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Generalized Baire Category and Differential Inclusions in Banach Spaces

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1. INTRODUCTION

Let E be a separable reflexive Banach space and consider the Cauchy problem

$$\dot{x}(t) \in F(x(t)), \qquad x(0) = x_0 \in E, \tag{1.1}$$

where F is a Hausdorff continuous multifunction with closed, bounded values. In this paper we prove the local existence of a solution of (1.1), assuming that the convex closure of $F(x_0)$ has finite codimension. More precisely, we assume the existence of a closed affine subspace $E_0 \subseteq E$ with finite codimension, such that the interior of $E_0 \cap \overline{\text{co}}\, F(x_0)$ relative to E_0 is nonempty. Two special cases deserve mention. If the interior of $\overline{\text{co}}\, F(x_0)$ is nonempty, the above condition holds with $E_0 = E$. On the other hand, if E is finite dimensional, every continuous multifunction F satisfies our condition. Indeed, one can always select an element $y_0 \in F(x_0)$ and set $E_0 = \{y_0\}$. The present result therefore contains the theorems of De Blasi and Pianigiani [7] and of Filippov [8], both as special cases.

For a map F whose values are convex sets with finite codimension, the Cauchy problem (1.1) was recently studied by A. Cortesi [4]. To remove the convexity assumption, we rely on a generalized version of Baire's category theorem, which will also be proved in this paper. Together with (1.1), we consider the problem

$$\dot{x}(t) \in \overline{\text{co}}\, F(x(t)), \qquad x(0) = x_0. \tag{1.2}$$

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The family S of all solutions of (1.2), on a suitable interval $[0, T]$, is nonempty and closed in the metric of uniform convergence. We will show that the set S_F of all $x \in S$ which are solutions of (1.1) is a subset of second category in S (in a generalized sense). This will imply that S_F is nonempty.

The possibility of using a category argument, in order to prove the existence of solutions of a Cauchy problem, was first suggested by a paper of Cellina [3]. This research program was pursued in a series of articles by De Blasi and Pianigiani [5–7]. A remarkable feature of their results is that the compactness assumptions on F, which are present in most of the previous papers [2, 12, 14, 15], can be entirely avoided here.

In the current literature, solutions of a Cauchy problem are often obtained by means of the Contraction Mapping Principle or by an application of Schauder's fixed point theorem. Compared with these other techniques for proving existence, Baire's theorem reaches a much stronger conclusion: the solution set is not only nonempty, but everywhere dense. In practice, this additional feature is often an inconvenience, because it severely restricts the range of applicability of the method. For instance, it is known [13] that the set S_F of solutions of (1.1) may not be dense on the set S of solutions of (1.2). Therefore, S_F is not of second category in S, in general, and a straightforward application of Baire's theorem is doomed to fail. To avoid this difficulty, in [7] the authors choose a closed subset $S^* \subseteq S$ (by an educated guess) and prove that S_F contains a subset of second category in S^*. In the present paper, this same technical problem is solved by using a more flexible version of Baire's theorem. Under suitable assumptions, we prove that a sequence of open sets has a nonempty intersection, even in cases where the intersection is not everywhere dense. Moreover, no preliminary guessing of a set $S^* \subseteq S$ contained in the closure of S_F will be needed.

The paper consists of 11 sections. Notations and basic definitions are contained in Section 2. Our main theorem, together with an equivalent result, is stated in Section 3. A "multivalued" version of Baire's theorem, with compact sets playing the role of points, is proved in Section 4. Section 5 contains a review of techniques and results from [7], for later use. An outline of the basic ideas involved in the proof of the main theorem can be found at the beginning of Section 6. The actual proof is given in Section 11, while Sections 7–10 contain a number of preliminary technical results.

2. Notations and Basic Definitions

In what follows, we write $\bar{A}$ and A^c for the closure and the complement of the set A, respectively, while $A \setminus B$ denotes a set-theoretic difference. If A

is a subset of a metric space (S, d) and $x \in S$, $d(x, A)$ denotes the distance between x and A. For $r>0$ we set $\bar{B}(A, r)=\{x \in S: d(x, A) \leqslant r\}$ and $B(A, r)=\{x \in S: d(x, A)<r\}$.

Throughout this paper, E is a separable and reflexive Banach space, with norm $\|\cdot\|$. Let X be a subset of E. The closed convex hull of X is written $\overline{\mathrm{co}}\, X$, while, for $\lambda \in \mathbb{R}$ and $Y \subseteq E$, $\lambda X=\{\lambda x; x \in X\}$, $X+Y=\{x+y: x \in X, y \in Y\}$. If X is closed and convex, a point $x_0 \in X$ is strongly exposed if there exists a continuous linear functional $\psi: E \rightarrow \mathbb{R}$ such that (i) $\psi(x)<\psi(x_0)$ for every $x \in X$ with $x \neq x_0$; (ii) if $(x_n)_{n \geqslant 1}$ is a sequence in X and $\psi(x_n) \rightarrow \psi(x_0)$, then $x_n \rightarrow x_0$. If X is also bounded, the set $\exp X$ of its strongly exposed points is nonempty and X is the closed convex hull of $\exp X$ (see Theorem 4 in [11]). Notice that every strongly exposed point is also an extremal point of X.

Let E_0 be a closed affine subspace of E and let X be a subset of E_0. By $\operatorname{int}_{E_0} X$ we denote the set $\{x \in X: \exists \varepsilon>0$ such that $B(x, \varepsilon) \cap E_0 \subseteq X\}$. If E_X is the intersection of all closed affine subspaces of E which contain X, we define the relative interior of X as $\operatorname{rel\,int} X=\operatorname{int}_{E_X} X$. When $X \subseteq E$ is convex, we say that X has finite codimension if there exists an affine subspace E_0 with finite codimension such that $\operatorname{int}_{E_0}(X \cap E_0) \neq \varnothing$.

Let X, Y be nonempty, bounded subsets of a Banach space Z. We define the Hausdorff distance h between X and Y as $h(X, Y)=\inf\{\varepsilon>0: X \subseteq B(Y, \varepsilon), Y \subseteq B(X, \varepsilon)\}$: $h(\cdot, \cdot)$ is a metric on the space of the nonempty closed and bounded subsets of Z. Let S be a metric space. A multivalued function $G: S \rightarrow Z$ with nonempty values is

(1) Hausdorff-upper semicontinuous (h-u.s.c.) if $\forall x_0 \in S$, $\forall \varepsilon>0$, $\exists \delta>0$ such that $x \in B(x_0, \delta)$ implies $G(x) \subseteq B(G(x_0), \varepsilon)$;

(2) Hausdorff-lower semicontinuous (h-l.s.c.) if $\forall x_0 \in S$, $\forall \varepsilon>0$, $\exists \delta>0$ such that $x \in B(x_0, \delta)$ implies $G(x_0) \subseteq B(G(x), \varepsilon)$;

(3) Hausdorff-continuous if $\forall x_0 \in S$, $\forall \varepsilon>0$, $\exists \delta>0$ such that $x \in B(x_0, \delta)$ implies $h(G(x_0), G(x))<\varepsilon$. This holds iff G is both h-u.s.c. and h-l.s.c.

The graph of G is the set $\{(x, y) \in S \times Z; y \in G(x)\}$. We recall that a multifunction G with closed graph and values contained in a compact set is h-u.s.c. (see [1, Corollary 1.1.1]). Finally, if $a, b \in \mathbb{R}$ we write $a \wedge b$ for $\min[a, b]$. The Lebesgue measure in $\mathbb{R}^d$ is denoted by $\operatorname{meas}(\cdot)$.

3. Statement of the Main Result

THEOREM 3.1. *Let E be a separable reflexive Banach space. Let $F: E \rightarrow E$ be a Hausdorff continuous multifunction with nonempty, closed, bounded* *values. If* $\overline{\text{co}}\, F(x_0)$ *has finite codimension, then the Cauchy problem* (1.1) *has a Carathéodory solution on some positive interval* $[O, T]$.

A more precise version of this result will be actually proved. By the assumption on $\overline{\text{co}}\, F(x_0)$, there exists a closed affine subspace $E_0 \subseteq E$ with finite codimension and a point $y_0 \in \text{int}_{E_0}(E_0 \cap \overline{\text{co}}\, F(x_0))$. Define the vector space $E' = E_0 - y_0 \subseteq E$. Let E'' be an algebraic supplement of E' in E, and denote by π', π'' the canonical projections of E onto E', E'', respectively. Since E' is closed and E'' is finite dimensional, the projections π', π'' are continuous, hence $E = E' \oplus E''$ is actually a topological sum. Clearly, $\pi'(y_0) \in \text{int}_{E'}(\pi' \circ \overline{\text{co}}\, F(x_0))$. We claim that $\pi'(y_0) \in \text{int}_{E'}(\pi' \circ \overline{\text{co}}\, F(x))$ for every x in a neighborhood of x_0.

