1 Basic ODE Theory

A first order ordinary differential equation has the form
\[ \dot{x}(t) = f(t, x(t)). \] (1.1)
Here \( \dot{x} = dx/dt \) is the derivative w.r.t. time. The equation is autonomous if \( f = f(x) \) does not depend on time. Given an initial data
\[ x(t_0) = x_0, \] (1.2)
by a solution of the Cauchy problem (1.1)-(1.2) we mean a continuously differentiable function \( t \mapsto x(t) \) defined on some open interval \( ]a, b[ \) with \( a < t_0 < b \), which satisfies the ODE at every time \( t \in ]a, b[ \), and the initial condition at time \( t = t_0 \).

**Local existence:** If \( f \) is continuous, then the Cauchy problem (1.1)-(1.2) has a solution (possibly not unique), defined for \( t \in ]t_0 - \delta, t_0 + \delta[ \), for \( \delta > 0 \) small.

**Uniqueness:** If, in addition, \( f \) has a continuous partial derivative \( \frac{\partial f}{\partial x} \), then the solution is unique.

**Global existence:** If there exists a constant \( C \) such that
\[ |f(t, x)| \leq C(1 + |x|) \quad \text{for all } t, x, \]
(i.e., if \( f \) grows at most like a polynomial of degree one in \( x \)), then the Cauchy problem has a solution defined for all times \( t \in \mathbb{R} \).

**Example 1 (non-existence).** The Cauchy problem
\[ \dot{x} = f(x) = \begin{cases} -1 & \text{if } x \geq 0, \\ +1 & \text{if } x < 0, \end{cases}, \quad x(0) = 0, \]
has no solution defined for \( t > 0 \). Here the function \( f \) is not continuous.

**Example 2 (non-uniqueness).** The Cauchy problem
\[ \dot{x} = f(x) = \frac{3}{2} x^{1/3}, \quad x(0) = 0 \]
has two distinct solutions, namely

\[ x(t) \equiv 0, \quad x(t) = t^{3/2}. \]

Here the function \( f \) is not continuously differentiable.

**Example 3 (blow up in finite time).** The Cauchy problem

\[ \dot{x} = f(x) = 1 + x^2, \quad x(0) = 0 \]

has the (unique) solution \( x(t) = \tan t \). This is defined only for \( t \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[ \), not for all times \( t \in \mathbb{R} \). Here the function \( f \) is not bounded by a polynomial of degree one.

Figure 1: Left: a Cauchy problem with no solution (Example 1). Center: a Cauchy problem with multiple solutions (Example 2). Right: a solution blowing up at \( t = \pi/2 \) (Example 3).

**Approximate solutions by the Euler method.** Fix a time step \( \Delta t > 0 \) and consider the times

\[ t_k = t_0 + k\Delta t \quad k = 1, 2, 3 \ldots \]

By induction on \( k \), we then define

\[ x_{k+1} = x_k + f'(x_k) \Delta t. \]

Then the piecewise affine function

\[ x(t) = x_k + f'(x_k)(t - t_k) \quad t \in [t_k, t_{k+1}] \]

provides an approximate solution to the Cauchy problem (1.1)-(1.2).

## 2 Phase Line Diagrams

Consider the autonomous ODE

\[ \dot{x} = f(x) \quad (2.1) \]

where \( f \) is continuously differentiable. We think of \( x(t) \) as the *position* of a point moving on the line, and \( \dot{x}(t) \) as its *velocity*. 

---

2
If $x(\cdot)$ is a solution of (2.1), the range of the map $t \mapsto x(t)$ is called the **orbit**. In other words, the orbit is the set of values $\{x(t); \ t \in \mathbb{R}\}$.

**Remark.** If $x(\cdot)$ is a solution of the autonomous ODE (2.1), then, for every constant $c$, the function $y(t) = x(t - c)$ is another solution, having the same orbit. It is essentially the same solution, shifted in time.

The **phase line diagram** is obtained by

- marking the *equilibrium points* where $f = 0$,
- adding arrows marking the direction of motion.

![Phase line diagram](image)

Figure 2: A phase line diagram. Here $x_1$, $x_2$, and $x_3$ are equilibrium points. The point $x_2$ is stable while $x_1$ and $x_3$ are unstable.

**Definition (stable equilibrium).** Let $x^*$ be an equilibrium point, so the $f(x^*) = 0$. We say that $x^*$ is *stable* if for every $\varepsilon > 0$ we can find $\delta > 0$ such that for any solution $x(\cdot)$ one has:

$$
\text{if } |x(0) - x^*| < \delta \quad \text{then } |x(t) - x^*| < \varepsilon \quad \text{for all } t \geq 0.
$$

We say that $x^*$ is **asymptotically stable** if $x^*$ is stable and moreover

$$
\lim_{t \to \infty} x(t) = x^*.
$$

If the equilibrium point $x^*$ is not stable, we say it is **unstable**.

**Intuitive meaning:** $x^*$ is a stable equilibrium if and only if every solution which starts close to $x^*$ remains close to $x^*$ at all positive times $t \in [0, +\infty[$.

**Theorem 1 (stable and unstable equilibrium points).**

- If $f(x^*) = 0$, $f(x) > 0$ for $x \in [x^* - \delta, x^*[$ and $f(x) < 0$ for $x \in ]x^*, x^* + \delta]$, then $x^*$ is asymptotically stable.
- If $f(x^*) = 0$ but $f(x) < 0$ for $x \in [x^* - \delta, x^*[$, or $f(x) > 0$ for $x \in ]x^*, x^* + \delta]$, then $x^*$ is unstable.
3 Bifurcations

Given the two ODEs
\[
\begin{align*}
\dot{x} &= f(x), \\
\dot{x} &= g(x),
\end{align*}
\] (3.1) (3.2)

we say that their phase diagrams are equivalent if there exists a one-to-one continuous function \( \varphi : \mathbb{R} \to \mathbb{R} \) mapping orbits of \( f \) onto orbits of \( g \), preserving the orientation.

