Self-consistent Feedback Stackelberg Equibria for Infinite Horizon Stochastic Games

Alberto Bressan and Yilun Jiang

Department of Mathematics, Penn State University,
University Park, PA 16802, USA.

e-mails: axb62@psu.edu, yxj141@math.psu.edu

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Abstract

The paper introduces a concept of "self consistent" Stackelberg equilibria for stochastic games in infinite time horizon, where the two players adopt feedback strategies and have exponentially discounted costs. The analysis is focused on games in continuous time, described by a controlled Markov process with finite state space. Results on the existence and uniqueness of such solutions are provided. As an intermediate step, a detailed description of the structure of the best reply map is achieved, in a "generic" setting. Namely: for all games where the cost functions and the transition coefficients of the Markov chain lie in open dense subset of a suitable space $\mathcal{C}^k$. Under generic assumptions, we prove that a self-consistent Stackelberg equilibrium exists, provided that either (i) the leader is far-sighted, i.e. his exponential discount factor is sufficiently small, or (ii) the follower is narrow-sighted, i.e. his discount factor is large enough.

1 Introduction

Because of their usefulness in economic modeling, Stackelberg games have been the topic of several investigations [2, 3, 4, 6, 9, 12, 13, 14]. In the present paper we consider a stochastic game for two players in infinite time horizon, with exponentially discounted costs. Our main focus will be on Stackelberg equilibrium solutions in feedback form. The state of the system is denoted by $x \in X$, and can take either discrete or continuum values. We assume that the two players (the leader and the follower, respectively) choose time-independent feedback strategies

$$u_1(x) \in U_1\,, \qquad u_2(x) \in U_2,$$

which affect the random evolution of the system. Two main settings can be considered:

1 - A discrete state space. Here $X = \{1, 2, 3, \ldots, N\}$. Given the feedback controls $u_k : X \mapsto U_k$, $k = 1, 2$, we assume that the evolution is described by a Markov process in continuous time, with transition probabilities

$$\text{Prob.}\Big\{x(t+\varepsilon)=j \;\Big|\; x(t)=i\Big\} \;=\; \phi_{ij}(u_1(i),u_2(i))\,\varepsilon+o(\varepsilon), \qquad\qquad j\neq i, \tag{1.1}$$

for some given functions $\phi_{ij}\geq 0$.

2 - A continuum state space. In this case $X=\mathbb{R}^d$. Given the controls u_1,u_2, we assume that the evolution is described by a diffusion process of the form

$$dx \;=\; f\big(x,u_1(x),u_2(x)\big)\,dt+\sigma(x)\,dW, \tag{1.2}$$

where f is a smooth function and W denotes standard Brownian motion [10, 11].

In both cases, the expected costs to the two players take the form

$$J_k \;\doteq\; E^{\mu}\left[\int_0^{+\infty} r_k\,e^{-r_k t}L_k\Big(x(t),u_1(x(t)),u_2(x(t))\Big)\,dt\right], \qquad\qquad k=1,2. \tag{1.3}$$

Here E^{μ} denotes the expectation for a given probability measure μ on the initial state $x(0)$. Moreover, r_1,r_2 are the exponential discount factors.

We now review the concept of **Stackelberg equilibrium solution**, based on the notion of **best reply**. This covers both cases of discrete and continuum state space.

Definition 1.1 *We say that $u_2^*:X\mapsto U_2$ is a* **best reply** *for Player 2 to a strategy $u_1^*:X\mapsto U_1$ implemented by Player 1 if, for every initial datum x_0, the control u_2^* is an optimal feedback for the following stochastic optimization problem.*

$$\textit{Minimize:}\quad E^{x_0}\left[\int_0^{+\infty} r_2e^{-r_2t}L_2\big(x(t),u_1^*(x(t)),u_2(t)\big)\,dt\right], \tag{1.4}$$

subject to the dynamics (1.1) or (1.2), with $u_1=u_1^*(x)$ and with initial condition

$$x(0) \;=\; x_0. \tag{1.5}$$

The set of best replies for Player 2 will be denoted by $\mathcal{R}_2(u_1^*)$.

Note that Player 2 solves a stochastic optimal control problem. The same feedback control u_2^* will thus be optimal simultaneously for every initial datum $x_0\in X$, and hence also for every probability distribution μ on the initial data.

Definition 1.2 *We say that a pair of functions $u_1^*:X\mapsto U_1$, $u_2^*:X\mapsto U_2$ is a* **feedback Stackelberg equilibrium** *for the game with dynamics (1.1) or (1.2), cost functions (1.3), and probability distribution μ on the initial data, if the following holds.*

(i) $u_2^*\in\mathcal{R}_2(u_1^*)$.

(ii) For every other feedback control $u_1^\sharp:X\mapsto U_1$ of the leader, and every optimal reply $u_2^\sharp\in\mathcal{R}_2(u_1^\sharp)$ of the follower, one has

$$E^{\mu}\left[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u_1^*(x(t)), u_2^*(t)\big)\, dt\right] \;\leq\; E^{\mu}\left[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u_1^\sharp(x(t)), u_2^\sharp(t)\big)\, dt\right]. \tag{1.6}$$

Notice that here we assume that the control of the leader $u_1 = u_1(x)$ is assigned only as a function of the state x. This is very different from a feedback $u_1 = u_1(x, u_2(x))$ which also depends on the instantaneous control chosen by the follower. For alternative concepts of Stackelberg equilibria, see for example [3, 4].

The main issue motivating the present paper is the dependence of the Stackelberg equilibrium on the initial probability distribution μ. In the case of an optimal control problem, or a Nash equilibrium to a differential game, the same feedback solution remains valid for every initial datum. However, this is no longer true for Stackelberg equilibria, which are not "time consistent", in general. In other words, if the leader were allowed to restart the game at time $\tau > 0$ from the state $x(\tau) \neq x_0$, he would likely choose a feedback control $u = \widehat{u}_1(x) \neq u_1^*(x)$, different from the original one.

A typical example occurs in a bank regulation problem. At a time when the overall banking system is healthy, the government (the leading player) announces a policy of "no intervention", ruling out any bailout. This should discourage bank managers (the followers) from taking excessive risks to boost their profits. However, if at a later time some large banks are at risk of bankruptcy, the government may wish to reverse its own policy and provide a bailout.

Aim of the present paper is to propose a concept of feedback equilibrium which does not make reference to any initial probability distribution, but depends only on the dynamics of the stochastic system and the cost functions to the players.

Definition 1.3 *Consider the stochastic game with dynamics (1.1) or (1.2), and cost functions (1.3). We say that a pair of functions (u_1^*, u_2^*), with $u_k^* : X \mapsto U_k$, $k = 1, 2$, is a* **self-consistent Stackelberg equilibrium in feedback form** *if the conditions (i)-(ii) in Definition 1.2 hold, for some probability μ on the set of initial data which is* invariant *for the Markov process (1.1) with transition intensities*

$$\phi_{ij}\big(u_1^*(i), u_2^*(i)\big), \tag{1.7}$$

or, respectively, for the diffusion process

$$dx \;=\; f\big(x, u_1^*(x), u_2^*(x)\big)\, dt + \sigma(x)\, dW. \tag{1.8}$$

Remarks. 1. The above definition can model a situation where the leading player (a far-sighted legislator) either does not know or is not much concerned with the present state of the system. Rather, he wants to draft a regulation which, after a short transient period, will be recognized as "best possible" at future times.

2. Definition 1.3 could be strengthened by requiring that the measure μ is the unique, asymptotically stable invariant probability distribution for the Markov process (1.1) with transition intensities (1.7), or for the diffusion process (1.8), respectively.

3. An alternative interpretation goes as follows. Consider a government (the leading player) announcing a legislation u_1^* which affects a large number of agents (industrial companies, financial institutions, etc.). If each of these agents acts independently, implementing a best reply u_2^*, the random states of these agents will evolve in time according to (1.7) or (1.8). Call μ^t the random distribution of these agents at time t, and assume that, as $t \to +\infty$, one has the asymptotic convergence $\mu^t \to \mu$. If the leading player is a far-sighted legislator, he can choose a feedback u_1^* which is optimal w.r.t. the asymptotic probability distribution μ. This feedback may not be optimal at the initial time $t = 0$, but it will get closer and closer to an optimal one as $t \to +\infty$.

4. Up to this stage, we tacitly assumed that the feedback strategies could be arbitrary measurable functions $u_k : X \mapsto U_k$, as long as the stochastic evolution equations (1.1) are well defined.

This assumption is entirely appropriate for $u_2^*(\cdot)$, which is the optimal feedback reply of the follower and can thus be determined by solving an equation of Hamilton-Jacobi type. On the other hand, if $u_1(\cdot)$ models a strategy publicly announced by a legislator, one may wish to restrict the choice of u_1 within a finite-dimensional set $\mathcal{U}_1$ of functions which can be more easily implemented in practice. For example, think of a central bank who announces the future values $u_1 = u_1(x)$ of the prime rate as a function of leading economic indicators $x = (x_1, \ldots, x_d)$. In a realistic setting, this function $x \mapsto u_1(x)$ must have a very simple structure. Say, piecewise constant or piecewise affine, with a small number of jumps.

If the feedback control $u_1(\cdot)$ is restricted to a small subset $\mathcal{U}_1$ of all measurable maps $X \mapsto U_1$, the concept of feedback Stackelberg equilibrium (u_1^*, u_2^*) remains valid with one obvious modification. Namely, in Definition 1.2 one should now require that $u_1^* \in \mathcal{U}_1$, and moreover (1.6) holds for every feedback control $u_1^\sharp \in \mathcal{U}_1$. Our Definition 1.3 of self-consistent equilibrium can be modified accordingly.

In the remainder of this paper we focus on the model (1.1) with discrete state space. Self-consistent equilibria for the diffusion process (1.2) will be studied in a separate paper.

Section 2 reviews some basic properties of the best reply map and of Stackelberg equilibria. Toward the existence of a self-consistent equilibrium, we consider the composite map

$$\mu \;\mapsto\; (u_1^*, u_2^*) \;\mapsto\; \mu^\infty\,, \tag{1.9}$$

where

- μ is a probability distribution on the initial state,
- (u_1^*, u_2^*) is a Stackelberg equilibrium solution in feedback form, relative to the initial distribution μ,
- μ^∞ is the asymptotic probability distribution as $t \to +\infty$ for the dynamical system (1.1), with feedback controls u_1^*, u_2^*.

According to Definition 1.3, (u_1^*, u_2^*) is a self-consistent Stackelberg equilibrium if $\mu^\infty = \mu$. To prove the existence of self-consistent equilibrium, under suitable assumptions we will show that the composite map in (1.9) is single-valued and has a fixed point. Three results in this direction will be provided.

In Section 3 we derive conditions that ensure that the best reply map $u_1 \mapsto \mathcal{R}_2(u_1)$ as well as the map $\mu \mapsto (u_1^*, u_2^*)$ in (1.9) are always single-valued. The existence of a fixed point for (1.9), stated in Theorem 3.1, is then a consequence of Brouwer's theorem. Unfortunately, this approach requires quite restrictive convexity assumptions, which are not always easy to check.

The analysis in the remaining sections follows an entirely different direction. Namely, we seek results on the existence of a self-consistent Stackelberg equilibrium which are valid for a *generic game*. That means: our results hold true for all triples (L_1, L_2, ϕ) of cost functions and transition functions, in an open dense set of a suitable space of continuously differentiable maps.

Our analysis shows that composite map (1.9) is a strict contraction, and hence has unique fixed point, in two main cases:

(i) The follower is narrow-sighted. Namely, the exponential discount factor r_2 is sufficiently large.

(ii) The best reply for the follower is single-valued, and the leader is far-sighted Namely, the discount factor r_1 is sufficiently small.

As an intermediate step, in Section 4 we first consider a one-shot game where the leader and the follower choose controls $x, y \in [0,1]^N$, while the cost functions have the form

$$F(x,y), \qquad\qquad G_\varepsilon(x,y) \;=\; \sum_{i=1}^{N} G_i(x_i,y_i) + \varepsilon\widetilde{G}(x,y), \tag{1.10}$$

respectively. Here we assume $F \in \mathcal{C}^2(\mathbb{R}^{2N})$, $G_i \in \mathcal{C}^3(\mathbb{R}^2)$, $\widetilde{G} \in \mathcal{C}^3(\mathbb{R}^{2N})$. Building upon the analysis in [7], we provide a detailed description of the best reply map for the follower, first for $\varepsilon = 0$, then for $\varepsilon > 0$ small, valid in a generic setting. Namely, our results apply to cost functions $G_i, \widetilde{G}$ in an open dense subset of $\mathcal{C}^3$. In all these cases, the Stackelberg equilibrium is unique and is stable w.r.t. small perturbations of the cost functions. These results, and the techniques developed to achieve them, can have independent interest.

In Section 5 the previous analysis is applied to the case of feedback equilibria stochastic game with dynamics (1.1) and cost functions (1.3). The case of a myopic follower corresponds to the one-shot game (1.10) with $\varepsilon = 0$, while $\varepsilon = r_2^{-1} > 0$ small corresponds to the case of a narrow-sighted follower. Under generic assumptions on the cost functions L_1, L_2 in (1.3), and on the transition functions ϕ_{ij} in (1.1), we prove that the map (1.9) is single-valued. Indeed, for $r_2 >> 1$ large enough, this map is a strict contraction. Its fixed point yields the unique self-consistent Stackelberg equilibrium.

Finally, in Section 6 we consider a far-sighted leader. Namely, we take discount factors $r_2 > 0$ arbitrary but $r_1 > 0$ suitably small. Assuming that the best reply map of the follower is single-valued, we show that, under generic conditions on the cost L_1 to the leader, a unique self-consistent Stackelberg equilibrium exists. Again, the proof is achieved by showing that the composed map in (1.9) is a strict contraction.

All our main results apply to games with *generic* dynamics and cost functions. Roughly speaking, they apply to "almost all" games, in a topological sense. As our analysis shows, by removing a small subset of possible cases, much stronger results can then be proved.

On the other hand, we should point out that there can be models of practical significance which do not satisfy our generic assumptions. For example, for generic transition functions ϕ_{ij}, any choice of the controls u_1, u_2 yields a Markov process (1.1) which is recurrent: starting from any state $i \in \{1, \ldots, N\}$ one can reach any other state j with positive probability. This condition is not satisfied whenever an absorbing state is present. For example, in a financial model, bankruptcy can be an irreversible state. This setting would not be covered by our results. There is indeed a trade-off between the quality of the results and the range of models to whom they apply.

2 Feedback equilibria with discrete state space

In this section we consider a continuous-time Markov process with finitely many states $i \in \{1, ..., N\}$ and transition functions ϕ_{ij} as in (1.1). To simplify our notation, from now on we denote by u, v the controls of the leader and of the follower, respectively. We assume that these controls take values in the sets $U_1 = U_2 = [0,1]$. A pair of feedbacks controls $u, v : \{1, \ldots, N\} \mapsto [0,1]$ can thus be identified with two vectors

$$u \;=\; (u_1, \ldots, u_N), \qquad v \;=\; (v_1, \ldots, v_N). \tag{2.1}$$

For any given pair of feedback controls $(u, v) \in [0,1]^N \times [0,1]^N$ and any initial state $x(0) = i \in \{1, \ldots, N\}$, the expected cost $J_{2,i}$ to Player 2 is

$$J_{2,i} \;\doteq\; E^{\{x(0)=i\}} \left[\int_0^{+\infty} r_2 e^{-r_2 t} L_2\Big(x(t), u_{x(t)}, v_{x(t)}\Big)\, dt \right]. \tag{2.2}$$

To shorten the notation, from now on we shall write

$$L_{2,i} \;=\; L_2(i, u_i, v_i), \qquad \phi_{ij} \;=\; \phi_{ij}(u_i, v_i), \qquad \phi_i \;=\; \sum_{j \neq i} \phi_{ij}\,.$$

Calling τ the first time when the system jumps from state i to some other state, we have the identities

$$\begin{aligned} J_{2,i} \;&=\; \int_0^\infty \left(\int_0^\tau r_2 e^{-r_2 t} L_{2,i}\, dt + e^{-r_2\tau} \sum_{j\neq i} \frac{\phi_{ij}}{\phi_i} J_{2,j} \right) \phi_i e^{-\phi_i \tau}\, d\tau \\ &=\; \int_0^\infty \left((1 - e^{-r_2\tau}) L_{2,i} + e^{-r_2\tau} \sum_{j\neq i} \frac{\phi_{ij}}{\phi_i} J_{2,j} \right) \phi_i e^{-\phi_i \tau}\, d\tau \\ &=\; L_{2,i} + \frac{1}{\phi_i + r_2} \left(\sum_{j\neq i} \phi_{ij} J_{2,j} - \phi_i L_{2,i} \right), \end{aligned} \tag{2.3}$$

$$J_{2,i} - \sum_{j\neq i} \frac{\phi_{ij}}{\phi_i + r_2} J_{2,j} \;=\; \frac{r_2}{\phi_i + r_2} L_{2,i}\,, \tag{2.4}$$

or equivalently:

$$J_{2,i} \;=\; L_{2,i} + \frac{1}{r_2} \sum_{j\neq i} \phi_{ij} (J_{2,j} - J_{2,i}). \tag{2.5}$$

From (2.5) it follows the limit

$$\lim_{r_2 \to +\infty} J_{2,i} \;=\; L_{2,i}\,. \tag{2.6}$$

Here and in the sequel, it will be convenient to adopt a vector notation and write

$$\mathbf{J_k} \;\doteq\; (J_{k,1}, \ldots, J_{k,N})^T, \qquad \mathbf{L_k} \doteq (L_{k,1}, \ldots, L_{k,N})^T, \qquad k = 1, 2. \tag{2.7}$$

Notice that (2.5) is a system of linear equations for the components of the column vector $\mathbf{J_2}$, namely

$$\Big(\mathbf{I} - \frac{1}{r_2}\mathbf{\Phi}\Big)\mathbf{J_2} \;=\; \mathbf{L_2}, \tag{2.8}$$

where $\mathbf{\Phi}$ is the matrix with components

$$\Phi_{ij} \;=\; \begin{cases} -\phi_i & \text{if} \quad i = j, \\ \phi_{ij} & \text{if} \quad i \neq j. \end{cases} \tag{2.9}$$

Notice that $\mathbf{\Phi}$ is the generator matrix of the continuous-time finite-state Markov process at (1.1).

