Dynamic Stability of the Nash Equilibrium for a Bidding Game

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Abstract

A one-sided limit order book is modeled as a noncooperative game for several players. An external buyer asks for an amount $X > 0$ of a given asset. This amount will be bought at the lowest available price, as long as the price does not exceed a upper bound $\overline{P}$. One or more sellers offer various quantities of the asset at different prices, competing to fulfill the incoming order. The size X of the order and the maximum acceptable price $\overline{P}$ are not a priori known, and thus regarded as random variables. In this setting, we prove that a unique Nash equilibrium exists, where each seller optimally prices his assets in order to maximize his own expected profit.

Furthermore, a dynamics is introduced, assuming that each player gradually adjusts his pricing strategy in reply to the strategies adopted by all other players. In the case of (i) infinitely many small players or (ii) two large players with one dominating the other, we show that the pricing strategies asymptotically converge to the Nash equilibrium.

Keywords: bidding game, limit order book, optimal pricing strategy, Nash equilibrium, asymptotic stability.

AMS subject classifications: 49K21, 49J21, 91A06, 91A13, 91A60.

1 Introduction

A bidding game related to a continuum model of the limit order book was recently considered in [4], proving the existence and uniqueness of a Nash equilibrium and determining the optimal strategies for the various agents. In the basic model, it is assumed that an external buyer asks for a random amount $X > 0$ of a given asset. This external agent will buy the amount X at the lowest available price, as long as this price does not exceed an upper bound $\overline{P}$. One or more sellers offer various quantities of this same asset at different prices, competing to fulfill the incoming order, whose size is not known a priori.

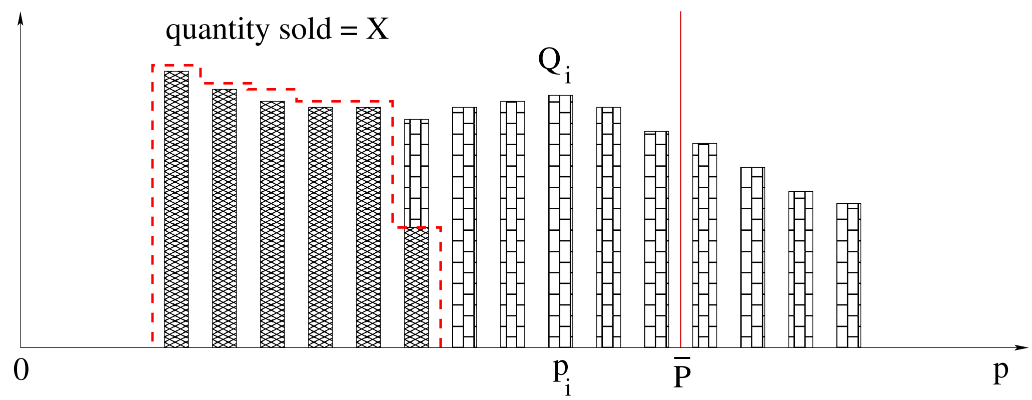

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Figure 1: The height of the various columns indicates the amount of asset offered for sale at the various prices. The buyer will buy a random quantity X, at the lowest possible price, as long as this price does not exceed the (random) upper bound $\overline{P}$.

Having observed the prices asked by his competitors, each player must determine an optimal strategy, maximizing his expected payoff. Because of the presence of the other sellers and of the upper bound $\overline{P}$, asking a higher price for an asset reduces the probability of selling it.

Aim of the present paper is to advance the analysis in BF1 [4] in three main directions:

(i) Consider a more realistic model where the maximum acceptable price $\overline{P}$ is a random variable, not known a priori.

(ii) Study the average reduction in the asked price, resulting from the competition among sellers.

(iii) Introduce a dynamics in the pricing strategies and study the asymptotic stability of the Nash equilibrium.

We assume that the random variable X, describing the amount of stock that the external agent wants to buy, has a distribution function

$$Prob.\{X \geq s\} \;=\; \psi(s) \tag{1.1}$$

psidef

for which the following holds.

(A1) *The map $s \mapsto \psi(s)$ is continuously differentiable and satisfies*

$$\psi(0) \;=\; 1\,, \qquad \psi(+\infty) \;=\; 0, \qquad \psi'(s) \;<\; 0 \qquad \textit{for all } s > 0\,, \tag{1.2}$$

psia1

$$(\ln \psi(s))'' \;\geq\; 0 \quad \textit{for all } s > 0\,. \tag{1.3}$$

typea

For example, the probability distributions determined by

$$\psi_1(s) \;=\; e^{-\lambda s} \qquad\qquad \lambda > 0\,, \tag{1.4}$$

pr1

$$\psi_2(s) \;=\; \frac{1}{(1+s)^\alpha} \qquad\qquad \alpha > 0\,,$$

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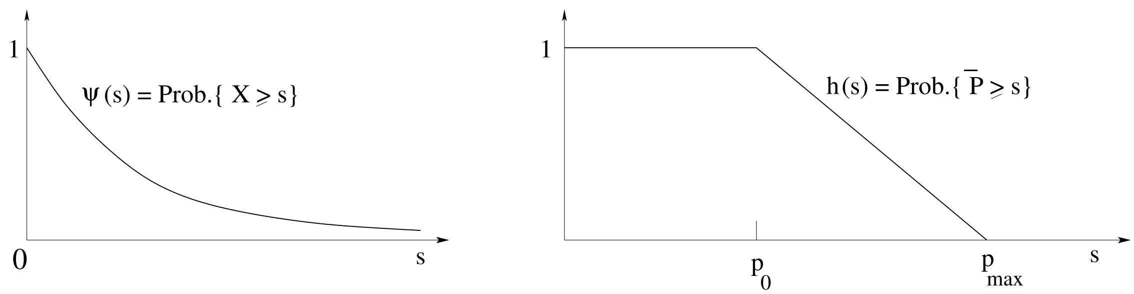

Figure 2: Left: a probability distribution for the random variable X, describing the size of the incoming order. Right: a probability distribution for the random variable $\overline{P}$, describing the maximum price that the buyer is willing to pay.

satisfy (1.3), while $\psi_3(s) = e^{-s^2}$ does not.

Differently from [4, 5, 6], we here assume that the maximum price $\overline{P}$ that the buyer is willing to pay is not known a priori. We thus model $\overline{P}$ as a random variable, independent of X, with a distribution function

$$h(s) \doteq Prob.\{\overline{P} \geq s\} \tag{1.5}$$ vpdef

which satisfies the following assumptions.

(A2) *The function $s \mapsto h(s)$ is continuous, continuously differentiable for $s \in]p_0, p_{max}[$, and satisfies*

$$\left\{\begin{array}{rclll} h(s) & = & 1 & \textit{for} & s \leq p_0\,, \\ h(s) & = & 0 & \textit{for} & s \geq p_{max}\,, \\ h'(s) & < & 0 & \textit{for} & p_0 < s < p_{max}\,, \\ (\ln h(s))'' & < & 0 & \textit{for} & p_0 < s < p_{max}\,. \end{array}\right. \tag{1.6}$$ ha1

For example, one may assume that the random variable $\overline{P}$ is uniformly distributed over the interval $[p_0, p_{\max}]$. This leads to

$$h(s) = \left\{\begin{array}{cll} 1 & \text{if} & s \leq p_0\,, \\ (p_{max} - s)/(p_{max} - p_0) & \text{if} & s \in [p_0, p_{max}]\,, \\ 0 & \text{if} & s \geq p_{max}\,. \end{array}\right. \tag{1.7}$$ pr3

In our model we assume that the i-th player owns a total amount κ_i of asset, which will be labelled by the variable $\beta \in [0, \kappa_i]$. He can put all of it on sale at a given price, or offer different portions at different prices. His strategy will be described by a nondecreasing function $\phi_i : [0, \kappa_i] \mapsto I\!R$. Here $\phi_i(\beta)$ is the price asked for the asset β.

