad \Delta\varphi + g(x, \varphi) = \varphi u \quad x \in \Omega,$$

together with the Neumann boundary conditions (1.1).

In order to define an optimization problem for the steady state solution (1.2), we consider the cost

$$(1.3) \quad \int_{\Omega} c(x) u(x) dx.$$

Here $c(x)$ is the cost for a unit of fishing effort at the location $x \in \Omega$. Of course, the simplest choice here is $c(x) = \text{constant}$. However, one may have a cost $c(x)$ which increases with the distance of the

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point x from the coastal city where fishing company has its base. In addition, if a region $\Omega_0 \subset \Omega$ is set aside as a marine park where no fishing is permitted, this can be modeled by setting $c(x) = +\infty$ for every $x \in \Omega_0$.

More generally, we consider a net profit which depends on the total fish caught, minus the harvesting cost:

$$(1.4) \quad J(\varphi, u) = \int_{\Omega} \varphi(x)u(x)dx - \int_{\Omega} c(x)u(x)dx.$$

The function $u = u(x)$ describes the harvesting strategy. It is reasonable to assume that it satisfies constraints of the form

$$(1.5) \quad u(x) \geq 0, \quad \int_{\Omega} b(x)u(x)dx \leq 1,$$

for some non-negative function $b(\cdot)$. The second constraint determines the maximum amount of harvesting power within the capabilities of the company. In practice, this may depend on the number of fishermen and on the size of fishing boats available.

With minor modifications, all of our analysis remains valid for somewhat more general cost functionals of the type

$$J(\varphi, u) = \int_{\Omega} \varphi(x)u(x)dx - \Psi \left(\int_{\Omega} c(x)u(x)dx \right),$$

assuming that Ψ is a non decreasing, convex, and lower semicontinuous function such that $\Psi(0) = 0$ and $\Psi'(0) = 1$.

Our main interest is in the existence and the qualitative properties of optimal solutions. This problem has two distinct features:

- (i) In general, a given strategy $u(\cdot)$ does not determine a unique solution to the nonlinear elliptic problem (1.2), (1.1). In particular, one always has the trivial solution $\varphi \equiv 0$. For this reason, it is convenient to consider “optimal pairs” (u, φ) , where u is an admissible harvesting strategy satisfying the constraints (1.5), while φ is a positive solution of the corresponding elliptic problem (1.2), (1.1).
- (ii) Since the cost functional (1.4) has only linear growth w.r.t. u , there is no guarantee that the optimal strategy $u(\cdot)$ will lie in the space $\mathbf{L}^1(\Omega)$. Indeed, existence of optimal solutions will be proved within the larger space of bounded Radon measures supported on the closure $\bar{\Omega}$ of the domain.

By deriving suitable necessary conditions for optimality, one can then understand whether the optimal measure μ is absolutely continuous w.r.t. Lebesgue measure. In the one-dimensional case, necessary conditions were obtained in [7] by means of the Pontryagin maximum principle. In the multidimensional case, we expect that optimality conditions can be derived by similar techniques as in [8]. When a marine park is present, so that $c(x) = +\infty$ for $x \in \Omega_0$, the optimal measure μ may concentrate a positive amount of mass along the boundary $\partial\Omega_0$. An example where this holds was constructed in [7].

The paper is organized as follows. In Section 2 we show the existence of a positive solution for the problem (1.2), (1.1), assuming that $u \in \mathcal{C}_c^\infty(\Omega)$. Section 3 introduces a suitable definition of solution in the case where u is replaced by a measure μ supported on the closure $\bar{\Omega}$. Finally, in Section 4 the existence of an optimal measure-valued control is established.

Problems of optimal harvesting of a marine resource, governed by a semilinear elliptic equation, have been the subject of several investigations [2, 9, 11, 12, 15]. We remark that a quadratic harvesting cost such as

$$\int_{\Omega} c(x)u^2(x)dx$$

is entirely natural from a mathematical point of view, and guarantees that the optimal strategy is described by a function $u^{opt} \in \mathbf{L}^2(\Omega)$. However, the linear cost (1.3) provides a more realistic model. We also remark that most of the theory of elliptic equations with measure-valued right hand side is

concerned with Dirichlet boundary conditions. In our fishery model, setting $\varphi(x) = 0$ for $x \in \partial\Omega$ would mean that the fish is instantly killed as soon as it touches the shore. Of course this is not the case. The Neumann boundary conditions (1.1) yield a more appropriate model.

Finally, we remark that in these optimization problems governed by a linear (as in [8]) or by a semilinear elliptic equation it is natural to assume that the measure μ vanishes on every set of capacity zero. In connection with our harvesting problem, this can be explained as follows. Assume $\mu = \mu_0 + \mu_1$, where μ_0 is a positive measure concentrated on a set of capacity zero. As shown in [10], one should have $\varphi = -\infty$ on the support of the measure $\varphi\mu_0$. Since in our case we know a priori that the fish density is $\varphi \geq 0$, we conclude that $\varphi\mu_0$ must be the zero measure, hence $\varphi = 0$ on the support of μ_0 . If $\mu_0 \neq 0$, then the harvesting strategy μ_1 yields a strictly better payoff than μ , because fishing at a location where the fish density is zero increases the cost without increasing the payoff. For this reason, all of our analysis will be concerned with measures which vanish on sets of zero capacity.

In the one-dimensional case, the existence and a characterization of measure-valued optimal controls were proved in [6, 7]. The multi-dimensional case, studied in the present paper, requires a more careful analysis, relying on Sobolev space theory. For the basic theory of elliptic equations with measure-valued right hand side we refer to [3, 4, 10].

2. AN ELLIPTIC PROBLEM WITH SMOOTH COEFFICIENTS

As a first step in the analysis, we derive some estimates on the solution to an elliptic boundary value problem with smooth coefficients. Consider the semilinear elliptic problem with Neumann boundary conditions

$$(2.1) \quad \begin{cases} -\Delta\varphi + \beta(x)\varphi = g(x, \varphi), & \text{in } \Omega, \\ \partial_{\mathbf{n}}\varphi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\partial_{\mathbf{n}}$ denotes the derivative in the direction of the outer normal to the boundary $\partial\Omega$. The following assumptions will be used.