Remark 2.1 For any given feedback control vectors u, v, if the continuous-time Markov chain is irreducible and positive recurrent, there exists a unique probability distribution

$$\mu^\infty \;\in\; \Delta_N \;\doteq\; \Big\{(p_1, p_2, ..., p_N)\,;\;\; p_i \geq 0,\;\; \sum p_i = 1\Big\}, \tag{2.10}$$

which is stationary, i.e. it satisfies $\mu^\infty \cdot \mathbf{\Phi} = \mathbf{0}$. For the basic theory of continuous-time Markov chains we refer to [16].

The uniqueness of the stationary distribution μ^∞ is a prerequisite for our analysis. Throughout the paper we thus make the following assumption:

(IR) *For any fixed pair of controls $(u, v) \in [0,1]^N \times [0,1]^N$, the continuous Markov chain defined in (1.1) is irreducible and positive recurrent.*

For any feedback control $u \in [0,1]^N$ chosen by the leader, the value function of the follower is defined as

$$V_2(i) \;=\; \min_v \; E^{\{x(0)=i\}}\left[\int_0^{+\infty} r_2\, e^{-r_2 t} L_2\big(x(t), u(x(t)), v(t)\big)\,dt\right]. \tag{2.11}$$

By a standard argument, V_2 satisfies the dynamic programming equation

$$V_2(i) \;=\; \min_{\omega \in [0,1]} \left\{ L_2(i, u_i, \omega) + \frac{1}{r_2}\sum_{j \neq i} \phi_{ij}(u_i, \omega)\Big(V_2(j) - V_2(i)\Big)\right\} \tag{2.12}$$

for every $i \in \{1, ..., N\}$. Without loss of generality, the minimization in (2.11) can be restricted to optimal controls of feedback type: $v \in [0,1]^N$. A feedback control v^* is optimal if it satisfies

$$v_i^* \;=\; \arg\min_{\omega \in [0,1]} \left\{ L_2(i, u_i, \omega) + \frac{1}{r_2}\sum_{j \neq i} \phi_{ij}(u_i, \omega)\Big(V_2(j) - V_2(i)\Big)\right\} \tag{2.13}$$

for every $i = 1, \dots, N$. Notice that in this case the same feedback control v^* is optimal for every probability distribution $\mu \in \Delta_N$ on the initial data. We denote by $\mathcal{R}_2(u) \subseteq [0,1]^N$ the set of feedback controls v^* for the follower which satisfy (2.13). These are the *best replies* for the follower to the strategy u chosen by the leader.

Remark 2.2 Assume that the Markov chain is reducible. For example, starting from state 1 one can reach states 2,3, but never state 4. Then, if the initial state is $x(0) \in \{1, 2, 3\}$, we can arbitrarily choose the control v_4 and still obtain a best reply for the follower. These kind of controls are here ruled out, if they do not satisfy (2.13).

Lemma 2.1 *Consider a game with dynamics (1.1) and cost functions (1.3), where the functions ϕ_{ij} and L_2 are continuous. Then the best reply map $u \mapsto \mathcal{R}_2(u)$ is a multivalued function with closed graph.*

Proof. We first observe that the set $\mathcal{R}_2(u)$ of best replies is independent of the probability distribution μ on the initial state. Indeed, for any given feedback control u of the leader, the best reply solves a standard optimal control problem for the follower.

Now consider a sequence of feedback controls $(u^n)_{n\geq 1}$ for the leading player, with $u^n \to u \in [0,1]^N$, as $n \to \infty$. Calling V_2^n, V_2 the corresponding value functions, we then have the convergence $V_2^n(i) \to V_2(i)$ for all $i \in \{1, \dots, N\}$.

Assume that $v^n \in \mathcal{R}_2(u^n)$ is a best reply, for every $n \geq 1$. Moreover, assume that $v_i^n \to v_i$ for every component $i \in \{1, \dots, N\}$ as $n \to \infty$. Then the assumption

$$v_i^n \;=\; \arg\min_{\omega\in[0,1]} \left\{ L_2(i, u_i^n, \omega) + \frac{1}{r_2} \sum_{j\neq i} \phi_{ij}(u_i^n, \omega)\Big(V_2^n(j) - V_2^n(i)\Big) \right\}$$

by continuity implies

$$v_i \;=\; \arg\min_{\omega\in[0,1]} \left\{ L_2(i, u_i, \omega) + \frac{1}{r_2} \sum_{j\neq i} \phi_{ij}(u_i, \omega)\Big(V_2(j) - V_2(i)\Big) \right\},$$

showing that the graph of $\mathcal{R}_2$ is closed. □

Lemma 2.2 *Consider a game with dynamics (1.1) and cost functions (1.3). Assume that:*

(i) Every transition function ϕ_{ij} is affine w.r.t. v_i. Namely, it has the form

$$\phi_{ij}(u_i, v_i) \;=\; a_{ij}(u_i) + b_{ij}(u_i)v_i\,, \qquad \textit{with} \;\; a_{ij}, b_{ij} \in \mathcal{C}^3(\mathbb{R}).$$

(ii) For every fixed $i \in \{1, \dots, N\}$, and $u_i \in [0,1]$, the function $L_2(i, u_i, \cdot)$ is strictly convex in the variable v_i.

Then for every $u \in [0,1]^N$, the set of best replies $\mathcal{R}_2(u)$ is a singleton.

Proof. Under the above assumptions (i)-(ii), for every i the map

$$\omega \;\mapsto\; L_2(i,u_i,\omega)+\frac{1}{r_2}\sum_{j\neq i}\phi_{ij}(u_i,\omega)\Big(V_2(j)-V_2(i)\Big) \tag{2.14}$$

is strictly convex, being the sum of a strictly convex and an affine function. Its minimum is thus attained at a single point $v_i^* \in [0,1]$. □

Lemma 2.3 *In addition to the assumptions of Lemma 2.2, assume that the cost functions $L_2(i,\cdot,\cdot)$ are $\mathcal{C}^3$ and satisfy*

$$\lim_{\omega\to 0+} L_2(i,u_i,\omega) \;=\; \lim_{\omega\to 1-} L_2(i,u_i,\omega) \;=\; +\infty. \tag{2.15}$$

Then the best reply map $u\mapsto \mathcal{R}_2(u)$ is single-valued and has $\mathcal{C}^2$ regularity.

Proof. 1. Given $u\in[0,1]^N$, let $v^*(u)$ be the unique best reply of the follower. Under the assumptions (i)-(ii) in Lemma 2.2, by (2.15), the minimizer of (2.14) cannot be achieved at the boundary of $[0,1]$, i.e $v_i^*(u)\notin\{0,1\}$. As a result, $v^*(u)$ remains strictly in the interior of $[0,1]^N$.

2. It remains to show that the best reply map $u\mapsto v^*(u)$ has $\mathcal{C}^2$ regularity. To fix the ideas, consider the initial distribution $\mu=(\frac{1}{N},\ldots,\frac{1}{N})\in\Delta_N$. Then the cost function for the follower can be expressed as

$$G(u,v)\doteq \mu^T\cdot \mathbf{C}^{-1}(u,v)\cdot \mathbf{L}_2(u,v),$$

where

$$\mathbf{C}(u,v) \;\doteq\; \mathbf{I}-\frac{1}{r_2}\mathbf{\Phi}(u,v), \tag{2.16}$$

and $\mathbf{\Phi}$ is the matrix at (2.9). For any fixed u, the feedback control of the follower is independent of the probability distribution μ on the initial state. By the necessary conditions for optimality, the unique minimizer $v^*(u)$ for the map $G(u,\cdot)$ is characterized by the equation

$$\nabla_v G(u,v^*(u)) \;=\; 0. \tag{2.17}$$

The regularity assumptions on L_2 and ϕ imply that $G\in\mathcal{C}^3$. By the implicit function theorem, to prove that the map $u\to v^*(u)$ has $\mathcal{C}^2$ regularity it thus suffices to check that the Hessian matrix D_v^2G of second order derivatives of G w.r.t. $v=(v_1,\ldots,v_N)$ is everywhere strictly positive definite. Toward this goal, observe that for every $i\in\{1,\ldots,N\}$ we have the implication

$$[\mathbf{C}^{-1}\mathbf{L}_2]_{v_i} \;=\; \mathbf{0} \qquad\Longrightarrow\qquad [\mathbf{C}^{-1}]_{v_i}\mathbf{L}_2 \;=\; -\mathbf{C}^{-1}\mathbf{L}_{2,v_i}\;.$$

We now compute

$$[\mathbf{C}^{-1}]_{v_i} \;=\; \mathbf{C}^{-1}\mathbf{C}_{v_i}\mathbf{C}^{-1}$$

where, by (2.16) and (2.9), the matrix $\mathbf{C}_{v_i}$ has the form

$$[\mathbf{C}_{v_i}]_{j,k}=\begin{cases} 0 & \text{if} \quad j\neq i,\\[2mm] -\dfrac{1}{r_2}\partial_{v_i}\phi_{ik} & \text{if} \quad j=i, k\neq i,\\[2mm] \dfrac{1}{r_2}\displaystyle\sum_{p\neq i}\partial_{v_i}\phi_{ip} & \text{if} \quad j=k=i.\end{cases}$$

Since ϕ_{ij} is affine w.r.t. v_i, all second derivatives vanish: $\mathbf{C}_{v_iv_j} = \mathbf{0}$. At the point $(u, v^*(u))$, the second order derivatives can thus be computed as

$$\begin{aligned}
[\mathbf{C}^{-1}\mathbf{L}_2]_{v_i,v_j} &= [\mathbf{C}^{-1}]_{v_i,v_j}\mathbf{L}_2 + [\mathbf{C}^{-1}]_{v_i}\mathbf{L}_{2,v_j} + [\mathbf{C}^{-1}]_{v_j}\mathbf{L}_{2,v_i} + \mathbf{C}^{-1}\mathbf{L}_{2,v_iv_j} \\
&= [\mathbf{C}^{-1}]_{v_j}\mathbf{C}_{v_i}\mathbf{C}^{-1}\mathbf{L}_2 + \mathbf{C}^{-1}\mathbf{C}_{v_i}[\mathbf{C}^{-1}]_{v_j}\mathbf{L}_2 + [\mathbf{C}^{-1}]_{v_i}\mathbf{L}_{2,v_j} + [\mathbf{C}^{-1}]_{v_j}\mathbf{L}_{2,v_i} + \mathbf{C}^{-1}\mathbf{L}_{2,v_iv_j} \\
&= -\mathbf{C}^{-1}\mathbf{C}_{v_j}\mathbf{C}^{-1}\mathbf{L}_{2,v_i} - \mathbf{C}^{-1}\mathbf{C}_{v_i}\mathbf{C}^{-1}\mathbf{L}_{2,v_j} + \mathbf{C}^{-1}\mathbf{C}_{v_i}\mathbf{C}^{-1}\mathbf{L}_{2,v_j} \\
&\qquad + \mathbf{C}^{-1}\mathbf{C}_{v_j}\mathbf{C}^{-1}\mathbf{L}_{2,v_i} + \mathbf{C}^{-1}\mathbf{L}_{2,v_iv_j} \\
&= \mathbf{C}^{-1}\mathbf{L}_{2,v_iv_j}\,.
\end{aligned} \tag{2.18}$$

By observing that

$$\mathbf{L}_{2,v_iv_j} = \begin{cases} (0, \ \dots \ ,0) & \text{if} \quad i \neq j, \\ \left(0, \ \dots, \ 0, \ \frac{\partial^2}{\partial v_i \partial v_i} L_2(i, u_i, v_i), \ 0, \ \dots, \ 0\right)^T & \text{if} \quad i = j, \end{cases}$$

we conclude that the Hessian matrix of the cost function G with respect to v at the point $(u, v^*(u))$ is diagonal. Namely,

$$D_v^2 G(u, v^*(u)) \;=\; \mathrm{diag}(d_1(u), d_2(u) \dots, d_n(u)),$$

where the diagonal element $d_i(u)$ is defined as

$$d_i(u) \;=\; \frac{\partial^2}{\partial v_i^2} L_2(i, u_i, v_i^*(u)) \cdot \frac{1}{N} \sum_{j=1}^{N} [\mathbf{C}^{-1}]_{j,i}\,.$$

Looking at form (2.16) we see that, for any $r_2 > 0$, the matrix $\mathbf{C}$ is strictly diagonally dominant with positive diagonal elements and nonpositive off-diagonal elements. Hence $\mathbf{C}$ is a nonsingular M-matrix [15]. In turn, this implies that $\mathbf{C}^{-1}$ has nonnegative elements and the thus $\sum_{j=1}^{N} [\mathbf{C}^{-1}]_{j,i} > 0$ for all $i \in \{1, \dots, N\}$.

On the other hand, since $L_2(i, u_i, v_i)$ is strictly convex with respect to v_i, $D_v^2 G(u, v^*(u))$ is strictly positive definite. This completes the proof. □

Remark 2.3 In the above lemmas, for simplicity the controls u_i, v_i were assumed to take values in $[0, 1]$. The same results clearly remain valid more generally if they take values in compact convex sets $U_1, U_2 \subset \mathbb{R}^d$, and $L_2(i, u_i, \omega) \to +\infty$ as ω approaches the boundary of U_2.

We conclude this section by briefly reviewing the existence of Stackelberg equilibrium solutions, for a game with finite state space $X = \{1, \dots, N\}$. For the basic theory of multifunctions we refer to [1] or the Appendix in [5].

Theorem 2.1 *Consider the game with dynamics (1.1) and exponentially discounted payoffs (1.3). For every probability distribution μ on the initial state, there exists a Stackelberg equilibrium in feedback form. Calling $\mathcal{S}(\mu)$ the set of all these equilibria, the multifunction $\mu \mapsto \mathcal{S}(\mu)$ is upper semicontinuous with compact values.*

Proof. 1. Let an initial probability distribution $\mu \in \Delta_N$ as in (2.10) be given. To construct a Stackelberg equilibrium, consider a sequence (u^n, v^n) with $v^n \in \mathcal{R}_2(u^n)$ for all $n \geq 1$, and assume

$$\begin{aligned} &\lim_{n\to\infty} E^\mu\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u^n(x(t)), v^n(x(t))\big)\, dt\Big] \\ &\quad = \inf_u \inf_{v\in\mathcal{R}_2(u)} E^\mu\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u(x(t)), v(x(t))\big)\, dt\Big]. \end{aligned}$$

By possibly taking a subsequence we can assume $(u^n, v^n) \to (u^*, v^*) \in [0,1]^{2N}$. The upper semicontinuity of the multifunction $\mathcal{R}_2$, proved in Lemma 2.1, now implies $v^* \in \mathcal{R}_2(u^*)$. Moreover the continuity of the expected value yields

$$\begin{aligned} &E^\mu\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u^*(x(t)), v^*(x(t))\big)\, dt\Big] \\ &\quad = \inf_u \inf_{v\in\mathcal{R}_2(u)} E^\mu\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u(x(t)), v(x(t))\big)\, dt\Big]. \end{aligned}$$

Hence (u^*, v^*) provides a Stackelberg equilibrium.