In the first part of the paper we prove the existence of a unique Nash equilibrium [9], where the pricing strategy of each player yields the maximum expected payoff, given the strategies adopted by all other players. An explicit formula for these pricing strategies is provided.

We also consider the limit of these Nash equilibria, as the number of players $n \to \infty$, while the total amount of assets put on sale remains bounded. This part of our analysis extends the earlier results in [4] to the case where the upper bound $\overline{P}$ is a random variable.

In Section 4 we study how the average asked price decreases as a result of the competition among sellers. As shown in Theorem 5, the pricing strategies in the Nash equilibria satisfy:

- If n competing agents put on sale different amounts $\kappa_1 \leq \cdots \leq \kappa_n$ of asset, the average price is larger then in the case where each agent offers the same amount $(\kappa_1 + \cdots + \kappa_n)/n$.
- For n competing agents, each putting on sale the same amount κ/n of asset, the average price decreases as either n or κ increase.
- For a fixed number of sellers $n \geq 2$, if each agent has the same amount κ/n of asset to put on sale, the average price approaches p_0 as $\kappa \to \infty$.

In Section 5 we introduce a dynamics, describing how the pricing strategies may evolve in time, if they are away from a Nash equilibrium. More precisely, let

$$J_i(\phi(\beta), \beta) \;=\; [\text{profit from the sale of asset } \beta] \times [\text{probability of selling asset } \beta]$$

be the expected payoff for the i-th player, achieved by putting asset β on sale at price $\phi(\beta)$. If $\frac{\partial}{\partial \phi} J_i(\phi(\beta), \beta) \neq 0$, then this expected payoff can be increased by suitably modifying the asked price $\phi(\beta)$. We thus consider the following systems of evolution equations, corresponding to a gradient flow:

$$\frac{\partial}{\partial t}\phi_i(\beta, t) \;=\; \frac{\partial}{\partial \phi} J_i(\phi_i(\beta, t), \beta) \qquad\qquad i = 1, \ldots, n. \tag{1.8}$$

Notice that, if $\phi(\beta)$ yields the maximum expected payoff, then the necessary conditions yield $\frac{\partial}{\partial \phi} J_i(\phi(\beta), \beta) \;=\; 0$, and the right hand side of (1.8) vanishes. In case of a Nash equilibrium, this is true for every $\beta \in [0, \kappa_i]$ and every i.

Our main concern is the asymptotic behavior of solutions to the system (1.8). In the cases of (i) infinitely many small players and (ii) two large players, with initial strategies satisfying a specific inequality assumption, we prove that, as $t \to \infty$, the pricing strategies asymptotically converge to the unique Nash equilibrium. On the other hand, for any number $n \geq 2$ of players, if the initial strategies have disjoint price ranges, we show that the solution to the system (1.8) converges to a different limit.

In addition to the classical paper [9], for an introduction to non-cooperative games and Nash equilibria we refer to [3, 8, 14, 15].

In the case where sellers have different beliefs about the fundamental value of the asset and on the distribution of the random order X, the equilibrium pricing strategies have been studied in [5, 6].

In the literature on mathematical finance, various models of the limit order book have been recently studied, mainly from the point of view of the agents who submit the limit orders. In [11, 13, 7] prices range over a discrete set of values, while in [10, 12, 1] prices are continuous and the shape of the limit order book is described by a density function. An important achievement of these models is that, as soon as the shape of the limit order book is given, this in turn determines a corresponding *price impact function*, describing how the bid and ask prices change after the execution of a market order.

In the present model, as well as in [4, 5, 6], prices are allowed to vary in a continuum of values but the shape of the limit order book is not given *a priori*. Indeed, this shape can be endogenously determined as the unique Nash equilibrium, resulting from the optimal pricing strategies implemented by the selling agents.

2 The optimization problem for a single player

Consider an agent offering an amount κ of assets for sale. By a **pricing strategy** we mean any nondecreasing function $\phi : [0,\kappa] \mapsto I\!R$. Here $\phi(\beta)$ is the price asked for asset $\beta \in [0,\kappa]$. We assume that this new seller competes with several other sellers already present on the market. To model this situation, we consider the nondecreasing function

$$p \;\mapsto\; \Phi(p) \;=\; [\text{total amount of assets put on sale at price} \leq p \text{ by all other agents}]. \tag{2.1}$$

Phi

In this case, is the new seller adopts the pricing strategy ϕ, his expected payoff will be

$$\begin{aligned} J(\phi) \;&=\; \int_0^\kappa [\text{profit from the sale of asset } \beta] \times [\text{probability of selling asset } \beta]\; d\beta \\ &=\; \int_0^\kappa [\phi(\beta) - p_0] \cdot \Big[\psi(\beta + \Phi(\phi(\beta))) \cdot h(\phi(\beta))\Big]\, d\beta\,. \end{aligned} \tag{2.2}$$

po2

Remark 1. We regard p_0 as the fundamental value of the asset. To every agent, keeping the asset or selling it at price p_0 is indifferent. A profit is achieved only by selling at a higher price.

Remark 2. In the case where two or more sellers put a positive amount of assets for sale at exactly the same price p^*, one needs to specify who sells it first. In our model, this happens when Φ has an upward jump at p^*, and the set $\{\beta\,;\;\; \phi(\beta) = p^*\}$ has positive measure. By taking Φ left continuous at p^* we model the case where the new seller has priority (i.e., his assets priced at p^* are sold before those of the other agents). By taking Φ right continuous at p^* we model the case where the other agents have priority.

In the case of a Nash equilibrium, however, this situation never happens. Indeed, since in our model the prices range continuously over the interval $[p_0, p_{max}]$, the agent which does not have priority can always improve his expected payoff by selling at a slightly lower price $p^* - \varepsilon$.

In this section we derive necessary and sufficient conditions in order that the pricing strategy ϕ be optimal. From the modeling assumptions (A2) it it obvious that an optimal strategy should satisfy

$$p_0 \;<\; \phi(\beta) \;<\; p_{max}\,. \tag{2.3}$$

p0max

Indeed, selling at price $\leq p_0$ can only produce a loss, while the probability of selling at price $\geq p_{max}$ is zero. In addition, if the function $p \mapsto \Phi(p)$ is smooth, for each $\beta \in [0,\kappa]$, the optimal price $\phi(\beta)$ will satisfy the necessary condition

$$\frac{\partial}{\partial \phi}[(\phi - p_0) \cdot \psi(\beta + \Phi(\phi)) \cdot h(\phi)] \;=\; 0. \tag{2.4}$$

nc2

Introducing the function

$$G^\beta(p) \;\doteq\; -\frac{\psi\Big(\beta + \Phi(p)\Big)}{\psi'\Big(\beta + \Phi(p)\Big)} \cdot \left(\frac{1}{p - p_0} + \frac{h'(p)}{h(p)}\right), \tag{2.5}$$

Gbd2

we see that (2.4) is equivalent to

$$\Phi'(\phi(\beta)) \; = \; G^{\beta}(\phi(\beta)). \tag{2.6}$$ nc0

Remark 3. From the assumptions (A1)-(A2) it follows that the function

$$Q(p) \; = \; \frac{1}{p-p_0} + \frac{h'(p)}{h(p)}$$

is strictly decreasing on the open interval $]p_0, p_{max}[$ and there exists a unique point $p^* \in]p_0, p_{max}[$ such that

$$Q(p^*) \; = \; 0. \tag{2.7}$$ p*

Moreover, on the interval $[p_0, p^*]$ where $Q \geq 0$, the assumption (1.3) implies

$$\frac{\partial}{\partial \beta} \, G^{\beta}(p) \; \geq \; 0\,. \tag{2.8}$$ Gb+

In the special case where $\psi(s) = e^{-\lambda s}$, the formula (2.5) simplifies to

$$G(p) \; \doteq \; \frac{1}{\lambda} \left(\frac{1}{p-p_0} + \frac{h'(p)}{h(p)} \right) . \tag{2.9}$$ Gdef

Notice that in this case the right hand side is independent of β. We also observe that, if h is the function in (1.7), then $p^* = (p_0 + p_{max})/2$.