(A1) *The domain $\Omega \subset \mathbb{R}^N$ is bounded, open, connected, and has a smooth boundary.*

(A2) *The nonlinear source term can be written as $g(x, s) = f(x, s)s$, where $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function satisfying*

$$(2.2) \quad \partial_s f(x, s) < 0, \quad \text{for all } (x, s) \in \bar{\Omega} \times \mathbb{R},$$

$$(2.3) \quad f(x, s) > 0, \quad \text{if and only if } s < h(x),$$

and $h : \bar{\Omega} \mapsto \mathbb{R}$ is a smooth nonnegative function.

(A3) *The function $\beta \in \mathcal{C}_c^\infty(\Omega)$ is non-negative and satisfies*

$$(2.4) \quad \int_{\Omega} \beta(x) dx < \int_{\Omega} f(0, x) ds.$$

By **(A2)**, there exists a constant M such that

$$(2.5) \quad g(x, s) \leq M, \quad h(x) \leq M \quad \text{for all } x \in \bar{\Omega}, s \geq 0.$$

Lemma 2.1. *Under the assumptions **(A1)**–**(A3)**, the boundary value problem (2.1) has a positive smooth bounded solution $\varphi = \varphi(x)$, such that*

$$(2.6) \quad 0 < \delta \leq \varphi(x) \leq M \quad \text{for all } x \in \bar{\Omega},$$

where M is the constant in (2.5).

Proof. By (2.3), $g(x, M) = f(x, M)M \leq 0$ for all $x \in \bar{\Omega}$. Hence the constant function identically equal to M is a supersolution of (2.1).

In order to construct a subsolution, we consider the functional

$$I(u) \doteq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} [\beta(x) - f(x, 0)] u^2 dx, \quad u \in H^1(\Omega),$$

and the manifold of codimension 1

$$\mathcal{B} \doteq \left\{ u \in H^1(\Omega); \int_{\Omega} u^2 dx = 1 \right\}.$$

Since β and $f(x, \cdot)$ are smooth, the following facts are clear:

- (i) I_n is a continuous, quadratic functional on $H^1(\Omega)$.
- (ii) I_n is coercive on \mathcal{B} . Namely, if $\{u_k\}_k \subset \mathcal{B}$ and $\lim_{k \rightarrow \infty} \|u_k\|_{H^1(\Omega)} = \infty$, then $\lim_{k \rightarrow \infty} I(u_k) = \infty$.
- (iii) I is weakly lower semicontinuous in $H^1(\Omega)$. Namely

$$u_k \rightharpoonup u \text{ weakly in } H^1(\Omega) \quad \implies \quad I(u) \leq \liminf_{k \rightarrow \infty} I(u_k).$$

Therefore, restricted to \mathcal{B} , the functional I is bounded from below. Since $\mathcal{B} \subset H^1(\Omega)$ is weakly closed, by taking the limit of a minimizing sequence we obtain a function $u \in \mathcal{B}$ such that

$$l \doteq \inf_{w \in \mathcal{B}} I(w) = I(u).$$

The necessary conditions for optimality yield the existence of a scalar Lagrange multiplier η such that the Gateaux derivative of I at u satisfies

$$DI(u) - \eta u = 0.$$

Observing that I is positively homogeneous of degree 2, we obtain

$$\eta = \left. \frac{d}{d\lambda} I(\lambda u) \right|_{\lambda=1} = \left. \frac{d}{d\lambda} [\lambda^2 I(u)] \right|_{\lambda=1} = 2I(u).$$

This means that u and η solve the linear eigenvalue problem

$$\begin{cases} -\Delta u + \beta(x)u = f(x, 0)u + \eta u, & \text{on } \Omega, \\ \partial_{\mathbf{n}} u = 0, & \text{on } \partial\Omega, \end{cases}$$

where

$$\eta = 2I(u).$$

Since the map with constant value $\frac{1}{\sqrt{m_N(\Omega)}}$ belongs to \mathcal{B} , by (2.4) we obtain

$$(2.7) \quad \eta = 2 \min_{w \in \mathcal{B}} I(w) \leq 2I\left(\frac{1}{\sqrt{m_N(\Omega)}}\right) = \frac{1}{m_N(\Omega)} \left(\int_{\Omega} \beta dx - \int_{\Omega} f(x, 0) dx \right) < 0.$$

Since the domain Ω is connected, classical results on linear elliptic eigenvalue problems [13] yield

$$(2.8) \quad u \in \mathbf{L}^{\infty}(\Omega), \quad 0 < c \leq u,$$

for some positive constant c . As subsolution for the problem (2.1) we now choose the function εu , where $\varepsilon > 0$ is a sufficiently small constant. Choosing ε small enough, we achieve

$$(2.9) \quad 0 < \varepsilon u \leq M.$$

Next, observe that

$$\begin{aligned} -\Delta(\varepsilon u) + \beta \varepsilon u - g(x, \varepsilon u) &= \varepsilon(-\Delta u + \beta u) - \varepsilon u f(x, \varepsilon u) \\ &= \varepsilon(u f(x, 0) + \eta u) - \varepsilon u f(x, \varepsilon u) = \varepsilon u \left((f(x, 0) - f(x, \varepsilon u)) + \eta \right). \end{aligned}$$

Due to the boundedness of u , the continuity of $f(x, \cdot)$, and (2.7), we can choose $\varepsilon > 0$ so small that

$$(f(x, 0) - f(x, \varepsilon u)) + \eta \leq \frac{\eta}{2} < 0, \quad x \in \Omega.$$

Therefore εu is a subsolution of (2.1). By (2.9) there exists a solution φ of (2.1) such that

$$0 < c\varepsilon \leq \varepsilon u \leq \varphi \leq M.$$

Since all coefficients of the equation are smooth, it is clear that the solution is smooth as well. \square

For future use, we derive here an inequality valid for solutions φ of (2.1). Let $G = G(t, x; y)$ be the Green function for the linear parabolic equation with Neumann boundary conditions

$$(2.10) \quad \begin{cases} w_t = \Delta w, & t > 0, \quad x \in \Omega, \\ \partial_{\mathbf{n}} w = 0, & t > 0, \quad x \in \partial\Omega. \end{cases}$$

As it is well known [14], for each fixed $y \in \Omega$ the function $G(\cdot, \cdot; y)$ provides a solution to (2.10) such that

$$(2.11) \quad \int_{\Omega} G(t, x; y) dx = 1, \quad \lim_{t \downarrow 0} \int_{\Omega} G(t, x; y) \phi(x) dx = \phi(y)$$

for every $\phi \in \mathcal{C}(\overline{\Omega})$. The solution of (2.10) with a continuous initial data $w(0, x) = \phi(x)$ is thus given by

$$w(t, x) = \int_{\Omega} G(t, x; y) \phi(y) dy \quad t > 0, \quad x \in \Omega.$$