2. We now prove the upper semicontinuity of the map $\mu \mapsto \mathcal{S}(\mu)$, and show that it takes compact values. Consider any convergent sequence $(\mu^n, u^n, v^n) \to (\mu^*, u^*, v^*)$ where $\mu^n \in \Delta_N, (u^n, v^n) \in \mathcal{S}(\mu^n)$. Notice that here we again use the upper semicontinuity of $\mathcal{R}_2$ to guarantee $v^* \in \mathcal{R}_2(u^*)$. By the definition of $\mathcal{S}(\mu)$, we have

$$\begin{aligned} &E^{\mu^n}\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u^n(x(t)), v^n(x(t))\big)\, dt\Big] \\ &\quad = \inf_u \inf_{v\in\mathcal{R}_2(u)} E^{\mu^n}\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u(x(t)), v(x(t))\big)\, dt\Big]. \end{aligned}$$

Since the expected cost J_1 defined in (1.3) is continuous with respect to μ, u, v, we achieve

$$\begin{aligned} &E^{\mu^*}\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u^*(x(t)), v^*(x(t))\big)\, dt\Big] \\ &\quad = \inf_u \inf_{v\in\mathcal{R}_2(u)} E^{\mu^*}\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u(x(t)), v(x(t))\big)\, dt\Big]. \end{aligned}$$

Hence the map $\mu \mapsto \mathcal{S}(\mu)$ is upper semicontinuous. For any fixed $\mu \in \Delta_N$ and any sequence $(u^n, v^n) \in \mathcal{S}(\mu)$, by possibly taking a subsequence, we can assume $(u^n, v^n) \to (u^*, v^*)$ with $v^* \in \mathcal{R}_2(u^*)$. By the same argument used in step **1**, $(u^*, v^*) \in \mathcal{S}(\mu)$. Since $(u, v) \in [0,1]^N \times [0,1]^N$ which is a bounded set, the upper semicontinuous map $\mu \mapsto \mathcal{S}(\mu)$ has compact values. □

3 Existence of Self-Consistent Stackelberg equilibria

As before, we consider a finite state space $X = \{1, \ldots, N\}$, so that a feedback control can be identified with a vector $u \in [0,1]^N$. Moreover, a probability distribution on X is identified with a point on the unit simplex Δ_N at (2.10).

For convenience, we shall denote by $\mathcal{S}(\mu)$ the set of all the Stackelberg equilibria (u, v) for the given initial distribution μ. Moreover, for any subset $\mathcal{S} \subseteq [0,1]^N \times [0,1]^N$ we define the family of invariant probability distributions

$$\mu^\infty(\mathcal{S}) \doteq \Big\{\mu \in \Delta_N\,;\quad \mu \text{ is an invariant distribution for the Markov chain (1.1)}$$
$$\text{generated by some } (u,v)\in\mathcal{S}\Big\}.$$

According to Definition 1.3, to construct a self-consistent Stackelberg equilibrium one needs to find a fixed point of the (possibly multivalued) transformation

$$\mu \;\mapsto\; \mathcal{S}(\mu) \;\mapsto \mu^\infty(\mathcal{S}(\mu)) \;\doteq\; \Psi(\mu). \tag{3.1}$$

In order to prove the existence of a fixed point for the above multifunction $\Psi : \Delta_N \mapsto \Delta_N$, a natural approach is to show that every $\Psi(\mu)$ is single-valued, and then use Brouwer's fix point theorem. In this section we examine some cases where this approach is successful.

Lemma 3.1 *Suppose that, for any given controls $(u,v) \in [0,1]^N \times [0,1]^N$, the continuous time Markov chain at (1.1) satisfies (IR). Then the map Ψ defined at (3.1) is an upper semicontinuous multifunction with nonempty, compact values.*

Proof. By Lemma 2.2 the multifunction $\mu \to \mathcal{S}(\mu)$ is upper semicontinuous with nonempty, compact values. It thus remains to check the continuity of the single-valued map $(u,v)\mapsto \mu^\infty(u,v)$. For any fixed pair of controls $(u,v)\in[0,1]^N\times[0,1]^N$, thanks to the assumption (**IR**), the kernel of $\mathbf{\Phi}$ in (2.9) has dimension 1. Therefore the invariant distribution $\mu^\infty(u,v)$ is the unique solution to the linear system

$$\mathbf{p}\cdot\mathbf{\Phi}(u,v) \;=\; \mathbf{0} \qquad \text{with constraint} \quad \mathbf{p}\cdot\mathbf{e}^T = 1,$$

where $\mathbf{e}$ is the row vector with all elements equal to 1. An equivalent way to express the constraint is

$$\mathbf{p}\cdot\mathbf{E} \;=\; \mathbf{e},$$

where $\mathbf{E}$ is an $n\times n$ matrix with all elements equal to 1. This yields

$$\mathbf{p} \;\doteq\; \mathbf{e}\cdot(\mathbf{\Phi}+\mathbf{E})^{-\mathbf{1}}. \tag{3.2}$$

We claim that $\mathbf{\Phi}+\mathbf{E}$ is invertible. Indeed, for any fixed pair of controls (u,v), the matrix $\mathbf{\Phi}$ has rank $N-1$, while $\mathbf{\Phi}\cdot\mathbf{e}^T=\mathbf{0}$. Thus the linear system $\mathbf{x}\cdot\mathbf{\Phi}=\mathbf{e}$ has no solution due to the fact for any linear transformation, the right null space is orthogonal to the row space. Using also the fact that $\mathbf{x}\cdot\mathbf{E}=\mathbf{e}\sum_{i=1}^N x_i$, we conclude that the system $\mathbf{x}\cdot(\mathbf{\Phi}+\mathbf{E})=\mathbf{0}$ has only the trivial solution. Hence $\mathbf{\Phi}+\mathbf{E}$ is invertible and the formula (3.2) is meaningful.

Since the jump rates $\phi_{ij}(u_i,v_i)$ depend continuously on the controls u_i, v_i, the invertible matrix $(\mathbf{\Phi}+\mathbf{E})(u,v)$ is a continuous function of u and v. In turn, the asymptotic probability distribution $\mu^\infty(u,v)$ depends continuously on u,v. This completes the proof of the upper semicontinuity of Ψ. □

Toward the existence of a self-consistent Stackelberg equilibrium, we shall assume that the best reply map satisfies the conclusions of Lemma 2.3, namely

(A1) *For every $u\in[0,1]^N$, the best reply $R_2(u)=\{v^*(u)\}$ is a singleton. Furthermore, the map $u\mapsto v^*(u)$ has $\mathcal{C}^2$ regularity.*

For every $u \in [0,1]^N$ and any initial state $x(0) = i \in \{1, \ldots, N\}$, the expected cost to the leader is

$$J_1(i,u) \doteq E^{x(0)=i}\left[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\Big(x(t), u_{x(t)}, v^*_{x(t)}(u)\Big)\, dt\right].$$

In analogy with (2.5), this expected cost is computed by

$$J_1(i,u) = L_1(i, u_i, v_i^*(u)) + \frac{1}{r_1}\sum_{j\neq i}\phi_{ij}(u_i, v_i^*(u)\Big)\big(J_1(j,u) - J_1(i,u)\big).$$

The above identities can be written in vector form

$$\left(\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}\right)\cdot \mathbf{J_1} = \mathbf{L_1}.$$

Therefore

$$\mathbf{J_1} = \left(\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}\right)^{-1}\mathbf{L_1}\,. \tag{3.3}$$

For any initial distribution $\mu \in \Delta_N$, the expected cost to the leader is now computed by the inner product $\mu \cdot \mathbf{J_1}(u, v^*(u))$. Notice that this is a linear combination of the components $J_1(i,u)$ with coefficients given by the components of the vector $\mu \in \Delta_N$. To prove the uniqueness of the Stackelberg equilibrium $S(\mu)$, for every initial distribution $\mu \in \Delta_N$, it now suffices to require the strict convexity of the maps $u \mapsto J_1(i,u)$. for every $i \in \{1, ..., N\}$.

To better understand this convexity requirement, consider the Jacobian matrix of the best reply map $u = (u_1, u_2, ..., u_N) \mapsto v^*(u) = (v_1^*, v_2^*, ..., v_N^*)$, namely

$$\widetilde{\mathbf{J}}_{\mathbf{2}}(i,j) \doteq \frac{\partial v_i^*(u)}{\partial u_j}.$$

In addition, consider the matrix

$$\mathbf{A} \doteq \left(\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}\right)^{-1}.$$

For every $k \in \{1, ..., N\}$, the Jacobian matrix of partial derivatives of the expected cost $J_1(k,u)$ starting at state k is computed by

$$\widetilde{J}_k(i,j) = \frac{\partial}{\partial u_j}\left(\frac{\partial(\mathbf{A}\cdot\mathbf{L_1})_k}{\partial u_i} + \sum_{k=1}^N \frac{\partial(\mathbf{A}\cdot\mathbf{L_1})_k}{\partial v_k}\widetilde{\mathbf{J}}_{\mathbf{2}}(k,i)\right). \tag{3.4}$$

Assuming the positive definiteness of the above matrices $\widetilde{J}_k$, we can prove a first result on the existence of self-consistent equilibria.

Theorem 3.1 *Consider the game with dynamics (1.1) and exponentially discounted payoffs (1.3). Assume that* **(A1)** *holds and moreover, for every $k \in \{1, ..., N\}$ and $u \in [0,1]^N$, the matrix $\widetilde{J}_k$ in (3.4) is strictly positive definite, namely*

$$w^T \widetilde{J}_k w > 0 \qquad \textit{for all } \; w \in \mathbb{R}^N \setminus \{0\}. \tag{3.5}$$

Then a self-consistent Stackelberg equilibrium exists.

Proof. By (3.5), $\widetilde{J}_1(k,u,v^*(u))$ is strictly convex with respect to u for all $k \in \{1,2,...,N\}$ and all $\mu \in \Delta_N$. Hence the set $S(\mu)$ of Stackelberg equilibria reduces to a single point. Lemma 3.1 further implies that the map Ψ in (3.1) is a continuous, single-valued map from Δ_N to itself. By Brouwer's theorem, Ψ has a fixed point μ^*. Therefore, the corresponding feedback controls $(u^*, v^*) \in S(\mu^*)$ determine a self-consistent Stackelberg equilibrium. □

Unfortunately, the computation of the matrix $\widetilde{J}_k$ in (3.4) can be far from easy. In particular, it requires finding the inverse of the matrix $\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}$. Moreover, in several cases the best reply map can be multi-valued. When this happens, there is no reason for the set of best replies to be convex, or even connected. It is thus unlikely that Kakutani's fixed point theorem can be of use in this setting. For all these reasons, a different approach will be developed in the remaining sections of the paper.

4 Generic structure of the best reply map

Our eventual goal is to understand the detailed structure of the best reply map $u \mapsto \mathcal{R}_2(u)$, with

$$u = (u_1, \ldots, u_N) \in [0,1]^N, \qquad\qquad \mathcal{R}_2(u) \subseteq [0,1]^N.$$

In the myopic case this map is decoupled. Indeed, for each $i \in \{1, \ldots, N\}$,

$$v_i \in \mathcal{R}_{2,i}(u_i) \doteq \arg\min_{\omega\in[0,1]} L_2(i, u_i, \omega)$$

depends only on the component u_i. Under generic assumptions on the cost function L_2, the structure of the multifunctions $\mathcal{R}_{2,i}$ has been analyzed in [7]. These results are here used as a starting point, to describe the best reply map in the more general case of a short-sighted follower with discount factor $r_2 >> 1$.

We begin by reviewing the results in [7]. Consider a one-shot Stackelberg game where the leader and the follower have cost functions $F(x,y)$ and $G(x,y)$. Here $x \in [0,1]$ and $y \in [0,1]$ are the strategies chosen by the leader and by the follower, respectively. The best reply map is the multifunction

$$x \mapsto R(x) \doteq \left\{ y^* \in [0,1]\,;\; G(x,y^*) = \min_{y\in[0,1]} G(x,y) \right\}. \tag{4.1}$$

In turn, the goal of the leader is to minimize the cost function F, restricted to the graph of R, namely

$$\min \left\{ F(x,y)\,; \quad x \in [0,1], \quad y \in R(x) \right\}. \tag{4.2}$$

For a generic cost function $G \in \mathcal{C}^3(\mathbb{R}^2)$, the structure of the map $R(\cdot)$ was described in [7].

Theorem 4.1 *There exists an open dense subset $\mathcal{G} \subset \mathcal{C}^3(\mathbb{R}^2)$ of cost functions such that, if $G \in \mathcal{G}$ then the best reply map (4.1) has the following structure.*

There exists finitely many points $0 = x_0 < x_1 < \cdots < x_\nu = 1$, and functions $\varphi_k \in \mathcal{C}^2(\mathbb{R})$ such that

$$\left\{ (x,y)\,; \quad y \in R(x)\,, \quad x \in [0,1] \right\} = \bigcup_{k=1}^{\nu} \left\{ (x, \varphi_k(x))\,; \quad x \in [x_{k-1}, x_k] \right\}. \tag{4.3}$$

For each $k = 1, \ldots, \nu - 1$, one has

$$G\big(x_k, \varphi_k(x_k)\big) \;=\; G\big(x_k, \varphi_{k+1}(x_k)\big), \qquad \frac{d}{dx} G\big(x, \varphi_k(x)\big)\bigg|_{x=x_k} > \frac{d}{dx} G\big(x, \varphi_{k+1}(x)\big)\bigg|_{x=x_k}. \tag{4.4}$$

Either $\varphi_k(x_k) \neq \varphi_{k+1}(x_k)$, or else

$$\varphi_k(x_k) \;=\; \varphi_{k+1}(x_k) \;\in\; \{0, 1\}, \qquad \varphi_k'(x_k) \;\neq\; \varphi_{k+1}'(x_k). \tag{4.5}$$

Moreover, three cases can arise:

(i) $\varphi_k(x) \equiv 0$ *and* $G_y(x, 0) > 0$ *for all* $x \in]x_{k-1}, x_k[$,

(ii) $\varphi_k(x) \equiv 1$ *and* $G_y(x, 1) < 0$ *for all* $x \in]x_{k-1}, x_k[$,

(iii) $0 < \varphi_k(x) < 1$ *for all* $x \in]x_{k-1}, x_k[$ *. In this case one has* $G_y(x, \varphi_k(x)) = 0$ *and* $G_{yy}(x, \varphi_k(x)) > 0$ *for all* $x \in [x_{k-1}, x_k]$.

The above theorem is proved by introducing finitely many systems of linear equations involving G and its derivatives. We then define $\mathcal{G}$ to be the set of all functions $G \in \mathcal{C}^3$ for which each of these systems has no solution. The density and openness of the set $\mathcal{G} \subset \mathcal{C}^3(\mathbb{R}^2)$ now follow from two facts:

(i) Each of the above systems contains more equations than variables. Hence, for a dense set of functions G, it does not have any solution.

(ii) Each equation is formulated in terms of G and its derivatives up to third order. By a continuity argument, the set of functions for which these equations have a solution is closed in $\mathcal{C}^3$, and its complement is open.

For all details we refer to [7].

Next, we consider a game where the strategies of the leader and the follower both range in $[0, 1]^N$. We start with the simple case where the cost function has the form

$$G(x, y) \;=\; \sum_{i=1}^{N} G_i(x_i, y_i). \tag{4.6}$$

Let $x_i \mapsto R_i(x_i) \subseteq [0, 1]$ be the best reply map corresponding to the cost function G_i. Then the vector-valued reply map $x \mapsto R(x) \subseteq [0, 1]^N$ has the product structure:

$$R(x_1, \ldots, x_N) \;=\; R_1(x_1) \times \cdots \times R_N(x_N). \tag{4.7}$$

By Theorem 4.1, if $G_i \in \mathcal{G}$ for every $i = 1, \ldots, N$, then the structure of the multifunction R can be immediately described in terms of the maps R_i.

In the remainder of this section we analyze the best reply maps for a family of perturbed cost functions of the form

$$G_\varepsilon(x, y) \;=\; \sum_{i=1}^{N} G_i(x_i, y_i) + \varepsilon \widetilde{G}(x, y) + o(\varepsilon), \tag{4.8}$$

where $G_i \in \mathcal{G}$ for every $i = 1, \ldots, N$, while $\widetilde{G} \in \mathcal{C}^3(\mathbb{R}^{2N})$ and $o(\varepsilon)$ denotes an additional term whose $\mathcal{C}^3$ norm vanishes faster than ε. Namely, $\varepsilon^{-1}\|o(\varepsilon)\|_{\mathcal{C}^3} \to 0$ as $\varepsilon \to 0$.