We now consider an ODE
\[
\dot{x} = f(r, x),
\] (3.3)

depending on the additional parameter \( r \). If the phase diagram changes type as \( r \) crosses some critical value \( r^* \), we say that a bifurcation occurs.
Theorem 2 (necessary conditions). Let $f$ be continuously differentiable. If a bifurcation happens at $r = r^*$, then one of these two possibilities must occur:

1 (standard bifurcation). There exists a value $x^*$ such that

$$
\begin{align*}
    f(r^*, x^*) &= 0, \\
    \frac{\partial f}{\partial x}(r^*, x^*) &= 0.
\end{align*}
$$

(3.4)

2 (bifurcation at infinity). The curve $\Gamma = \{(r, x); f(r, x) = 0\}$ has a vertical asymptote. Namely, there exist points $(r_n, x_n)$ such that

$$
\lim_{n \to \infty} r_n = r^*, \quad \lim_{n \to \infty} x_n = \pm \infty, \quad f(r_n, x_n) = 0 \quad \text{for all } n.
$$

(3.5)

Figure 5: A “proof” of Theorem 2. Assume that at $r^*$ the function $f$ has exactly two zeroes: $f(r^*, x_1^*) = f(r^*, x_2^*) = 0$. If $\frac{\partial f}{\partial x}(r^*, x_1^*) \neq 0$, and $\frac{\partial f}{\partial x}(r^*, x_2^*) \neq 0$, then near these two points we can use the implicit function theorem. For every $r$ close to $r^*$, the function $f(r, \cdot)$ still has exactly two zeroes: $f(r, x_1(r)) = f(r, x_2(r)) = 0$, for some continuous functions $x_1(r)$ and $x_2(r)$. Therefore, the phase diagram for $\dot{x} = f(r, x)$ is equivalent to the phase diagram for $\dot{x} = f(r^*, x)$. No bifurcation occurs.

Two ways to find bifurcations (Fig. 6)

1 - (Geometric method): In the $r$-$x$ plane, draw the curve $\Gamma = \{(r, x); f(r, x) = 0\}$. This separates the two regions where $f > 0$ and where $f < 0$. Along each vertical line where $r=$constant, draw the phase diagram of the ODE (3.3). Check for which values of $r$ this diagram changes type (see fig. 6).

2 - (Analytic method): Find (i) solutions to the system of equations (3.4), and (ii) vertical asymptotes for the curve $\Gamma$ of points where $f(r, x) = 0$. 

Figure 6: Here the phase diagrams (drawn along vertical lines) change type when the parameter $r$ crosses the values $r_1^*$ and $r_2^*$. The solid portion of the curve where $f(r, x) = 0$ corresponds to stable equilibrium points. The dotted portion corresponds to unstable equilibria. The bifurcation point $(r_1^*, x_1^*)$ provides a solution to the system of equations (3.4). At $r_2^*$ the bifurcation occurs “at infinity”.

4 Special types of bifurcations. Normal forms.

Let $(r^*, x^*)$ be a point where (3.4) holds. Different types of bifurcations can occur. They are classified according to their normal forms (providing the simplest example of each type of bifurcation).

**Saddle-node bifurcation.** Two branches of equilibrium points (where $f(r, x) = 0$), one stable and one unstable, join together and disappear as $r$ crosses the critical value $r^*$.

Normal form: \[ f(r, x) = r + x^2. \]

This bifurcation occurs at a point $P^* = (r^*, x^*)$ where (3.4) holds, together with

\[ \frac{\partial f}{\partial r}(P^*) \neq 0, \quad \frac{\partial^2 f}{\partial x^2}(P^*) \neq 0. \]  

(4.1)

Indeed, if (4.1) holds, then near $P^*$ the equation $f(r, x) = 0$ implicitly defines a function $r = r(x)$, having a local max (or a local min) at $x = x^*$.

**Transcritical bifurcation.** Two branches of equilibrium points cross each other at $r = r^*$, exchanging equilibrium properties (the stable one becomes unstable, and vice-versa).

Normal form: \[ f(r, x) = (r - x)x. \]
This kind of bifurcation can only occur at a point \( P^* = (r^*, x^*) \) where (3.4) holds, together with
\[
\frac{\partial f}{\partial r}(P^*) = 0.
\] (4.2)

In this case, the Taylor approximation for \( f \) near \( P^* \) takes the form
\[
f(r, x) \approx \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (P^*) \cdot (x - x^*)^2 + \frac{\partial^2 f}{\partial r \partial x} (P^*) \cdot (r - r^*) (x - x^*) + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} (P^*) \cdot (r - r^*)^2
\]
\[
= a X^2 + b RX + c R^2 \quad (X = x - x^*, \ R = r - r^*). 
\]

A transcritical bifurcation occurs when the quadratic form
\[
Q(R, X) = a X^2 + b RX + c R^2
\]
vanishes along two lines:
\[
X = \lambda_1 R, \quad X = \lambda_2 R,
\]
\[
\lambda_1, \ \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

This happens provided that \( a \neq 0 \) and \( b^2 - 4ac > 0 \).

Figure 7: Two types of bifurcations. The shaded regions are the sets where \( f(r, x) > 0 \).
Pitchfork bifurcation. Three branches of equilibrium points join together at \( r = r^* \). Two of them disappear.