The following theorem extends the necessary condition (2.6) to the case where Φ is a nondecreasing function. Since the proof is the same as for Theorem 4.2 in [4], we omit details.

Theorem 1 (necessary conditions for optimality). *Let the functions ψ, h satisfy the assumptions (A1)-(A2), and let $\Phi : I\!R_+ \mapsto I\!R_+$ be a nondecreasing map. If $\phi : [0, \kappa] \mapsto [p_0, p_{max}]$ is an optimal pricing strategy, then for almost every $\beta \in [0, \kappa]$, setting $p \doteq \phi(\beta)$ one has*

$$\limsup_{\varepsilon \to 0-} \frac{\Phi(p+\varepsilon) - \Phi(p)}{\varepsilon} \; \leq \; G^{\beta}(p) \; \leq \; \liminf_{\varepsilon \to 0+} \frac{\Phi(p+\varepsilon) - \Phi(p)}{\varepsilon}\,. \tag{2.10}$$ nc12

To obtain the existence and an explicit description of the Nash equilibrium, the following result will be used.

Theorem 2 (sufficient conditions for optimality). *Let the functions ψ, h satisfy the assumptions (A1)-(A2), and let $\Phi : I\!R_+ \mapsto I\!R_+$ be a nondecreasing map with $\Phi(p_0) = 0$. Let $\phi : [0, \kappa] \mapsto [p_0, p_{max}]$ be a pricing strategy such that, for a.e. $\beta \in [0, \kappa]$, the following holds.*

- *The function $\Phi(\cdot)$ is Lipschitz continuous on $[p_0, \phi(\beta)]$. Moreover, its derivative satisfies*

$$\Phi'(p) \; \leq \; G^{\beta}(p) \qquad \textit{for a.e.} \;\; p \leq \phi(\beta), \tag{2.11}$$ Phi'1

$$\Phi'(p) \; \geq \; G^{\beta}(p) \qquad \textit{for a.e.} \;\; p > \phi(\beta). \tag{2.12}$$ Phi'2

Then $\phi(\cdot)$ is optimal.

Proof. 1. For any given $\beta \in [0,\kappa]$, consider the map

$$p \;\mapsto\; J(p,\beta) \;=\; (p-p_0)\cdot\psi(\beta+\Phi(p))\cdot h(p), \tag{2.13}$$

describing the expected payoff achieved by putting the asset β on sale at price p. We observe that

$$J(p_0,\beta) \;=\; J(p_{max},\beta) \;=\; 0.$$

Moreover, since Φ is nondecreasing and can have only upward jumps, while ψ is decreasing, the map (2.13) can only have downward jumps. More precisely, for any $p_0 \le p_1 < p_2 \le p_{max}$,

$$\begin{aligned} J(p_2,\beta) \;&\le\; J(p_1,\beta) \\ &+\int_{p_1}^{p_2}\Big[\psi(\beta+\Phi(p))h(p)+(p-p_0)\psi'(\beta+\Phi(p))\Phi'(p)h(p)+(p-p_0)\psi(\beta+\Phi(p))h'(p)\Big]\,dp\,. \end{aligned} \tag{2.14}$$

Notice that equality holds as long as $p_2 \le \phi(\beta)$, because by assumption Φ is Lipschitz continuous on $[p_0,\phi(\beta)]$.

2. For a.e. β, by (2.11) the integrand in (2.14) is ≥ 0 for $p \in [0,\phi(\beta)]$. Hence the map $p \;\mapsto\; J(p,\beta)$ is Lipschitz continuous and nondecreasing on $[0,\phi(\beta)]$.

On the other hand, by (2.12) the integrand in (2.14) is ≤ 0 for $p \in [\phi(\beta), p_{max}]$. Hence the map $p \;\mapsto\; J(p,\beta)$ is nonincreasing on $[\phi(\beta), p_{max}]$, possibly with downward jumps.

We conclude that, for a.e. $\beta \in [0,\kappa]$, the function $J(\cdot,\beta)$ achieves its global maximum at $p = \phi(\beta)$. This implies the optimality of the pricing strategy $\phi(\cdot)$. □

3 Nash equilibria

3.1 Finitely many competing sellers.

We now consider n sellers competing against each other. We assume that i-th agent has an amount κ_i of assets to offer for sale. His pricing strategy will be described by the function $\phi_i : [0,\kappa_i] \mapsto \mathbb{R}$. For every $i \in \{1,\ldots,n\}$, let

$$\Phi_i(p) \;\doteq\; \sum_{j\neq i} \sup\;\{\beta\in[0,\kappa_j]\,;\quad \phi_j(\beta) < p\} \tag{3.1}$$

be the total amount of assets offered by all other agents $j \neq i$ at price $< p$. Then the expected payoff for agent i is

$$J_i(\phi_i) \;=\; \int_0^{\kappa_i}[\phi_i(\beta)-p_0]\cdot\Big[\psi(\beta+\Phi_i(\phi_i(\beta)))\cdot h(\phi_i(\beta))\Big]\,d\beta\,. \tag{3.2}$$

Definition. *An n-tuple of pricing strategies $(\phi_1,\ldots,\phi_n)$ is a* **Nash equilibrium** *if each ϕ_i yields the maximum expected payoff (3.2) to the i-th player, given the function Φ_i determined by the strategies of all other players.*

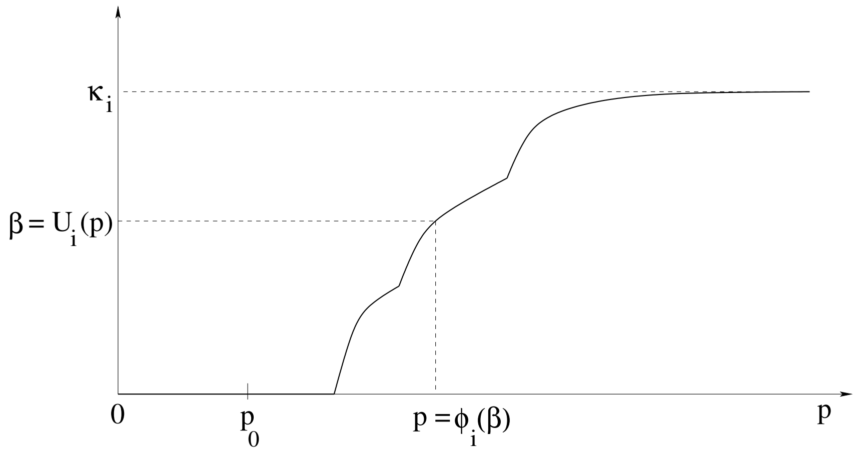

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Figure 3: The functions $p \mapsto U_i(p)$ and $\beta \mapsto \phi_i(\beta)$ are generalized inverses of each other.

Notice that the optimal strategy $\beta \mapsto \phi_i(\beta)$ for the i-th player must satisfy the necessary condition

$$\frac{\partial}{\partial \phi}\left[(\phi - p_0) \cdot \psi(\beta + \Phi_i(\phi)) \cdot h(\phi)\right] \;=\; 0, \tag{3.3}$$

nc3

for a.e. $\beta \in [0, \kappa_i]$. Of course, this is the same as (2.4), with Φ replaced by Φ_i.