4.1 Regular Stratified Domains.

Definition 4.1 *Let $V \subset \mathbb{R}^d$ be a compact set. Given $k \geq 1$, by a $\mathcal{C}^k$-**stratification** of V we mean a finite collection $\mathcal{S} = \{\mathcal{M}_{ij}\,;\ \ i = 0, \ldots, d,\ j = 1, \ldots, N_i\}$ of manifolds $\mathcal{M}_{ij} \subset \mathbb{R}^d$ with the following properties.*

(i) The manifolds $\mathcal{M}_{ij}$ provide a disjoint covering of V. Namely

$$V \;=\; \bigcup_{i,j} \mathcal{M}_{ij}\,, \qquad\qquad (i,j) \neq (i',j') \quad \Longrightarrow \quad \mathcal{M}_{ij} \cap \mathcal{M}_{i'j'} \;=\; \emptyset. \tag{4.9}$$

(ii) Each $\mathcal{M}_{ij}$, $1 \leq j \leq N_i$ is an embedded manifold of class $\mathcal{C}^k$, with dimension i. More precisely, every point $\bar{x} \in \mathcal{M}_{ij}$ has a neighborhood $\mathcal{N}_{\bar{x}}$ with the following property. There exist a $\mathcal{C}^k$ map $\varphi : \mathbb{R}^d \mapsto \mathbb{R}^{d-i}$ such that

$$rank(D\varphi(x)) \;=\; d - i \quad \text{for every } x, \qquad\qquad \mathcal{M}_{ij} \cap \mathcal{N}_{\bar{x}} \;=\; \{x;\ \ \varphi(x) = 0\} \cap \mathcal{U}_{\bar{x}}.$$

Example 4.1 Consider the best reply map in (4.7), obtained as a product of the best replies R_i. This map can be associated with a stratification of the hypercube $[0,1]^N$ as follows.

For each $i = 1, \ldots, N$, as in (4.3) assume that the best reply map R_i is described by

$$\Big\{(x,y)\,;\ \ y \in R_i(x)\,,\ \ x \in [0,1]\Big\} \;=\; \bigcup_{k=1}^{\nu(i)} \Big\{(x, \varphi_{i,k}(x))\,;\ \ x \in [\xi_{i,k-1}, \xi_{i,k}]\Big\}, \tag{4.10}$$

for suitable points

$$0 \;=\; \xi_{i,0} \;<\; \xi_{i,1} < \ldots < \xi_{i,\nu(i)} \;=\; 1.$$

For each i, the interval $[0,1]$ can be written as a disjoint union of single points and open intervals:

$$\begin{aligned}[0,1] \;&=\; \{0\} \,\cup\,]\xi_{i,0}, \xi_{i,1}[\, \cup \{\xi_{i,1}\} \,\cup\, \cdots \,\cup\, \{\xi_{i,\nu(i)-1}\} \,\cup\,]\xi_{i,\nu(i)-1}, \xi_{i,\nu(i)}[\, \cup\, \{1\} \\ &=\; \bigcup_{\ell=1}^{1+2\nu(i)} V_{i,\ell}\,.\end{aligned} \tag{4.11}$$

Notice that $\nu(i) + 1$ of the sets on the right hand side of (4.11) are singletons, and $\nu(i)$ of these are open intervals.

In turn, this yields a stratification of the cube $[0,1]^N$ in terms of manifolds of the form

$$\mathcal{M}_\ell \;=\; V_{1,\ell(1)} \times V_{2,\ell(2)} \times \cdots \times V_{N,\ell(N)}\,. \tag{4.12}$$

A covering of $[0,1]^N$ is obtained by taking all possible choices of multi-indices

$$\ell \;=\; (\ell(1), \ldots, \ell(N)), \qquad \text{with} \quad \ell(i) \in \{1, 2, \ldots, 1 + 2\nu(i)\} \quad \text{for every } i.$$

The total number of these submanifolds is $\prod_{i=1}^N (1 + 2\nu(i))$. The dimension of the manifold $\mathcal{M}_\ell$ is

$$\dim(\mathcal{M}_\ell) \;=\; \#\Big\{i\,;\ \ V_{i,\ell(i)} \text{ is an open interval}\Big\} \;\in\; \{0, 1, 2, \ldots, N\}.$$

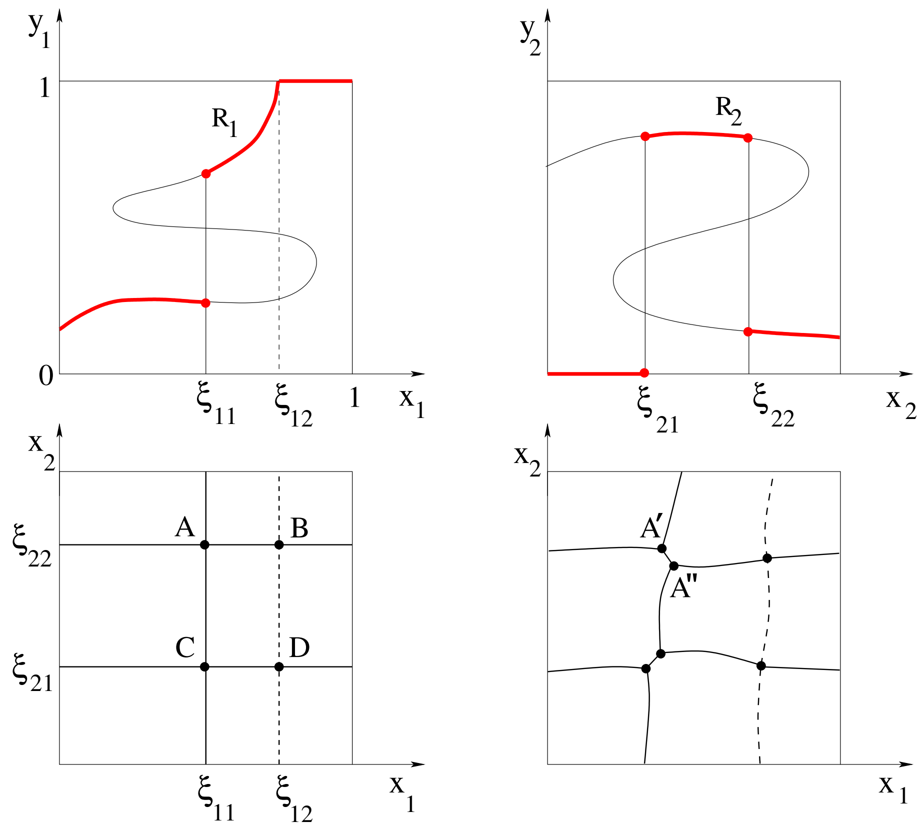

Figure 1: Above: for a myopic follower, the best reply maps are decoupled. We can thus consider separately the graphs of the best reply maps $x_1 \mapsto R_1(x_1)$ and $x_2 \mapsto R_2(x_2)$. The thin lines denote the solutions to $\nabla_y G_i = 0$. The thick lines denote the global minima of the functions $G_i(x_i, \cdot)$. For x_1 in a neighborhood of ξ_{11}, the function $G(x_1, \cdot)$ has two local minima, which coincide at $x_1 = \xi_{11}$. The best reply map R_1 is two-valued at $x_1 = \xi_{11}$ and has a kink at $x_1 = \xi_{12}$. The best reply map R_2 is two-valued at $x_2 = \xi_{21}$ and at $x_2 = \xi_{22}$. Still for a myopic follower, we can look at the best reply map R as a multivalued map from $[0,1]^2$ into itself. The figure al the bottom left shows the stratification of $[0,1]^2$ corresponding to this map: $R(x_1, x_2) = R_1(x_1) \times R_2(x_2)$. Notice that R is 4-valued at A and at C, and it is 2-valued along on the lines $x_1 = \xi_{11}$, $x_2 = \xi_{12}$ and $x_2 = \xi_{22}$. Moreover, it has a kink along the line $x_1 = \xi_{12}$. Bottom right: the stratification of $[0,1]^2$ induced by the best reply map R_ε corresponding to a small, smooth perturbation of the cost function to the follower. Notice that, for a generic perturbation $\widetilde{G}$ in (4.8), the point A is replaced by the two points A' and A''.

The best reply map R in (4.7) can now be described in terms of this stratification. Consider any manifold $\mathcal{M}_\ell$ of dimension N. By construction, this is the cartesian product of N open intervals, say

$$\mathcal{M}_\ell \;=\;]\xi_{1,\ell(1)-1},\, \xi_{1,\ell(1)}[\;\times\; \cdots \;\times\;]\xi_{N,\ell(N)-1},\, \xi_{N,\ell(N)}[\,. \tag{4.13}$$

Restricted to $\mathcal{M}_\ell$, the map R is single-valued, and coincides with a smooth function. Indeed, by (4.10) we have

$$R(x_1,\ldots,x_N) \;=\; \Big\{ \big(\varphi_{1,\ell((1)}(x_1)\,,\; \varphi_{2,\ell(2)}(x_2)\,,\; \ldots\;,\, \varphi_{N,\ell(N)}(x_N)\big) \Big\}. \tag{4.14}$$

The graph of R is now obtained by taking the closure of its restriction on the N-dimensional open boxes $\mathcal{M}_\ell$ in (4.13).

We observe that, along manifolds of dimension $< N$, the best reply can be multi-valued. The cardinality of $R(x)$ is computed as follows. For each $i = 1,\ldots,N$, split

$$\{0,1,2,\ldots,\nu(i)\} \;=\; J_i \cup K_i\,,$$

where

- J_i is the set of indices $j \in \{1,2,\ldots,\nu(i)-1\}$ such that R_i has a jump at ξ_{ij}. Namely, the set $R_i(\xi_{ij})$ contains two distinct values.
- K_i is the set of indices j such that either $\xi_{ij} \in \{0,1\}$ or else $R_i(\xi_{ij}) \in \{0,1\}$. Notice that, if the second alternative holds, then in a neighborhood of ξ_{ij} the best reply map R_i is single-valued, with a kink at ξ_{ij}.

For example, for the best reply maps R_1, R_2 shown at the top of Fig. 1, one has

$$J_1 = \{\xi_{11}\}, \qquad K_1 = \{0,\xi_{12},1\}, \qquad J_2 = \{\xi_{21},\xi_{22}\}, \qquad K_2 = \{0,1\}.$$

Given any manifold $\mathcal{M}_\ell$ of the form (4.12), for all $x = (x_1,\ldots,x_N) \in \mathcal{M}_\ell$ the cardinality of $R(x)$ is

$$\#R(x) \;=\; 2^q, \qquad \text{where} \qquad q \;=\; \#\Big\{ i\,;\;\; V_{i,\ell(i)} = \{\xi_{ij}\} \;\text{ for some } j \text{ with } \xi_{ij} \in J_i \Big\}.$$

4.2 Structure of the best reply map, for a generic perturbation.

Next, consider a one-shot Stackelberg game where the leader and the follower choose their strategies $x, y \in [0,1]^N$. Let $F, G_\varepsilon : [0,1]^{2N} \mapsto \mathbb{R}$ be respectively the cost functions for the leader and for the follower, with G_ε as in (4.8).

Our goal is to prove that, under generic assumptions on the functions $F, G_i, \widetilde{G}$, for every $\varepsilon > 0$ sufficiently small the Stackelberg equilibrium is unique and can be characterized by a set of $2N$ independent scalar equations in the variables $x_1,\ldots,x_N, y_1,\ldots,y_N$.

To fix the ideas, assume that, in the unperturbed case (4.6), the game has a unique Stackelberg equilibrium, attained at (x^*, y^*) with $x^* \in \mathcal{M}_\ell = V_{1,\ell(1)}\times\cdots\times V_{N,\ell(N)}$. In general, the manifold

$\mathcal{M}_\ell$ will have dimension

$$\dim(\mathcal{M}_\ell) \;=\; N-p-q, \qquad \begin{cases} p \;=\; \#\Big\{i\,;\;\; V_{i,\ell(i)}=\{\xi_{ij}\} \quad \text{for some } j=\ell(i)\in K_i\Big\}, \\ q \;=\; \#\Big\{i\,;\;\; V_{i,\ell(i)}=\{\xi_{ij}\} \quad \text{for some } j=\ell(i)\in J_i\Big\}. \end{cases} \tag{4.15}$$

To help the reader, we first explain the main ideas with the aid of some figures. For simplicity, let $N=2$ and assume that, when the cost function G has the decoupled form (4.6), the best reply map induces the stratification shown in Fig. 1, bottom left. In particular: for every (x_1,x_2) in a neighborhood of the point $A=(x_1^*,x_2^*)\doteq(\xi_{11},\xi_{22})$, the equation

$$\nabla_y G(x,y) \;=\; 0 \tag{4.16}$$

has four distinct solutions corresponding to local minima of $G(x,\cdot)$ (together with other solutions, which do not yield local minima of G). For notational convenience, these four solutions of (4.16) will be denoted by $x\mapsto y_{\mathcal{I}}(x)$, where $\mathcal{I}\subseteq\{1,2\}$.

We wish to understand what happens to the best reply map when (4.6) is replaced by (4.8), for a generic perturbation $\widetilde{G}$ and for $\varepsilon>0$ small enough. In this case, in a neighborhood of the point $A=(x_1^*,x_2^*)\doteq(\xi_{11},\xi_{22})$, the equation (4.16) still has four distinct solutions $x\mapsto y^\varepsilon_{\mathcal{I}}(x)$ corresponding to local minima of the function G_ε in (4.8). The four domains where each of the functions $y^\varepsilon_{\mathcal{I}}$ yields a global minimum of G_ε will be denoted by

$$\Omega^\varepsilon_{\mathcal{I}} \;\doteq\; \Big\{x\,;\;\; G_\varepsilon(x,y^\varepsilon_{\mathcal{I}}(x)) \;=\; \min_{\mathcal{I}'\subseteq\{1,2\}} G_\varepsilon(x,y^\varepsilon_{\mathcal{I}'}(x))\Big\}. \tag{4.17}$$

When $\varepsilon=0$, these four domains have a rectangular shape, as shown in Fig. 1, bottom left. At the point A, all four functions $G(x^*,y_{\mathcal{I}}(x^*))$ coincide, and the best reply map is thus quadruple-valued. On the other hand, for a generic perturbation $\widetilde{G}$, when $\varepsilon>0$ the domains $\Omega^\varepsilon_{\mathcal{I}}$ will have a more complicated shape, shown in Fig. 1, bottom right. In particular, the point A splits into 2 distinct points $A'_\varepsilon, A''_\varepsilon$, where the best reply map R_ε is triple-valued.

We now analyze the general case $N\geq 1$. For clarity of exposition, we first consider the case where $p=0$, so that all singularities of the best reply map through $x^*=(x_1^*,\ldots,x_N^*)$ are jumps. Recalling (4.15), let $\mathcal{J}\subseteq\{1,\ldots,N\}$ be the subset of indices i such that $\ell(i)\in J_i$. These are the components i such that the best reply map for G_i has a jump at x_i^*. According to (4.15), its cardinality is $\#\mathcal{J}=q$. For notational convenience, if $i\in\mathcal{J}$ we define the functions y_i^-,y_i^+ in a neighborhood of the jump point x_i^* so that

$$R_i(x_i) \;=\; \begin{cases} \{y_i^-(x_i)\} & \text{if} \quad x_i<x_i^*, \\ \{y_i^+(x_i)\} & \text{if} \quad x_i>x_i^*, \\ \{y_i^-(x_i),\,y^+(x_i)\} & \text{if} \quad x_i=x_i^*. \end{cases} \tag{4.18}$$

On the other hand, if $i\notin\mathcal{J}$, hence x_i^* is not a jump point, we simply write $R_i(x_i)=\{y_i(x_i)\}$ for x_i close to x_i^*. For every subset $\mathcal{I}\subseteq\mathcal{J}$, we consider the function

$$y_{\mathcal{I}}(x) \;\doteq\; (y_1,\ldots,y_N), \qquad \text{where} \qquad y_i \;=\; \begin{cases} y_i^+(x) & \text{if} \;\; i\in\mathcal{I}, \\ y_i^-(x) & \text{if} \;\; i\in\mathcal{J}\setminus\mathcal{I}, \\ y_i(x) & \text{if} \;\; i\notin\mathcal{J}. \end{cases} \tag{4.19}$$

Consider the 2^q linear functions of the variable $z = (z_1, \ldots, z_N)$ defined as follows. For every subset $\mathcal{I} \subseteq \mathcal{J}$, define

$$\begin{aligned}\Lambda_{\mathcal{I}}(z) &= \sum_{i\in\mathcal{I}} \left[\frac{d}{dx_i} G_i(x_i, y_i^+(x_i))\right]_{x_i=x_i^*} z_i + \sum_{i\in\mathcal{J}\setminus\mathcal{I}} \left[\frac{d}{dx_i} G_i(x_i, y_i^-(x_i))\right]_{x_i=x_i^*} z_i \\ &\quad + \sum_{i\notin\mathcal{J}} \left[\frac{d}{dx_i} G_i(x_i, y_i(x_i))\right]_{x_i=x_i^*} z_i\,.\end{aligned} \tag{4.20}$$

By construction, for $x \approx x^*$ and every $\mathcal{I} \subseteq \mathcal{J}$ we have the linear approximations

$$G(x, y_{\mathcal{I}}(x)) = G(x^*, y_{\mathcal{I}}(x^*)) + \Lambda_{\mathcal{I}}(x - x^*) + \mathcal{O}(1)\cdot |x - x^*|^2. \tag{4.21}$$

On the other hand, for $\varepsilon > 0$, the corresponding approximation for the perturbed function G_ε in (4.8) takes the form

$$G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x)) = G(x^*, y_{\mathcal{I}}(x^*)) + \Lambda_{\mathcal{I}}(x - x^*) + \varepsilon\widetilde{G}(x^*, y_{\mathcal{I}}(x^*)) + \mathcal{O}(1)\cdot (|x - x^*| + \varepsilon)^2. \tag{4.22}$$

By assumption, at the point $x = x^*$ all the 2^q functions $G(x, y_{\mathcal{I}}(x))$, $\mathcal{I} \subseteq \mathcal{J}$, coincide.