LEMMA 3.2. *Let* Y_0 *be a closed convex subset of a Banach space* E'. *If* $B(y_0, \rho) \subseteq Y_0$, *then* $B(y_0, \rho/3) \subseteq Y$ *for every closed convex set* $Y \subseteq E'$ *with* $h(Y, Y_0) < \rho/3$. *In particular,* Y *has nonempty interior.*

Proof. Assume, on the contrary, that $h(Y, Y_0) < \rho/3$, but $\omega \notin Y$ for some $\omega \in B(y_0, \rho/3)$. Let ϕ be a continuous linear functional on E', with unit norm, which separates ω from Y:

$$\phi(y_0) + \rho/3 > \phi(\omega) > \phi(y), \qquad \forall y \in Y.$$

Choose $y_1 \in B(y_0, \rho)$ such that $\phi(y_1) > \phi(y_0) + 2\rho/3$. This implies

$$\|y_1 - y\| \geq \phi(y_1) - \phi(y) > (\phi(y_0) - 2\rho/3) - \phi(\omega) > \rho/3,$$

for every $y \in Y$, hence $h(Y, Y_0) \geq \rho/3$, a contradiction. ∎

Since the map F is Hausdorff continuous, the same holds for the maps $x \to \overline{\text{co}}\, F(x)$ and $x \to \pi' \circ \overline{\text{co}}\, F(x)$. The previous lemma thus implies that the set of points $\{x \in E : \pi'(y_0) \in \text{int}_{E'}(\pi' \circ \overline{\text{co}}\, F(x))\}$ is open. Consider the following set of assumptions:

(A1) $x_0 = 0 \in E$,

(A2) $E = E' \oplus E''$, with continuous projections $\pi' \colon E \to E'$, $\pi'' \colon E \to E''$,

(A3) $\|x\| = \max\{\|\pi'(x)\|_{E'}, \|\pi''(x)\|_{E''}\}$, E'' being a finite-dimensional euclidean space,

(A4) $F(x) \subset B(0, M-1)$, $\forall x \in B(0, 2\tilde{\rho})$,

(A5) $\exists \omega' \in E' \colon \omega \in \text{int}_{E'}(\pi' \circ \overline{\text{co}}\, F(x))$, $\forall x \in B(0, 2\tilde{\rho})$,

(A6) $0 < T \leq \tilde{\rho}/M$.

In the setting of Theorem 3.1, by possibly translating the origin and using an equivalent norm, the previous remarks indicate that it is always possible to choose suitable $E', E'', \omega', \tilde{\rho}, M,$ and T such that (A1)–(A6) hold. Therefore, Theorem 3.1 is an immediate consequence of the following more precise result.

THEOREM 3.3. *Let $F: E \to E$ be a Hausdorff continuous multifunction with closed bounded values, on the separable reflexive Banach space E. If the assumptions* (A1)–(A6) *hold, then* (1.1) *has a Carathéodory solution defined on* $[0, T]$.

4. A GENERALIZED CATEGORY THEOREM

Let S be a complete metric space, and $\mathscr{F} = \{K_i : i \in \mathscr{I}\}$ $(\mathscr{I} \neq \varnothing)$ be a family of nonempty compact subsets of S.

DEFINITION 4.1. A set $R \subseteq S$ is $\mathscr{F}$-rare iff for every $K \in \mathscr{F}$, $\varepsilon > 0$ there exists $K' \in \mathscr{F}$ such that $K' \subseteq B(K, \varepsilon) \backslash \bar{R}$.

DEFINITION 4.2. A set $M \subseteq S$ is $\mathscr{F}$-meager iff it is the union of countably many $\mathscr{F}$-rare sets.

In the special case where $\mathscr{F} = \{\{x\} : x \in S\}$ is the family of all singletons, a subset $V \subseteq S$ is $\mathscr{F}$-rare ($\mathscr{F}$-meager) iff V is rare (meager) in the usual sense. The following is thus an extension of Baire's Category Theorem.

THEOREM 4.3. *The complement of an $\mathscr{F}$-meager set M in S is nonempty. More precisely,*

$$\overline{S \backslash M} \cap K \neq \varnothing, \qquad \forall K \in \mathscr{F}. \tag{4.1}$$

Proof. It suffices to show that for every $K \in \mathscr{F}$, $\varepsilon > 0$,

$$S \backslash M \cap B(K, \varepsilon) \neq \varnothing. \tag{4.2}$$

Let $M = \bigcup_{n=1}^{\infty} R_n$, each R_n being $\mathscr{F}$-rare. Choose $K_1 \in \mathscr{F}$ such that $K_1 \subseteq B(K, \varepsilon) \backslash \bar{R}_1$. Since K_1 is compact and the complement of $B(K, \varepsilon)$ is closed, the distance

$$\delta = \inf\{d(x, y) : x \in K_1,\ y \in B^c(K, \varepsilon) \cup \bar{R}_1\}$$

is strictly positive. Therefore, setting $\delta_1 = \min\{\delta/3, 1\}$ we have

$$B(K_1, 2\delta_1) \subseteq B(K, \varepsilon) \backslash \bar{R}_1. \tag{4.3}$$

By induction, construct a sequence of compact sets $K_n \in \mathscr{F}$ and radii δ_n such that $0 < \delta_n \leq 1/n$ and

$$B(K_n, 2\delta_n) \subseteq B(K_{n-1}, \delta_{n-1}) \backslash \bar{R}_n. \tag{4.4}$$

Of course (4.4) implies

$$\bar{B}(K_n, \delta_n) \subseteq B(K_{n-1}, \delta_{n-1}) \backslash \bar{R}_n \subseteq B(K_{n-1}, \delta_{n-1}). \tag{4.5}$$

Denoting by $\alpha(V)$ the Kuratowski measure of noncompactness of a set V, we have

$$\alpha(\bar{B}(K_n, \delta_n)) \leq 2\delta_n \leq 2/n.$$

By Kuratowski's theorem [10, p. 412], the intersection $D = \bigcap_{n=1}^{\infty} \bar{B}(K_n, \delta_n)$ is a compact nonempty set. From (4.3) and (4.4) it now follows that

$$D \subseteq B(K, \varepsilon) \backslash \left(\bigcup_{n=1}^{\infty} R_n \right),$$

proving the theorem. ∎

EXAMPLE. Let $S = \mathbb{R}$ and let $\mathscr{F}$ be the family of compact sets consisting of the interval $[0, 1]$ together with all singletons $\{x\}$ with $x \geq 1$. Then the set $R = (-\infty, 1]$ is $\mathscr{F}$-rare, according to Definition 4.1. Its complement $R^c = (1, +\infty)$ is not dense on $\mathbb{R}$, but the intersection $\overline{R^c} \cap K$ is nonempty, for all $K \in \mathscr{F}$.

5. A Set of Upper Semicontinuous Functionals

We now introduce a family of upper semicontinuous functionals $\phi: E \times E \to [-\infty, 1]$ which, loosely speaking, measure the distance of a vector $u \in E$ from the set of extreme points of the convex set $\overline{\text{co}}\, F(x)$. Both the definition and the main properties of these functionals are taken from [7].

By $\mathscr{K}(E)$ we denote the family of all nonempty closed, bounded subsets of E. Let C be any closed ball in E. If $X \in \mathscr{K}(E)$ and $u \in \overline{\text{co}}\, X$, define

$$\alpha(C, X) = \inf\{\|y - x\| : y \in C, x \in X\} \wedge 1, \tag{5.1}$$

$$\gamma_C(u, X) = \begin{cases} 0, & \text{if } C \cap \overline{\text{co}}\, X = \varnothing, \\ \sup\{\lambda \in [0, 1] : u \in \lambda(C \cap \overline{\text{co}}\, X) + (1 - \lambda)\, \overline{\text{co}}\, X\}, & \text{if } C \cap \overline{\text{co}}\, X \neq \varnothing, \end{cases} \tag{5.2}$$

$$d_C(u, X) = \alpha(C, X) \cdot \gamma_c(u, X). \tag{5.3}$$

LEMMA 5.1. *If* $X \in \mathscr{K}(E)$ *and* $u \in \exp \overline{\text{co}}\, X$, *then* $d_C(u, X) = 0$. *Moreover,* $\exp \overline{\text{co}}\, X \neq \emptyset$ *and* $\overline{\text{co}} \exp \overline{\text{co}}\, X = \overline{\text{co}}\, X$.

For the proof, see Proposition 4.7 in [7].

LEMMA 5.2. *Let* $X_0 \in \mathscr{K}(E)$, $\varepsilon > 0$ *be given. Then there exists* $\delta > 0$ *such that*

$$d_C(u, X_0) \geqslant d_C(u, X) - \varepsilon$$

for every $X \in \mathscr{K}(E)$ *satisfying* $\overline{\text{co}}\, X \subseteq \overline{\text{co}}\, X_0$, $h(X, X_0) < \delta$ *and every* $u \in \overline{\text{co}}\, X$.

Indeed, the assumption $\overline{\text{co}}\, X \subseteq \overline{\text{co}}\, X_0$ implies $\gamma_C(u, X) \leqslant \gamma_C(u, X_0)$. This, together with the continuity of the map $X \to \alpha(C, X)$ w.r.t. the Hausdorff metric, yields the result.

Let F be the multifunction considered at (1.1). For any closed ball $C \subseteq E$, define

$$\phi_C(u, x) = \begin{cases} d_C(u, F(x)), & \text{if } u \in \overline{\text{co}}\, F(x), \\ -\infty, & \text{if } u \notin \overline{\text{co}}\, F(x). \end{cases} \tag{5.4}$$

LEMMA 5.3. *The map* $\phi_C \colon E \times E \to [-\infty, 1]$ *is upper semicontinuous. For any fixed* x, *the map* $u \to \phi_C(u, x)$ *is concave.*

Indeed, the upper semicontinuity of γ_C and the concavity of the map $u \to \gamma_C(u, X)$ (for $u \in \overline{\text{co}}\, X$) were proved in Propositions 4.3 and 4.2 in [7]. Since the map F is Hausdorff-continuous and $\alpha(C, X)$ depends continuously on X, the above statements are clear.