To determine these equilibrium strategies, it is convenient to introduce the functions

$$U_i(p) \;=\; [\text{amount put on sale by } i\text{-th agent at price} < p]\,, \tag{3.4}$$

Uidef

$$u_i(p) \;\doteq\; U_i'(p), \qquad\qquad U(p) \;\doteq\; \sum_{i=1}^n U_i(p).$$

Notice that the U_i provides a generalized inverse to the function $\phi_i : [0, \kappa_i] \mapsto I\!R$ describing the strategy of the i-th player (see Fig. 3). Indeed, up to sets of measure zero, one has

$$U(p) \;=\; \sup\{\beta\,;\;\; \phi_i(\beta) < p\}\,, \qquad\qquad \phi_i(\beta) \;=\; \sup\{p\,;\;\; U_i(p) < \beta\}.$$

Let $0 < \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ be the amounts of asset put on sale by the various players. We will show that the Nash equilibrium strategies are obtained as follows.

STEP 1. Construct a piecewise smooth function $p \mapsto U(p)$ on the half-open interval $[p_0, p^*[$, by solving the family of ODEs

$$U'(p) \;=\; -\frac{n-j+1}{n-j} \cdot \frac{\psi(U(p))}{\psi'(U(p))} \cdot \left(\frac{1}{p - p_0} + \frac{h'(p)}{h(p)}\right), \qquad\qquad p \in [p_j, p_{j+1}]\,. \tag{3.5}$$

U'

with terminal condition

$$U(p^*-) \;=\; \kappa^* \;\doteq\; \sum_{i=1}^n \kappa_i - (\kappa_n - \kappa_{n-1}). \tag{3.6}$$

Up*

Here the points $p_0 < p_1 \leq p_2 \leq \cdots \leq p_n$ are inductively determined by

$$p_n \;=\; p^*, \qquad\qquad \frac{U(p_{j+1}) - U(p_j)}{n-j+1} \;=\; \kappa_j - \kappa_{j-1}\,, \qquad j = 1, \ldots, n-1\,. \tag{3.7}$$

pj

For notational convenience, we here define $\kappa_0 = 0$.

STEP 2. For $i = 1, \ldots, n-1$ the optimal strategy U_i is Lipschitz continuous and satisfies

$$U_i'(p) \;=\; \begin{cases} \dfrac{U'(p)}{n-j+1} & p \in [p_j, p_{j+1}], \qquad 1 \leq j \leq i, \\[2ex] 0 & p \notin [p_1, p_{i+1}]. \end{cases} \tag{3.8}$$

Moreover,

$$U_n(p) \;=\; \begin{cases} U_{n-1}(p) & p < p_n\,, \\ \kappa_n & p \geq p_n\,. \end{cases} \tag{3.9}$$

In other words, Player n puts an amount $\kappa_n - \kappa_{n-1}$ of assets for sale all at the price $p_n = p^*$, while his remaining assets are priced in the same way as Player $n-1$.

Theorem 3. *Let the assumptions (A1)-(A2) hold. Then the bidding game has a unique Nash equilibrium. The corresponding functions $U_1, \ldots, U_n$ in (3.4) are determined by (3.5)–(3.9).*

Proof. 1. We begin by proving that the function U and the points p_i are uniquely determined by the equations (3.5)–(3.7). For this purpose, we shall use backward induction on $i = n, n-1, \ldots, 2, 1$.

The first step is to solve the backward Cauchy problem

$$U'(p) \;=\; -2 \cdot \frac{\psi\big(U(p)\big)}{\psi'\big(U(p)\big)} \cdot \left(\frac{1}{p-p_0} + \frac{h'(p)}{h(p)}\right) \tag{3.10}$$

for $p < p_n = p^*$, with terminal condition $U(p_n) = \kappa^*$ defined at (3.6). Observe that the right hand side of (3.10) is strictly positive. Moreover, since the function $\frac{1}{p-p_0}$ is not integrable, we have

$$\lim_{p \to p_0+} U(p) \;=\; -\infty. \tag{3.11}$$

Therefore, there exists a unique point p_{n-1} such that

$$\frac{U(p_n) - U(p_{n-1})}{2} \;=\; \kappa_{n-1} - \kappa_{n-2}\,.$$

This provides the first inductive step.

Next, assume that U has been constructed on the interval $[p_{j+1}, p^*]$. If $j = 0$ we are done. Otherwise, the function U can be extended backwards on the additional interval $[p_j, p_{j+1}]$ by solving the Cauchy problem

$$U'(p) \;=\; -\frac{n-j+1}{n-j} \cdot \frac{\psi\big(U(p)\big)}{\psi'\big(U(p)\big)} \cdot \left(\frac{1}{p-p_0} + \frac{h'(p)}{h(p)}\right) \tag{3.12}$$

for $p < p_{j+1}$, with terminal condition at $p = p_{j+1}$ provided by the inductive step. As before, the solution U of this ODE is strictly increasing and satisfies (3.11). Hence there exists a unique point p_j such that

$$\frac{U(p_{j+1}) - U(p_j)}{n-j+1} \;=\; \kappa_j - \kappa_{j-1}\,.$$

This achieves the inductive step of our construction.

By induction, we thus obtain a function $p \mapsto U(p)$, defined for $p \in [p_1, p^*]$, with

$$p_0 \;<\; p_1 \;\leq\; p_2 \;\leq\; p_n \;=\; p^*\,, \qquad\qquad U(p_1) \;=\; 0.$$

We then set

$$\begin{cases} U(p) &= 0 \qquad\qquad \text{if}\ \ p \leq p_1\,,\\ U(p) &= \kappa \qquad\qquad \text{if}\ \ p \leq p^*\,. \end{cases}$$

2. We now show that the bidding strategies $U_1, \ldots, U_n$ in (3.4), determined by (3.5)–(3.9) provide a Nash equilibrium. Fix any $i \in \{1, \ldots, n\}$ and consider the function

$$\Phi_i(p) \;\doteq\; U(p) - U_i(p)\,. \tag{3.13}$$

Phiid

According to our construction, the i-th player puts his asset $\beta \in [0, \kappa_i]$ on sale at a price $\phi(\beta)$ which satisfies

$$\Phi_i(\phi(\beta)) + \beta \;=\; U(\phi(\beta)).$$

We claim that this price is optimal. Indeed, the sufficient conditions in Theorem 2 are satisfied.

To fix the ideas, assume first that $1 \leq i < n$. Then $\phi(\beta) \in [p_1, p_{i+1}]$. Moreover,

$$\Phi_i'(p) \;=\; 0 \qquad\qquad p \;<\; p_1\,.$$

For $p \in [p_1, \phi(\beta)]$, since $U_i(p) \leq \beta$, by (A1) we have

$$\begin{aligned}\Phi_i'(p) \;&=\; -\frac{\psi\Big(\Phi_i(p) + U_i(p)\Big)}{\psi'\Big(\Phi_i(p) + U_i(p)\Big)} \cdot \left(\frac{1}{p - p_0} + \frac{h'(p)}{h(p)}\right)\\[2ex] &\leq\; -\frac{\psi\Big(\Phi_i(p) + \beta\Big)}{\psi'\Big(\Phi_i(p) + \beta\Big)} \cdot \left(\frac{1}{p - p_0} + \frac{h'(p)}{h(p)}\right) \;=\; G^\beta(p).\end{aligned}$$

Finally, for $p \geq \phi(\beta)$, since $U_i(p) \leq \beta$ we have

$$\begin{aligned}\Phi_i'(p) \;&=\; -\frac{\psi\Big(\Phi_i(p) + U_i(p)\Big)}{\psi'\Big(\Phi_i(p) + U_i(p)\Big)} \cdot \left(\frac{1}{p - p_0} + \frac{h'(p)}{h(p)}\right)\\[2ex] &\geq\; -\frac{\psi\Big(\Phi_i(p) + \beta\Big)}{\psi'\Big(\Phi_i(p) + \beta\Big)} \cdot \left(\frac{1}{p - p_0} + \frac{h'(p)}{h(p)}\right) \;=\; G^\beta(p).\end{aligned}$$

By Theorem 2, ϕ is an optimal strategy.