Let now $\varphi \in \mathcal{C}^2(\Omega)$ be any function such that

$$(2.12) \quad \begin{cases} \Delta\varphi \geq -M, & \text{in } \Omega, \\ \partial_{\mathbf{n}}\varphi = 0, & \text{on } \partial\Omega. \end{cases}$$

In particular, φ could be the solution to the elliptic problem (2.1) constructed in Lemma 2.1. For any $t > 0$, consider the averaged function

$$(2.13) \quad \varphi^{(t)}(y) \doteq \int_{\Omega} G(t, x; y) \varphi(x) dx.$$

Using the boundary conditions in (2.10) and (2.12) to integrate by parts, by the first equations in (2.11) and (2.12) one obtains

$$(2.14) \quad \begin{aligned} \frac{d}{dt} \varphi^{(t)}(y) &= \frac{d}{dt} \int_{\Omega} G(t, x; y) \varphi(x) dx = \int_{\Omega} G_t(t, x; y) \varphi(x) dy \\ &= \int_{\Omega} \Delta G(t, x; y) \varphi(x) dx = \int_{\Omega} G(t, x; y) \Delta\varphi(x) dx \geq -M. \end{aligned}$$

By (2.11), for every $y \in \Omega$ we have

$$(2.15) \quad \lim_{t \downarrow 0} \varphi^{(t)}(y) = \varphi(y).$$

Together with (2.14), this implies

$$(2.16) \quad \varphi(y) \leq \varphi^{(t)}(y) + Mt, \quad \text{for all } y \in \Omega, \quad t > 0.$$

3. AN ELLIPTIC PROBLEM WITH MEASURE-VALUED COEFFICIENTS

In general, an optimal strategy for the harvesting problem may be a measure, not necessarily absolutely continuous w.r.t. Lebesgue measure. We thus need to study the semilinear elliptic problem with Neumann boundary conditions

$$(3.1) \quad \begin{cases} -\Delta\varphi + \varphi\mu = g(x, \varphi) & x \in \Omega, \\ \partial_{\mathbf{n}}\varphi = 0 & x \in \partial\Omega, \end{cases}$$

where μ is a nonnegative, bounded Radon measure on the closure $\overline{\Omega}$. A solution to (3.1) need not be smooth, or even continuous. In the following, we shall consider solutions of (3.1) in distributional sense.

Definition 3.1. *Let $\varphi : \overline{\Omega} \mapsto \mathbb{R}$ be an upper semicontinuous, non-negative function whose pointwise values are determined by*

$$(3.2) \quad \varphi(y) = \lim_{t \downarrow 0} \varphi^{(t)}(y) = \lim_{t \downarrow 0} \int_{\Omega} G(t, x; y) \varphi(x) dx,$$

with G as in (2.11)–(2.13). We say that φ is a solution of (3.1) if

$$\varphi \in \mathbf{L}^{\infty}(\Omega) \cap H^1(\Omega),$$

and for every test function $\phi \in \mathcal{C}^2(\mathbb{R}^N)$ one has

$$(3.3) \quad \int_{\Omega} \nabla\phi \cdot \nabla\varphi dx + \int_{\overline{\Omega}} \phi\varphi d\mu = \int_{\Omega} \phi g(x, \varphi) dx.$$

Under the assumptions **(A1)**–**(A2)** on the domain Ω and on the source term g , made in Section 2, our next result provides the existence and uniqueness of a nontrivial solution to the measure-valued elliptic problem (3.1).

Theorem 3.1. *Let the assumptions **(A1)**–**(A2)** hold. Assume that*

$$(3.4) \quad \begin{aligned} &\mu \text{ is a bounded nonnegative Radon measure on } \overline{\Omega}, \\ &\mu(A) = 0, \quad \text{for every } A \subset \overline{\Omega} \text{ with zero capacity,} \\ &\mu(\overline{\Omega}) < \int_{\Omega} f(x, 0) dx. \end{aligned}$$

Then the boundary value problem (3.1) has a unique positive bounded solution such that

$$(3.5) \quad 0 < \varphi(x) \leq M \quad \text{for a.e. } x \in \Omega,$$

where M is the constant in (2.5). Moreover, one has

$$(3.6) \quad \int_{\overline{\Omega}} \varphi d\mu = \int_{\Omega} g(x, \varphi) dx \leq M m_N(\Omega),$$

$$(3.7) \quad \int_{\Omega} |\nabla\varphi|^2 dx + \int_{\overline{\Omega}} \varphi^2 d\mu = \int_{\Omega} \varphi g(x, \varphi) dx \leq M^2 m_N(\Omega),$$

where m_N denotes the N -dimensional Lebesgue measure.

Proof. 1. By assumption, the boundary $\partial\Omega$ is smooth. Hence, by choosing $\varepsilon > 0$ small enough, for every point $x \in \mathbb{R}^N$ with distance $d(x, \partial\Omega) \leq \varepsilon$, the perpendicular projection $\pi(x) \in \partial\Omega$ is well defined. For $r \geq 0$, consider the open sets

$$\begin{aligned}\Omega_r &\doteq \left\{ x \in \mathbb{R}^N ; d(x, \Omega) < \varepsilon \right\}, \\ \Omega_{-r} &\doteq \left\{ x \in \Omega ; d(x, \partial\Omega) > \varepsilon \right\}.\end{aligned}$$

We can choose $\varepsilon > 0$ sufficiently small so that the projection $x \mapsto \pi(x)$ is smooth on a neighborhood of the closed set $\overline{\Omega}_\varepsilon \setminus \Omega_{-\varepsilon}$.

Let $\chi \in C_c^\infty(\Omega_\varepsilon)$ be a smooth function such that

$$\begin{aligned}\chi(x) &\in [0, 1] && \text{for all } x \in \Omega_\varepsilon, \\ \chi(x) &= 1 && \text{for all } x \in \overline{\Omega}.\end{aligned}$$

By a reflection across the boundary of Ω , we now define a bounded, linear extension operator $E : H^1(\Omega) \mapsto H_0^1(\Omega_\varepsilon)$ by setting

$$(3.8) \quad E\phi(x) \doteq \begin{cases} \phi(x) & \text{if } x \in \overline{\Omega}, \\ \chi(x)\phi(2\pi(x) - x) & \text{if } x \in \Omega_\varepsilon \setminus \overline{\Omega}. \end{cases}$$

2. We shall approximate the measure μ by a sequence of measures μ_n having smooth density w.r.t. Lebesgue measure, and study the regularized problems. Notice that we can extend μ to a positive measure on Ω_ε simply by setting $\mu(\Omega_\varepsilon \setminus \overline{\Omega}) = 0$.