Let $(u_i)_{i \geqslant 1}$ be a sequence of points dense on E. The family of all closed balls centered at some u_i with positive rational radius is countable, hence it can be arranged into a sequence, say $(C_j)_{j \geqslant 1}$. Define the functional

$$\phi(u, x) = \sum_{j=1}^{\infty} 2^{-j} \phi_{C_j}(u, x).$$

LEMMA 5.4. *The map* $\phi \colon E \times E \to [-\infty, 1]$ *is upper semicontinuous. Moreover*

$$\phi(u, x) > 0, \qquad \forall u \in \overline{\text{co}}\, F(x) \backslash F(x). \tag{5.6}$$

Proof. The first statement follows from Lemma 5.3. To prove (5.6), let $u \in \overline{\text{co}}\, F(x) \backslash F(x)$. Since $F(x)$ is closed, there exists a rational $\rho > 0$ such that $B(u, 4\rho) \cap F(x) = \emptyset$. The density of the sequence $(u_i)_{i \geqslant 1}$ implies that $\|u_k - u\| < \rho$ for some integer k. By construction, $\bar{B}(u_k, 2\rho) = C_j$ for some j. Observe that $\bar{B}(u_k, 2\rho) \subseteq B(u, 3\rho)$, hence $\alpha(C_j, F(x)) \geqslant \min\{\rho, 1\} > 0$. Moreover, $u \in C_j \cap \overline{\text{co}}\, F(x)$, hence $\gamma_{C_j}(u, F(x)) = 1$. This implies $\phi_{C_j}(u, x) > 0$, hence $\phi(u, x) > 0$. ∎

6. A Family of Differential Inclusions with Memory

To motivate the definition of "n-level feedback" given below, a few words of introduction are in order. In a naive attempt to prove the existence of solutions of (1.1), one may consider the closed, nonempty set S of all solutions of (1.2), and try to show that the subset S_F of all $x \in S$ which are solutions of (1.1) is of second Baire category in S. This will be the case if, for every integer $j, n \geqslant 1$, the set of solutions

$$S_{jn} = \left\{ x \in S \colon \int_0^T \phi_{C_j}(\dot{x}(t), x(t))\, dt < 1/n \right\} \tag{6.1}$$

is an open dense subset of S. Indeed, (5.6) implies $S_F \supseteq (\bigcap_{j,n} S_{jn})$.

Unfortunately, this naive approach fails when F is not Lipschitz continuous: a well-known counterexample by Pliś (see [9]) shows that S_F need not be dense on S. In [7], this difficulty is overcome by considering not the whole set of solutions S of (1.2), but a suitable closed subset $S^* \subseteq S$ with the property that $S_{jn} \cap S^*$ is dense in S^* for all $j, n \geqslant 1$. S^* is defined as the closure of the set of all polygonal solutions of (1.2) whose derivative lies a.e. in the interior of $\overline{\text{co}}\, F(x)$.

In the present case, however, the interior of $\overline{\text{co}}\, F(x)$ may be empty, and a substantially different line of proof is needed. We shall define a family $\{\mathscr{E}_i \colon i \in \mathscr{I}\}$ of differential inclusions with memory, with the following properties. Each $\mathscr{E}_i$ has a nonempty compact set K_i of solutions, with $K_i \subseteq S$. If $\mathscr{F} = \{K_i \colon i \in \mathscr{I}\}$ is the collection of all such sets of solutions, for every $j, n \geqslant 1$ the set $S \backslash S_{jn}$ is $\mathscr{F}$-rare. Our "multivalued" category theorem will therefore imply

$$\hat{S} = \bigcap_{j,n} S_{jn} = \left\{ x \in S \colon \int_0^T \phi(\dot{x}(t), x(t))\, dt = 0 \right\} \neq \varnothing. \tag{6.2}$$

By (5.6), every $x \in \hat{S}$ is a solution of (1.1). This will establish Theorem 3.3.

The next definition was inspired by Filippov's construction of piecewise linear approximate solutions [1, p. 112]. Here $\tilde{B}$ denotes the closed ball $\bar{B}(0, \tilde{\rho}) \subseteq E$.

Definition 6.1. A n-level feedback $\mathscr{E}$ on $[0, T] \times \tilde{B}$ is a triple $(\mathscr{P}, \mathscr{G}, f)$ consisting of

(1) $n+1$ finite partitions of $[0, T]$

$$\mathscr{P}_k = \{t_0^k, t_1^k, \ldots, t_{p_k}^k\}, \qquad k = 1, \ldots, n+1,$$

with $t_i^k = iT/p_k$, p_{k+1} being an integer multiple of p_k, for all $k \leqslant n$;

(2) n finite open coverings of $\tilde{B}$

$$\mathscr{G}_k = \{V_1^k, ..., V_{q_k}^k\}, \qquad k = 1, ..., n;$$

(3) a finite set of continuous vector fields on E

$$x \to f[i_1, ..., i_n, j](x)$$

with $i_k \in \{1, ..., q_k\}$, $1 \leqslant k \leqslant n$, $j \in \{0, ..., p_{n+1} - 1\}$, each vector field $f[\cdots]$ being defined on some open subset of E.

For $t \in [0, T]$, $1 \leqslant k \leqslant n+1$, define

$$\tau^k(t) = \max\{t_i^k : 0 \leqslant i \leqslant p_k,\ t_i^k \leqslant t\}, \tag{6.3}$$

$$\sigma_{n+1}(t) = \max\{l: 0 \leqslant l \leqslant p_{n+1},\ t_l^{n+1} \leqslant t\}. \tag{6.4}$$

DEFINITION 6.2. An absolutely continuous map $y: [0, T] \to E$ is a solution of $\mathscr{E}$ if there exist n functions

$$\sigma_k: \{t_0^k, ..., t_{p_k-1}^k\} \to \{1, ..., q_k\}, \qquad 1 \leqslant k \leqslant n,$$

such that

(i) $y(t_i^k) \in \overline{V_{\sigma_k(t_i^k)}^k}$, $\forall i, k$,

(ii) $\dot{y}(t) = f[\sigma_1(\tau^1(t)), \sigma_2(\tau^2(t)), ..., \sigma_n(\tau^n(t)), \sigma_{n+1}(t)](y(t))$

a.e. on $[0, T]$.

Intuitively, the velocity $\dot{y}(t)$ of a solution of the feedback $\mathscr{E}$ is determined by two things: (1) the time t, measured by the value $j = \sigma_{n+1}(t)$ of a digital clock, (2) the position of y at the nodes $\tau^1(t), ..., \tau^n(t)$ of the partitions $\mathscr{P}_1, ..., \mathscr{P}_n$ immediately preceding t. More precisely, on the interval $[t_i^k, t_{i+1}^k)$ of the kth partition $\mathscr{P}_k$, $\dot{y}(t)$ depends on the set V_l^k of the covering $\mathscr{G}_k$ whose closure contains $y(t_i^k)$. If $y(t_i^k)$ is contained in more than one set, for example $y(t_i^k) \in \overline{V_2^k} \cap \overline{V_5^k}$, then different choices are possible (in the example, $\sigma_k(t_i^k) = 2$ or $\sigma_k(t_i^k) = 5$). However, one has to stick with the same choice throughout the interval $[t_i^k, t_{i+1}^k)$. In the following, the short notation $[\mathbf{i}, j] = [i_1, ..., i_n, j]$ will be used.

DEFINITION 6.3. Under the assumptions (A1)–(A6) in Section 3, the n-level feedback $\mathscr{E} = (\mathscr{P}, \mathscr{G}, f)$, on $[0, T] \times \tilde{B}$, is admissible for (1.1) if, for all $\mathbf{i}$, j,

(i) $f[\mathbf{i}, j](x) \in \text{rel int}\ \overline{\text{co}}\ F(x)$, $\forall x \in \text{Dom}\ f[\mathbf{i}, j]$;

(ii) the domain of $f[\mathbf{i}, j](\cdot)$ is an open neighborhood of $\bar{B}(V_{i_n}^n, MT/p_n) \cap \tilde{B}$;

(iii) the projection of $f[\mathbf{i}, j](x)$ on E' is constant and lies in the interior of $\pi' \circ \overline{\text{co}}\, F(x)$, i.e.,

$$f[\mathbf{i}, j](x) = \pi' \circ f[\mathbf{i}, j](x) + \pi'' \circ f[\mathbf{i}, j](x)$$
$$= f'[\mathbf{i}, j] + f''[\mathbf{i}, j](x),$$

with

$$f'[\mathbf{i}, j] \in \text{int}_{E'}(\pi' \circ \overline{\text{co}}\, F(x)), \qquad \forall x \in \text{Dom}\, f[\mathbf{i}, j].$$

7. Solutions of Admissible Feedbacks

Definition 7.1. Let $\mathscr{E} = (\mathscr{P}, \mathscr{G}, f)$ be an n-level feedback. We call $S_{\mathscr{E}}$ the set of all solutions $y(\cdot)$ of $\mathscr{E}$ such that $y(0) = 0$. For $\delta \in [0, 1]$, $S_{\mathscr{E}}^{\delta}$ denotes the set of all absolutely continuous z: $[0, T] \to E$ such that $z(0) = 0$ and there exist n functions β_k: $\{t_0^k, ..., t_{p_k - 1}^k\} \to \{1, ..., q_k\}$, $1 \leqslant k \leqslant n$, such that

$$z(t_i^k) \in \overline{V^k_{\beta_k(t_i^k)}}, \qquad \forall i, k \tag{7.1}$$

$$\dot{z}(t) \in f[\beta_1(\tau^1(t)), ..., \beta_n(\tau^n(t)), \beta_{n+1}(t)](z(t))$$
$$+ (\bar{B}(0, \delta) \cap E'') \qquad \text{a.e. on } [0, T]. \tag{7.2}$$

Here, as in (6.4), $\beta_{n+1}(t) = \max\{i: t_i^{n+1} \leqslant t\}$. Intuitively, $S_{\mathscr{E}}^{\delta}$ represents a set of δ-approximate solutions of the feedback $\mathscr{E}$. Notice that the "error" $\dot{z}(t) - f[\cdots](z(t))$ must be inside the finite dimensional space E''. Of course, $S_{\mathscr{E}} = S_{\mathscr{E}}^0$.