3. The uniqueness of the Nash equilibrium is proved in the same way as in [4]. For this reason, we only summarize the main steps of the proof.

Consider any Nash equilibrium, and let $U_1, \ldots, U_n$ describe the strategies of the various sellers, as in (3.4). Let $U(p) \doteq \sum_i U_i(p)$. The same arguments used in Lemma 8.1 in [4] yield:

(i) The map $p \mapsto U(p)$ is Lipschitz continuous on the half-open interval $[p_0, p^*[$ and constant for $p > p^*$, possibly with a jump at p^*.

(ii) For all except at most one index $i \in \{1, \ldots, n\}$, the function U_i is globally Lipschitz continuous.

(iii) There exists a minimum asking price p_A and a constant $\delta_0 > 0$ such that

$$\begin{array}{rcll} U(p) & = & 0 & \text{for all } p \leq p_A\,, \\ U'(p) & \geq & \delta_0 & \text{for a.e. } \ p \in [p_A\,,\, p^*]\,. \end{array} \tag{3.14}$$

Next, by Rademacher's theorem every function U_i is a.e. differentiable on the interval $[p_A\,,\, p^*]$. For any p, consider the subset of indices

$$\mathcal{I}(p) \ \doteq \ \{i\,;\ \ U_i'(p) > 0\}$$

and call $N(p) = \# I(p)$ the cardinality of this set. This function is a.e. well defined, and Lebesgue measurable. From the necessary conditions it follows that the function U satisfies the ODE

$$U'(p) \ = \ -\frac{N(p)}{N(p)-1} \cdot \frac{\psi\big(U(p)\big)}{\psi'\big(U(p)\big)} \cdot \left(\frac{1}{p-p_0} + \frac{h'(p)}{h(p)}\right), \qquad\qquad p \in [p_A\,,\, p^*]\,. \tag{3.15}$$

As in the proof of Theorem 8.2 in [4] one can show that, for each $i \in \{1, \ldots, n\}$, the set of prices where the i-th player offers assets for sale is an interval of the form $[p_A,\, p_{i+1}]$. Moreover,

$$p_A \ \doteq \ p_1 \ \leq p_2 \ \leq \ \cdots \ \leq p_n \ = \ p_{n+1} \ = \ p^*.$$

As a consequence, the functions U_i, $i = 1, \ldots, n$, are uniquely determined by the ODEs (3.5) and (3.8), together with the equations (3.6), (3.7), and (3.9). This achieves the proof of uniqueness. For all details, we refer to [4]. □

Example 1. In the special case where ψ, h are given by (1.4) and (1.7), for any $0 < \kappa_1 \leq \cdots \leq \kappa_n$ the Nash equilibrium solution is determined by the equations

$$\begin{cases} U_i(p) \ = \ 0, & \text{if } \ p \leq p_1, \\ U_i(p) \ = \ \kappa_i, & \text{if } \ p \geq p_{i+1}\,, \\ U_i'(p) \ = \ \dfrac{G(p)}{n-k}, & \text{if } \ p_k < p < p_{k+1}, \quad k \leq i\,, \end{cases} \tag{3.16}$$

with $G(p)$ as in (2.9). Moreover, the points $p_1 \leq \cdots \leq p_n$ are inductively determined by the identities

$$p_n \ = \ p^* \ = \ \frac{p_0 + p_{max}}{2}\,, \qquad\qquad \int_{p_j}^{p_{j+1}} \frac{G(p)}{n-j}\, dp \ = \ \kappa_j - \kappa_{j-1} \qquad (j = 1, \ldots, n-1)\,.$$

The case $n = 4$ is illustrated in Fig. 4.

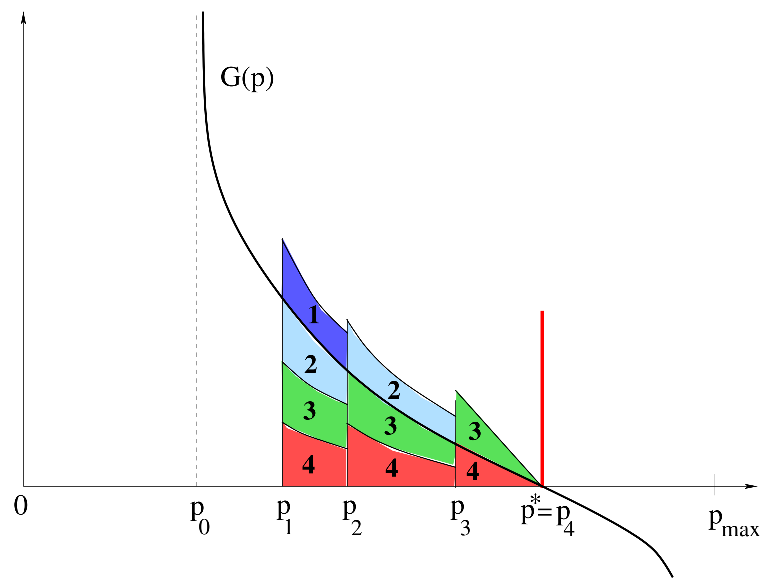

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Figure 4: The optimal strategies $u_1, \ldots, u_4$ in a Nash equilibrium, assuming $\psi(s) = e^{-\lambda s}$. Areas of the regions $1, 2, 3, 4$ are proportional to the amount of asset put on sale by Players $1, 2, 3, 4$ at the given prices. Notice that, for every i, an optimality condition holds: $u_i(p) > 0 \implies \sum_{j \neq i} u_j(p) = G(p)$.

3.2 Infinitely many competing sellers.

We consider here the limiting case where the number of sellers approaches infinity while the total amount of asset on sale remains bounded. More precisely, for each $n \geq 1$, consider amounts

$$0 \;<\; \kappa_1^{(n)} \;\leq\; \kappa_2^{(n)} \;\leq\; \cdots \;\leq\; \kappa_n^{(n)},$$

and assume that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \kappa_j^{(n)} \;=\; \kappa, \qquad\qquad \lim_{n\to\infty} \sup_{1\leq j\leq n} \kappa_j^{(n)} \;=\; 0. \tag{3.17}$$ limn

Let $U^{(n)}(p)$ be the total amount of asset put on sale at price $\leq p$ (by all players combined) in a Nash equilibrium. If the limits (limn 3.17) hold, then we will show that as $n \to \infty$ one has the uniform convergence

$$U^{(n)}(p) \;\to\; U^{\sharp}(p). \tag{3.18}$$ UUs

The function $U^{\sharp}$ can be characterized as the unique Lipschitz continuous function such that

$$\begin{cases} U(p) &=& 0 & \text{if} \quad p \leq p_A\,, \\ U(p) &=& \kappa & \text{if} \quad p \geq p^*\,, \end{cases} \tag{3.19}$$ U1

$$U'(p) \;=\; -\,\frac{\psi\big(U(p)\big)}{\psi'\big(U(p)\big)} \cdot \left(\frac{1}{p - p_0} + \frac{h'(p)}{h(p)} \right) \qquad \text{for a.e. } p \in [p_A, p^*], \tag{3.20}$$ U2

for a suitable value $p_A \in [p_0, p^*]$. Notice that the above equations imply that the map

$$\begin{aligned} p \;\mapsto\; & [\text{profit from the sale of an asset at price } p] \times [\text{probability of selling the asset}] \\ =\; & [p - p_0] \cdot \Big[\psi(U^{\sharp}(p)) \cdot h(p)\Big] \end{aligned} \tag{3.21}$$ ip

is constant over the interval $p \in [p_A, p^*]$. We can thus regard $U^\sharp(\cdot)$ as describing the price distribution in a Nash equilibrium with infinitely many small players.