Given a general perturbation $\widetilde{G}$ as in (4.8), consider the 2^q-tuple of numbers

$$\lambda_{\mathcal{I}} = \widetilde{G}\big(x^*, y_{\mathcal{I}}(x^*)\big). \tag{4.23}$$

Under generic conditions we expect that, in a neighborhood of x^*, the domains

$$\Omega^\varepsilon_{\mathcal{I}} \doteq \Big\{x\,;\;\; G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x)) = \min_{\mathcal{I}'\subseteq\mathcal{J}} G_\varepsilon(x, y^\varepsilon_{\mathcal{I}'}(x))\Big\} \tag{4.24}$$

will produce the same stratification as the polytopes

$$\overline{\Omega}_{\mathcal{I}} \doteq \Big\{x \in \mathbb{R}^N\,;\;\; \Lambda_{\mathcal{I}}(x) + \lambda_{\mathcal{I}} = \min_{\mathcal{I}'\subseteq\mathcal{J}} \Lambda_{\mathcal{I}'}(x) + \lambda_{\mathcal{I}'}\Big\}. \tag{4.25}$$

This motivates the following

Definition 4.2 *Given a finite family of linear functions $\{\Lambda_{\mathcal{I}}\}_{\mathcal{I}\subseteq\mathcal{J}}$, we say that a 2^q-tuple of real numbers $(\lambda_{\mathcal{I}})_{\mathcal{I}\subseteq\mathcal{J}}$ is* generic *if, for any collection of $q+2$ distinct sets $\mathcal{I}_1, \mathcal{I}_2, \ldots \mathcal{I}_{q+2} \subseteq \mathcal{J}$, the linear system of $q+1$ equations for the q variables $z = (z_i)_{i\in\mathcal{J}}$*

$$\Lambda_{\mathcal{I}_1}(z) + \lambda_{\mathcal{I}_1} = \Lambda_{\mathcal{I}_2}(z) + \lambda_{\mathcal{I}_2} = \cdots = \Lambda_{\mathcal{I}_{q+2}}(z) + \lambda_{\mathcal{I}_{q+2}} \tag{4.26}$$

has no solutions.

Remark 4.1 According to (4.20), the linear functions $\Lambda_{\mathcal{I}}$ are defined on the entire space $\mathbb{R}^N$. However, for any $\mathcal{I}, \mathcal{I}' \subseteq \mathcal{J}$, the difference $\Lambda_{\mathcal{I}}(z) - \Lambda_{\mathcal{I}'}(z)$ does not depend on the components z_i with $i \notin \mathcal{J}$. With a slight abuse of notation, one can thus regard (4.26) as a set of equations for the q variables $z = (z_i)_{i\in\mathcal{J}}$ only. This fact plays a key role in the proof of the next lemma.

Lemma 4.1 *Given the 2^q linear functions $\Lambda_{\mathcal{I}}$ in (4.20), the subset $\mathcal{S}$ of generic 2^q-tuples $(\lambda_{\mathcal{I}})_{\mathcal{I}\subseteq\mathcal{J}}$ is open and dense.*

As a consequence, for all functions $\widetilde{G}$ in an open dense subset of $\mathcal{C}^3(\mathbb{R}^{2N})$, the values in (4.23) satisfy $(\lambda_{\mathcal{I}})_{\mathcal{I}\subseteq\mathcal{J}} \in \mathcal{S}$.

Proof. For any collection of distinct subsets $\mathcal{I}_1, \mathcal{I}_2, \ldots \mathcal{I}_{q+2} \subseteq \mathcal{J}$, the system (4.26) is equivalent to

$$\Lambda_{\mathcal{I}_1}(z) - \Lambda_{\mathcal{I}_i}(z) \;=\; \lambda_{\mathcal{I}_1} - \lambda_{\mathcal{I}_i}, \qquad\qquad i \in \{2, \ldots, q+2\}, \tag{4.27}$$

which is a linear system of $q+1$ equations and q variables. Hence the set $\widetilde{\mathcal{S}}$ of 2^q-tuples $(\lambda_{\mathcal{I}})_{\mathcal{I}\subseteq\mathcal{J}}$ for which (4.27) has no solutions is dense in $\mathbb{R}^{2^q}$. Furthermore, by continuity, $\widetilde{\mathcal{S}}$ is also open. By definition, $\mathcal{S}$ is the intersection of all such open dense sets $\widetilde{\mathcal{S}}$, corresponding to all finite collections of distinct subsets $\mathcal{I}_1, \mathcal{I}_2, \ldots \mathcal{I}_{q+2} \subseteq \mathcal{J}$. Therefore, $\mathcal{S}$ itself is open and dense. The second statement of the lemma follows immediately. □

4.3 Generic structure of the Stackelberg equilibrium

In a Stackelberg game, the goal of the leading player is to minimize the cost function $F = F(x, y)$ restricted to the graph of the best reply map for the follower. More specifically, when the cost function G_ε for the follower has the form (4.8), the leader seeks

$$\min \bigl\{F(x, y)\,; \quad x \in [0,1]^N, \quad y \in R_\varepsilon(x)\bigr\}. \tag{4.28}$$

Here

$$R_\varepsilon(x) \;\doteq\; \bigl\{y^* \in [0,1]^N\,; \quad G_\varepsilon(x, y^*) \le G_\varepsilon(x, y) \text{ for all } \; y \in [0,1]^N\bigr\} \tag{4.29}$$

denotes the best reply map. Under generic assumptions on $F, G_1, \ldots, G_N$, and $\widetilde{G}$, we want to show that, for all $\varepsilon \in [0, \varepsilon_0]$ sufficiently small, the constrained optimization problem (4.28) has a unique global minimizer $(x^*_\varepsilon, y^*_\varepsilon)$. Moreover, this minimizer depends Lipschitz continuously on ε and is stable w.r.t. small perturbations of the cost function F.

Theorem 4.2 *Consider a cost function $G(x, y)$ of the form (4.6) where for every $i = 1, \ldots, N$, the function $G_i \in \mathcal{G}$ satisfies all generic properties listed in Theorem 4.1. Then there exists $\varepsilon_0 > 0$ and a open dense subset of functions $(F(x, y), \widetilde{G}(x, y)) \in \mathcal{C}^2 \times \mathcal{C}^3$ such that the problem (4.28) has a unique minimizer $(x^*_\varepsilon, y^*_\varepsilon) \in [0,1]^N$ for all $\varepsilon \in [0, \varepsilon_0]$ sufficiently small.*

Proof. 1. Call

$$\mathcal{A} \;\doteq\; \{(x, y)\,; \quad y \in R(x)\}, \qquad\qquad \mathcal{A}_\varepsilon \;\doteq\; \{(x, y)\,; \quad y \in R_\varepsilon(x)\} \tag{4.30}$$

the graphs of the best reply maps corresponding to the cost functions G and G_ε, respectively. Let $F \in \mathcal{C}^2(\mathbb{R}^{2N})$ be given. Let $(x^*, y^*) \in [0,1]^{2N}$ be a Stackelberg equilibrium for the game with costs F, G, so that

$$F(x^*, y^*) \;=\; \min_{(x,y)\in\mathcal{A}} F(x, y).$$

For clarity of exposition, we first assume that $(x^*, y^*) \in]0,1[^{2N}$ is an interior point. The modifications required to treat the general case where some of the components x^*_i, y^*_i take values 0 or 1 will be discussed at the end. As before, call $\mathcal{J} \subset \{1, \ldots, N\}$ the set of indices such that the best reply map $x_i \mapsto R_i(x_i)$ has a jump at x^*_i. Recalling the notation introduced at (4.19), assume

$$y^* \;=\; y_{\mathcal{I}^*}(x^*),$$

for some $\mathcal{I}^* \subseteq \mathcal{J}$. For every $\mathcal{I} \subseteq \mathcal{J}$, define the composite map $F_{\mathcal{I}}(x) \doteq F(x, y_{\mathcal{I}}(x))$. Then the first order necessary conditions for a constrained local minimum imply

$$\frac{\partial}{\partial x_i} F_{\mathcal{I}^*}(x^*) \quad \begin{cases} \geq 0 & \text{for} \quad i \in \mathcal{I}^*, \\ \leq 0 & \text{for} \quad i \in \mathcal{J} \setminus \mathcal{I}^*, \\ = 0 & \text{for} \quad i \notin \mathcal{J}. \end{cases} \tag{4.31}$$

As shown in [7], by performing an arbitrarily small $\mathcal{C}^2$ modification of F we can assume that

(a1) The point (x^*, y^*) where the constrained global minimum is attained is unique.

(a2) The necessary conditions in (4.31) are satisfied, with all inequalities being strict.

(a3) The $(N-q) \times (N-q)$ Hessian matrix of second partial derivatives

$$\left(\frac{\partial^2}{\partial x_j \partial x_k} F_{\mathcal{I}^*}(x^*) \right)_{j,k \notin \mathcal{J}}$$

is strictly positive definite.

Indeed, all this can be achieved by replacing F with

$$F(x,y) + \Big[a_0 \big(|x - x^*|^2 + |y - y^*|^2 \big) + \sum_{k=1}^{N} a_k (x_k - x_k^*) \Big] \varphi(x,y). \tag{4.32}$$

Here $a_0 > 0$ and $a_k \in \mathbb{R}$ can be chosen arbitrarily small, while $\varphi \in \mathcal{C}_c^\infty$ is a cutoff function such that $\varphi(x,y) = 1$ for $(x,y) \in [0,1]^{2N}$.

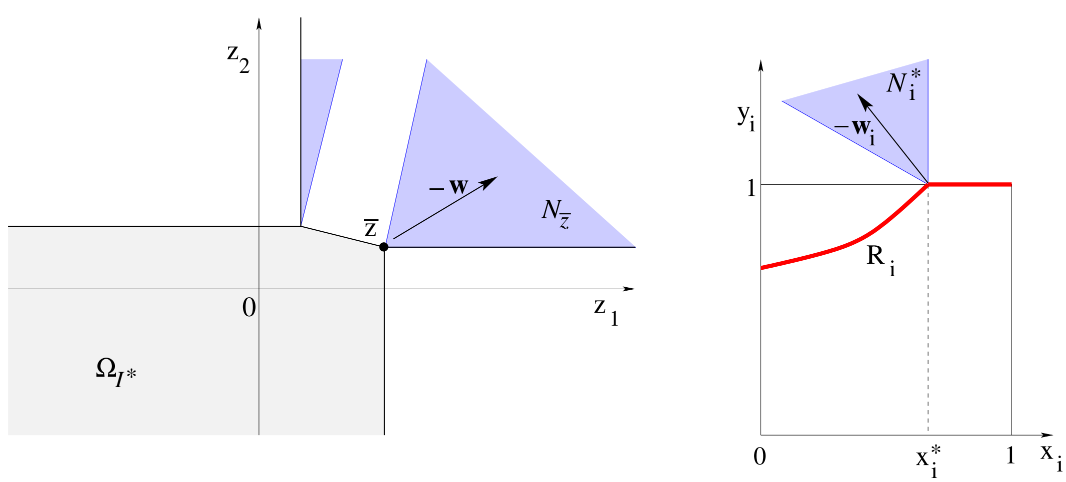

Figure 2: Left: the polytope $\overline{\Omega}_{\mathcal{I}^*}$ defined at (4.33). Under generic assumptions, there exists a unique point $\overline{z} \in \overline{\Omega}_{\mathcal{I}^*}$ which minimizes the inner product with $\mathbf{w}$. Notice that the vector $-\mathbf{w}$ lies in the interior of the normal cone $\mathcal{N}_{\overline{z}}$ at $\overline{z}$. Right: a point x_i^* where the best reply map $x_i \mapsto R_i(x_i)$ has a kink. By performing an arbitrarily small perturbation of the cost function F, the negative gradient $-\mathbf{w}_i \doteq -(F_{x_i}, F_{y_i})$ is contained in the interior of the normal cone $\mathcal{N}_i^*$.

2. Assume that the perturbation $\widetilde{G}$ satisfies the conclusion of Lemma 4.1. Recalling Remark 4.1, consider the polytope (see Fig. 2, left)

$$\overline{\Omega}_{\mathcal{I}^*} \doteq \Big\{ z = (z_i)_{i\in\mathcal{J}}\,;\quad \Lambda_{\mathcal{I}^*}(z) + \lambda_{\mathcal{I}^*} = \min_{\mathcal{I}\subseteq\mathcal{J}} \Lambda_{\mathcal{I}}(z) + \lambda_{\mathcal{I}} \Big\} \subset \mathbb{R}^q . \tag{4.33}$$

We claim that this polytope is unbounded, with nonempty interior. Indeed, consider the vector

$$\mathbf{e} = (e_j)_{j\in\mathcal{J}}\,, \qquad \text{where} \qquad e_j = \begin{cases} +1 & \text{if } j \in \mathcal{I}^*, \\ -1 & \text{if } j \notin \mathcal{I}^*. \end{cases}$$

In view of (4.20) and the strict inequality in (4.4), it follows

$$\big\langle \mathbf{e}\,,\, \Lambda_{\mathcal{I}^*} - \Lambda_{\mathcal{I}} \big\rangle > 0.$$

Hence, for all $c > 0$ large enough, the vector $c\mathbf{e}$ lies in the interior of $\overline{\Omega}_{\mathcal{I}^*}$.

Next, consider the vector of partial derivatives at the point x^*

$$\mathbf{w} \doteq \left(\frac{\partial}{\partial x_i} F_{\mathcal{I}^*}(x^*, y_{\mathcal{I}^*}(x^*)) \right)_{i\in\mathcal{J}} \in \mathbb{R}^q. \tag{4.34}$$

According to the generic assumption (a2), all of these derivatives are nonzero. In view of (4.31), with all inequalities being strict, and by the structure of the $\Lambda_{\mathcal{I}}$, the we achieve the existence of a point $\overline{z} \in \overline{\Omega}_{\mathcal{I}^*}$ such that

$$\langle \overline{z}, \mathbf{w}\rangle = \min_{z\in\overline{\Omega}_{\mathcal{I}^*}} \langle z, \mathbf{w}\rangle. \tag{4.35}$$

By a further, small $\mathcal{C}^2$ perturbation of the function F, still of the form (4.32), we can assume that the constrained minimum is achieved at a unique point $\overline{z}$, and moreover

$$-\mathbf{w} \in \text{int}\, \mathcal{N}_{\overline{z}}\,. \tag{4.36}$$

Here $\text{int}\,\mathcal{N}_{\overline{z}}$ denotes the interior of the outer normal cone to the polytope $\overline{\Omega}_{\mathcal{I}^*}$ at the point $\overline{z}$. Since we are assuming that the values $\lambda_{\mathcal{I}}$ in (4.23) satisfy the conclusion of Lemma 4.1, we can identify a family of q subsets $\mathcal{I}_1, \ldots, \mathcal{I}_q \subseteq \mathcal{J}$ such that the point $\overline{z}$ is characterized by the q linearly independent equations

$$\Lambda_{\mathcal{I}^*}(z) + \lambda_{\mathcal{I}^*} = \Lambda_{\mathcal{I}_k}(z) + \lambda_{\mathcal{I}_k}\,, \qquad k = 1, \ldots, q\,. \tag{4.37}$$

Moreover, there exist unique coefficients $\bar{c}_k > 0$ such that

$$-\langle z, \mathbf{w}\rangle = \sum_{i=1}^{q} \bar{c}_k (\Lambda_{\mathcal{I}_k}(z) - \Lambda_{\mathcal{I}^*}(z)) \qquad \text{for all } z \in \mathbb{R}^q. \tag{4.38}$$

3. Given the function $G = \sum_i G_i$, F, and $\widetilde{G}$ as above, we claim that, for all $\varepsilon \geq 0$ sufficiently small, the Stackelberg equilibrium for the cost functions F, G_ε is achieved at a unique point $(x^*_\varepsilon, y^\varepsilon_{\mathcal{I}^*}(x^*_\varepsilon))$, continuously depending on ε. The main ingredients of the proof are as follows:

- Thanks to (a1), by a continuity and compactness argument, for all $\varepsilon > 0$ small enough the minimum must be attained at some point $(x_\varepsilon, y^\varepsilon_{\mathcal{I}^*}(x_\varepsilon))$ in a small neighborhood of (x^*, y^*).