Proposition 7.2. *Let $\mathscr{E} = (\mathscr{P}, \mathscr{G}, f)$ be an admissible n-level feedback for* (1.1), *according to Definition* 6.3. *Then the set $S_{\mathscr{E}}$ of its solutions is nonempty and compact. Moreover, the multivalued map $\delta \to S_{\mathscr{E}}^{\delta}$ from* $[0, 1]$ *into* $C^0([0, T]; E)$ *has closed graph and compact values* (*hence it is Hausdorff-upper semicontinuous*).

Proof. To show that $S_{\mathscr{E}} \neq \varnothing$, we shall construct a solution y of E piecewise on the intervals $[t_j^{n+1}, t_{j+1}^{n+1}] = I_j$ of the partition $\mathscr{P}_{n+1}$, by induction on j. Since every $\mathscr{G}_k$ is a covering, for each $k \in \{1, ..., n\}$ we can choose an index $\sigma_k(0) \in \{1, ..., q_k\}$ such that $0 \in V^k_{\sigma_k(0)}$. By conditions (ii) and (iii) in Definition 6.3 together with (A4) in Section 3, the vector field $f[\sigma_1(0), ..., \sigma_n(0), 0](\cdot)$ is defined on an open neighborhood of the set $\bar{B}(V^n_{\sigma_n(0)}, MT/p_n) \cap \tilde{B}$ and takes values in a finite-dimensional subset of the ball $B(0, M-1)$. Therefore, the Cauchy problem

$$\dot{u}(t) = f[\sigma_1(0), ..., \sigma_n(0), 0](u(t)), \qquad u(0) = 0 \in E$$

has a solution $y(t)$, defined on the interval $[0, t_1^{n+1}]$. Now assume, by induction, that the function y has been defined up to the nodal point $t_j^{n+1} \in \mathscr{P}_{n+1}$, $0 < t_j^{n+1} < T$. In addition, assume that the values $\sigma_k(t_i^k)$ have already been chosen for all nodal points $t_i^k < t_j^{n+1}$. Define $l(j) = \min\{k: t_j^{n+1} \in \mathscr{P}_k\}$. If $k < l(j)$, then $\tau^k(t_j^{n+1}) < t_j^{n+1}$ and $\sigma_k(\tau^k(t_j^{n+1}))$ has already been defined. If $l(j) \leqslant k \leqslant n$, choose $\sigma_k(t_j^{n+1})$ such that $y(t_j^{n+1}) \in V^k_{\sigma_k(t_j^{n+1})}$. This is possible because $\mathscr{G}_k$ is a covering. The admissibility of the feedback $\mathscr{E}$ implies that the Cauchy problem

$$\dot{u}(t) = f[\sigma_1(\tau^1(t)), ..., \sigma_n(\tau^n(t)), \sigma_{n+1}(t)](u(t))$$
$$u(t_j^{n+1}) = y(t_j^{n+1})$$

has a solution $u(\cdot)$ on the interval $I_j = [t_j^{n+1}, t_{j+1}^{n+1}]$. Extend the function y by setting $y(t) = u(t)$ on I_j. By (A4) and (A6) in Section 3, y cannot escape from the ball $\tilde{B}$ within time T. By induction, the function y can be defined on the whole interval $[0, T]$. Our construction implies that (i) and (ii) in Definition 6.2 are both satisfied, hence y is a solution of $\mathscr{E}$.

To prove the second statement, assume that $\delta_m \to \delta$ and that $z_m \to z$ in $C^0([0, T]; E)$ with $z_m \in S_{\mathscr{E}}^{\delta_m}$ for all $m \geqslant 1$. By possibly taking a subsequence we can assume that the discrete maps $\beta_1, ..., \beta_n$ which appear in Definition 7.1 are the same for all m. In particular, since $z_m(t_i^k) \in V^k_{\beta_k(t_i^k)}$, the uniform convergence of the sequence z_m to z implies

$$z(t_i^k) \in \overline{V^k_{\beta_k(t_i^k)}} \tag{7.3}$$

at all nodal points $t_i^k \in \mathscr{P}_k$, $k = 1, ..., n$. Moreover, for $t \in [t_j^{n+1}, t_{j+1}^{n+1}]$,

$$z_m(t) = \int_0^{t_1^{n+1}} f[\beta_1(0), ..., \beta_n(0), 0](z_m(s))\, ds + \cdots +$$
$$\int_{t_j^{n+1}}^t f[\beta_1(\tau^1(t)), ..., \beta_n(\tau^n(t)), j](z_m(s))\, ds + \phi_m(t),$$

where ϕ_m is a Lipschitz continuous function from $[0, T]$ into the finite-dimensional space E'', with $\phi_m(0) = 0$ and with Lipschitz constant δ_m. Therefore, Ascoli's theorem provides a subsequence, say $\phi_{m'}$, converging to some function ϕ, with Lipschitz constant δ. Using the continuity of the vector fields $f[\mathbf{i}, j]$ and recalling (7.3), we conclude that $z \in S_{\mathscr{E}}^{\delta}$. This implies that the map $\delta \to S_{\mathscr{E}}^{\delta}$ has closed graph. In particular, each set $S_{\mathscr{E}}^{\delta}$ is closed, and such is also $S_{\mathscr{E}} = S_{\mathscr{E}}^0$. Since all functions $z \in S_{\mathscr{E}}^{\delta}$ are uniformly Lipschitz continuous and take values inside a finite-dimensional subspace of E, the compactness of $S_{\mathscr{E}}^{\delta}$ follows again from Ascoli's theorem. ∎

8. A Closure Theorem

Let S be the family of all solutions of (1.2). We will show that the sets S_{jn} defined at (6.1) are relatively open in S.

THEOREM 8.1. *For any closed ball $C \subseteq E$ and any $\delta > 0$, the set*

$$R_{C,\delta} = \left\{ x \in S \colon \int_0^T \phi_C(\dot{x}(t), x(t))\, dt \geqslant \delta \right\}$$

is closed in $C^0([0, T]; E)$.

Proof. Let $(x_n)_{n \geqslant 1}$ be a sequence in $R_{C,\delta}$ such that $x_n \to x$ uniformly on $[0, T]$. The corresponding sequence of derivatives $\dot{x}_n$ converges weakly to $\dot{x}$ in $L^2([0, T]; E)$. By Mazur's theorem, there exists a sequence of convex combinations

$$v_n = \sum_{k=1}^{r_n} \lambda_{n,k} \dot{x}_{n+k}$$

of the $\dot{x}_n$ which converges to $\dot{x}$ strongly in $L^2([0, T]; E)$ and pointwise a.e. on $[0, T]$. Fix $\varepsilon > 0$. Using the theorems of Lusin and Egorov we obtain a compact subset $J \subseteq [0, T]$ such that $\operatorname{meas}(J) > T - \varepsilon$, the restriction of $\dot{x}$ to J is continuous, and v_n converges to $\dot{x}$ uniformly on J. By the upper semicontinuity of d_C and the compactness of the set $\{(\dot{x}(t), x(t)) \colon t \in J\}$, there exists $\eta > 0$ such that

$$d_C(\dot{x}(t), F(x(t))) \geqslant d_C(v, \bar{B}(F(x(t)), \eta)) - \varepsilon \tag{8.1}$$

whenever $t \in J$, $\|v - \dot{x}(t)\| < \eta$. Using Lemma 5.2 and the continuity of F, we can find $\rho > 0$ such that $t \in J$, $\|y - x(t)\| \leqslant \rho$ imply $F(y) \subseteq \bar{B}(F(x(t)), \eta)$ and

$$d_C(u, \bar{B}(F(x(t)), \eta)) \geqslant d_C(u, F(y)) - \varepsilon \tag{8.2}$$

for all $u \in \overline{\text{co}}\, F(y)$. Assume that $\|v_n(t) - \dot{x}(t)\| < \eta$ and $\|x_n(t) - x(t)\| < \rho$ for all $n \geqslant N$. The concavity of the map $u \to d_C(u, \bar{B}(F(x(t)), \eta))$ together with (8.1), (8.2) yields

$$\begin{aligned} d_C(\dot{x}(t), F(x(t))) &\geqslant d_C(v_n(t), \bar{B}(F(x(t)), \eta)) - \varepsilon \\ &\geqslant \sum_{k=1}^{r_n} \lambda_{n,k}\, d_C(\dot{x}_{n+k}(t), \bar{B}(F(x(t)), \eta)) - \varepsilon \qquad (8.3) \\ &\geqslant \sum_{k=1}^{r_n} \lambda_{n,k}\, d_C(\dot{x}_{n+k}(t), F(x_{n+k}(t))) - 2\varepsilon \end{aligned}$$

whenever $n \geqslant N$, $t \in J$. Since $\phi_C(\dot{x}_{n+k}(t), x_{n+k}(t)) = d_C(\dot{x}_{n+k}(t), F(x_{n+k}(t))) \in [0, 1]$ for a.e. $t \in [0, T]$, from (8.3) we obtain

$$\begin{aligned}\int_0^T \phi_C(\dot{x}(t), x(t))\, dt &\geqslant \int_J d_C(\dot{x}(t), F(x(t)))\, dt \\ &\geqslant \int_J \sum_{k=1}^{r_n} \lambda_{n,k}[d_C(\dot{x}_{n+k}(t), F(x_{n+k}(t))) - 2\varepsilon]\, dt \\ &\geqslant \sum_{k=1}^{r_n} \lambda_{n,k}\left[\int_0^T \phi_C(\dot{x}_{n+k}(t), x_{n+k}(t))\, dt - \varepsilon\right] - 2\varepsilon T \\ &\geqslant \delta - \varepsilon - 2\varepsilon T.\end{aligned}$$

Since ε was arbitrary, the theorem is proved. ▮

9. Some Geometric Lemmas

Let $Q \subseteq B(0, M)$ be a closed, convex subset of the Banach space E with finite codimension. Observe that the relative interior of Q is nonempty and dense on Q. Moreover, if V is a closed convex neighborhood of a point $\omega \in Q$, then $\omega \in \operatorname{rel\,int} Q$ if and only if $\omega \in \operatorname{rel\,int}(V \cap Q)$. Throughout this section, we refer to the decomposition $E = E' \oplus E''$, with E' closed and E'' finite-dimensional, assuming that (A2), (A3) in Section 3 hold. We remark that, if $A \subseteq \mathbb{R}^d$ is compact and convex, then $\operatorname{rel\,int} A$ is the smallest convex subset of A whose closure is A.