This kind of bifurcation usually occurs when \( f(r, x) \) is an odd function of \( x \), so that

\[
f(r, -x) = -f(r, x).
\]

Normal forms:

\[
\begin{align*}
  f(r, x) &= rx + x^3 \quad \text{(subcritical)}, \\
  f(r, x) &= rx - x^3 \quad \text{(supercritical)}.
\end{align*}
\]

Figure 8: Examples of pitchfork bifurcations. The shaded regions are the sets where \( f(r, x) > 0 \). In the subcritical case, the new branches appear for \( r < r^* \). In the supercritical case, the new branches appear for \( r > r^* \).

5 Linear systems in the plane

Consider the linear homogeneous system

\[
\begin{align*}
  \dot{x} &= ax + by, \\
  \dot{y} &= cx + dy,
\end{align*}
\]

where the upper dot denotes a derivative w.r.t. time. Using vector notation, this can be written as

\[
\dot{x} = Ax,
\]

with \( x = \begin{pmatrix} x \\ y \end{pmatrix}, \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). The characteristic polynomial of the matrix \( A \) is

\[
p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc).
\]

The roots of this polynomial are the eigenvalues of \( A \). We can write \( p(\lambda) \) as

\[
p(\lambda) = \lambda^2 - \tau\lambda + \Delta,
\]
where

\[
\begin{align*}
\tau &= a + d = \text{trace of } A, \\
\Delta &= ad - bc = \text{determinant of } A.
\end{align*}
\]

The eigenvalues of \( A \) are thus computed by

\[
\lambda_1, \lambda_2 = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}.
\] (5.5)

Eigenvalues \( \lambda_1, \lambda_2 \) are

- real
  - \( \lambda_1 < \lambda_2 < 0 \) stable node
  - \( \lambda_1 < 0 < \lambda_2 \) saddle
  - \( 0 < \lambda_1 < \lambda_2 \) unstable node
- complex conjugate
  - \( \lambda_{1,2} = \alpha \pm i\beta \)
  - \( \alpha < 0 \) stable spiral
  - \( \alpha > 0 \) unstable spiral

Figure 9: The phase portrait for the system (5.1) can be of five main types, depending on the eigenvalues \( \lambda_1, \lambda_2 \) of the matrix \( A \).

Figure 10: The two parameters \( \Delta = \text{det}(A) \) and \( \tau = \text{trace}(A) \) determine the eigenvalues \( \lambda_1, \lambda_2 \), and hence the phase portrait.

The phase portrait of the system (5.1) essentially depends on the eigenvalues \( \lambda_1, \lambda_2 \) of the matrix \( A \). Five main cases arise, described in Fig. 9. By (5.5), these cases occur when \((\Delta, \tau)\) lies in one of the five regions shown in Fig. 10.
6 Computing the general solution

**CASE 1**: The matrix $A$ has two real, distinct eigenvalues $\lambda_1 < \lambda_2$. This happens whenever $\tau^2 - 4\Delta > 0$.

Solving the equations

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2,$$

we can determine two linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.

The system (5.2) then has the two special solutions

$$\phi_1(t) = e^{\lambda_1 t} \mathbf{v}_1, \quad \phi_2(t) = e^{\lambda_2 t} \mathbf{v}_2.$$

Since the system is linear, any linear combination of these two solutions is still a solution. The general solution thus has the form

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2. \quad (6.1)$$

If an initial datum $\mathbf{x}(0) = \mathbf{x}_0$ is given, we choose the coefficients $c_1, c_2$ in (6.1) so that

$$\mathbf{x}(0) = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{x}_0. \quad (6.2)$$

Note that the constants $c_1, c_2$ can be uniquely determined by (6.2), because the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent.

Depending on the signs of $\lambda_1, \lambda_2$, three cases can occur (Fig. 11).

**Case 1a (stable node)**: $\lambda_1 < \lambda_2 < 0$. As $t \to +\infty$, all trajectories flow into the origin. The component along $\mathbf{v}_1$ decays faster, and trajectories are asymptotically tangent to $\mathbf{v}_2$.

**Case 1b (unstable node)**: $0 < \lambda_1 < \lambda_2$. As $t \to +\infty$, trajectories flow away from the origin, becoming arbitrarily large. For negative times, as $t \to -\infty$, trajectories approach the origin. The component along $\mathbf{v}_2$ decays faster, and trajectories are asymptotically tangent to $\mathbf{v}_1$.

**Case 1c (saddle)**: $\lambda_1 < 0 < \lambda_2$. The zero solution is unstable. As $t \to +\infty$ the component along $\mathbf{v}_1$ approaches zero, while the component along $\mathbf{v}_2$ becomes arbitrarily large. On the
other hand, as \( t \to -\infty \), the \( v_1 \)-component becomes large, while the \( v_2 \) component approaches zero.

**CASE 2:** The matrix \( A \) has two complex conjugate eigenvalues \( \lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta \). This happens whenever \( \tau^2 - 4\Delta < 0 \).

Given the complex eigenvalue \( \lambda = \alpha + i\beta \), we compute a complex eigenvector \( w = w_1 + i w_2 \) such that

\[
Aw = \lambda w
\]

Hence

\[
\phi(t) = e^{\lambda t}w = e^{\alpha t + i\beta t}(w_1 + i w_2)
\]

is a complex-valued solution of the system (5.2). Recalling that

\[
e^{\alpha t + i\beta t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t} \cdot (\cos \beta t + i \sin \beta t),
\]

and taking the real part and the imaginary part of \( \phi(t) \) we obtain two real-valued solutions:

\[
Re[\phi(t)] = e^{\alpha t} \cdot (\cos \beta t w_1 - \sin \beta t w_2),
\]

\[
Im[\phi(t)] = e^{\alpha t} \cdot (\sin \beta t w_1 + \cos \beta t w_2).
\]

Since the system is linear, any linear combination of these two solutions is still a solution. The general solution thus has the form

\[
x(t) = c_1 e^{\alpha t} \cdot (\cos \beta t w_1 - \sin \beta t w_2) + c_2 e^{\alpha t} \cdot (\sin \beta t w_1 + \cos \beta t w_2).
\]

If an initial datum \( x(0) = x_0 \) is given, we choose the coefficients \( c_1, c_2 \) in (6.4) so that

\[
x(0) = c_1 w_1 + c_2 w_2 = x_0.
\]

Depending on the sign of \( \alpha \), two cases can occur (Fig. 12).