- By (a2)-(a3) and the uniqueness of the point $\overline{z}$ in (4.35), the necessary conditions for the optimality of the point x_ε yield a system of N linearly independent equations. By the implicit function theorem, the map $\varepsilon \mapsto x_\varepsilon$ is thus uniquely defined, and has $\mathcal{C}^1$ regularity.

We now turn to details. Recall that, for every $\mathcal{I} \subseteq \mathcal{J}$ the equations (4.18)-(4.19) determine a function $y_{\mathcal{I}}$ on a neighborhood of x^*. All these 2^q functions satisfy the implicit equation $\nabla_y G(x, y_{\mathcal{I}}(x)) = 0$. Since all functions G_i in (4.8) satisfy the conclusions of Theorem 4.1, by continuity, for all $\varepsilon \geq 0$ small enough the Hessian of the map $y \mapsto G_\varepsilon(x, y)$ is strictly positive in a neighborhood of (x^*, y^*). By the implicit function theorem we thus obtain 2^q distinct functions $y^\varepsilon_{\mathcal{I}}$, all defined on a neighborhood of x^*. Each of them satisfies the implicit equation

$$\nabla_y G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x)) \;=\; 0 \;\in\; \mathbb{R}^N. \tag{4.39}$$

Next, in connection with the subsets $\mathcal{I}_1, \ldots, \mathcal{I}_q$ in (4.37), define the $(N-q)$-dimensional manifold

$$\mathcal{M}_\varepsilon \;=\; \Big\{(x,y)\,;\quad y = y^\varepsilon_{\mathcal{I}^*}(x)\,,\quad G_\varepsilon(x, y^\varepsilon_{\mathcal{I}^*}(x)) \;=\; G_\varepsilon(x, y^\varepsilon_{\mathcal{I}_k}(x)),\quad k = 1, \ldots, q\Big\}. \tag{4.40}$$

Notice that this is well defined for $\varepsilon \geq 0$ sufficiently small and x in a neighborhood of x^*. It will be convenient to use the notation $x = (z, \zeta) \in \mathbb{R}^q \times \mathbb{R}^{N-q}$, where

$$z \;=\; (x_i)_{i\in\mathcal{J}}\,, \qquad \zeta \;=\; (x_i)_{i\notin\mathcal{J}}\,. \tag{4.41}$$

Accordingly, we write $x^* = (z^*, \zeta^*)$. We can now solve the $N+q$ equations in (4.40), expressing the variables z, y as functions of ζ. This yields a parameterization of $\mathcal{M}_\varepsilon$ of the form

$$\zeta \;\mapsto\; \Big(x^\varepsilon(\zeta),\, y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta))\Big). \tag{4.42}$$

In the next steps we will show that, for $\varepsilon \geq 0$ sufficiently small, the point $(x^*_\varepsilon, y^*_\varepsilon)$ where the Stackelberg equilibrium is achieved can be determined as the unique minimizer of the function F restricted to the manifold $\mathcal{M}_\varepsilon$.

4. When $\varepsilon = 0$, the point $(x^*, y^*) = (x^*, y_{\mathcal{I}^*}(x^*))$ is the unique minimizer of F constrained to the admissible set $\mathcal{A}$ in (4.30). For $\varepsilon > 0$ small, define the set corresponding to (4.33) as

$$\Omega^\varepsilon_{\mathcal{I}^*} \;\doteq\; \Big\{(x,y);\quad y = y^\varepsilon_{\mathcal{I}^*}(x)\,,\quad G_\varepsilon(x, y^\varepsilon_{\mathcal{I}^*}(x)) = \min_{\mathcal{I}\subseteq\mathcal{J}} G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x))\Big\}. \tag{4.43}$$

By continuity, for $\varepsilon > 0$ small enough any minimizer of F on $\mathcal{A}_\varepsilon$ must lie in a small neighborhood of (x^*, y^*). This already implies that $(x^*_\varepsilon, y^*_\varepsilon) \in \Omega^\varepsilon_{\mathcal{I}^*}$.

Recalling (4.40), we now claim that $\mathcal{M}_\varepsilon \subset \Omega^\varepsilon_{\mathcal{I}^*}$, restricted to a neighborhood of (x^*, y^*). In other words, we claim that along $\mathcal{M}_\varepsilon$ one has

$$G_\varepsilon(x, y^\varepsilon_{\mathcal{I}^*}(x)) \;\leq\; G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x)) \qquad \text{for all } \mathcal{I} \subseteq \mathcal{J}. \tag{4.44}$$

Using the notation introduced at (4.42), for $\mathcal{I} \notin \{\mathcal{I}^*, \mathcal{I}_1, \ldots, \mathcal{I}_q\}$ we have

$$\lim_{\varepsilon\to 0+} \frac{z^\varepsilon(\zeta^*) - z^*}{\varepsilon} \;=\; \overline{z}\,, \tag{4.45}$$

$$
\begin{aligned}
&\lim_{\varepsilon\to 0+} \frac{G_\varepsilon\big(x^\varepsilon(\zeta^*), y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta^*))\big) - G_\varepsilon\big(x^\varepsilon(\zeta^*), y^\varepsilon_{\mathcal{I}}(x^\varepsilon(\zeta^*))\big)}{\varepsilon} \\
&\quad = \ D_yG(x^*,y^*)\cdot \lim_{\varepsilon\to 0+} \frac{y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta^*)) - y^\varepsilon_{\mathcal{I}}(x^\varepsilon(\zeta^*))}{\varepsilon} + \Big[\widetilde{G}(x^*, y_{I^*}(x^*)) - \widetilde{G}(x^*, y_I(x^*))\Big] \\
&\quad = \ \big(\Lambda_{\mathcal{I}^*}(\bar z) - \lambda_{\mathcal{I}^*}\big) - \big(\Lambda_{\mathcal{I}}(\bar z) - \lambda_{\mathcal{I}}\big) \ < \ 0,
\end{aligned}
\tag{4.46}
$$

because we are assuming that $\widetilde{G}$ satisfies the generic conditions stated in Lemma 4.1. If we replace ζ^* with a nearby value ζ, the limit (4.45) will take some value z close to $\overline{z}$, while the limit (4.46) remains strictly negative. This proves (4.44), as long as $\varepsilon > 0$ remains sufficiently small.

5. Calling $F^\varepsilon_{\mathcal{I}^*}(x) \doteq F(x, y^\varepsilon_{\mathcal{I}^*}(x))$, the first order necessary conditions for the optimality of the point x^*_ε yield

$$
-\nabla F^\varepsilon_{\mathcal{I}^*}(x^*_\varepsilon) \ \in \ \mathcal{N}_{x^*_\varepsilon}, \tag{4.47}
$$

where $\mathcal{N}_{x^*_\varepsilon}$ denotes the outer normal cone to $\Omega^\varepsilon_{\mathcal{I}^*}$ at the point x^*_ε.

To determine this normal cone, we observe that by the previous step only the q constraints

$$
G_\varepsilon(x, y^\varepsilon_{\mathcal{I}_k}(x)) \ \geq \ G_\varepsilon(x, y^\varepsilon_{\mathcal{I}^*}(x)), \qquad\qquad k = 1, \ldots, q, \tag{4.48}
$$

can be active in a neighborhood of x^*. For $\mathcal{I} \subset \mathcal{J}$ call $\mathbf{v}^\varepsilon_{\mathcal{I}}(x)$ the gradient of the composed map

$$
x \ \mapsto \ G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x)).
$$

As $x \to x^*$ and $\varepsilon \to 0+$ we have the convergence

$$
\mathbf{v}^\varepsilon_{\mathcal{I}}(x) \ \to \ \mathbf{v}_{\mathcal{I}},
$$

where, as in (4.20),

$$
\mathbf{v}_{\mathcal{I}} \ = \ (v_1, \ldots, v_N), \qquad\qquad v_i \ = \ \begin{cases} \Big[\frac{d}{dx_i} G_i(x_i, y_i^+(x_i))\Big]_{x_i = x_i^*} & \text{if} \quad i \in \mathcal{I}\,, \\ \Big[\frac{d}{dx_i} G_i(x_i, y_i^-(x_i))\Big]_{x_i = x_i^*} & \text{if} \quad i \in \mathcal{J}\setminus\mathcal{I}\,, \\ \Big[\frac{d}{dx_i} G_i(x_i, y_i(x_i))\Big]_{x_i = x_i^*} & \text{if} \quad i \notin \mathcal{J}\,. \end{cases}
$$

The necessary conditions (4.47) can be written as

$$
-\nabla F^\varepsilon_{\mathcal{I}^*}(x^*_\varepsilon) \ = \ \sum_{k=1}^q c^\varepsilon_k \Big(\mathbf{v}^\varepsilon_{\mathcal{I}_k}(x^*_\varepsilon) - \mathbf{v}^\varepsilon_{\mathcal{I}^*}(x^*_\varepsilon)\Big), \tag{4.49}
$$

where the coefficients of the linear combination satisfy $c^\varepsilon_k \geq 0$ for every k. Moreover, $c^\varepsilon_k > 0$ only if the corresponding constraint in (4.48) is satisfied as an equality.

As $\varepsilon \to 0$ we have $x^*_\varepsilon \to x^*$ and $c^\varepsilon_k \to \bar c_k$, where $\bar c_k$ are the coefficients in the linear combination (4.38). By our generic assumption, the q vectors $\mathbf{v}_{\mathcal{I}_k} - \mathbf{v}_{\mathcal{I}^*}$ are linearly independent and all $\bar c_k$ are strictly positive. Hence the same holds for the vectors $\mathbf{v}^\varepsilon_{\mathcal{I}_k}(x^*_\varepsilon) - \mathbf{v}^\varepsilon_{\mathcal{I}^*}(x^*_\varepsilon)$ and the constants c^ε_k, for $\varepsilon > 0$ small enough. This implies that all constraints in (4.48) are satisfied as equalities, hence $(x^*_\varepsilon, y^*_\varepsilon) \in \mathcal{M}_\varepsilon$.

Finally, (a3) implies that the map $\zeta \mapsto F\big(x^\varepsilon(\zeta), y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta))\big)$ is strictly convex in a neighborhood of ζ^*, hence it has a unique local minimum. By the previous arguments, for $\varepsilon > 0$ small this provides the unique global minimum of F on the entire domain $\mathcal{A}_\varepsilon$.

6. We now discuss the modifications needed to handle the case where the myopic equilibrium is attained at a point $(x^*, y^*) = (x_1^*, \ldots, x_N^*, y_1^*, \ldots, y_N^*)$ on the boundary of the domain $[0,1]^{2N}$. To fix the ideas, consider the sets of indices

$$\begin{array}{lll} \mathcal{J}_0 \doteq \{i\,;\;\; x_i^* = 0\}, & \qquad & \mathcal{J}_0' \doteq \{i\,;\;\; y_i^* = 0\},\\ \mathcal{J}_1 \doteq \{i\,;\;\; x_i^* = 1\}, & \qquad & \mathcal{J}_1' \doteq \{i\,;\;\; y_i^* = 1\}. \end{array}$$

As shown by the analysis in [7], for $G_1, \ldots, G_N$ in an open dense subset of $\mathcal{C}^3([0,1]^{2N}$, all these sets are disjoint.

For every $i \in \mathcal{J}_0 \cap \mathcal{J}_1$, the necessary conditions for optimality imply that at $x = x^*$ the gradient of F satisfies

$$\frac{\partial}{\partial x_i} F(x, y_{\mathcal{I}^*}(x)) \;=\; \begin{cases} \dfrac{\partial F}{\partial x_i} + \dfrac{\partial F}{\partial y_i} \cdot \dfrac{\partial}{\partial x_i} y_{\mathcal{I}^*} \;\geq\; 0 & \text{if} \quad x_i^* = 0,\\[2mm] \dfrac{\partial F}{\partial x_i} + \dfrac{\partial F}{\partial y_i} \cdot \dfrac{\partial}{\partial x_i} y_{\mathcal{I}^*} \;\leq\; 0 & \text{if} \quad x_i^* = 1. \end{cases} \tag{4.50}$$

By possibly performing a further small perturbation of F, always of the form (4.32), we can assume that all the above inequalities are strict.

Similarly, for each $i \in \mathcal{J}_0' \cup \mathcal{J}_1'$, call $\mathcal{A}_i \subset [0,1]^2$ the graph of the best reply map $u_i \mapsto R_i(u_i)$. Let $\mathcal{N}_i^* \subset \mathbb{R}^2$ be the normal cone to $\mathcal{A}_i$ at the point (x_i^*, y_i^*). Since F attains its minimum at (x^*, y^*) restricted to $\mathcal{A}_1 \times \cdots \times \mathcal{A}_N$, the necessary conditions for optimality imply

$$-\mathbf{w}_i \;\in\; \mathcal{N}_i^*\,, \qquad\qquad \mathbf{w}_i \;\doteq\; (F_{x_i}, F_{y_i})(x^*, y^*) \;\in\; \mathbb{R}^2. \tag{4.51}$$

By possibly performing a further small perturbation of F, as in (4.32), we can assume that $-\mathbf{w}_i$ lies strictly in the interior, namely (see Fig. 2, right)

$$-\mathbf{w}_i \;\in\; \text{int}\; \mathcal{N}_i^*. \tag{4.52}$$

As before, in a neighborhood of x^* the implicit equation (4.39) determines 2^q distinct functions $x \mapsto y^\varepsilon_{\mathcal{I}}(x)$, with $\mathcal{I} \subseteq \mathcal{J}$. In the present case, however, these maps may not take values inside $[0,1]^N$. Because of (4.50), for $\varepsilon > 0$ small enough, the minimum of F on the graph $\mathcal{A}_\varepsilon$ of the best reply map will be attained at a point $(x_\varepsilon^*, y_\varepsilon^*)$ with

$$\begin{array}{llll} x^*_{\varepsilon,i} = 0 & \text{for } i \in \mathcal{J}_0, & \qquad y^*_{\varepsilon,i} = 0 & \text{for } i \in \mathcal{J}_0',\\ x^*_{\varepsilon,i} = 1 & \text{for } i \in \mathcal{J}_1\,, & \qquad y^*_{\varepsilon,i} = 1 & \text{for } i \in \mathcal{J}_1'\,. \end{array} \tag{4.53}$$

Let $\mathcal{M}_\varepsilon$ be the manifold in (4.40). The same arguments used in the previous steps now show that the global minimum of F restricted to $\mathcal{A}_\varepsilon$ is attained on the sub-manifold $\mathcal{M}_\varepsilon' \subset \mathcal{M}_\varepsilon$ consisting of all points $(x, y) \in \mathcal{M}_\varepsilon$ which satisfy the additional p identities in (4.53). Notice that $\mathcal{M}_\varepsilon'$ has dimension

$$\dim(\mathcal{M}_\varepsilon') \;=\; \dim(\mathcal{M}_\varepsilon) - p \;=\; N - p - q, \qquad p \;=\; |\mathcal{J}_0| + |\mathcal{J}_1| + |\mathcal{J}_0'| + |\mathcal{J}_1'|, \quad q \;=\; |\mathcal{J}|.$$

Indeed, it consists of all points (x, y) which satisfy the $N + q$ equations

$$y = y^\varepsilon_{\mathcal{I}^*}(x)\,, \qquad G_\varepsilon(x, y^\varepsilon_{\mathcal{I}^*}(x)) \;=\; G_\varepsilon(x, y^\varepsilon_{\mathcal{I}_k}(x)), \quad k = 1, \ldots, q, \tag{4.54}$$

together with the p equations in (4.53). As in (4.41), it is convenient to use the notation $x = (z, \zeta) \in \mathbb{R}^{p+q} \times \mathbb{R}^{N-p-q}$, where

$$z \;=\; (x_i)_{i\in\widehat{\mathcal{J}}}\,, \qquad \zeta \;=\; (x_i)_{i\in\mathcal{J}^\dagger}\,, \qquad \widehat{\mathcal{J}} \;=\; \mathcal{J}\cup\mathcal{J}_0\cup\mathcal{J}_1\cup\mathcal{J}'_0\cup\mathcal{J}'_1\,, \quad \mathcal{J}^\dagger \;=\; \{1, \ldots, N\}\setminus\widehat{\mathcal{J}}.$$

Accordingly, we write $x^* = (z^*, \zeta^*)$. We can now solve the $N+p+q$ equations in (4.53)-(4.54), expressing the variables z, y as functions of ζ. Namely

$$x \;=\; x^\varepsilon(\zeta) \;=\; (z^\varepsilon(\zeta), \zeta), \qquad y \;=\; y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta)). \tag{4.55}$$

This yields a parameterization of $\mathcal{M}'_\varepsilon$ in terms of the $N - p - q$ variables ζ_i, $i \in \mathcal{J}^\dagger$. The unique point $(x^*_\varepsilon, y^*_\varepsilon) \in \mathcal{M}'_\varepsilon \subset [0, 1]^N$, where the global constrained minimum is attained, is determined by the additional $N - p - q$ equations

$$\nabla_\zeta F\Big(x^\varepsilon(\zeta), y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta)\Big) \;=\; 0. \tag{4.56}$$

Assuming that the Hessian matrix of second partial derivatives

$$\left(\frac{\partial^2}{\partial x_j \partial x_k} F_{\mathcal{I}^*}(x^*)\right)_{j,k\in\mathcal{J}^\dagger} \tag{4.57}$$

is strictly positive definite in a neighborhood of (x^*, y^*), we obtain a unique point of global minimum $(x^*_\varepsilon, y^*_\varepsilon)$, for all $\varepsilon > 0$ sufficiently small. □

Corollary 4.1 *There exists an open dense set $\mathcal{S}$ of functions $(F, G_1, \ldots, G_N, \widetilde{G}) \in \mathcal{C}^2(\mathbb{R}^{2N}) \times [\mathcal{C}^3(\mathbb{R}^2)]^N \times \mathcal{C}^3(\mathbb{R}^{2N})$ for which the following holds. The game with costs F and G_ε as in (4.8) admits a unique Stackelberg equilibrium $(x^*_\varepsilon, y^*_\varepsilon)$ for all $\varepsilon > 0$ small enough. Moreover, the map $\varepsilon \mapsto (x^*_\varepsilon, y^*_\varepsilon)$ has $\mathcal{C}^1$ regularity.*

Proof. Let any $(F, G_1, \ldots, G_N, \widetilde{G}) \in \mathcal{C}^2(\mathbb{R}^{2N}) \times [\mathcal{C}^3(\mathbb{R}^2)]^N \times \mathcal{C}^3(\mathbb{R}^{2N})$ be given. By an arbitrarily small $\mathcal{C}^3$ perturbation we can assume that $G_1, \ldots, G_N \in \mathcal{G}$, where $\mathcal{G}$ is the set of functions satisfying the conclusion of Theorem 4.1. Next, we can slightly modify the functions F and $\widetilde{G}$ so that the conclusion of Theorem 4.2 holds. This already proves the density of the set of functions $(F, G_1, \ldots, G_N, \widetilde{G})$ which satisfy the conclusion of the corollary.