Lemma 9.1. *If* $y_0 \in \operatorname{rel\,int}(Q \cap (y_0 + E''))$ *and* $\pi'(y_0) \in \operatorname{int}_{E'}(\pi'(Q))$, *then* $y_0 \in \operatorname{rel\,int} Q$. *Conversely, if* $y_0 \in \operatorname{rel\,int} Q$ *and* $\operatorname{int}_{E'}(\pi'(Q)) \neq \varnothing$, *then* $\pi'(y_0) \in \operatorname{int}_{E'}(\pi'(Q))$.

Proof. The set $A = (y_0 + E'') \cap \operatorname{rel\,int} Q$ is clearly a convex subset of $(y_0 + E'') \cap Q$. By our previous remark, to prove the first statement it suffices to show that A is dense on $(y_0 + E'') \cap Q$. Let $\omega_0 \in (y_0 + E'') \cap Q$, $\varepsilon > 0$. By the second assumption, there exists $\delta > 0$ such that $B(\pi'(y_0), \delta) \cap E' \subseteq \pi'(Q)$. Choose $\omega_1 \in \operatorname{rel\,int} Q$ such that $\|\omega_1 - \omega_0\| < \varepsilon$. Since the projections π', π'' have norm 1, this implies $\pi'(\omega_0 + (\delta/\varepsilon)(\omega_0 - \omega_1)) \in B(\pi'(\omega_0), \delta) \cap E' \subseteq \pi'(Q)$. Hence $\pi'(\omega_0 + (\delta/\varepsilon)(\omega_0 - \omega_1)) = \pi'(\omega_2)$ for some $\omega_2 \in Q$. Consider the convex combination $\omega = (\delta\omega_1 + \varepsilon\omega_2)/(\varepsilon + \delta)$. An easy computation shows that $\omega \in A$ and $\|\omega - \omega_0\| \leqslant (\delta\|\omega_1 - \omega_0\| + \varepsilon\|\omega_2 - \omega_0\|)/(\varepsilon + \delta) \leqslant \varepsilon + 2M\varepsilon/(\varepsilon + \delta)$. Since δ and M are fixed while $\varepsilon > 0$ is arbitrary, the density of A on $(y_0 + E'') \cap Q$ is proved.

Conversely, assume $y_0 \in \operatorname{rel\,int} Q$ and $B(\pi'(y_1), \rho) \cap E' \subseteq \pi'(Q)$ for some $y_1 \in Q$, $\rho > 0$. Choose $y_2 \in Q$ and $\lambda > 0$ such that $y_0 = \lambda y_1 + (1 - \lambda)\, y_2$.

The convexity of $\pi'(Q)$ now implies $\pi'(Q) \supseteq (\lambda B(\pi'(y_1), \rho) \cap E') + (1-\lambda)\,\pi'(y_2) = B(\pi'(y_0), \lambda\rho) \cap E'$. ▮

In the following, if Ω is a bounded subset of $\mathbb{R}^d$ with positive Lebesgue measure, we consider the barycenter of Ω:

$$\mathrm{bar}(\Omega) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega x\, dx.$$

If $A \subseteq \mathbb{R}^d$ is compact and convex, $\mathrm{bar}(B(A, 1)) \in A$. Moreover, the map $A \to \mathrm{bar}\,(B(A, 1))$ is Lipschitz continuous w.r.t. the Hausdorff metric [1, p. 77].

LEMMA 9.2. *Let $A \subseteq \mathbb{R}^d$ be compact and convex. Then $b = \mathrm{bar}(B(A, 1)) \in \mathrm{rel\,int}\, A$.*

Proof. It suffices to prove that, if ψ is a linear functional on $\mathbb{R}^d$ and

$$\psi(b) = \max\{\psi(x) \colon x \in A\}, \tag{9.1}$$

then ψ is constant on A. Call Γ the hyperplane $\{x \in \mathbb{R}^d \colon \psi(x) = \psi(b)\}$ and let $S(x)$ be the point symmetric to x w.r.t. Γ, i.e., $S(x) = x + 2(\pi(x) - x)$, $\pi(x)$ being the perpendicular projection of x on Γ. Define

$$B^+ = \{x \in B(A, 1) \colon \psi(x) \geq \psi(b)\},$$
$$B_1 = B^+ \cup S(B^+), \qquad B_2 = B(A, 1) \backslash B_1.$$

Notice that, if (9.1) holds, then $B_1 \subseteq B(A, 1)$.

If $\mathrm{meas}(B_2) > 0$, using well-known properties of the barycenter and the linearity of ψ, we obtain

$$\psi(b) = \frac{\mathrm{meas}(B_1)}{\mathrm{meas}(B(A, 1))} \cdot \psi(b_1) + \frac{\mathrm{meas}(B_2)}{\mathrm{meas}(B(A, 1))} \cdot \psi(b_2), \tag{9.2}$$

where b_1, b_2 are the barycenters of B_1, B_2, respectively. By symmetry, $\psi(b_1) = \psi(b)$. Since $\psi(x) < \psi(b)$ for every $x \in B_2$, we have $\psi(b_2) < \psi(b)$ and (9.2) yields a contradiction. Therefore $\mathrm{meas}(B_2) = 0$, from which one easily deduces $B_2 = \varnothing$ and $A \subseteq \Gamma$, completing the proof. ▮

LEMMA 9.3. *Let $x \to Q(x)$ be a Hausdorff continuous multifunction defined on an open set $U \subseteq E$, with closed convex values contained in the ball $B(0, M) \subseteq E = E' \oplus E''$. Assume $y_0 \in Q(x_0)$ and $\pi'(y_0) \in \mathrm{int}_{E'}\, \pi'(Q(x_0))$. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\pi'(y_0) \in \mathrm{int}_{E'}\, \pi'(Q(x) \cap \bar{B}(y_0, \varepsilon)), \qquad \forall x \in B(x_0, \delta). \tag{9.3}$$

Proof. By Lemma 3.2 and the Hausdorff continuity of Q, there exist $\rho', \delta' > 0$ such that

$$E' \cap \bar{B}(\pi'(y_0), 2\rho') \subseteq \pi'(Q(x)), \qquad \forall x \in B(x_0, \delta').$$

Choose $\rho \in (0, \rho')$ such that $\rho + 2M\rho/\rho' < \varepsilon$, and $\delta \in (0, \delta')$ such that

$$Q(x) \cap B(y_0, \rho) \neq \varnothing, \qquad \forall x \in B(x_0, \delta).$$

Let $z \in E'$, $\|z - \pi'(y_0)\| \leqslant \rho$. The lemma will be proved by showing that, if $\|x - x_0\| < \delta$, then $z = \pi'(y)$ for some $y \in Q(x) \cap \bar{B}(y_0, \varepsilon)$. Choose any $y_1 \in Q(x) \cap \bar{B}(y_0, \rho)$. If $\pi'(y_1) = z$ we are done. Otherwise, define

$$z' = z + \frac{z - \pi'(y_1)}{\|z - \pi'(y_1)\|}\,\rho'. \tag{9.4}$$

Since $z' \in \bar{B}(\pi'(y_1), \rho' + \rho)$, $z' = \pi'(y_2)$ for some $y_2 \in Q(x)$. From (9.4) we deduce $z = \pi'(y)$ with

$$y = (\|z - \pi'(y_1)\|\; y_2 + \rho' y_1)/(\rho' + \|z - \pi'(y_1)\|).$$

The choice of ρ implies

$$\begin{aligned}\|y - y_0\| &\leqslant \frac{\|z - \pi'(y_1)\|}{\rho'}\,\|y_2 - y_0\| + \|y_1 - y_0\| \\ &\leqslant \frac{2M\rho}{\rho'} + \rho < \varepsilon,\end{aligned}$$

proving (9.3). ∎

LEMMA 9.4. *With the same assumptions of Lemma* 9.3, *for every* $\varepsilon > 0$ *there exists a neighborhood* V *of* x_0 *such that the map*

$$x \to A(x) = Q(x) \cap \bar{B}(y_0, \varepsilon) \cap (y_0 + E'')$$

is Hausdorff continuous on V.