**Case 2a (stable spiral point):** If \( \alpha < 0 \), trajectories are spirals converging to the origin as \( t \to +\infty \).

**Case 2b (unstable spiral point):** If \( \alpha > 0 \), trajectories are spirals moving away from the origin as time increases.
7  Borderline cases

The previous classification covered the five main cases in Fig. 9. These correspond to the 5 regions in Fig. 10. It remains to study the borderline cases, when $(\Delta, \tau)$ lies along one of the boundaries between these regions.

**CASE 3:** Assume that $\tau \neq 0$ but $\Delta = 0$. This happens at the boundary between saddles and (stable or unstable) nodes. The general solution still has the form (6.1), but now one of the eigenvalues is zero. Two possibilities arise (Fig. 13).

**Case 3a:** $\lambda_1 < \lambda_2 = 0$. The general solution is

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 v_2.$$  \hspace{1cm} (7.1)

As $t \to +\infty$, the component along $v_1$ approaches zero, while the component along $v_2$ remains constant. The origin is neutrally stable.

**Case 3b:** $0 = \lambda_1 < \lambda_2$. The general solution is

$$x(t) = c_1 v_1 + c_2 e^{\lambda_2 t} v_2.$$  \hspace{1cm} (7.2)

As $t \to +\infty$, the $v_1$-component remains constant, while the $v_2$-component becomes arbitrarily large (unless it is identically zero). The origin is unstable.

![Figure 13: Left: a phase portrait when the eigenvalues are $\lambda_1 < \lambda_2 = 0$. Right: a phase portrait when $0 = \lambda_1 < \lambda_2$.](image)

**CASE 4:** Assume that the matrix $A$ has a double eigenvalue $\lambda = \lambda_1 = \lambda_2$. This happens when $\tau^2 - 4\Delta = 0$ (at the boundary between nodes and spirals). The equilibrium point $x^* = 0$ is stable if $\lambda < 0$ and unstable if $\lambda > 0$.

**Case 4a (symmetrical node).** The matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is diagonal. The system (5.2) thus reduces to $\dot{x} = \lambda x$. The general solution is $x(t) = e^{\lambda t} x_0$.

**Case 4b (degenerate node).** The matrix $A$ is not diagonal. In this case there exists only one linearly independent eigenvector

$$A v_1 = \lambda v_1.$$ \hspace{1cm} (7.3)
However, we can find an auxiliary vector \( v_2 \) such that \( v_1 \) and \( v_2 \) are linearly independent and
\[
A v_2 = \lambda v_2 + v_1. \tag{7.4}
\]

Thanks to (7.3)-(7.4), the two functions
\[
\phi_1(t) = e^{\lambda t} v_1, \quad \phi_2(t) = e^{\lambda t} v_2 + t e^{\lambda t} v_1,
\]
are both solution to the system (5.2). The general solution is thus
\[
x(t) = c_1 e^{\lambda t} v_1 + c_2 (e^{\lambda t} v_2 + t e^{\lambda t} v_1).
\]

- If \( \lambda < 0 \), as \( t \to +\infty \) all solutions approach the origin, tangentially to the vector \( v_1 \).
- If \( \lambda > 0 \), as \( t \to +\infty \) solutions become arbitrarily large. On the other hand, as \( t \to -\infty \), all solutions approach the origin, tangentially to the vector \( v_1 \).

![Figure 14: A stable, degenerate node. Here \( v_1 \) is an eigenvector of the matrix \( A \).](image)

**CASE 5 (center):** Assume \( \tau = 0, \Delta > 0 \) (at the boundary between stable and unstable spiral).

In this case the eigenvalues are purely imaginary \( \lambda_1 = i\beta, \lambda_2 = -i\beta \). Since \( \alpha = 0 \), by (7.2) the general solution has the form (Fig. 15)
\[
x(t) = c_1 (\cos \beta t \, w_1 - \sin \beta t \, w_2) + c_2 (\sin \beta t \, w_1 + \cos \beta t \, w_2). \tag{7.5}
\]

All trajectories are periodic with period \( 2\pi/\beta \).

### 8 Nonlinear systems

A general system of two ODEs has the form
\[
\begin{align*}
\dot{x} &= f(x, y), \\
\dot{y} &= g(x, y).
\end{align*} \tag{8.1}
\]
The phase portrait is a planar diagram showing the orbits and the direction of motion. Of particular interest are the equilibria (also called steady states) and the periodic orbits.

To sketch the phase portrait, it is useful to sketch the nullclines. These are the curves where \( f(x, y) = 0 \), hence the vector \( \left( \begin{array}{c} f \\ g \end{array} \right) \) is vertical, and the curves where \( g(x, y) = 0 \), hence the vector \( \left( \begin{array}{c} f \\ g \end{array} \right) \) is horizontal.

## 9 Conservative systems

The system (8.1) is conservative if there exists a non-constant function \( E(x, y) \) such that

\[
\frac{d}{dt} E(x(t), y(t)) = 0
\]

for every solution of (8.1). This means that \( E \) is constant in time along every trajectory.