Since μ is zero on all sets with zero capacity, following [10] we can find measures $\tilde{\mu}$ and $\bar{\mu}$ such that

$$(3.9) \quad \mu = \tilde{\mu} + \bar{\mu}, \quad \tilde{\mu}, \bar{\mu} \geq 0, \quad \tilde{\mu} \in \mathbf{L}^1(\Omega_\varepsilon), \quad \bar{\mu} \in H^{-1}(\Omega_\varepsilon).$$

More precisely, there exists functions $\Phi_0, \Phi_1, \dots, \Phi_N$ such that

$$\Phi_0 \in \mathbf{L}^1(\Omega_\varepsilon), \quad \Phi_i \in \mathbf{L}^2(\Omega_\varepsilon), \quad i = 1, \dots, N,$$

and, for every bounded function $\phi \in H^1(\Omega_\varepsilon)$, one has

$$(3.10) \quad \int_{\Omega_\varepsilon} \phi d\mu = \int_{\Omega_\varepsilon} \Phi_0 \phi dx - \sum_{i=1}^N \int_{\Omega_\varepsilon} \Phi_i \phi_{x_i} dx.$$

By shifting the measure μ strictly inside the domain $\Omega_{-\varepsilon/n}$ and taking a mollification, we obtain a sequence of smooth approximating measures μ_n . This procedure yields sequences of functions $\Phi_{0,n}, \dots, \Phi_{N,n} \in C_c^\infty(\Omega)$ such that

$$(3.11) \quad \beta_n \doteq \Phi_{0,n} + \sum_{i=1}^N (\Phi_{i,n})_{x_i} \in C_c^\infty(\Omega),$$

$$(3.12) \quad \beta_n \geq 0, \quad \int_{\Omega} \beta_n(x) dx \leq \int_{\Omega} f(0, x) dx - \delta$$

for some $\delta > 0$ and all $n \geq 1$. Moreover, as $n \rightarrow \infty$, one has the convergence

$$(3.13) \quad \|\Phi_{0,n} - \Phi_0\|_{\mathbf{L}^1(\Omega_\varepsilon)} \rightarrow 0, \quad \|\Phi_{i,n} - \Phi_i\|_{\mathbf{L}^2(\Omega_\varepsilon)} \rightarrow 0 \quad i = 1, \dots, N.$$

3. Applying Lemma 2.1, for every $n \geq 1$ we obtain the existence of a strictly positive smooth solution $\varphi_n : \bar{\Omega} \mapsto]0, M]$ to the problem

$$(3.14) \quad \begin{cases} -\Delta\varphi + \beta_n(x)\varphi = g(x, \varphi), & \text{in } \Omega, \\ \partial_{\mathbf{n}}\varphi = 0, & \text{on } \partial\Omega. \end{cases}$$

We claim that

$$(3.15) \quad \int_{\Omega} |\nabla\varphi_n|^2 dx \leq M^2 m_N(\Omega), \quad \text{for all } n \geq 1.$$

Indeed, since $\varphi_n \leq M$ and $g \leq M$, multiplying the first equation in (3.14) by φ_n and integrating over Ω one obtains

$$\begin{aligned} 0 &= - \int_{\Omega} \varphi_n \Delta\varphi_n dx + \underbrace{\int_{\Omega} \varphi_n^2 \beta_n dx}_{\geq 0} - \int_{\Omega} \varphi_n g(x, \varphi_n) dx \\ &\geq \int_{\Omega} |\nabla\varphi_n|^2 dx - M^2 m_N(\Omega). \end{aligned}$$

4. Let $E\varphi_n \in H_0^1(\Omega_\varepsilon)$ be the extension on φ_n , defined as in (3.8). Thanks to the previous estimates, by possibly taking a subsequence and relabeling, we can assume the strong convergence

$$(3.16) \quad \|E\varphi_n - \varphi\|_{\mathbf{L}^2(\Omega_\varepsilon)} \rightarrow 0$$

and the weak convergence

$$(3.17) \quad E\varphi_n \rightharpoonup \varphi \quad \text{in } H_0^1(\Omega_\varepsilon),$$

for some function $\varphi \in H_0^1(\Omega_\varepsilon)$. In addition, we can assume the pointwise convergence

$$(3.18) \quad \lim_{n \rightarrow \infty} E\varphi_n(x) = \varphi(x) \quad \text{for a.e. } x \in \Omega_\varepsilon$$

and the convergence of the averages

$$(3.19) \quad \lim_{n \rightarrow \infty} \varphi_n^{(t)}(x) \rightarrow \varphi^{(t)}(x)$$

for every fixed $t > 0$, uniformly for $x \in \bar{\Omega}$.

Indeed, for every fixed time t , the function $G(t, \cdot; \cdot)$ admits a Lipschitz continuous extension defined on $\bar{\Omega} \times \bar{\Omega}$. Hence the maps $x \mapsto \varphi_n^{(t)}(x)$ are uniformly Lipschitz continuous as well. Applying Ascoli's compactness theorem, by taking a subsequence and relabeling, we can achieve (3.19) for every rational $t > 0$. By continuity, the convergence holds for every (possibly irrational) $t > 0$ as well.

In the remainder of the proof we will show that the restriction of φ to the closed domain $\bar{\Omega}$ provides a solution of (3.1), in the sense of Definition 3.1.

5. We claim that the limit (3.2) exists, for every $x \in \bar{\Omega}$, and defines an upper semicontinuous function.

Indeed, for every $n \geq 1$ and $x \in \bar{\Omega}$ the function $t \mapsto \varphi_n^{(t)}(x) + Mt$ is nondecreasing. Taking the limit as $n \rightarrow \infty$, this implies that the map $t \mapsto \varphi^{(t)}(x) + Mt$ is nondecreasing as well. Hence the function

$$(3.20) \quad \bar{\varphi}(x) \doteq \inf_{t>0} \left(\varphi^{(t)}(x) + Mt \right) = \lim_{t \downarrow 0} \varphi^{(t)}(x)$$

is well defined for every $x \in \bar{\Omega}$. Moreover, $\bar{\varphi}(x) = \varphi(x)$ at every Lebesgue point of φ , hence almost everywhere.