Proof. The map A has closed graph and its values are compact subsets of the finite-dimensional affine space $y_0 + E''$. Hence A is Hausdorff upper semicontinuous on its domain. To show that A is also lower semicontinuous, we use Lemma 9.3 and choose an open neighborhood V of x_0 such that

$$\pi'(y_0) \in \operatorname{int}_{E'} \pi'(Q(x) \cap \bar{B}(y_0, \varepsilon/2)), \qquad \forall x \in V. \tag{9.5}$$

Fix any $\tilde{x} \in V$ and any $\tilde{y} \in A(\tilde{x})$. Since $A(\tilde{x})$ is compact, it is enough to prove that, for every $\eta > 0$, there exists $\delta > 0$ such that

$$d(\tilde{y}, A(x)) \leqslant \eta, \qquad \forall x \in B(\tilde{x}, \delta). \tag{9.6}$$

By (9.5), there exists $y \in Q(\tilde{x}) \cap \bar{B}(y_0, \varepsilon/2)$ such that $\pi'(y) = \pi'(\tilde{y}) = \pi'(y_0)$. Choose $\lambda > 0$ sufficiently small so that, setting $y_\lambda = \lambda y + (1-\lambda)\, \tilde{y} \in B(y_0, \varepsilon)$, one has $\|y_\lambda - \tilde{y}\| < \eta/2$. Choose $\eta' \in (0, \eta/2)$ so small that $B(y_\lambda, \eta') \subset B(y_0, \varepsilon)$. A new application of Lemma 9.3 provides the existence of $\delta > 0$ such that $\|x - \tilde{x}\| < \delta$ implies

$$\pi'(y_0) \in \pi'(Q(x) \cap \bar{B}(y_\lambda, \eta')). \tag{9.7}$$

From (9.7) we deduce that, whenever $\|x - \tilde{x}\| < \delta$, there exists y such that

$$y \in Q(x) \cap \bar{B}(y_\lambda, \eta') \cap (y_0 + E'') \subseteq A(x).$$

Therefore $d(\tilde{y}, A(x)) \leqslant \|\tilde{y} - y_\lambda\| + \|y_\lambda - y\| < \eta/2 + \eta' < \eta$, proving (9.6). ▌

COROLLARY 9.5. *If the assumptions* (A1)–(A6) *in Section* 3 *hold, then there exists a continuous vector field* $\tilde{g}$: $B(0, 2\tilde{\rho}) \to E$ *such that, for all* x,

$$\tilde{g}(x) \in \operatorname{rel\,int} \overline{\operatorname{co}}\, F(x), \tag{9.8}$$

$$\pi'(\tilde{g}(x)) = \omega', \tag{9.9}$$

where ω' *is the point considered at* (A5).

Proof. For $x \in B(0, 2\tilde{\rho})$, set $A(x) = \overline{\operatorname{co}}\, F(x) \cap (\omega' + E'')$, and define $\tilde{g}(x) = \operatorname{bar}(B(A(x), 1) \cap (\omega' + E''))$, the barycenter being taken with respect to the Lebesgue measure on the affine, finite-dimensional space $\omega' + E''$. The Lemmas 9.1 and 9.2 now imply (9.8), while the continuity of $\tilde{g}$ follows from Lemma 9.4. ▌

LEMMA 9.6. *Let all of the assumptions in Theorem* 3.3 *hold. Let* $\varepsilon > 0$, $\xi \in B(0, 2\tilde{\rho})$, $v \in \operatorname{rel\,int} \overline{\operatorname{co}}\, F(\xi)$ *be given. For a fixed closed ball* $C \subset E$, *consider the function* d_C *defined at* (5.3). *Then there exist a neighborhood* V *of* ξ, *finitely many continuous functions* $g_1, ..., g_N$: $V \to E$, *and rational coefficients* $\lambda_1, ..., \lambda_N > 0$ *with the following properties*:

(i) $\sum_{k=1}^{N} \lambda_k = 1$,

(ii) $y_k(x) \in \operatorname{rel\,int} \overline{\operatorname{co}}\, F(x)$,

(iii) $g'_k = \pi'(g_k(x)) \in \operatorname{int}_{E'}(\pi' \circ \overline{\operatorname{co}}\, F(x))$ *is independent of* x,

(iv) $\sum_{k=1}^{N} \lambda_k g'_k = \pi'(v)$,

(v) $\|v - \sum_{k=1}^{N} \lambda_k g_k(x_k)\| < \varepsilon$,

(vi) $d_C(g_k(x), F(x)) < \varepsilon$,

for every $x, x_1, ..., x_N \in V$, $k = 1, ..., N$.

Proof. Call E_ξ the closed affine subspace generated by $\overline{\text{co}}\, F(\xi)$. Let z^* be a strongly exposed point of $\overline{\text{co}}\, F(\xi)$. By Lemma 5.1, $d_C(z^*, F(\xi)) = 0$. Using the upper semicontinuity of d_C, we can find a point $z \in \text{rel int}\, \overline{\text{co}}\, F(\xi)$, close to z^*, and $\delta > 0$ such that

$$B(z, \delta) \cap E_\xi \subset \overline{\text{co}}\, F(\xi), \tag{9.10}$$

$$d_C(y, F(\xi)) < \varepsilon, \qquad \forall y \in B(z, \delta) \cap E_\xi. \tag{9.11}$$

Since $v \in \text{rel int}\, \overline{\text{co}}\, F(\xi)$, $\pi'(v) \in \text{int}_{E'} \pi'(\overline{\text{co}}\, F(x))$. Moreover, there exist $y' \in \overline{\text{co}}\, F(\xi)$ and a rational number $\eta \in (0, 1]$ such that

$$v = \eta z + (1 - \eta)\, y'. \tag{9.12}$$

By Lemma 5.1, there exist finitely many elements $y'_2, ..., y'_N \in \exp \overline{\text{co}}\, F(\xi)$ and rational coefficients $\eta_2, ..., \eta_N > 0$ such that

$$\sum_{k=2}^{N} \eta_k = 1, \qquad \left\| y' - \sum_{k=2}^{N} \eta_k y'_k \right\| < \delta\eta.$$

Since $d_C(y'_k) = 0$ and d_C is upper semicontinuous, by choosing $y_k \in \text{rel int}\, \overline{\text{co}}\, F(\xi)$ sufficiently close to y'_k, we still have

$$\left\| y' - \sum_{k=2}^{N} \eta_k y_k \right\| < \delta\eta, \tag{9.13}$$

$$d_C(y_k, F(\xi)) < \varepsilon, \qquad k = 2, ..., N. \tag{9.14}$$

Define

$$y_1 = z + \frac{1}{\eta}\left(v - \eta z - (1 - \eta) \sum_{k=2}^{N} \eta_i y_i\right) \in B(z, \delta) \cap \overline{\text{co}}\, F(\xi),$$

$$\lambda_1 = \eta, \qquad \lambda_k = (1 - \eta)\, \eta_k, \qquad k = 2, ..., N.$$

By (9.13), (9.14), and the upper semicontinuity of d_C, there exists $\rho > 0$ and a neighborhood V' of ξ such that

$$\left\| v - \sum_{k=1}^{N} \lambda_k v_k \right\| < \varepsilon, \tag{9.15}$$

$$d_C(v_k, F(x)) < \varepsilon \tag{9.16}$$

for every $x \in V'$, $k \in \{1, ..., N\}$, $v_k \in \bar{B}(y_k, \rho)$. Using Lemma 9.4, we can find a neighborhood $V \subseteq V'$ of ξ such that the multifunctions

$$x \to A_k(x) = \overline{\text{co}}\, F(x) \cap \bar{B}(y_k, \rho) \cap (y_k + E'')$$

are defined and continuous on V. For $x \in V$, $k = 1, ..., N$, define

$$g_k(x) = \text{bar}(B(A_k(x), 1) \cap (y_k + E'')),$$

the barycenter being taken with respect to the Lebesgue measure on the finite-dimensional, affine space $y_k + E''$.

The continuity of g_k now follows from the Hausdorff continuity of A_k, condition (ii) is a consequence of Lemmas 9.1 and 9.2, while (v) and (vi) follow from (9.15), (9.16), respectively. By construction, the remaining conditions (i), (iii), and (iv) clearly hold. This completes the proof. ∎

10. An Approximation Theorem

Theorem 10.1. *Let $\mathscr{E} = (\mathscr{P}, \mathscr{G}, f)$ be an admissible n-level feedback for (1.1). Let C be a closed ball in E, $\delta > 0$. Then there exists an admissible $(n+1)$-level feedback $\hat{\mathscr{E}} = (\hat{\mathscr{P}}, \hat{\mathscr{G}}, \hat{f})$ such that*

$$S_{\hat{\mathscr{E}}} \subseteq B(S^{\delta}_{\mathscr{E}}, \delta)$$

and

$$\int_0^T \phi_C(\dot{y}(t), y(t))\, dt \leqslant \delta T, \qquad \forall y \in S_{\hat{\mathscr{E}}}.$$

Proof. Let E^* be the finite-dimensional space spanned by E'' and by the vector fields $f[\mathbf{i}, j]$ of $\mathscr{E}$. Set $\Omega = E^* \cap \tilde{B}$, where $\tilde{B} = \bar{B}(0, \tilde{\rho}) \subseteq E$, as usual. For every fixed $\xi \in \Omega$, denote by $\Gamma(\xi)$ the set of multiindexes $\mathbf{i} = (i_1, ..., i_n)$ such that

$$\xi \in \bar{B}(V^n_{i_n}, MT/p_n). \tag{10.1}$$

By the definition of admissible feedback, $\mathbf{i} \in \Gamma(\xi)$ implies that, for all j, the domain of $f[\mathbf{i}, j]$ is an open neighborhood of ξ. Applying Lemma 9.6 to all vector fields $f[\mathbf{i}, j]$ with $\mathbf{i} \in \Gamma(\xi)$, we deduce the existence of a radius $\rho_\xi > 0$, a finite set of vector fields $g_1, ..., g_N$ defined on $B(\xi, \rho_\xi)$, and rational coefficients $\lambda_1[\mathbf{i}, j], ..., \lambda_N[\mathbf{i}, j] \geqslant 0$ satisfying the following conditions:

(C1) $B(\xi, \rho_\xi) \subseteq \text{Dom}\, f[\mathbf{i}, j]$;

(C2) $\|f[\mathbf{i}, j](x) - f[\mathbf{i}, j](\xi)\| < \delta/2$;

(C3) $g_k(x) \in \text{rel int } \overline{\text{co}}\, F(x)$;

(C4) $g'_k = \pi' \circ g_k(x) \in \text{int}_{E'}(\pi' \circ \overline{\text{co}}\, F(x))$ is independent of x;

(C5) $d_C(g_k(x), F(x)) < \delta$;

(C6) $\sum_{k=1}^{N} \lambda_k[\mathbf{i}, j] = 1$;

(C7) $\sum_{k=1}^{N} \lambda[\mathbf{i}, j]\, g'_k = \pi' \circ f[\mathbf{i}, j](x) = f'[\mathbf{i}, j]$;

(C8) $\|f[\mathbf{i}, j](\xi) - \sum_{k=1}^{N} \lambda_k[\mathbf{i}, j]\, g_k(x_k)\| < \delta/2$

whenever $x, x_1, ..., x_N \in B(\xi, \rho_\xi)$, $\mathbf{i} \in \Gamma(\xi)$, $k = 1, ..., N$,

together with

(C9) $\forall i \in \{1, ..., q_n\}$, if $\xi \notin \bar{B}(V_i^n, MT/p_n)$, then $B(\xi, \rho_\xi) \cap \bar{B}(V_i^n, MT/p_n) = \varnothing$.