- A system having a stable or unstable node, or a stable or unstable spiral point, cannot be conservative.
- To show that a system is conservative, we need to find a function \( E = E(x, y) \) such that

\[
\frac{\partial E}{\partial x} \cdot \dot{x} + \frac{\partial E}{\partial y} \cdot \dot{y} = \frac{\partial E}{\partial x} \cdot f + \frac{\partial E}{\partial y} \cdot g = 0.
\]

In particular, this is true if

\[
\frac{\partial}{\partial x} E(x, y) = g(x, y), \quad \frac{\partial}{\partial y} E(x, y) = -f(x, y).
\]
Example. A system having the special form \[
\begin{align*}
\dot{x} &= f(y) \\
\dot{y} &= g(x)
\end{align*}
\] is always conservative. Indeed, we can choose functions \(F, G\) such that \(F'(y) = f(y), G'(x) = -g(x)\). Then \(E(x, y) = F(y) + G(x)\) is constant along every trajectory.

9.1 Mechanical systems with positional forces

The standard example of a conservative system comes from Newton’s law of dynamics
\[
m\ddot{x} = F(x) \quad \text{[mass \times acceleration] = [force]}
\]
where the force \(F\) is positional (depends only on \(x\), not on \(\dot{x}\)). This can be written as
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \frac{1}{m}F(x)
\end{align*}
\] (9.1)
Consider a potential function \(V\), such that \(V'(x) = -F(x)\). Then the total energy
\[
E(x, y) = my^2/2 + V(x) = \text{[kinetic energy] + [potential energy]}
\]
is conserved along every trajectory of (9.1). Indeed,
\[
\frac{d}{dt}E(x(t), y(t)) = \frac{\partial E}{\partial x} \cdot \dot{x} + \frac{\partial E}{\partial y} \cdot \dot{y} = V'(x) \cdot y + my \cdot \frac{F(x)}{m} = 0.
\]

In general, the two equations
\[
\frac{\partial}{\partial x}E(x, y) = a(x, y), \quad \frac{\partial}{\partial y}E(x, y) = b(x, y)
\]
can be solved for all \((x, y) \in \mathbb{R}^2\) if and only if
\[
\frac{\partial a}{\partial y} = \frac{\partial^2 E}{\partial y \partial x} = \frac{\partial^2 E}{\partial x \partial y} = \frac{\partial b}{\partial x}.
\]
Assuming that \(\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}\), the function \(E\) can be computed as follows.

STEP 1: Thinking \(y\) as a constant, find \(A(x, y)\) such that \(\frac{\partial}{\partial x}A(x, y) = a(x, y)\).

STEP 2: Find a function \(B(y)\) whose derivative satisfies \(B'(y) = b(x, y) - \frac{\partial}{\partial y}A(x, y)\).

Then \(E(x, y) = A(x, y) + B(y)\) satisfies (*)

\[\text{1In general, the two equations}\]
\[
\frac{\partial}{\partial x}E(x, y) = a(x, y), \quad \frac{\partial}{\partial y}E(x, y) = b(x, y)
\]
\[
\text{can be solved for all } (x, y) \in \mathbb{R}^2 \text{ if and only if}\]
\[
\frac{\partial a}{\partial y} = \frac{\partial^2 E}{\partial y \partial x} = \frac{\partial^2 E}{\partial x \partial y} = \frac{\partial b}{\partial x}.
\]
\[
\text{Assuming that } \frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}, \text{ the function } E \text{ can be computed as follows.}
\]
\[
\text{STEP 1: Thinking } y \text{ as a constant, find } A(x, y) \text{ such that } \frac{\partial}{\partial x}A(x, y) = a(x, y).
\]
\[
\text{STEP 2: Find a function } B(y) \text{ whose derivative satisfies } B'(y) = b(x, y) - \frac{\partial}{\partial y}A(x, y).
\]
\[
\text{Then } E(x, y) = A(x, y) + B(y) \text{ satisfies (*)}.
\]
9.2 Hamiltonian formulation

Introducing the variables

\[ q = x, \quad p = my = \text{momentum}, \]

the energy can be written as

\[ E(x, y) = H(q, p) = \frac{p^2}{2m} + V(q). \]

In terms of the Hamiltonian function \( H(p, q) \), the equations of motion (9.1) can be written as

\[
\begin{align*}
\dot{p} &= m\dot{y} = F(x) = -V'(q), \\
\dot{q} &= \dot{x} = \frac{p}{m},
\end{align*}
\]

(9.2)

For every solution of (9.2), one immediately checks that

\[
\frac{d}{dt}H(p(t), q(t)) = \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial q} \dot{q} = 0.
\]

10 Time reversible systems

A second order differential equation \( \ddot{x} = g(x, \dot{x}) \) is **time reversible** if, given any solution \( x(t) \), the function

\[ x^\flat(t) = x(-t) \]

(obtained by reversing the direction of time) is another solution.

Equivalently, the system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= g(x, y)
\end{align*}
\]

is **reversible** if, given any solution \((x(t), y(t))\), the function

\[ (x^\flat(t), y^\flat(t)) = (x(-t), -y(-t)) \]

is another solution. This is true provided that \( g \) is an *even* function of \( y \), namely

\[ g(x, y) = g(x, -y). \]

11 Equilibrium points

**Equilibria** are points \( P^* = (x^*, y^*) \) such that \( f(x^*, y^*) = g(x^*, y^*) = 0 \). These correspond to solutions constant in time.
To determine the stability of an equilibrium point \( P^* = (x^*, y^*) \), we compute the matrix of partial derivatives

\[
A = \begin{pmatrix}
\frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\
\frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*)
\end{pmatrix}.
\]  

(11.1)

Calling \( X = x - x^* \), \( Y = y - y^* \), we obtain the Taylor approximation

\[
\begin{pmatrix}
\dot{X} \\
\dot{Y}
\end{pmatrix} = A \begin{pmatrix}
X \\
Y
\end{pmatrix} + \text{higher order terms}
\]

Looking at this linear approximation, we obtain information about the stability of the point \( P^* \) for the original nonlinear system (1).