The openness of the set $\mathcal{G} \subset \mathcal{C}^3(\mathbb{R}^2)$ was proved in [7]. We also observe that the conditions (a1)–(a3) on F involve strict inequalities, hence are satisfied by an open set of functions. Finally, there is an open set of functions $\widetilde{G}$ which satisfy the conclusion of Lemma 4.1. Combining these facts, we conclude that the set $\mathcal{S}$ is open. □

5 Stackelberg equilibria with a narrow-sighted follower

Now we are ready to study the existence of self-consistent Stackelberg equilibria for a narrow-sighted follower, i.e. with discount factor $r_2 >> 1$. We first formulate the optimization problem for a myopic follower, which yields a cost function in diagonal form. Then, using the results proved in Section 4, we show the generic stability of the Stackelberg equilibrium w.r.t. perturbations of the cost functions L_1, L_2 for the leader and for the follower, respectively. In turn, this will yield the existence of a unique self-consistent Stackelberg equilibrium, as a fixed point of the transformation (1.9). We still denote by $u, v \in [0,1]^N$ the feedbacks adopted respectively by the leader and by the follower, as in (2.1).

Let $\mu = (\mu_1, \ldots, \mu_N) \in \Delta_N$ be a probability distribution on the initial state, and assume that $\mu_i > 0$ for every $i = 1, \ldots, N$. We define the cost function for a myopic follower as

$$G(u,v) \;=\; \sum_{i=1}^{N} \mu_i L_{2,i}(u_i, v_i). \tag{5.1}$$

In this case, the best reply $v^*(u) = (v_1^*, \ldots, v_N^*)(u)$ is the one that minimizes the instantaneous running cost, namely

$$v_i \;\in\; R_i(u_i) \;\doteq\; \Big\{\omega^* \in [0,1]\,;\;\; L_{2,i}(u_i, \omega^*) \;=\; \min_{\omega\in[0,1]} L_{2,i}(u_i, \omega)\Big\}. \tag{5.2}$$

Notice that this does not depend on the initial distribution μ.

On the other hand, if the system is initially in state i, the expected cost to the leading player is then

$$J_{1,i}(u,v) \;\doteq\; E^{x(0)=i}\left[\int_0^{+\infty} r_1 e^{-r_1 t} L_{1,x(t)}\big(u_{x(t)}, v_{x(t)}\big)\,dt\right]. \tag{5.3}$$

A similar computation as in (2.3)–(2.5) now yields

$$J_{1,i} \;=\; L_{1,i} + \frac{1}{r_1}\sum_{j\neq i} \phi_{ij}(J_{1,j} - J_{1,i}), \qquad i \in \{1, \ldots, N\}. \tag{5.4}$$

Recalling (2.7) we can write (5.4) in vector notation:

$$\mathbf{J_1}(u,v) \;=\; \Big(\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}(u,v)\Big)^{-1}\mathbf{L_1}(u,v). \tag{5.5}$$

Given the probability distribution μ on the initial state, the expected cost to the leader is

$$F(u,v) \;\doteq\; \sum_{i=1}^{N} \mu_i\, J_{1,i}(u,v). \tag{5.6}$$

Recalling (5.2), we say that a pair of feedback strategies $(u^*, v^*) = (u_1^*, \ldots, u_N^*, v_1^*, \ldots, v_N^*)$ is a **Stackelberg equilibrium** if (u^*, v^*) is an optimizer for the constrained minimization problem for the leader

$$\min_{(u,v)\in\mathcal{A}_{\text{myopic}}} F(u,v) \tag{5.7}$$

where

$$\mathcal{A}_{\text{myopic}} \;\doteq\; \Big\{(u,v) \in [0,1]^{2N}\;;\; v_i \in R_i(u_i),\; i = 1, \ldots, N\Big\}.$$

Next, we consider the case of a short-sighted follower, whose discount factor is $r_2 >> 1$. If the initial state is $x(0) = i$, the expected cost $J_{2,i}$ was computed at (2.2)–(2.8). More generally, given a probability distribution μ on the initial state, the expected cost is computed as

$$\sum_{i=1}^{N} \mu_i J_{2,i}(u,v) \;=\; \mu^T\Big(\mathbf{I} - \frac{1}{r_2}\mathbf{\Phi}(u,v)\Big)^{-1}\mathbf{L_2}(u,v).$$

Setting $\varepsilon = r_2^{-1}$, this can be written in the form

$$G_\varepsilon(u,v) \;=\; G(u,v) + \varepsilon \widetilde{G}(u,v) + o(\varepsilon), \tag{5.8}$$

where G is the myopic cost in (5.1), while

$$\widetilde{G}(u,v) \;=\; \sum_{i,j} \mu_i \Phi_{ij}(u,v) L_{2,j}(u,v). \tag{5.9}$$

In view of (5.8), we are thus in the same framework studied in Section 4. We shall use the results of the previous section to analyze the uniqueness and stability of the Stackelberg equilibrium, for generic cost functions L_1, L_2 and transition intensity functions ϕ_{ij}, in the case of a narrow-sighted follower with $r_2 >> 1$.

Remark 5.1 It is important to observe that, while the cost functions G, G_ε for the follower depend on the initial probability distribution μ, the best reply does not. Indeed, for every $\varepsilon = r_2^{-1} > 0$ the follower solves a stochastic optimization problem, and an optimal feedback $v^*_\varepsilon = (v^*_{\varepsilon,1}, \ldots, v^*_{\varepsilon,N})$ is simultaneously optimal for every $\mu \in \Delta_N$. When studying the generic structure of the best reply map for the follower, it is thus not restrictive to assume that the probability distribution on the initial state is

$$\overline{\mu} \;=\; \left(\frac{1}{N}, \;\ldots\;, \frac{1}{N}\right). \tag{5.10}$$

On the other hand, one should be aware that, for $\varepsilon > 0$, the optimal strategy of the leader does depend on the initial distribution $\mu \in \Delta_N$, in general.

5.1 Generic stability of the Stackelberg equilibrium.

Our next goal in this section is to show that, for an open dense set of functions $L_{1,i} \in \mathcal{C}^2(\mathbb{R}^2)$ and $L_{2,i} \in \mathcal{C}^3(\mathbb{R}^2)$, and transition functions $\phi_{ij} \in \mathcal{C}^3(\mathbb{R}^2)$, the results in Theorem 4.2 can be applied to the cost functions G, F, G_ε in (5.1), (5.6), and (5.8).

Theorem 5.1 *There exists an open dense set $\mathcal{F}$ of cost functions $L_{1,i} \in \mathcal{C}^2(\mathbb{R}^2)$, $L_{2,i} \in \mathcal{C}^3(\mathbb{R}^2)$ and transition functions $\phi_{ij} \in \mathcal{C}^2(\mathbb{R}^2)$ with $\phi_{ij}(u_i, v_i) \geq 0$, such that, for $(L_1, L_2, \phi) \in \mathcal{F}$, the game with cost functions F, G in (5.6), (5.1), modeling a myopic follower, has a unique Stackelberg equilibrium.*

In addition, for every $(L_1, L_2, \phi) \in \mathcal{F}$ and every probability distribution $\mu \in \Delta_N$ on the initial state, there exists $\varepsilon_0 > 0$ small enough so that, for any $0 < \varepsilon \leq \varepsilon_0$, the game with cost functions F, G_ε in (5.6), (5.8), modeling a narrow-sighted follower, has a unique Stackelberg equilibrium.

All these solutions are stable w.r.t. small perturbations of the cost functions $L_{1,i} \in \mathcal{C}^2$, $L_{2,i} \in \mathcal{C}^3$, and of the transition functions $\phi_{ij} \in \mathcal{C}^3$, respectively.

Proof. 1. Let any $(\mathbf{L}_1, \mathbf{L}_2, \phi) \in \mathcal{C}^2 \times \mathcal{C}^3 \times \mathcal{C}^2$ be given, with $\phi_{ij} \geq 0$ for all $i, j \in \{1, \ldots, N\}$.

By Theorem 4.1 there exists an open dense set $\mathcal{G} \subset \mathcal{C}^3(\mathbb{R}^2)$ such that, for $G_i \in \mathcal{G}$, the best reply maps $u_i \mapsto R_i(u_i)$ have the regularity properties listed in (4.3)–(4.4). We now observe that, in the myopic case, the cost function (5.1) for the follower has the diagonal form (4.6), with $G_i(u_i, v_i) = \mu_i L_{2,i}(u_i, v_i)$. Hence, by an arbitrarily small $\mathcal{C}^3$ perturbation of each of the functions $L_{2,i}$ we can achieve $G_i \in \mathcal{G}$ for $i = 1, \ldots, N$.

2. Let now (u^*, v^*) be a (possibly non-unique) Stackelberg equilibrium. According to the analysis in Section 4, there exists an open dense set of functions $(F, \widetilde{G}) \in \mathcal{F} \subset \mathcal{C}^2 \times \mathcal{C}^3$ such that the conclusion of Theorem 4.2 holds. In the steps **3** $-$ **5** of the proof we will show that, by performing arbitrarily small perturbations of the running costs $L_{1,i}, L_{2,i}$ and of the transition functions ϕ_{ij}, the corresponding cost functions $(F, \widetilde{G}) \in \mathcal{F}$ satisfy the conclusion of Theorem 4.2, with F given by (5.6) and $\widetilde{G}$ as in (5.9).

3. As observed in Remark 5.1, to study the best reply map it is not restrictive to assume that the initial probability distribution is $\overline{\mu} = (\frac{1}{N}, \ldots, \frac{1}{N})$. As in Section 4, call $\mathcal{J} \subset \{1, \ldots, N\}$ the set of indices i such that the best reply map R_i has a jump at $u_i = u_i^*$, say from v_i^- to v_i^+. Setting $q = |\mathcal{J}|$, for each of the 2^q subsets $\mathcal{I} \subseteq \mathcal{J}$, define the function $u \mapsto v_{\mathcal{I}}(u)$ as in (4.19), with (x, y) replaced by (u, v). Moreover, recalling (2.9), consider the 2^q linear functions $\Lambda_{\mathcal{I}}(z)$ as in (4.20), and the scalar numbers

$$\lambda_{\mathcal{I}} \doteq \sum_{i,j} \frac{1}{N} \Phi_{ij}(u_i^*, v_i^{\pm}) L_{2,j}(u_j^*, v_j^{\pm}), \tag{5.11}$$

with the understanding that we are taking the value v_k^+ for $k \in \mathcal{I}$ and the value v_k^- for $k \in \mathcal{J} \setminus \mathcal{I}$, while $v_k^- = v_k^+$ for $k \notin \mathcal{J}$.

By slightly changing the transition functions ϕ_{ij} at the points $(u_i^*, v_i^{\pm})$, and by possibly performing a further small perturbation of the cost functions to the follower, we can ensure that the numbers $\lambda_{\mathcal{I}}$ in (5.11) satisfy the generic conditions in Definition 4.2.

4. According to (5.5), for every initial probability distribution $\mu \in \Delta_N$, the cost function for the leader can be written as

$$F^{\mu}(u, v) = \sum_{i=1}^{N} c_i(u, v) L_{1,i}(u_i, v_i), \tag{5.12}$$

where

$$c_i(u, v) = \sum_{k=1}^{N} \mu_k \left[\left(\mathbf{I} - \frac{1}{r_1} \mathbf{\Phi}(u, v) \right)^{-1} \right]_{ki} > 0.$$

Notice that $c_i(u, v)$ is independent of the cost function $\mathbf{L}_1$ and only depends on μ and on the transition functions $\phi_{ij}(u, v)$. The strict positivity of c_i follows from the fact that the Markov chain is irreducible and positive recurrent.

5. We now analyze how a small $\mathcal{C}^\infty$ perturbation of the functions $L_{1,i}$ affects the cost function $F = F^{\mu}$ in (5.12).

Fix a cutoff function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ such that

$$\varphi(x, y) \in [0, 1], \qquad \varphi(x, y) = 1 \quad \text{for all } (x, y) \in [0, 1]^2. \tag{5.13}$$

Assume that the cost functions $L_{1,i}$ for the leader are replaced by

$$L^\sharp_{1,i}(u_i,v_i) \;=\; L_{1,i}(u_i,v_i)+\Big(\varepsilon'\big(|u_i-u_i^*|^2+|v_i-v_i^*|^2\big)+a_i(u_i-u_i^*)+b_i(v_i-v_i^*)\Big)\,\varphi(u_i,v_i), \quad (5.14)$$

for some small constants $\varepsilon'>0$ and $a_i,b_i\in\mathbb{R}$, $i=1,\ldots,N$. Calling

$$F^\sharp(u,v) \;=\; \sum_{i=1}^N c_i(u,v)L^\sharp_{1,i}(u_i,v_i),$$

we find

$$\begin{cases} \partial_{u_i}F^\sharp(u^*,v^*)-\partial_{u_i}F(u^*,v^*) \;=\; c_i(u^*,v^*)a_i\,,\\ \partial_{v_i}F^\sharp(u^*,v^*)-\partial_{v_i}F(u^*,v^*) \;=\; c_i(u^*,v^*)b_i\,. \end{cases} \quad (5.15)$$

Since $c_i>0$, from (5.15) it follows that, by taking any $\varepsilon'>0$, the conditions (a1) and (a3) in the proof of Theorem 4.2 are satisfied. Moreover, by suitably choosing the constants a_i,b_i in (5.14), we can perform arbitrary modifications of the gradient of the function F at the point (u^*,v^*), and achieve condition (a2) as well. Finally, by a further small modification of the gradient of F, we obtain that the vector $\mathbf{w}$ in (4.34) satisfies (4.35). This establishes our claim made in step **2.**

6. We now recall that the set $\mathcal{F}$ of functions $(F,G,\widetilde{G})$ which satisfy the conclusion of Theorem 4.2 is open in the spaces $\mathcal{C}^2\times\mathcal{C}^3\times\mathcal{C}^3$. In turn, the family of triples

$$(L_{1,i},\,L_{2,i},\,\phi_{ij})_{i,j=1,\ldots,N} \;\in\; \mathcal{C}^2\times\mathcal{C}^3\times\mathcal{C}^3, \quad (5.16)$$

for which the corresponding functions defined at (5.6), (5.1), and (5.9) satisfy $(F,G,\widetilde{G})\in\mathcal{F}$, is open. This completes the proof. □

5.2 Self-Consistent Stackelberg equilibria with a narrow-sighted follower.

Relying on the previous analysis we now prove

Theorem 5.2 *There exists an open dense set $\mathcal{F}\subset\mathcal{C}^2\times\mathcal{C}^3\times\mathcal{C}^3$ such that, if the cost functions and the transition intensities satisfy $(L_1,L_2,\phi_{ij})\in\mathcal{F}$, then for every discount factor $r_2>0$ sufficiently large the stochastic game with dynamics (1.1) and cost functions (1.3) admits a unique self-consistent Stackelberg equilibrium.*

Proof. 1. Let $(L_{1,i},L_{2,i},\phi_{ij})\in\mathcal{F}$ be given, where $\mathcal{F}$ is the open dense set considered in Theorem 5.1. Let $(u^*,v^*)\in[0,1]^{2N}$ be the unique Stackelberg equilibrium corresponding to a myopic follower, i.e. to $\varepsilon=0$. We observe that in this case both the leader and the follower solve an optimal control problem, hence the feedbacks u^*,v^* are simultaneously optimal for every probability distribution on the initial data.