Repeat the above construction for every $\xi \in \Omega$. The family $\{B(\xi, \rho_\xi/2) : \xi \in \Omega\}$ is an open covering of the compact set Ω. Let $\{B(\xi_l, \rho_l/2) : l = 2, ..., \hat{q}_{n+1}\}$ be a finite subcovering. Choose an integer multiple $\hat{p}_{n+1}$ of p_{n+1} such that

$$T/\hat{p}_{n+1} < \min\{\rho_l/2M;\ l = 2, ..., \hat{q}_{n+1}\} \wedge \delta/2M. \tag{10.2}$$

Call g_k^l, $\lambda_k^l[\mathbf{i}, j]$ respectively the vector fields and the rational coefficients constructed in connection with the point ξ_l, $l = 2, ..., \hat{q}_{n+1}$, $k = 1, ..., N_l$.

We are now ready to define the feedback $\hat{\mathscr{E}}$.

Partitions. Set $\hat{\mathscr{P}}_k = \mathscr{P}_k$ for $1 \leqslant k \leqslant n$ and define

$$\hat{\mathscr{P}}_{n+1} = \{\hat{t}_i^{n+1} : i = 0, ..., \hat{p}_{n+1}\}$$

with $\hat{t}_i^{n+1} = iT/\hat{p}_{n+1}$. Recalling that the coefficients λ_k^l are all rational, there exist integers p and $m_k^l[\mathbf{i}, j]$ such that

$$\lambda_k^l[\mathbf{i}, j] = m_k^l[\mathbf{i}, j]/p \tag{10.3}$$

for all $k, l, \mathbf{i}, j$. Set $\hat{p}_{n+2} = p \cdot \hat{p}_{n+1}$ and define

$$\hat{\mathscr{P}}_{n+2} = \{\hat{t}_i^{n+2} : i = 0, ..., \hat{p}_{n+2}\}, \qquad \text{with} \quad \hat{t}_i^{n+2} = iT/\hat{p}_{n+2}.$$

Coverings. Set $\hat{\mathscr{G}}_k = \mathscr{G}_k$ iff $k = 1, ..., n$. To construct the last covering $\hat{\mathscr{G}}_{n+1}$, for $l = 2, ..., \hat{q}_{n+1}$ define $\hat{V}_l^{n+1} = B(\xi_l, \rho_l/2)$. The above sets $\hat{V}_l^{n+1}$ now cover Ω. To obtain a covering of $\tilde{B}$, one more set must be added. The compactness of Ω implies that the distance

$$\rho_1 = \inf\left\{\|x - y\| : x \in \Omega,\ y \notin \bigcup_l B(\xi_l, \rho_l/2)\right\}$$

is strictly positive. Define

$$\hat{V}_1^{n+1} = \{x \in E: \|x\| < 2\tilde{\rho}, d(x, \Omega) > \rho_1/2\}. \tag{10.4}$$

Then $\hat{\mathscr{G}}_{n+1} = \{\hat{V}_l^{n+1}; l = 1, ..., \hat{q}_{n+1}\}$ is an open covering of $\tilde{B}$.

Functions. Set $\lambda_0^l[\mathbf{i}, j] = 0$ for every $\mathbf{i}, j, l$, and define $\hat{\tau}^k(t) = \max\{\hat{t}_i^k: 0 \leqslant i \leqslant p_k, \hat{t}_i^k \leqslant t\}$ as in (6.3). Call $\omega = T/\hat{p}_{n+2}$ the length of the intervals of the partition $\hat{\mathscr{P}}_{n+2}$. If $\mathbf{i} = (i_1, ..., i_n)$ is a multi-index, $l \in \{1, ..., \hat{q}_{n+1}\}$ and $r \in \{0, ..., \hat{p}_{n+2} - 1\}$ to define $\hat{f}[\mathbf{i}, l, r]$ we consider two cases.

Case 1. $l \geqslant 2$ and $\mathbf{i} \in \Gamma(\xi_l)$, i.e., $\xi_l \in \bar{B}(V_{i_n}^n, MT/p_n)$. Since the new partition $\hat{\mathscr{P}}_{n+1}$ is finer than the old partition $\mathscr{P}_{n+1}$, there exists a unique $j = j(r) \in \{0, ..., p_{n+1} - 1\}$ such that $I = [\hat{\tau}^{n+1}(r\omega), \hat{\tau}^{n+1}(r\omega) + T/\hat{p}_{n+1}) \subseteq [t_j^{n+1}, t_{j+1}^{n+1})$ (i.e., $t_j^{n+1} = \tau^{n+1}(r\omega)$). By conditions (C1) and (C8), on the ball $B(\xi_l, \rho_l)$ the vector field $f[\mathbf{i}, j]$ can be approximated by a convex combination of the vector fields g_k^l with coefficients $\lambda_k^l[\mathbf{i}, j]$, $k = 1, ..., N_l$. To "track" a solution of the differential equation $\dot{x}(t) = f[\mathbf{i}, j](x(t))$ on the interval I, a solution $y(\cdot)$ of the new feedback $\hat{\mathscr{E}}$ should satisfy the equations $\dot{y}(t) = g_k^l(y(t))$ on subintervals $I_k \subseteq I$ whose lengths are proportional to $\lambda_k^l[\mathbf{i}, j]$. With this in mind, we define

$$\gamma[\mathbf{i}, l, r] = \max\left\{k \geqslant 1: \hat{\tau}^{n+1}(r\omega) + \frac{T}{\hat{p}_{n+1}} \sum_{m=0}^{k-1} \lambda_m^l[\mathbf{i}, j] \leqslant r\omega\right\}, \tag{10.5}$$

$$\hat{f}[\mathbf{i}, l, r](x) = g_{\gamma[\mathbf{i}, l, r]}^l(x), \qquad \forall x \in B(\xi_l, \rho_l). \tag{10.6}$$

Otherwise stated, $\hat{f}[\mathbf{i}, l, r] = g_k^l$ whenever $[r\omega, (r+1)\omega] \subseteq I_k$, with

$$I_k = \left[\hat{\tau}^{n+1}(r\omega) + \frac{T}{\hat{p}_{n+1}} \sum_{m=0}^{k-1} \lambda_m^l[\mathbf{i}, j(r)], \hat{\tau}^{n+1}(r\omega) + \frac{T}{\hat{p}_{n+1}} \sum_{m=0}^{k} \lambda_m^l[\mathbf{i}, j(r)]\right]. \tag{10.7}$$

Case 2. $l = 1$ or $\mathbf{i} \notin \Gamma(\xi_l)$. In this case we set

$$\hat{f}[\mathbf{i}, l, r](x) = \tilde{g}(x), \qquad \forall x \in B(0, 2\tilde{\rho}),$$

where $\tilde{g}$ is the vector field constructed in Corollary 9.5.

Let us show that the $(n+1)$-level feedback $\hat{\mathscr{E}}$ is admissible for (1.1). If

$\hat{f}[\mathbf{i}, l, j] = g_k^l$, as in Case 1, then the properties (i) and (iii) in Definition 6.3 follow from (C3) and (C4), respectively. Moreover, (10.2) implies

$$\mathrm{Dom}(g_k^l) = B(\xi_l, \rho_l) \supseteq \bar{B}(B(\xi_l, \rho_l/2), MT/\hat{p}_{n+1}),$$

which proves (ii). In Case 2, $\hat{f}[\mathbf{i}, l, r] = \tilde{g}$ and all conditions (i)–(iii) are clearly satisfied. Hence $\hat{\mathscr{E}}$ is admissible. By Proposition 7.2, the set of solutions $S_{\hat{\mathscr{E}}}$ is therefore nonempty and compact.

Now consider any $y \in S_{\hat{\mathscr{E}}}$. Define $\sigma_{n+2}(t) = \max\{r : \hat{t}_r^{n+2} \leqslant t\}$ and let $\sigma_k : \{\hat{t}_0^k, ..., \hat{t}_{pk-1}^k\} \to \{1, ..., \hat{q}_k\}$, $k = 1, ..., n+1$ be such that

$$y(\hat{t}_i^k) \in \overline{V_{\sigma_k(\hat{t}_i^k)}^k}, \qquad 1 \leqslant k \leqslant n+1,\ 0 \leqslant i \leqslant \hat{p}_k, \tag{10.8}$$

$$\dot{y}(t) = \hat{f}[\sigma_1(\hat{\tau}^1(t)), ..., \sigma_{n+1}(\hat{\tau}^{n+1}(t)), \sigma_{n+2}(t)](y(t)) \tag{10.9}$$

a.e. on $[0, T]$. We claim that the vector fields $\hat{f}[\cdot,\cdot]$ wich actually occur in (10.9) are all defined according to Case 1. This will be a consequence of the next lemma. To shorten the notation, we write $\sigma_k(t) = \sigma_k(\hat{\tau}^k(t))$.