**Theorem (Stability by linearization).** Let \( P^* = (x^*, y^*) \) be an equilibrium point, and let \( A \) be the matrix in (11.1).

- If both eigenvalues of \( A \) have strictly negative real part, then \( P^* \) is asymptotically stable.
- If at least one of the eigenvalues of \( A \) has strictly positive real part, then \( P^* \) is unstable.

In some cases, the linearization does not provide conclusive information about stability. In alternative one can use

**Theorem (Lyapunov stability).** Let \( P^* = (x^*, y^*) \) be an equilibrium point for the system (1). Assume that there exists a continuously differentiable function \( V(x, y) \) such that

(i) \( V \) has a strict local minimum at \( P^* \),

(ii) \( \frac{d}{dt} V(x(t), y(t)) \leq 0 \) along every trajectory.

Then \( P^* \) is a stable equilibrium.

12 Periodic orbits

A non-constant solution of the ODE

\[
\dot{x} = f(x)
\]  

(12.1)

is **periodic** if there exists a period \( T \) such that \( x(t + T) = x(t) \) for every \( t \).

12.1 Non-existence of periodic orbits

Given the planar system (8.1), if there exists a function \( V(x, y) \) such that

\[
\frac{d}{dt} V(x(t), y(t)) < 0
\]
Figure 16: Proving the Lyapunov stability theorem. For every constant \( c \), the curve where \( V(x, y) = c \) acts as a barrier. Trajectories that are inside the shaded region where \( V < c \) cannot get outside.

for every non-constant solution, then there cannot be any periodic orbit.

**Example.** If we can find a function \( V(x, y) \) such that \( \left( \frac{f}{g} \right) = -\nabla V \), then the system (1) is called a **gradient system**. This means

\[
\begin{align*}
    f(x, y) &= -\frac{\partial}{\partial x} V(x, y), \\
    g(x, y) &= -\frac{\partial}{\partial y} V(x, y).
\end{align*}
\]

In this case, for every non-constant trajectory we have

\[
\frac{d}{dt} V(x(t), y(t)) = \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial y} \dot{y} = -\left( \frac{\partial V}{\partial x} \right)^2 - \left( \frac{\partial V}{\partial y} \right)^2 < 0.
\]

As a consequence, a gradient system cannot have any periodic orbit.

**12.2 Existence of periodic orbits**

A domain \( \Omega \subset \mathbb{R}^2 \) is **positively invariant** for the ODE (6) if every trajectory that starts in \( \Omega \) remains inside \( \Omega \) for all positive times. In other words,

\[
x(0) \in \Omega \quad \implies \quad x(t) \in \Omega \quad \text{for all } t \geq 0.
\]
Theorem (Poincaré-Bendixon). Assume that

(i) The vector field \( f \) is continuously differentiable.

(ii) \( \Omega \subset \mathbb{R}^2 \) is a closed, bounded domain, which is positively invariant for the ODE (12.1).

(iii) \( \Omega \) does not contain any equilibrium point.

Then \( \Omega \) contains a periodic orbit.

**Sketch of the proof.**

STEP 1. Let \( t \mapsto x(t) \) be any trajectory contained inside \( \Omega \), for all \( t \geq 0 \).

**Definition.** The \( \omega \)-limit set of \( x(\cdot) \) is the set of all points \( \bar{x} \) such that

\[
\bar{x} = \lim_{n \to \infty} x(t_n) \quad \text{for some sequence of times} \quad t_n \to +\infty.
\] (12.2)

This set is nonempty. Indeed, choosing \( t = 1, 2, 3, \ldots \) we obtain a sequence of points \( x(1), x(2), x(3), \ldots \) all inside \( \Omega \). Since \( \Omega \) is closed and bounded, this sequence has a convergent subsequence. Therefore (12.2) holds for some subsequence of integer times \( t_1 < t_2 < t_3 < \cdots \).

STEP 2. Let now \( \bar{x} \) be a point in the \( \omega \)-limit set of \( x(\cdot) \), so that (12.2) holds. By assumption, \( f(\bar{x}) \neq 0 \). Hence we can draw a segment \( S \) through the point \( \bar{x} \) perpendicular to the velocity vector \( f(\bar{x}) \), as shown in Fig. 17.

For each \( x \in S \), we define \( \phi(x) \) to be the first point where the trajectory starting at \( x \) returns on \( S \) (Fig. 17, right). This first-return map \( \phi \) is usually called the “Poincaré map”.

Call \( t_1 \) the first time when the trajectory \( x(\cdot) \) crosses \( S \). Let \( t_2 \) be the time of the second crossing, etc... (see Fig. 17, left). A topological argument shows that the sequence \( x(t_1), x(t_2), x(t_3), \ldots \) converges monotonically to \( \bar{x} \).

We now have

\[
\phi(\bar{x}) = \lim_{n \to \infty} \phi(x(t_n)) = \lim_{n \to \infty} x(t_{n+1}) = \bar{x}.
\]

This means that the trajectory starting at \( \bar{x} \) returns exactly at the same point \( \bar{x} \) after some time. Hence it is periodic.
Two ways to use the Poincaré-Bendixon theorem:

**Method 1.** Write the system (8.1) in polar coordinates, using the identities

\[ r\dot{r} = x\dot{x} + y\dot{y}, \quad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}. \]

Then find two radii \(0 < r_1 < r_2\) so that the annular region \(\Omega = \{ r_1 \leq \sqrt{x^2 + y^2} \leq r_2 \}\) satisfies the assumptions of the theorem. For this purpose, we need to check two things:

(i) The set \(\Omega\) is positively invariant. This is true if

\[ r = r_1 \quad \implies \quad \dot{r} \geq 0 \]
\[ r = r_2 \quad \implies \quad \dot{r} \leq 0. \]

(ii) The vector field has no zeroes inside \(\Omega\). In other words: there is no point \(P^* = (r^*, \theta^*) \in \Omega\) such that \(\dot{r}(P^*) = 0\) and \(\dot{\theta}(P^*) = 0\).