Theorem 5. *Under the assumptions (A1)-(A2), consider a sequence of Nash equilibria, where as $n \to \infty$ the limits (3.17) hold. Then the corresponding price distributions $U^{(n)}$ converge uniformly to the function $U^\sharp$, defined as the solution to (3.19)-(3.20).*

Proof. 1. For each n, the function $U^{(n)}$ is constructed according to (3.5)–(3.7). Therefore

$$U^{(n)}(p) \;=\; \sum_{j=1}^{n} \kappa_j^{(n)} \;\to\; \kappa \;=\; U^\sharp(p) \qquad \text{for} \;\; p \geq p^*.$$

Moreover, $U^{(n)}$ can have a jump at p^*. However, by the second assumption in (3.17), the size of this jump goes to zero. Indeed,

$$U^{(n)}(p^*-) \;=\; \sum_{j=1}^{n} \kappa_j^{(n)} \;-\; \big(\kappa_n^{(n)} - \kappa_{n-1}^{(n)}\big) \;\to\; \kappa\,.$$

Comparing (3.5) with (3.20), we observe that

$$\lim_{n\to\infty} U^{(n)}(p^*-) \;=\; U^\sharp(p^*), \qquad\qquad \frac{d}{dp}U^{(n)}(p) \;\geq\; \frac{d}{dp}U^\sharp(p),$$

for every $p < p^*$ where both $U^{(n)}$ and $U^\sharp$ are strictly positive. This already implies

$$\limsup_{n\to\infty} U^{(n)}(p) \;\leq\; U^\sharp(p) \qquad\qquad p \in [p_0, p^*]\,.$$

2. Given $\varepsilon > 0$, we can find integers m, N large enough so that

$$1 \;\leq\; \frac{m+1}{m} \;\leq\; 1+\varepsilon\,, \qquad\qquad \sum_{j=1}^{n-m} \kappa_j^{(n)} \;\geq\; \kappa - \varepsilon \quad \text{for all} \;\; n > N\,. \tag{3.22}$$

Call V_ε the solution to

$$V'(p) \;=\; (1+\varepsilon)\frac{\psi\big(V(p)\big)}{\psi'\big(V(p)\big)} \cdot \left(\frac{1}{p-p_0} + \frac{h'(p)}{h(p)}\right), \qquad\qquad V(p^*) \;=\; \kappa - \varepsilon\,. \tag{3.23}$$

We claim that

$$V_\varepsilon(p) \;\leq\; U^{(n)}(p) \qquad\qquad \text{for all } n \geq N,\;\; p \in [p_0, p^*]. \tag{3.24}$$

Indeed, recalling (3.5)–(3.7), let

$$p_1^{(n)} \;\leq\; p_2^{(n)} \;\leq\; \cdots \;\leq\; p_n^{(n)} \;=\; p^*$$

the points determined in the construction of $U^{(n)}$. By the second inequality in (3.22), for every $n > N$ we have

$$V_\varepsilon(p_{n-m}^{(n)}) \;\leq\; V_\varepsilon(p^*) \;=\; \kappa - \varepsilon \;\leq\; U^{(n)}(p_{n-m}^{(n)})\,.$$

Moreover, the first inequality in (3.22) implies

$$\frac{d}{dp}U^{(n)}(p) \;\leq\; \frac{d}{dp}V_\varepsilon(p).$$

Hence (see Fig. 5, right)

$$V_\varepsilon(p) \;\leq\; U^{(n)}(p)$$

for every $p \leq p^*$. We now observe that, as $\varepsilon \to 0$, the function $p \mapsto \max\{V_\varepsilon(p), 0\}$ converges to $U^\sharp$ uniformly on $[p_0, p^*]$. This implies

$$\liminf_{n\to\infty} U^{(n)}(p) \;\geq\; U^\sharp(p) \qquad\qquad p \in [p_0, p^*]\,,$$

completing the proof. □

4 Price reduction resulting from the competition

In this section we prove some inequalities, showing how the average price asked for the asset decreases as a result of the competition between sellers.

To fix the ideas, consider n sellers, offering the amounts $\kappa_1 \leq \cdots \leq \kappa_n$ of asset for sale. Let $\beta \mapsto \phi_i(\beta)$ be the corresponding Nash equilibrium pricing strategies. Calling

$$\kappa \;=\; \kappa_1 + \;\cdots\; + \kappa_n$$

the total amount of asset for sale, the average asked price is

$$A(\kappa_1, \ldots, \kappa_n) \;=\; \frac{1}{\kappa}\sum_{i=1}^{n}\int_0^{\kappa_i} \phi_i\beta)\, d\beta\,. \tag{4.1}$$ avp

In the special case where $\kappa_1 = \cdots = \kappa_n = \kappa/n$, we write

$$A_n(\kappa) \;\doteq\; A\Big(\frac{\kappa}{n}, \ldots, \frac{\kappa}{n}\Big). \tag{4.2}$$ ank

Theorem 5. *Assume that the functions ψ, h satisfy (A1)-(A2). For the Nash equilibrium strategies, the following holds.*

(i) For any given $\kappa_1, \ldots, \kappa_n$ and $\kappa = \sum_i \kappa_i$, one has

$$A_n(\kappa) \;\leq\; A(\kappa_1, \ldots, \kappa_n) \;\leq\; p^*. \tag{4.3}$$ ne1

(ii) For any $m > n$ one has

$$A_m(\kappa) \;\leq\; A_n(\kappa). \tag{4.4}$$ ne2

(iii) In the case where $\psi(s) = e^{-\lambda s}$, for any $n \geq 2$ one has

$$\kappa \;<\; \kappa' \qquad\Longrightarrow\qquad A_n(\kappa') \;<\; A_n(\kappa), \tag{4.5}$$ kk’

$$\lim_{\kappa\to\infty} A_n(\kappa) \;=\; p_0\,. \tag{4.6}$$ n3

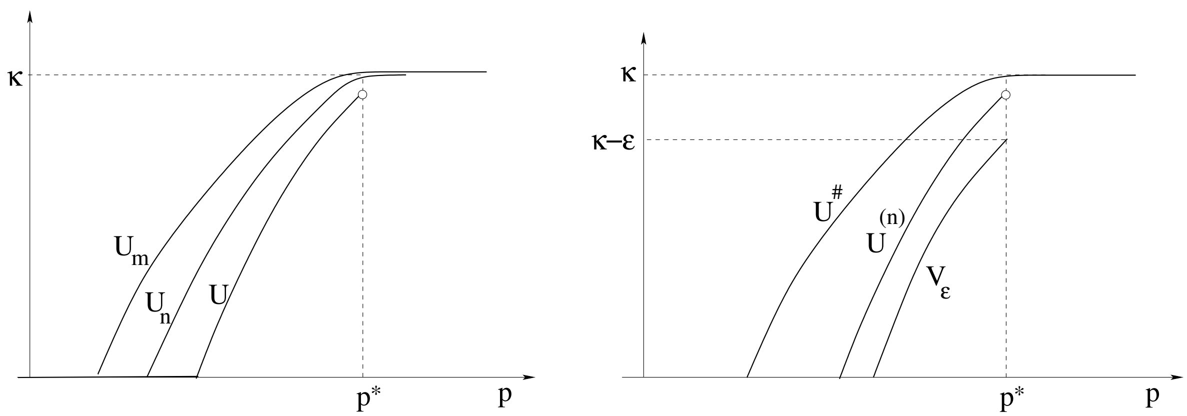

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Figure 5: Left: Comparing the price distributions $U \leq U_n \leq U_m$, in the proof of (4.3)-(4.4). Right: comparing the distribution functions $V_\varepsilon \leq U^{(n)}$, in the proof of Theorem 5.