LEMMA 10.2. *In the above setting, one has*

$$y(\hat{t}_i^{n+1}) \in \Omega, \qquad \forall i \in \{0, ..., \hat{p}_{n+1}\} \tag{10.10}$$

$$\xi_{\sigma_{n+1}(t)} \in \bar{B}(V_{\sigma_n(t)}^n, MT/\hat{p}_n), \qquad \forall t \in [0, T]. \tag{10.11}$$

The statements (10.10), (10.11) will be proved together, by induction on $i \in \{0, ..., \hat{p}_{n+1}\}$. Assume that (10.10) holds for all $i \leqslant \imath$ and that (10.11) holds whenever $t \in [0, \hat{t}_\imath^{n+1})$. In particular, for $t \in I_\imath = [\hat{t}_\imath^{n+1}, \hat{t}_{\imath+1}^{n+1})$, (10.8) and (10.10) imply $\sigma_{n+1}(t) = \sigma_{n+1}(\hat{t}_\imath^{n+1}) \geqslant 2$, because the closure of V_1^{n+1} does not intersect Ω. Assume that (10.11) fails for some $t \in I_\imath$. Then the condition (C9) implies

$$\bar{B}(\xi_{\sigma_{n+1}(t)}, \rho_{\sigma_{n+1}(t)}) \cap \bar{B}(V_{\sigma_n(t)}^n, MT/\hat{p}_n) = \varnothing.$$

We now have

$$y(\hat{\tau}^n(t)) \in \overline{V_{\sigma_n(t)}^n},$$

$$y(\hat{\tau}^{n+1}(t)) \in \bar{B}(\xi_{\sigma_{n+1}(t)}, \rho_{\sigma_{n+1}(t)}/2) \subseteq B(\xi_{\sigma_{n+1}(t)}, \rho_{\sigma_{n+1}(t)}),$$

together with the Lipschitz condition

$$\|y(\hat{\tau}^{n+1}(t)) - y(\hat{\tau}^n(t))\| \leqslant M \cdot (\hat{\tau}^{n+1}(t) - \hat{\tau}^n(t)) \leqslant MT/\hat{p}_n,$$

reaching a contradiction. This establishes (10.11) for all $t \in [0, \hat{t}_{\imath+1}^{n+1})$. In particular, we now know that, for $t \in [\hat{t}_\imath^{n+1}, \hat{t}_{\imath+1}^{n+1})$, the vector fields $\hat{f}[\cdot,\cdot]$ occurring in (10.9) are defined according to Case 1. To prove (10.10), observe that the conditions (C4) and (C7) together with the definitions (10.5), (10.6) imply

$$\pi'(y(\hat{t}_{i+1}^{n+1}) - y(\hat{t}_i^{n+1})) = \frac{T}{\hat{p}_{n+1}} \sum_{k=1}^{N_l} \lambda_k^l[\mathbf{i}, j]\, g_k^l$$

$$= \frac{T}{\hat{p}_{n+1}} f^l[\mathbf{i}, j] \in E^*, \qquad (10.12)$$

where $[\mathbf{i}, j] = [\sigma_1(\hat{t}_i^{n+1}), ..., \sigma_n(\hat{t}_i^{n+1}), \tau^{n+1}(\hat{t}_i^{n+1})]$, $l = \sigma_{n+1}(\hat{t}_i^{n+1})$.

Because of the inductive hypothesis, (10.12) implies $y(\hat{t}_{i+1}^{n+1}) \in \Omega$ because no solution $y \in S_{\hat{\mathscr{E}}}$ can escape from $\tilde{B} = \bar{B}(0, \tilde{\rho})$ within time T. By induction, the lemma is proved. ∎

We now return to the proof of Theorem 10.1. By Lemma 10.2, Case 2 never applies, hence for a.e. $t \in [0, T]$, $\dot{y}(t) = g_k^l(y(t))$ for some k, l depending on t. Condition (C5) now yields

$$\int_0^T \phi_C(\dot{y}(t), y(t))\, dt = \int_0^T d_C(g_{k(t)}^{l(t)}(y(t)), F(y(t)))\, dt \leqslant \delta T.$$

It remains to prove that $y \in B(S_{\mathscr{E}}^{\delta}, \delta)$. Define $z(\cdot)$ as the polygonal function which coincides with y at each node $\hat{t}_i^{n+1}$ of the partition $\hat{\mathscr{P}}_{n+1}$ and is linear on each interval $[\hat{t}_{i+1}^{n+1}, \hat{t}_i^{n+1}]$; $i \in \{1, ..., \hat{p}_{n+1}\}$. Since y has Lipschitz constant M, (10.2) implies

$$\|y(t) - z(t)\| \leqslant \delta, \qquad \forall t \in [0, T].$$

We claim that $z \in S_{\mathscr{E}}^{\delta}$. For $1 \leqslant k \leqslant n$, set $\beta_k = \sigma_k$. By construction, for $k \leqslant n$ the coverings $\mathscr{G}_k$ in the feedbacks $\mathscr{E}$ and $\hat{\mathscr{E}}$ coincide. Hence (7.1) is an immediate consequence of (6.5). To prove (7.2), fix an arbitrary interval $I_i = [\hat{t}_i^{k+1}, \hat{t}_{i+1}^{k+1})$ of the partition $\hat{\mathscr{P}}_{n+1}$. For $t \in I_i$, call

$$[\mathbf{i}, j] = [i_1, ..., i_n, j] = [\beta_1(\tau^1(t)), ..., \beta_n(\tau^n(t)), \beta_{n+1}(t)],$$

$$[\mathbf{i}, l, \sigma_{n+2}(t)] = [\sigma_1(\hat{\tau}^1(t)), ..., \sigma_n(\hat{\tau}^n(t)), \sigma_{n+1}(\hat{\tau}^{n+1}(t)), \sigma_{n+2}(t)].$$

For $t \in I_i$, both $y(t)$ and $z(t)$ remain inside $B(\xi_l, \rho_l)$. By the definitions, we have

$$\dot{y}(t) = \hat{f}[\mathbf{i}, l, \sigma_{n+2}(t)](y(t)), \qquad (10.13)$$

$$\dot{z}(t) = \frac{\hat{p}_{n+1}}{T} \int_{I_i} f[\mathbf{i}, l, \sigma_{n+2}(s)](y(s))\, ds$$

$$= \sum_{k=1}^{N_l} \lambda_k^l[\mathbf{i}, j] \left(\frac{1}{\text{meas}(I_k)} \int_{I_k} g_k^l(y(s))\, ds \right), \qquad (10.14)$$

with I_k defined at (10.7).

Using (C7) together with (10.5) and (10.6), from (10.14) we deduce

$$\pi'(\dot{z}(t)) = \sum_{k=1}^{N_l} \lambda_k^l[\mathbf{i}, j] \cdot \pi'(g_k^l(y(t))) = f'[\mathbf{i}, j]. \tag{10.15}$$

Moreover (C2) and (C8) imply

$$\begin{aligned} &\|f[\mathbf{i}, j][z(t)) - \dot{z}(t)\| \\ &\quad \leqslant \|f[\mathbf{i}, j](z(t)) - f[\mathbf{i}, j](\xi_l)\| \\ &\quad\quad + \left\| f[\mathbf{i}, j](\xi_l) - \sum_{k=1}^{N_l} \lambda_k^l[\mathbf{i}, j] \left(\frac{1}{\text{meas}(I_k)} \int_{I_k} g_k^l(y(s))\, ds \right) \right\| \\ &\quad \leqslant \delta/2 + \delta/2. \end{aligned} \tag{10.16}$$

Together, (10.15) and (10.16) yield (7.2). This completes the proof of Theorem 10.1. ∎

11. Completion of the Proof

Let $\{\mathscr{E}_i : i \in \mathscr{I}\}$ be the family of all admissible n-level feedbacks ($n \geqslant 1$) for the Cauchy problem (1.1), according to Definition 6.3. Notice that this family is nonempty. Indeed, using the assumptions (A1)–(A6) in Section 3, an admissible 1-level feedback $\mathscr{E} = (\mathscr{P}, \mathscr{G}, f)$ on $[0, T] \times \tilde{B}$ can be easily constructed by setting $\mathscr{P}_1 = \mathscr{P}_2 = \{0, T\}$, $\mathscr{G}_1 = \{V_1\} = \{B(0, 2\tilde{\rho})\}$ and letting f consist of the single vector field $f[1, 0] = \tilde{g}$ defined in Corollary 9.5. For every $i \in \mathscr{I}$, define the set of solutions $K_i = S_{\mathscr{E}_i}$, which is nonempty and compact by Proposition 7.2. Set $\mathscr{F} = \{K_i : i \in \mathscr{I}\}$. As in Section 6, let S be the family of all solutions of (1.2) and consider the sets S_{jn} defined at (6.1). By Theorem 8.1, the sets $R_{jn} = S \backslash S_{jn}$ are all closed. To prove that they are $\mathscr{F}$-rare, let $\mathscr{E}$ be any admissible n-level feedback and let $\varepsilon > 0$, $j, n \geqslant 1$ be given. By Proposition 7.2 there exists $\delta > 0$ such that $S_{\mathscr{E}}^\delta \subseteq B(S_{\mathscr{E}}, \varepsilon/2)$. Moreover, Theorem 10.1 provides an admissible $(n+1)$-level feedback $\hat{\mathscr{E}}$ such that $S_{\hat{\mathscr{E}}} \subseteq B(S_{\mathscr{E}}^\delta, \varepsilon/2) \subseteq B(S_{\mathscr{E}}, \varepsilon)$ and

$$\int_0^T \phi_{C_j}(\dot{y}(t), y(t))\, dt < 1/n, \qquad \forall y \in S_{\hat{\mathscr{E}}}.$$

This shows that the sets R_{jn} are $\mathscr{F}$-rare. Therefore, Theorem 4.3 implies that

$$\hat{S} = \bigcap_{j,n} S_{jn} \neq \varnothing.$$

Using Lemma 5.4, we conclude that every $y \in \hat{S}$ is actually a solution of (1.1). This completes the proof of Theorem 3.3.

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