**Method 2.** As a first step, find a closed, bounded set \(Q \subset \mathbb{R}^2\) (usually a rectangle, a triangle, or a trapezoid) which is positively invariant. This is true if at every boundary point the vector field \(\begin{pmatrix} f \\ g \end{pmatrix}\) is either tangent to the boundary or points toward the interior of \(Q\).

Next, check that all equilibrium points inside \(Q\) are unstable nodes or spirals (with eigenvalues having strictly positive real parts). By removing from \(Q\) a small neighborhood of each of these unstable equilibria, we then obtain a positively region \(\Omega\) which satisfies the assumptions of the theorem.
13 Index theory

Consider a continuous function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (i.e., a vector field on the plane), and a closed curve $\gamma$ in the plane. Assume that $f$ does not vanish at any point $x \in \gamma$.

The index of the closed curve $\gamma$ w.r.t. the vector field $f$ is defined as the number of counterclockwise rotations made by the unit vector $v(x) = \frac{f(x)}{|f(x)|}$, as $x$ moves one whole loop (counterclockwise) along $\gamma$.

This index is denoted as $I(f; \gamma)$. It is always an integer (positive, negative, or zero).

![Figure 19: The index of a closed curve $\gamma$ w.r.t. a vector field $f$.](image)

**Theorem.** If $I(\gamma; f) \neq 0$, then the vector field $f$ has a zero inside the region bounded by the curve $\gamma$.

Two special cases of this theorem are shown in Fig. 21.

**Corollary 1.** Assume that the domain $\Omega$ bounded by the curve $\gamma$ is positively invariant for the ODE (6). In other words, assume that at each point of $\gamma$ the vector field $f$ is either tangent or points toward the interior of $\Omega$. Then $f$ has a zero inside $\Omega$.

**Corollary 2.** Let $\gamma$ be a periodic orbit for the ODE (6). Then $f$ has a zero inside the region bounded by $\gamma$.

---

**Sketch of the proof.** Assume that $I(\gamma; f) \neq 0$ but $f(x) \neq 0$ for every point $x$ inside the region bounded by $\gamma$. We show that this leads to a contradiction.

As shown in Fig. 20, consider a family of closed curves $\gamma(s)$, depending continuously on the parameter $s \in [0, 1]$, such that $\gamma(1)$ coincides with the original curve $\gamma$, while $\gamma(s)$ shrinks to a point as $s \rightarrow 0$.

Since we are assuming that $f(x) \neq 0$, the unit vector $v(x) = \frac{f(x)}{|f(x)|}$ is well defined and varies continuously with $x$. Therefore, the map $s \mapsto I(\gamma(s); f)$ is continuous. Since it takes only integer values, this index must be constant.

However, when $\gamma(s)$ shrinks to a point, as $x$ moves along $\gamma(s)$ the unit vector $v(x)$ is almost constant, hence it cannot perform a full rotation. We thus conclude that $I(\gamma(s); f) = 0$ for $s \approx 0$.

We have thus reached a contradiction: $0 \neq I(\gamma(1); f) = I(\gamma(0); f) = 0$. □
Figure 20: Proving the Index Theorem. As \( s \) decreases from 1 to 0, the closed curve \( \gamma(s) \) shrinks to a point.

Figure 21: Left: the vector field \( f \) points always toward the interior of the region \( \Omega \) bounded by the closed polygonal curve \( \gamma \). Right: \( \gamma \) is a periodic orbit, and hence \( f \) is tangent to \( \gamma \) at every point. In both cases \( I(\gamma; f) = 1 \), and therefore \( f \) must have a zero inside the region bounded by \( \gamma \).
14 Bifurcations for nonlinear systems

A system of two ODEs depending on a parameter \( \mu \) has the form
\[
\begin{align*}
\dot{x} &= f(x, y, \mu), \\
\dot{y} &= g(x, y, \mu).
\end{align*}
\] (14.1)

We say that a **bifurcation** occurs if the phase portrait of the planar system changes, as \( \mu \) crosses a critical value \( \mu^* \). Namely

(i) new equilibrium points appear (or disappear), or else
(ii) a periodic orbit appears (or disappears).

15 Saddle-node and pitchfork bifurcations

These are bifurcations where the number of equilibrium points changes.

The values \( (x^*, y^*, \mu^*) \) where these bifurcations can occur are found by solving the system of equations
\[
\begin{align*}
f(x, y, \mu) &= 0, \\
g(x, y, \mu) &= 0, \\
\frac{\partial f}{\partial x}(x, y, \mu) \frac{\partial g}{\partial y}(x, y, \mu) - \frac{\partial f}{\partial y}(x, y, \mu) \frac{\partial g}{\partial x}(x, y, \mu) &= 0.
\end{align*}
\] (15.1)

The first two equations say that \( (x^*, y^*) \) is an equilibrium point. The third equation means that at this point the Jacobian matrix
\[
A = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}
\] (15.2)

has zero determinant, hence it is not invertible.