The representation (3.20) of the function $\bar{\varphi}$ as the pointwise infimum of a decreasing family of Lipschitz continuous functions shows that $\bar{\varphi}$ is upper semicontinuous.

6. To prove that φ solves (3.1), let $\phi \in \mathcal{C}^2(\mathbb{R}^N)$ be a test function. From (3.14) it follows

$$\begin{aligned}
(3.21) \quad 0 &= \int_{\Omega} \nabla \varphi_n \cdot \nabla \phi \, dx + \int_{\Omega} \varphi_n \beta_n \phi \, dx - \int_{\Omega} g(x, \varphi_n) \phi \, dx \\
&= \int_{\Omega} \nabla \varphi_n \cdot \nabla \phi \, dx + \int_{\Omega_\varepsilon} E\varphi_n \left(\Phi_{0,n} + \sum_{i=1}^N (\Phi_{i,n})_{x_i} \right) \phi \, dx - \int_{\Omega} g(x, \varphi_n) \phi \, dx \\
&= \int_{\Omega} \nabla \varphi_n \cdot \nabla \phi \, dx + \int_{\Omega_\varepsilon} E\varphi_n \Phi_{0,n} \phi \, dx - \int_{\Omega_\varepsilon} \sum_{i=1}^N (E\varphi_n)_{x_i} \Phi_{i,n} \phi \, dx - \int_{\Omega} g(x, \varphi_n) \phi \, dx.
\end{aligned}$$

Letting $n \rightarrow \infty$, using the weak convergence $E\varphi_n \rightharpoonup \varphi$ in $H^1(\Omega_\varepsilon)$, the uniform bounds $\|\varphi_n\|_{\mathbf{L}^\infty(\Omega)} \leq M$, and the strong convergence $\|\Phi_{0,n} - \Phi_0\|_{\mathbf{L}^1(\Omega)} \rightarrow 0$, $\|\Phi_{i,n} - \Phi_i\|_{\mathbf{L}^2(\Omega)} \rightarrow 0$, we conclude

$$\begin{aligned}
(3.22) \quad 0 &= \int_{\Omega} \nabla \varphi \cdot \nabla \phi \, dx + \int_{\Omega_\varepsilon} \varphi \Phi_0 \phi \, dx - \int_{\Omega_\varepsilon} \sum_{i=1}^N \varphi_{x_i} \Phi_i \phi \, dx - \int_{\Omega} g(x, \varphi) \phi \, dx \\
&= \int_{\Omega} \nabla \varphi \cdot \nabla \phi \, dx + \int_{\overline{\Omega}} \varphi \phi \, d\mu - \int_{\Omega} g(x, \varphi) \phi \, dx.
\end{aligned}$$

Indeed, since μ is supported on $\overline{\Omega}$, integrating w.r.t. the measure μ over Ω_ε or $\overline{\Omega}$ is the same. This shows that φ is a solution of (3.1).

7. Clearly, from (2.6) it follows $0 \leq \varphi \leq M$. We now prove that φ is strictly positive.

As a first step we show that

$$(3.23) \quad \varphi \not\equiv 0.$$

Assume by contradiction that $\varphi(x) = 0$ for all $x \in \Omega$. Then

$$(3.24) \quad \varphi_n \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega).$$

By (3.14) we have

$$-\frac{\Delta \varphi_n}{\varphi_n} = f(x, \varphi_n) - \beta_n.$$

Integrating over Ω and using the Neumann boundary conditions one obtains

$$(3.25) \quad \int_{\Omega} \frac{|\nabla \varphi_n|^2}{\varphi_n^2} \, dx = \int_{\Omega} \beta_n \, dx - \int_{\Omega} f(x, \varphi_n) \, dx.$$

Letting $n \rightarrow \infty$, thanks to (3.4), (2.6), and (3.24), one obtains

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{|\nabla \varphi_n|^2}{\varphi_n^2} \, dx = \mu(\overline{\Omega}) - \int_{\Omega} f(x, 0) \, dx < 0,$$

reaching to a contradiction. Therefore (3.23) is proved.

To prove that φ is strictly positive, we observe that

$$-\frac{\Delta \varphi_n}{\varphi_n} = -\operatorname{div} \left(\frac{\nabla \varphi_n}{\varphi_n} \right) - \frac{|\nabla \varphi_n|^2}{\varphi_n^2} = -\Delta(\log(\varphi_n)) - \frac{|\nabla \varphi_n|^2}{\varphi_n^2}.$$

yields

$$-\Delta(\log(\varphi_n)) = f(x, \varphi_n) - \beta_n + \frac{|\nabla \varphi_n|^2}{\varphi_n^2}.$$

Thanks to (3.11), (3.13), (2.6), and (3.25) the right hand side is bounded in $\mathbf{L}^1(\Omega)$, hence

$$\{\Delta(\log(\varphi_n))\}_{n \in \mathbb{N}} \text{ is bounded in } \mathbf{L}^1(\Omega).$$

By the analysis in [16], this implies

$$\{\log(\varphi_n)\}_{n \in \mathbb{N}} \text{ is bounded in } W^{1,q}(\Omega), \quad 1 \leq q < \frac{N}{N-1}.$$

Therefore there exists a function $\psi \in W^{1,q}(\Omega)$, $1 \leq q < \frac{N}{N-1}$ such that, passing to a subsequence,

$$\begin{aligned} \log(\varphi_n) &\rightharpoonup \psi \quad \text{weakly in } W^{1,q}(\Omega), \quad 1 \leq q < \frac{N}{N-1}, \\ \log(\varphi_n) &\rightarrow \psi \quad \text{strongly in } \mathbf{L}^q(\Omega), \quad 1 \leq q < \frac{N}{N-1} \text{ and a.e. in } \Omega. \end{aligned}$$

We now have

$$\psi(x) > -\infty \quad \text{and} \quad \varphi(x) = e^{\psi(x)} > 0 \text{ for a.e. } x \in \Omega$$

proving our claim.