Let μ^∞ the asymptotic stationary probability distribution, for the dynamics (1.1) corresponding to the feedbacks u^*,v^*. When $\varepsilon=0$, μ^∞ trivially achieves the unique self-consistent equilibrium.

2. Next, for $\varepsilon \geq 0$ small, consider the maps

$$\mu \ \mapsto\ (u^*_\varepsilon(\mu), v^*_\varepsilon(\mu)), \qquad\qquad (u,v)\ \mapsto\ \mu^\infty(u,v).$$

Here $(u^*_\varepsilon(\mu), v^*_\varepsilon(\mu))$ are the feedback controls corresponding to the unique Stackelberg equilibrium, with discount factors r_1 and $r_2 = \varepsilon^{-1}$, and with probability distribution μ assigned on the initial state. Moreover, $\mu^\infty(u,v) \in \Delta_N$ is the asymptotic probability distribution as $t \to +\infty$, for the Markov process (1.1) with transition functions $\phi_{ij}(u_i, v_i)$. Under generic assumptions on the functions $\phi_{ij} \geq 0$, this process has has a unique stationary distribution, for every $(u,v) \in [0,1]^{2N}$.

For $\varepsilon > 0$ small, we claim that

$$\frac{\partial}{\partial \mu}\big(u^*_\varepsilon(\mu), v^*_\varepsilon(\mu)\big) \ =\ \mathcal{O}(1)\cdot\varepsilon, \tag{5.17}$$

where the Landau symbol $\mathcal{O}(1)$ denotes a uniformly bounded function. Indeed, recalling the notation used at (4.56), we can write

$$u \ =\ u^\varepsilon(\zeta) \ =\ (z^\varepsilon(\zeta), \zeta), \qquad\qquad v \ =\ v^\varepsilon_{\mathcal{I}^*}(u^\varepsilon(\zeta)), \tag{5.18}$$

where

$$\zeta \ =\ (\zeta_i)_{i\in\mathcal{J}^\dagger}\,, \qquad\qquad \mathcal{J}^\dagger \doteq \{1,\ldots,N\}\setminus(\mathcal{J}\cup\mathcal{J}_0\cup\mathcal{J}_1\cup\mathcal{J}_0'\cup\mathcal{J}_1')$$

is the variable along the $(N-p-q)$-dimensional manifold $\mathcal{M}'_\varepsilon$.

This yields a parameterization of $\mathcal{M}'_\varepsilon$ in terms of the $N-p-q$ variables ζ_i, $i \in \mathcal{J}^\dagger$. The unique point $(u^*_\varepsilon(\mu), v^*_\varepsilon(\mu)) \in \mathcal{M}'_\varepsilon$, where the global constrained minimum of F^μ is attained, is determined by the additional $N-p-q$ equations

$$\nabla_\zeta F^\mu\big(u^\varepsilon(\zeta), v^\varepsilon_{\mathcal{I}^*}(u^\varepsilon(\zeta)\big) \ =\ 0. \tag{5.19}$$

In order to prove (5.17) it suffices to show that, at a point where (5.19) holds, one has

$$\frac{\partial}{\partial\mu_j}\frac{\partial}{\partial\zeta_i}F^\mu\big(u^\varepsilon(\zeta), v^\varepsilon_{\mathcal{I}^*}(u^\varepsilon(\zeta)\big) \ =\ \mathcal{O}(1)\cdot\varepsilon, \tag{5.20}$$

for every $i \in \mathcal{J}^\dagger$, $j \in \{1,\ldots,N\}$, and $\varepsilon \geq 0$ sufficiently small.

To prove (5.20), we first observe that in (5.11) the initial probability distribution μ enters linearly in the function F^μ. Moreover, the function $c_i(u,v)$ is $\mathcal{C}^3$ w.r.t. all variables, and all functions $L_{1,i}(u_i,v_i)$, $u^\varepsilon(\zeta)$ and $v^\varepsilon_{\mathcal{I}^*}(u)$ have $\mathcal{C}^2$ regularity. Thus we conclude that the left hand side of (5.20) is a $\mathcal{C}^1$ function of ζ.

When $\varepsilon = 0$, i.e. in the myopic case, the best reply is decoupled: $v^*(u) = (v_1^*(u_1),\ldots,v_N^*(u_N))$. As a result, under the best reply, at any state i the running cost and the dynamics depend only on u_i. Thus the leader's optimization problem becomes a standard stochastic optimal control problem and this shows that the Stackelberg equilibrium (u^*, v^*) is independent of the initial distribution μ. Under our generic assumptions, for $\varepsilon > 0$ small enough the best reply can now be written as

$$v^*_{\varepsilon,i}(u) \ =\ v^*_i(u_i) + \varepsilon\widetilde{v}_i(u) + o(\varepsilon), \tag{5.21}$$

where the second term depends simultaneously on all components of u. In turn, the optimal control for the leader has the expansion

$$u^*_\varepsilon(\mu) = u^* + \varepsilon\tilde{u}(\mu) + o(\varepsilon).$$

Combining above estimates, one achieves (5.20).

4. Next, we observe that the map $(u, v) \mapsto \mu^\infty(u, v)$ is Lipschitz continuous. By (5.17), for all $\varepsilon > 0$ sufficiently small the composed map

$$\mu \;\mapsto\; \mu^\infty\big(u^*_\varepsilon(\mu), v^*_\varepsilon(\mu)\big)$$

is a strict contraction and thus has a unique fixed point. By definition, this provides the self-consistent Stackelberg equilibrium. □

6 A far-sighted leader

In this section we consider the case where the discount factor $r_1 \geq 0$ of the leader is very small. Meanwhile, the discount factor $r_2 > 0$ is any positive number.

In this section we shall assume that all the conditions in Lemma 2.2 and Lemma 2.3 hold. In particular, these imply that the best reply map $u \mapsto v^*(u)$ is single-valued and satisfies the properties **(A1)** stated in Section 3.

For $(u, v) \in [0, 1]^N \times [0, 1]^N$, set

$$F^\infty(u, v) \;=\; \sum_{i=1}^{N} \mu_i^\infty(u, v) \cdot L_i(u_i, v_i). \tag{6.22}$$

where $\mu^\infty(u, v)$ is the asymptotic probability distribution for the dynamics (1.1), determined by the feedback controls (u, v).

Lemma 6.1 *Let ϕ and $\mathbf{L}_2$ satisfy the same assumptions as in Lemmas 2.2 and 2.3. Then there exists an open dense set of functions $\mathbf{L}_{1,i} \in \mathcal{C}^2(\mathbb{R}^2)$ such that the optimization problem*

$$\min_{u\in[0,1]^N} F^\infty(u, v^*(u)) \tag{6.23}$$

has a unique minimizer $u^* = (u^*_1, \ldots, u^*_N) \in [0, 1]^N$ in generic position. More precisely, the following implications hold:

$$\begin{cases} u^*_i = 0 & \Longrightarrow \quad \dfrac{\partial}{\partial u_i} F(u, v^*(u))\Big|_{u=u^*} > 0, \\ u^*_i = 1 & \Longrightarrow \quad \dfrac{\partial}{\partial u_i} F(u, v^*(u))\Big|_{u=u^*} < 0, \end{cases} \tag{6.24}$$

In addition, calling $I = \{i \in \{1, \ldots, N\}\,;\;\; 0 < u_i < 1\}$, the Hessian matrix of second derivatives

$$\left(\frac{\partial^2}{\partial u_i \partial u_j} F^\infty(u, v^*(u))\right)_{i,j\in I}$$

computed at $(u^*, v^*(u^*))$ is strictly positive definite.

Proof. Indeed, all the above conditions can be achieved by an arbitrarily small, smooth perturbation of the functions $L_{1,i}$, as in (5.14). In this case, one could even use a simpler kind of perturbations, having the form

$$L^\flat_{1,i}(u_i,v_i) \;=\; L_{1,i}(u_i,v_i) + \Big(\varepsilon'|u_i-u_i^*|^2 + a_i(u_i-u_i^*)\Big)\,\varphi(u_i,v_i). \tag{6.25}$$

□

Theorem 6.1 *Let ϕ and $\mathbf{L}_2$ satisfy the same assumptions as in Lemmas 2.2 and 2.3. Then there exists an open dense set of functions $\mathbf{L}_1 \in \mathcal{C}^2(\mathbb{R}^2)\times\ldots\times\mathcal{C}^2(\mathbb{R}^2)$ such that for all discount exponents $r_1>0$ sufficiently small a self-consistent Stackelberg equilibrium exists.*

Proof. For any initial probability distribution $\mu\in\Delta_N$ and $r_1>0$, define the function

$$J^\mu_{1,r_1}(u,v) \doteq E^\mu\Big[\int_0^\infty r_1 e^{-r_1 t} L_{1,x(t)}(u_{x(t)},v_{x(t)})\,dt\Big].$$

Recalling (3.3), we have

$$J^\mu_{1,r_1} \;=\; \mu\cdot r_1\Big(r_1\mathbf{I}-\mathbf{\Phi}\Big)^{-1}\cdot\mathbf{L}_1. \tag{6.26}$$

Call $\mathbf{A}$ the adjunct matrix of $\mathbf{\Phi}$, defined as the transpose of the cofactor matrix:

$$[\mathbf{A}]_{ij} = (-1)^{i+j}M_{ji}\,,$$

where M_{ij} is the (i,j)-minor of order $n-1$ of $\mathbf{\Phi}$. Since the continuous-time Markov chain is irreducible and positive recurrent, $\mathbf{\Phi}$ has rank $N-1$. As a result, both the left and right null space of $\mathbf{\Phi}$ have dimension 1, being spanned by the vectors $\mu^\infty = (\mu_1^\infty,\ldots,\mu_N^\infty)$ and $(1/N,\ldots,1/N)^T$, respectively. Since

$$\mathbf{\Phi}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{\Phi} = \mathbf{0},$$

for all $i,j=1,\ldots,N$ one has

$$[\mathbf{A}]_{ij} \;=\beta\,\mu_j^\infty,$$

for some constant β. To determine β, let

$$-\lambda_N \;\le\;\cdots\;\le\; -\lambda_2 \;<\; -\lambda_1 \;=\; 0$$

be the eigenvalues of $\mathbf{\Phi}$. For $r_1>0$ small we then have the expansion

$$r_1\Big(r_1\mathbf{I}-\mathbf{\Phi}(u,v)\Big)^{-1} \;=\; \frac{1}{\prod_{i=2}^N\lambda_i}\mathbf{A}(u,v) + r_1\widetilde{\mathbf{\Phi}}(u,v) + o(r_1),$$

for a suitable function $\widetilde{\mathbf{\Phi}}$, whose expression can be derived from $\mathbf{\Phi}$. Since

$$\mu^\infty\cdot(r_1\mathbf{I}-\mathbf{\Phi}) \;=\; r_1\mu^\infty \qquad\Longrightarrow\qquad \mu^\infty \;=\; \mu^\infty\cdot r_1(r_1\mathbf{I}-\mathbf{\Phi})^{-1},$$

in the limit $r_1\to 0+$ one has the implication

$$\mu^\infty = \frac{1}{\prod_{i=2}^N\lambda_i}\mu^\infty\cdot\mathbf{A} \qquad\Longrightarrow\qquad \beta = \prod_{i=1}^N\lambda_i.$$

Thus by setting $\varepsilon = r_1$, the cost function has the expansion

$$J^{\mu}_{1,\varepsilon}(u,v) \;=\; \frac{1}{\prod_{i=2}^{N}\lambda_i}\mu\cdot\mathbf{A}\cdot\mathbf{L}_1 + \varepsilon\mu\cdot\tilde{\mathbf{\Phi}}\cdot\mathbf{L}_1 + o(\varepsilon) \;=\; F^{\infty}(u,v) + \varepsilon\widetilde{F}(u,v) + o(\varepsilon),$$

where $\widetilde{F} = \mu\cdot\tilde{\mathbf{\Phi}}\cdot\mathbf{L}_1$.

Call u^* the minimizer in (6.23) and v^* the unique best reply $v^*(u^*)$. Due to the form of F^{∞} defined in (6.22) and the fact that $\mu_i^{\infty} > 0$, we can perturb the cost functions $L_{1,i}$ as in (6.25), and achieve arbitrary modifications of the gradient of the function F^{∞} at point (u^*, v^*). As a result, all conditions (a1)-(a3) in the proof of Theorem (4.2) will be satisfied.

By the stability results in [7], for all $r_1 = \varepsilon$ small enough, the minimization problem

$$\min_{u\in[0,1]^N} J^{\mu}_{1,\varepsilon}(u, v^*(u)) \tag{6.27}$$

has a unique global minimizer $u^*_{\varepsilon}(\mu)$.

When $\varepsilon = 0$, by construction $\mu^{\infty}(u^*, v^*)$ is the asymptotic probability distribution, hence this achieves the self-consistent equilibrium. Also when $\varepsilon = 0$, the Stackelberg equilibrium does not depend on the initial distribution μ, i.e. $u^*_0(\mu) = u^*$.

For any fixed $\mu \in \Delta_N$, the global minimizer $u^*_{\varepsilon}(\mu)$ of (6.27) is determined by the equations

$$\nabla_u J^{\mu}_{1,\varepsilon}\Big(u, v^*_{\varepsilon}(u)\Big) \;=\; 0.$$

Similar arguments as in the proof of (5.20), now yield

$$\frac{\partial}{\partial\mu_j}\frac{\partial}{\partial u_i} J^{\mu}_{1,\varepsilon}\Big(u^*_{\varepsilon}(\mu), v^*_{\varepsilon}(u^*_{\varepsilon}(\mu))\Big) \;=\; \mathcal{O}(1)\cdot\varepsilon, \tag{6.28}$$

for any pair $(i,j) \in \{1,\ldots,N\}^2$. Since in the limit $\varepsilon = 0$ the left hand side of (6.28) is zero, by continuity we achieve

$$\frac{\partial}{\partial\mu} u^*_{\varepsilon}(\mu) \;=\; \mathcal{O}(1)\cdot\varepsilon.$$

In turn, the composed map

$$\mu \;\mapsto\; u^*_{\varepsilon}(\mu) \;\mapsto\; \mu^{\infty}\Big(u^*_{\varepsilon}(\mu), v^*(u^*_{\varepsilon}(\mu))\Big)$$

is a strict contraction, hence it has a unique fixed point. □

References

[1] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[2] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, reprint of the second edition, SIAM, Philadelphia, 1999.

[3] A. Bensoussan, S. Chen, and S. P. Sethi, Feedback Stackelberg solutions of infinite-horizon stochastic differential games. In: Models and Methods in Economics and Management Science, 3–15, *Internat. Ser. Oper. Res. Management Sci.*, **198**, Springer, 2014.

[4] A. Bensoussan, S. Chen, and S. P. Sethi, The maximum principle for global solutions of stochastic Stackelberg differential games. *SIAM J. Control Optim.* **53** (2015), 1956–1981.

[5] A. Bressan and B. Piccoli, *Introduction to the Mathematical Theory of Control*, AIMS Series in Applied Mathematics, Springfield Mo. 2007.

[6] A. Bressan and D. Wei, Stackelberg solutions of feedback type for differential games with random initial data. *Dynamic Games & Appl.* **3** (2013), 341–358.

[7] A. Bressan and Y. Jiang, On the generic structure and stability of Stackelberg equilibria, *J. Optim. Theory Appl.*, to appear.

[8] E J. Dockner, S. Jorgensen, N V. Long, and G. Sorger, *Differential Games in Economics and Management Science*, Cambridge University Press, 2000.

[9] M. Jungers, E. Trélat, and H. Abou-Kandil, Min-max and min-min Stackelberg strategies with closed-loop information structure, *J. Dyn. Control Syst.* **17** (2011) 387–425.

[10] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, 1975.

[11] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York, 1993.

[12] T. Li and S. Sethi, A review of dynamic Stackelberg game models. *Discr. Contin. Dyn. Syst. - B*, **22** (2017), 125–159.

[13] J. Medanic, Closed-loop Stackelberg strategies in linear-quadratic problems, *IEEE Trans. Automat. Control* **23** (1978) 632–637.

[14] G P. Papavassilopoulos and J B. Cruz, Nonclassical control problems and Stackelberg games, *IEEE Trans. Automat. Control* **24** (1979) 155–166.

[15] R. J. Plemmons, M-matrix characterizations. I – Nonsingular M-matrices, *Linear Algebra Appl.* **18** (1977), 175–188.

[16] William J. Anderson, *Continuous-Time Markov Chains.* Springer-Verlag, New York, 1991.