Proof. 1. Let $U(p)$ be the total amount of asset put on sale at price $\leq p$, jointly by all players. Observe that, in a Nash equilibrium, this price alway ranges within the interval $[p_0, p^*]$. Hence

$$U(p_0) \;=\; 0, \qquad\qquad U(p^*) \;=\; \kappa\,,$$

and the second inequality in (4.3) is obvious. The average price is computed by the Stieltjes integral

$$A \;=\; \int_{p_0}^{p^*} p\, dU(p) \;=\; p^*\kappa - \int_{p_0}^{p^*} U(p)\, dp\,. \tag{4.7}$$ avp2

In the general case of n players, the function U is Lipschitz continuous for $p \in [p_0, p^*[$, possibly with a jump at $p = p^*$. Indeed,

$$U(p^*-) \;=\; \kappa - (\kappa_n - \kappa_{n-1})\,. \tag{4.8}$$ Upp

For $p < p^*$, according to (3.12) the function U satisfies the ODE

$$U'(p) \;=\; -\,\frac{n-j(p)+1}{n-j(p)} \cdot \frac{\psi\Big(U(p)\Big)}{\psi'\Big(U(p)\Big)} \cdot \left(\frac{1}{p-p_0} + \frac{h'(p)}{h(p)}\right) \tag{4.9}$$ ODEU

for some integer-valued function $j(p) \in \{1, \ldots, n-1\}$.

On the other hand, call $p \mapsto U_n(p)$ the total amount put on sale at price $\leq p$ in the case of n equal players, i.e. with $\kappa_1 = \cdots = \kappa_n = \kappa/n$. In this case, the function U_n is globally Lipschitz continuous and provides a solution to the Cauchy problem

$$U_n'(p) \;=\; -\,\frac{n}{n-1} \cdot \frac{\psi\Big(U(p)\Big)}{\psi'\Big(U(p)\Big)} \cdot \left(\frac{1}{p-p_0} + \frac{h'(p)}{h(p)}\right)\,, \qquad\qquad U_n(p^*) = \kappa\,. \tag{4.10}$$ ODEnn

Comparing (4.10) with (4.8)-(4.9), we conclude that $U(p) \leq U_n(p)$ for all $p \in [p_0, p^*]$, see Fig. 5, left. By (4.7) this implies the first inequality in (4.3).

2. If $m > n$, then the corresponding solutions of the Cauchy problem (4.10) satisfy

$$U_m(p) \;\geq\; U_n(p) \qquad\qquad p \in [p_0, p^*]\,.$$

See Fig. 5, left. By (4.7) this implies $A_m(\kappa) \leq A_n(\kappa)$.

3. To prove (4.5), assume $\psi(s) = e^{-\lambda s}$ and let $\kappa' > \kappa$. Choose $p_0 < p'_A < p_A < p^*$ so that

$$\int_{p_A}^{p^*} \frac{n}{n-1} G(p)\,dp \;=\; \kappa, \qquad\qquad \int_{p'_A}^{p^*} \frac{n}{n-1} G(p)\,dp \;=\; \kappa'.$$

with $G(p)$ given at (2.9). Then

$$\begin{aligned} A_n(\kappa') &= \frac{1}{\kappa'} \cdot \left(\int_{p'_A}^{p_A} + \int_{p_A}^{p^*} \right) \frac{n}{n-1} p\, G(p)\,dp \\ &= \frac{\kappa' - \kappa}{\kappa'} \cdot \left[\frac{1}{\kappa' - \kappa} \int_{p'_A}^{p_A} \frac{n}{n-1} p\, G(p)\,dp \right] + \frac{\kappa}{\kappa'} \cdot \left[\frac{1}{\kappa} \int_{p_A}^{p^*} \frac{n}{n-1} p\, G(p)\,dp \right] \\ &\leq \frac{\kappa' - \kappa}{\kappa'} p_A + \frac{\kappa}{\kappa'} A_n(\kappa) \;<\; A_n(\kappa). \end{aligned}$$

4. Finally, to prove (4.6), fix $\varepsilon > 0$ and define

$$\kappa_\varepsilon \;\doteq\; \int_{p_0+\varepsilon}^{p^*} \frac{n}{n-1} G(p)\,dp\,.$$

Then for any $\kappa > \kappa_\varepsilon$ we have

$$\begin{aligned} A_n(\kappa) &= \frac{1}{\kappa} \int_{p_A}^{p^*} \frac{n}{n-1} p\, G(p)\,dp \\ &= \frac{\kappa - \kappa_\varepsilon}{\kappa} \cdot \left[\frac{1}{\kappa - \kappa_\varepsilon} \int_{p_A}^{p_0+\varepsilon} \frac{n}{n-1} p\, G(p)\,dp \right] + \frac{\kappa_\varepsilon}{\kappa} \cdot \left[\frac{1}{\kappa_\varepsilon} \int_{p_0+\varepsilon}^{p^*} \frac{n}{n-1} p\, G(p)\,dp \right] \\ &\leq \frac{\kappa - \kappa_\varepsilon}{\kappa} (p_0 + \varepsilon) + \frac{\kappa_\varepsilon}{\kappa} p^*, \end{aligned} \tag{4.11}$$

for some $p_A < p_0 + \varepsilon$, depending on κ. As $\kappa \to \infty$, the right hand side of (4.11) converges to $p_0 + \varepsilon$. Therefore,

$$p_0 \;\leq\; \limsup_{\kappa \to \infty} A_n(\kappa) \;\leq\; p_0 + \varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrary, this proves (4.6). □

5 Dynamic stability of the Nash equilibrium

In this section we assume that each agent can gradually modify his own pricing strategy, in reply to the strategies adopted by all the other players. Our main interest is in the dynamic stability of the Nash equilibrium. To simplify the analysis, we shall henceforth assume that the the random buying order X has exponential distribution, so that $\psi(s) = e^{-\lambda s}$.

5.1 Infinitely many small players.

We first consider a model with a very large number of small players, each with a small quantity of assets. Let

$$U(p) \;=\; \int_{p_0}^{p} u(x)\,dx \tag{5.1}$$

U5

be the total amount of assets offered for sale at price $< p$. Then the expected payoff achieved by offering a unit amount of asset at price p is

$$J(p,U) \;=\; (p-p_0)\psi(U(p))h(p) \;=\; e^{-\lambda U(p)}(p-p_0)h(p).$$

If the map $p \mapsto J(p,U)$ is not constant, the agent pricing his asset at p may increase his payoff by varying the price according to

$$\dot p \;=\; \frac{d}{dp}J(p,U(p)) \;=\; e^{-\lambda U(p)}\Big[h(p) - \lambda(p-p_0)U'(p)h(p) + (p-p_0)h'(p)\Big]. \tag{5.2}$$

pt

From the above, we obtain a conservation law for the price density $u(p) \doteq U'(p)$, with flux $\Phi = \dot p \cdot u$, namely

$$u_t + \Big\{e^{-\lambda U(p)} \cdot \Big[h(p) - \lambda(p-p_0)h(p)u + (p-p_0)h'(p)\Big]u\Big\}_p \;=\; 0. \tag{5.3}$$

claw1

The characteristic speed is

$$e^{-\lambda U(p)}\Big[h(p) + (p-p_0)h'(p) - 2\lambda(p-p_0)h(p)u\Big].$$

Notice that (5.3) is a conservation law with strictly concave flux. Upward jumps provide admissible shocks, while downward jumps are not admissible. Steady states are those where the flux vanishes identically, so that

$$u(p) \;\in \{0,\, G(p)\} \qquad\qquad \text{for a.e. } p,$$

where G is the function defined at (2.9). Let $\kappa = \int u(p)\,dp$ be the total amount of assets offered for sale. The admissibility conditions imply that a unique steady state exists, namely

$$u^\sharp(p) \;=\; \begin{cases} G(p) & \text{if } \ p \ \in \ [p_A, p^*]\,, \\ 0 & \text{if } \ p \ \notin \ [p_A, p^*]\,. \end{cases} \tag{5.4}$$

uss

Here the points $p^*, p_A \in [p_0, p_{max}]$ are uniquely determined by the identities

$$G(p^*) \;=\; 0, \qquad\qquad \int_{p_A}^{p^*} G(p)\,dp \;=\; \kappa. \tag{5.5}$$

ip1*

If u is an entropy-admissible solution of (5.3), then the integrated function

$$U(t,p) \;\doteq\; \int_{p_0}^{p} u(t,x)\,dx \tag{5.6}$$

UUd

provides a viscosity solution [2] to the evolution equation

$$U_t + e^{-\lambda U(p)} \cdot \Big[h(p) - \lambda(p-p_0)h(p)U_p + (p-p_0)h'(p)\Big]U_p \;=\; 0. \tag{5.7}$$