8. We now prove the uniqueness of the positive solutions of (3.1) satisfying (3.5). The main idea is taken from [5]. Assume that there exist two positive solutions φ and $\tilde{\varphi}$ of (3.1), so that

$$-\frac{\Delta\varphi}{\varphi} + \mu = f(x, \varphi), \quad -\frac{\Delta\tilde{\varphi}}{\tilde{\varphi}} + \mu = f(x, \tilde{\varphi}),$$

and hence

$$-\frac{\Delta\varphi}{\varphi} + \frac{\Delta\tilde{\varphi}}{\tilde{\varphi}} = f(x, \varphi) - f(x, \tilde{\varphi}).$$

Multiplying by $\varphi^2 - \tilde{\varphi}^2$ and integrating over Ω , using **(A2)**, (3.5), and the Neumann boundary conditions one obtains

$$\begin{aligned} 0 &= \int_{\Omega} \left(-\frac{\Delta\varphi}{\varphi} + \frac{\Delta\tilde{\varphi}}{\tilde{\varphi}} \right) (\varphi^2 - \tilde{\varphi}^2) dx - \int_{\Omega} (f(x, \varphi) - f(x, \tilde{\varphi})) (\varphi^2 - \tilde{\varphi}^2) dx \\ &= \int_{\Omega} \left(-\varphi\Delta\varphi + \frac{\tilde{\varphi}^2}{\varphi}\Delta\varphi + \frac{\varphi^2}{\tilde{\varphi}}\Delta\tilde{\varphi} - \tilde{\varphi}\Delta\tilde{\varphi} \right) dx \\ &\quad - \int_{\Omega} (f(x, \varphi) - f(x, \tilde{\varphi})) (\varphi - \tilde{\varphi}) (\varphi + \tilde{\varphi}) dx \\ &= \int_{\Omega} \left(|\nabla\varphi|^2 - \frac{2\tilde{\varphi}}{\varphi}\nabla\tilde{\varphi} \cdot \nabla\varphi + \frac{\tilde{\varphi}^2}{\varphi^2}|\nabla\varphi|^2 - \frac{2\varphi}{\tilde{\varphi}}\nabla\varphi \cdot \nabla\tilde{\varphi} + \frac{\varphi^2}{\tilde{\varphi}^2}|\nabla\tilde{\varphi}|^2 + |\nabla\tilde{\varphi}|^2 \right) dx \\ &\quad + \underbrace{\int_{\partial\Omega} \left(-\varphi\partial_{\mathbf{n}}\varphi - \frac{\tilde{\varphi}^2}{\varphi}\partial_{\mathbf{n}}\varphi + \frac{\varphi}{\tilde{\varphi}}\partial_{\mathbf{n}}\tilde{\varphi} - \tilde{\varphi}\partial_{\mathbf{n}}\tilde{\varphi} \right) dS}_{=0} \\ &\quad - \int_{\Omega} \int_0^1 \underbrace{\partial_{\varphi}f(x, \theta\varphi + (1-\theta)\tilde{\varphi})}_{<0} (\varphi - \tilde{\varphi})^2 \underbrace{(\varphi + \tilde{\varphi})}_{>0 \text{ a.e.}} d\theta dx \\ &\geq \int_{\Omega} \left(\left[|\nabla\varphi|^2 - \frac{2\tilde{\varphi}}{\varphi}\nabla\tilde{\varphi} \cdot \nabla\varphi + \frac{\tilde{\varphi}^2}{\varphi^2}|\nabla\varphi|^2 \right] + \left[|\nabla\tilde{\varphi}|^2 - \frac{2\varphi}{\tilde{\varphi}}\nabla\tilde{\varphi} \cdot \nabla\varphi + \frac{\varphi^2}{\tilde{\varphi}^2}|\nabla\varphi|^2 \right] \right) dx \\ &\quad + \inf_{\Omega \times [0, M]} |\partial_{\varphi}f| \int_{\Omega} (\varphi - \tilde{\varphi})^2 (\varphi + \tilde{\varphi}) dx \\ &= \underbrace{\int_{\Omega} \left(\left[\nabla\varphi - \frac{\varphi}{\tilde{\varphi}}\nabla\tilde{\varphi} \right]^2 + \left[\nabla\tilde{\varphi} - \frac{\tilde{\varphi}}{\varphi}\nabla\varphi \right]^2 \right) dx}_{\geq 0} + \inf_{\Omega \times [0, M]} |\partial_{\varphi}f| \int_{\Omega} (\varphi - \tilde{\varphi})^2 (\varphi + \tilde{\varphi}) dx \\ &\geq \underbrace{\inf_{\Omega \times [0, M]} |\partial_{\varphi}f|}_{>0} \underbrace{\int_{\Omega} (\varphi - \tilde{\varphi})^2 (\varphi + \tilde{\varphi}) dx}_{\geq 0} \geq 0. \end{aligned}$$

Therefore $\varphi = \tilde{\varphi}$ for a.e. $x \in \Omega$, proving uniqueness of positive solutions.

9. We conclude by proving the estimates (3.6) and (3.7). Let $\varphi \in \mathbf{L}^\infty(\Omega) \cap H^1(\Omega)$ be the positive bounded solution of (3.1).

Using the function $\phi \equiv 1$ as test function in (3.3) we obtain

$$\int_{\overline{\Omega}} \varphi d\mu = \int_{\Omega} g(x, \varphi) dx,$$

therefore (3.6) follows from (3.5).

Due to the density of $\mathcal{C}^2(\overline{\Omega})$ in $\mathbf{L}^\infty(\Omega) \cap H^1(\Omega)$, we can find a sequence of test functions $\{\phi_\nu\}_{\nu \in \mathbb{N}} \subset \mathcal{C}^2(\mathbb{R}^N)$ such that

$$\phi_\nu \longrightarrow \varphi \quad \text{a.e. in } \overline{\Omega} \text{ and in } H^1(\Omega), \text{ as } \nu \rightarrow \infty.$$

By (3.3) one has

$$\int_{\Omega} \nabla \phi_\nu \cdot \nabla \varphi dx + \int_{\overline{\Omega}} \phi_\nu \varphi d\mu = \int_{\Omega} \phi_\nu g(x, \varphi) dx.$$

Letting $\nu \rightarrow \infty$ we obtain

$$\int_{\Omega} |\nabla \varphi|^2 dx + \int_{\overline{\Omega}} \varphi^2 d\mu = \int_{\Omega} \varphi g(x, \varphi) dx.$$