15.1 Saddle-node bifurcations.

As \( \mu \) crosses a critical value \( \mu^* \) a saddle and a (stable or unstable) node join together and disappear. The standard example is shown in Fig. 22:
\[
\begin{align*}
\dot{x} &= \mu - x^2, \\
\dot{y} &= -y.
\end{align*}
\]

For \( \mu > 0 \) there are two equilibrium points: a saddle at \( (-\sqrt{\mu}, 0) \) and a stable node at \( (\sqrt{\mu}, 0) \).

For \( \mu < 0 \) there is no equilibrium point.
15.2 Pitchfork bifurcations.

These typically occur at the origin and for odd systems, such that

\[ f(-x, -y) = -f(x, y), \quad g(-x, -y) = -g(x, y). \]

**Supercritical pitchfork:** As \( \mu \) crosses a critical value \( \mu^* \) a saddle and two stable nodes join together, leaving a single stable node.

The standard example is shown in Fig. 23:

\[
\begin{align*}
\dot{x} &= \mu x - x^3, \\
\dot{y} &= -y.
\end{align*}
\]

For \( \mu < 0 \) there is only one equilibrium point, namely a stable node at \((0, 0)\).

For \( \mu > 0 \) there are three equilibrium points: a stable node at \((-\sqrt{\mu}, 0)\), a saddle at \((0, 0)\), and another stable node at \((\sqrt{\mu}, 0)\).
Subcritical pitchfork: As $\mu$ crosses a critical value $\mu^*$ a saddle and two unstable nodes join together, leaving a single unstable node.

The standard example is is shown in Fig. 24:

$$\begin{cases} 
\dot{x} &= \mu x + x^3, \\
\dot{y} &= -y.
\end{cases}$$

For $\mu < 0$ there are three equilibrium points: a saddle at $(-\sqrt{\mu}, 0)$, a stable node at $(0, 0)$, and another saddle at $(\sqrt{\mu}, 0)$. For $\mu > 0$ there is only one equilibrium point, namely a saddle at $(0, 0)$.
16 Hopf bifurcations

These are bifurcations where a periodic orbit appears (or disappears).

The Hopf bifurcation occurs at an equilibrium point where

(i) The eigenvalues of the Jacobian matrix $A$ in (15.2) are purely imaginary,

(ii) As $\mu$ crosses a critical value $\mu^*$, the real part of the eigenvalues changes sign. The equilibrium point thus changes from an unstable spiral to stable spiral.

The values $(x^*, y^*, \mu^*)$ where a Hopf bifurcation can occur are found by solving the system of equations

$$
\begin{cases}
  f(x, y, \mu) = 0, \\
  g(x, y, \mu) = 0, \\
  \frac{\partial f}{\partial x}(x, y, \mu) + \frac{\partial g}{\partial y}(x, y, \mu) = 0.
\end{cases}
$$

The first two equations say that $(x^*, y^*)$ is an equilibrium point. The third equation implies that at this point the Jacobian matrix $A$ has zero trace.

16.1 Supercritical Hopf bifurcation.

As $\mu$ crosses a critical value $\mu^*$, a stable periodic orbit and an unstable equilibrium point join together, leaving a stable equilibrium point.

The standard example is shown in Fig. 25:

$$
\begin{align*}
  \dot{x} &= \mu x + y - x^3, \\
  \dot{y} &= -x + \mu y - y^3.
\end{align*}
$$

For $\mu < 0$ the origin is a stable spiral point, while for $\mu > 0$ the origin is an unstable spiral.

When $\mu = 0$, the origin is stable. Indeed, in polar coordinates we find

$$
r \dot{r} = x \dot{x} + y \dot{y} = -(x^4 + y^4) < 0.
$$

For $\mu > 0$ small, a topological argument yields the existence of a stable periodic orbit.
16.2 Subcritical Hopf bifurcation.

As $\mu$ crosses a critical value $\mu^*$, an unstable periodic orbit and a stable equilibrium point join together, leaving an unstable equilibrium point.

The standard example is shown in Fig. 26:

$$\begin{align*}
\dot{x} &= \mu x + y + x^3, \\
\dot{y} &= -x + \mu y + y^3.
\end{align*}$$

For $\mu < 0$ the origin is a stable spiral point, while for $\mu > 0$ the origin is an unstable spiral.

When $\mu = 0$, the origin is unstable. Indeed, in polar coordinates we find

$$r\dot{r} = x\dot{x} + y\dot{y} = x^4 + y^4 > 0.$$  

For $\mu < 0$ small, a topological argument yields the existence of an unstable periodic orbit.

Figure 26: A subcritical Hopf bifurcation. At the critical value $\mu^* = 0$ the origin is unstable. For $\mu < 0$ the origin is a stable spiral point and the phase portrait contains an unstable periodic orbit.
**Note:** To distinguish between supercritical and subcritical Hopf bifurcation, the key is to understand the stability of the equilibrium when $\mu = \mu^*$. In this case, the linearization always indicates that the equilibrium is a center, providing no useful information. To understand whether it is stable or not, we must look at higher order terms. Usually, this can be done writing the system in polar coordinates. Consider the ODE

$$
\frac{dr}{d\theta} = r \frac{x\dot{x} + y\dot{y}}{xy - y\dot{x}} = r \frac{x f(x, y) + y g(x, y)}{x f(x, y) - y f(x, y)}.
$$

(16.3)

with initial data

$$
r(0) = \varepsilon,
$$

with $\varepsilon > 0$ small. Check if $r(2\pi)$ is larger or smaller than $r(0)$. Depending on the direction of rotation, this yields the desired information about stability (Fig. 27).

Figure 27: Deciding the stability of the origin, using polar coordinates. Assume that, when $\varepsilon > 0$ is small enough, the solution of (16.3) satisfies $r(2\pi) > r(0) = \varepsilon$. Left: if trajectories rotate counterclockwise, then the origin is unstable. Right: if trajectories rotate clockwise, then the origin is stable.