Therefore (3.7) follows from (3.5). □

4. EXISTENCE OF AN OPTIMAL MEASURE-VALUED HARVESTING STRATEGY

In this final section we study the existence of an optimal pair (φ^*, μ^*) for the problem

$$(4.1) \quad \text{maximize:} \quad J(\varphi, \mu) \doteq \int_{\overline{\Omega}} \varphi(x) d\mu(x) - \int_{\overline{\Omega}} c(x) d\mu(x),$$

where φ is the unique strictly positive solution of

$$(4.2) \quad \begin{cases} \Delta \varphi(x) + g(x, \varphi) = \varphi \mu, & x \in \Omega, \\ \partial_{\mathbf{n}} \varphi = 0, & x \in \partial\Omega, \end{cases}$$

and μ is a non-negative Radon measure on $\overline{\Omega}$ which satisfies

$$(4.3) \quad \int_{\overline{\Omega}} b d\mu \leq 1.$$

Remark 4.1. Integrating the equation (4.2) over the domain $\overline{\Omega}$ and using the Neumann boundary conditions, one obtains the balance law

$$(4.4) \quad \underbrace{\int_{\Omega} \Delta \varphi dx}_{=0} + \int_{\Omega} g(x, \varphi) dx = \int_{\overline{\Omega}} \varphi d\mu.$$

According to (4.4), since no fish can enter or escape through the boundary of Ω , at an equilibrium configuration the total growth rate of the fish population must be exactly balanced by the total harvesting rate. In the following, instead of (4.1), we thus consider the equivalent problem

$$(4.5) \quad \text{maximize:} \quad J(\varphi, \mu) \doteq \int_{\Omega} g(x, \varphi) dx - \int_{\overline{\Omega}} c(x) d\mu(x).$$

In addition to the hypotheses **(A1)**-**(A2)** made in Section 2, we now assume

(A4) The functions $b : \bar{\Omega} \mapsto \mathbb{R}$ and $c : \bar{\Omega} \mapsto \mathbb{R}$ are both lower semi-continuous, and satisfy

$$(4.6) \quad b(x) \geq 0, \quad c(x) \geq c_0 > 0, \quad \text{for all } x \in \bar{\Omega},$$

for some positive constant c_0 .

Theorem 4.1. *Under the assumptions **(A1)**, **(A2)**, and **(A4)**, the constrained maximization problem (4.5), (4.2) admits an optimal solution.*

Proof. 1. Let $\{(\varphi_n, \mu_n)\}_{n \in \mathbb{N}}$ be a maximizing sequence, where μ_n is a positive measure on $\bar{\Omega}$ and φ_n is the corresponding positive solution of (3.1), according to Definition 3.1. Taking a sequence of test functions $\phi_\nu \in \mathcal{C}^2(\bar{\Omega})$ such that $\|\phi_\nu - \varphi_n\|_{H^1(\Omega)} \rightarrow 0$ as $\nu \rightarrow \infty$, from (3.3) we obtain

$$(4.7) \quad \begin{aligned} \int_{\Omega} |\nabla \varphi_n|^2 dx + \int_{\bar{\Omega}} |\varphi_n|^2 d\mu &= \lim_{\nu \rightarrow \infty} \left(\int_{\Omega} \nabla \phi_\nu \cdot \nabla \varphi_n dx + \int_{\bar{\Omega}} \phi_\nu \varphi_n d\mu \right) \\ &= \lim_{\nu \rightarrow \infty} \int_{\Omega} \phi_\nu g(x, \varphi_n) dx = \int_{\Omega} \varphi_n g(x, \varphi_n) dx \leq M^2 m_N(\Omega). \end{aligned}$$

This shows that the sequence $(\varphi_n)_{n \geq 1}$ is uniformly bounded in $H^1(\Omega)$.

2. By the assumptions, there exists a constant $c_0 > 0$ small enough so that

$$(4.8) \quad c(x) > c_0, \quad h(x) > c_0 \quad \text{for all } x \in \Omega.$$

We claim that is not restrictive to assume that

$$(4.9) \quad \text{supp}(\mu_n) \subseteq A_n \doteq \{x \in \bar{\Omega}; \varphi_n(x) \geq c_0\}.$$

Otherwise we could consider the measure

$$\mu'_n \doteq \chi_{A_n} \mu_n.$$

Clearly, μ'_n satisfies (4.3) and the function φ_n provides a subsolution to the equation (3.1), with μ_n replaced by μ'_n . Therefore, we can find a solution $\varphi'_n \geq \varphi_n$ of the same problem. We now have

$$\begin{aligned} J(\varphi_n, \mu_n) - J(\varphi'_n, \mu'_n) &= \int_{\bar{\Omega}} \varphi_n d\mu_n - \int_{\bar{\Omega}} \varphi'_n d\mu'_n - \int_{\bar{\Omega}} c d\mu_n + \int_{\bar{\Omega}} c d\mu'_n \\ &= \int_{\bar{\Omega}} \varphi_n d\mu_n - \int_{A_n} \varphi'_n d\mu_n - \int_{\bar{\Omega}} c d\mu_n + \int_{A_n} c d\mu_n \\ &\leq \int_{\bar{\Omega} \setminus A_n} \varphi_n d\mu_n - \int_{\bar{\Omega} \setminus A_n} c d\mu_n \leq \int_{\bar{\Omega} \setminus A_n} (\varphi_n - c) d\mu_n \leq 0. \end{aligned}$$

We can thus replace (φ_n, μ_n) with the pair (φ'_n, μ'_n) , which satisfies the additional condition (4.9).

3. For each $n \geq 1$, from (2.5) and (4.8)-(4.9) it follows

$$(4.10) \quad 0 < c_0 \leq \varphi_n(x) \leq M \quad \text{for all } x \in \bar{\Omega}.$$

Indeed, the upper bound in (4.10) is a consequence of (3.5). If the lower bound does not hold, consider the set $\Omega_0 = \{x \in \Omega; \varphi_n(x) < c_0\}$. Restricted to this set, the function φ_n provides a smooth solution to the elliptic equation

$$\Delta \varphi_n = -g(x, \varphi_n(x)) < 0,$$

with boundary conditions

$$\begin{aligned} \varphi_n(x) &= c_0 & \text{on } \partial\Omega_0 \cap \Omega, \\ \partial_{\mathbf{n}} \varphi_n(x) &= 0 & \text{on } \partial\Omega. \end{aligned}$$

Therefore, the minimum principle for superharmonic functions yields

$$\min_{x \in \Omega_0} \varphi_n(x) = \min_{x \in \Omega \cap \partial\Omega_0} \varphi_n(x) = c_0.$$

4. We now show that the total mass of the measures μ_n remains uniformly bounded. Indeed, from (4.4) and (4.10) it follows

$$c_0 \mu_n(\overline{\Omega}) \leq \int_{\overline{\Omega}} \varphi_n d\mu_n = \int_{\Omega} g(x, \varphi_n) dx \leq M m_N(\Omega).$$

Consider the positive measures $\nu_n \doteq \varphi_n \mu_n$. Clearly, all these measures are uniformly bounded. Thanks to the bounds (4.7), by possibly taking a subsequence and relabeling we can assume

$$(4.11) \quad \begin{aligned} \nu_n &\rightharpoonup \nu, & \mu_n &\rightharpoonup \mu & \text{in the sense of weak convergence of measures on } \overline{\Omega}, \\ \varphi_n &\longrightarrow \varphi & & & \text{strongly in } \mathbf{L}^2(\Omega), \text{ and a.e. in } \Omega, \\ \varphi_n &\rightharpoonup \varphi & & & \text{weakly in } H^1(\Omega), \end{aligned}$$

for some non-negative Radon measures ν, μ , and some function $\varphi \in H^1(\Omega)$. By (3.20), the pointwise values of each φ_n are determined by

$$(4.12) \quad \varphi_n(x) \doteq \inf_{t>0} \left(\varphi_n^{(t)}(x) + Mt \right) = \lim_{t \downarrow 0} \varphi_n^{(t)}(x),$$

where $\varphi_n^{(t)}$ are the averaged functions, defined as in (2.13). As in step 5 of the proof of Theorem 3.1, by taking a further subsequence and relabeling, we can assume that the Lipschitz continuous functions $\varphi_n^{(t)}$ converge to $\varphi^{(t)}$ uniformly on $\overline{\Omega}$, for every fixed $t > 0$. For each $x \in \overline{\Omega}$, the map $t \mapsto \varphi_n^{(t)}(x) + Mt$ is non-decreasing, for every $n \geq 1$. Hence the same holds for the map $t \mapsto \varphi^{(t)}(x) + Mt$. As in (3.2), (3.20), the pointwise values of φ are determined by

$$(4.13) \quad \varphi(x) \doteq \inf_{t>0} \left(\varphi^{(t)}(x) + Mt \right) = \lim_{t \downarrow 0} \varphi^{(t)}(x) = \lim_{t \downarrow 0} \int_{\Omega} G(t, y; x) \varphi(y) dy.$$

Being the pointwise infimum of a decreasing family of continuous maps, φ is upper semicontinuous. By (4.11), φ provides a weak solution to

$$\begin{cases} \Delta \varphi + g(x, \varphi) = \nu, & x \in \Omega, \\ \partial_{\mathbf{n}} \varphi = 0, & x \in \partial\Omega. \end{cases}$$

From the uniform bounds (4.10) it follows

$$(4.14) \quad 0 < c_0 \leq \varphi(x) \leq M \quad \text{for all } x \in \overline{\Omega}.$$

Defining

$$(4.15) \quad \varphi^* \doteq \varphi, \quad \mu^* \doteq \frac{\nu}{\varphi},$$

in the following two steps we will show that the pair (φ^*, μ^*) provides an optimal solution to our harvesting problem.

5. Since φ is an upper semicontinuous function satisfying $\varphi(x) \geq c_0 > 0$, the definition (4.15) is meaningful. We now establish the key inequality

$$(4.16) \quad \mu^* \leq \mu.$$

To prove that (4.16) holds, thanks to the upper semicontinuity of φ it suffices to show that

$$(4.17) \quad \int_{\overline{\Omega}} \frac{\phi}{\psi} d\nu \leq \int_{\overline{\Omega}} \phi d\mu, \quad \text{for every } \phi, \psi \in \mathcal{C}(\overline{\Omega}), \phi \geq 0, \psi > \varphi.$$

Since ψ is continuous on the compact set $\overline{\Omega}$, we can choose $t, \delta > 0$ small enough so that

$$(4.18) \quad \varphi(x) \leq \varphi^{(t)}(x) + Mt < \psi(x) - \delta \quad \text{for all } x \in \overline{\Omega}.$$

Since $\varphi_n \rightarrow \varphi$ in $\mathbf{L}^2(\Omega)$, as $n \rightarrow \infty$ the corresponding functions $\varphi_n^{(t)}$ converge to $\varphi^{(t)}$ uniformly on $\overline{\Omega}$. Hence for all n large enough we have

$$\varphi_n(x) \leq \varphi_n^{(t)}(x) + Mt \leq \varphi^{(t)}(x) + Mt + \delta < \psi(x) \quad \text{for all } x \in \overline{\Omega}.$$

This yields

$$\int_{\overline{\Omega}} \frac{\phi}{\psi} d\nu = \lim_{n \rightarrow \infty} \int_{\overline{\Omega}} \phi \frac{1}{\psi} d\nu_n = \lim_{n \rightarrow \infty} \int_{\overline{\Omega}} \phi \frac{\varphi_n}{\psi} d\mu_n \leq \lim_{n \rightarrow \infty} \int_{\overline{\Omega}} \phi d\mu_n = \int_{\overline{\Omega}} \phi d\mu.$$

6. We conclude by proving that the pair (φ^*, μ^*) is optimal. Since $(\varphi_n, \mu_n)_{n \geq 1}$ is a maximizing sequence, using (4.11), the lower semicontinuity of the cost function $c(\cdot)$, and then (4.15), we obtain

$$\begin{aligned} \sup_{(\varphi, \mu)} J(\varphi, \mu) &= \lim_{n \rightarrow \infty} J(\varphi_n, \mu_n) = \lim_{n \rightarrow \infty} \left(\int_{\Omega} g(x, \varphi_n) dx - \int_{\overline{\Omega}} c d\mu_n \right) \\ &\leq \int_{\Omega} g(x, \varphi^*) dx - \int_{\overline{\Omega}} c d\mu \leq \int_{\Omega} g(x, \varphi^*) dx - \int_{\overline{\Omega}} c d\mu^* = J(\varphi^*, \mu^*). \end{aligned}$$

Finally, the lower semicontinuity of $b(\cdot)$ yields

$$\int_{\overline{\Omega}} b d\mu^* \leq \int_{\overline{\Omega}} b d\mu \leq \liminf_{n \rightarrow \infty} \int_{\overline{\Omega}} b d\mu_n \leq 1.$$

This shows that the pair (φ^*, μ^*) is admissible, completing the proof. \square

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