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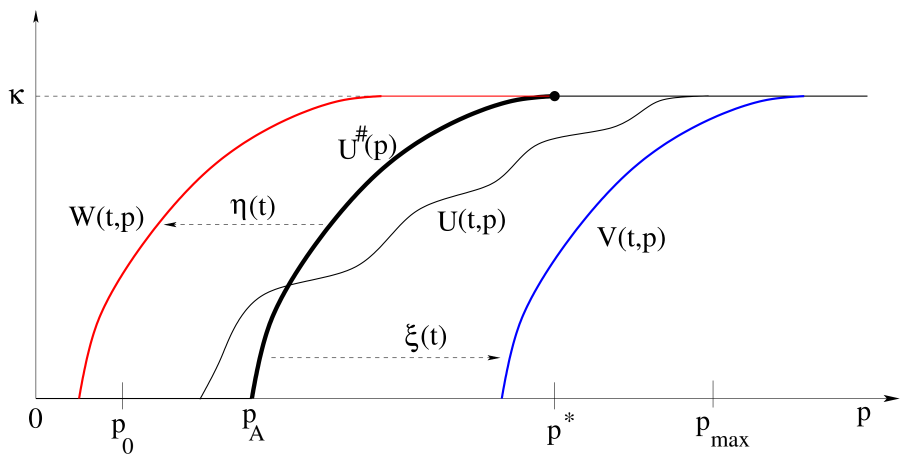

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Figure 6: In the proof of Theorem 6, the upper solution W and the lower solution V are obtained by shifting the graph of $U^\sharp$ to the left and to the right, respectively.

Theorem 6. *Let h satisfy the assumptions (A2). Let $u(0,p) = \bar{u}(p)$ be an initial data supported inside the open interval $]p_0, p_{max}[$. Then, as $t \to +\infty$, the solution of (5.7) converges in $\mathbf{L}^1$ to the function $u^\sharp$ defined by (5.4)-(5.5).*

Proof. 1. Consider the integrated function U in (5.6). By assumption, there exist $\kappa, \delta > 0$ such that at time $t = 0$ the initial data

$$\overline{U}(p) \;=\; \int_{p_0}^{p} \bar{u}(x)\,dx$$

satisfy

$$\begin{cases} \overline{U}(p) &= 0 \qquad \text{if} \quad p \leq p_0 + \delta\,, \\ \overline{U}(p) &= \kappa \qquad \text{if} \quad p \geq p_{max} - \delta\,. \end{cases} \tag{5.8}$$ Uin

We shall construct a subsolution V and a supersolution W of (5.7) with

$$V(t,p) \;\leq\; U(t,p) \;\leq\; W(t,p), \tag{5.9}$$ VW0

$$\lim_{t\to+\infty} V(t,p) \;=\; \lim_{t\to+\infty} W(t,p) \;=\; U^\sharp(p)\,. \tag{5.10}$$ LVW

A comparison argument will thus yield the convergence $U(t,\cdot) \to U^\sharp$ as $t \to \infty$.

2. As shown in Fig. 6, the lower and upper solutions V, W will have the form

$$V(t,p) \;=\; U^\sharp(p - \xi(t)), \qquad\qquad W(t,p) \;=\; U^\sharp(p + \eta(t)), \tag{5.11}$$ VWdef

for suitable functions ξ, η. As in Theorem 5, here $U^\sharp$ is the unique Lipschitz continuous function such that

$$h(p) - \lambda(p - p_0)h(p)U^\sharp_p(p) + (p - p_0)h'(p) \;=\; 0 \qquad\qquad p_A \;<\; p \;<\; p^*, \tag{5.12}$$ U8

$$\begin{cases} U(p) &= 0 \qquad \text{if} \quad p \leq p_A\,, \\ U(p) &= \kappa \qquad \text{if} \quad p \geq p^*\,, \end{cases} \tag{5.13}$$ U55

for a suitable value $p_A \in [p_0, p^*]$. We recall that (5.12) is equivalent to

$$U^\sharp_p(p) \doteq \frac{d}{dp}U^\sharp(p) = G(p) \doteq \frac{1}{\lambda}\left(\frac{1}{p-p_0} + \frac{h'(p)}{h(p)}\right) \qquad p_A < p < p^*. \tag{5.14}$$

By choosing

$$\xi(0) = p_{max} - p_0, \qquad \eta(0) = p^* - p_0\,,$$

we achieve

$$V(0,p) = 0 \leq U_0(p) \leq \kappa = W(0,p) \qquad \text{for all } p \in [p_0, p_{max}]. \tag{5.15}$$

3. For any $\xi > 0$, using (5.12)–(5.14) we obtain

$$\begin{aligned} I(\xi) &\doteq \inf_{p_0+\delta<p<p_{max}-\delta} \Big[h(p) - (p-p_0)h(p)G(p+\xi) + (p-p_0)h'(p)\Big] \\ &= \inf_{p_0+\delta<p<p_{max}-\delta} (p-p_0)h(p)\Big[G(p) - G(p+\xi)\Big] > 0\,. \end{aligned} \tag{5.16}$$

If we now choose the map $t \mapsto \xi(t)$ satisfying

$$\dot\xi(t) = -\,e^{-\lambda\kappa} I(\xi(t))\,, \tag{5.17}$$

then the function V in (5.11) will be a lower solution of (5.7) on the domain

$$\Omega \doteq [0,\infty[\, \times [p_0+\delta\,,\ p_{max}-\delta]. \tag{5.18}$$

Observing that $V \leq U$ on the parabolic boundary of Ω, i.e. on the set

$$\{0\} \times [p_0+\delta\,,\ p_{max}-\delta] \ \cup\ [0,\infty[\ \times\ \{p_0+\delta\} \cup\ [0,\infty[\ \times\ \{p_{max}-\delta\}, \tag{5.19}$$

we conclude that $V(t,x) \leq U(t,x)$ for all $(t,x) \in \Omega$.

4. Similarly, for any $\eta > 0$, using (5.12)–(5.14) we obtain

$$\begin{aligned} J(\xi) &\doteq \sup_{p_0+\delta<p<p_{max}-\delta} \Big[h(p) - (p-p_0)h(p)G(p-\eta) + (p-p_0)h'(p)\Big] \\ &= \sup_{p_0+\delta<p<p_{max}-\delta} (p-p_0)h(p)\Big[G(p) - G(p-\eta)\Big] < 0. \end{aligned} \tag{5.20}$$

If we now choose the map $t \mapsto \eta(t)$ satisfying

$$\dot\eta(t) = -\,e^{-\lambda\kappa} J(\eta(t))\,, \tag{5.21}$$

then the function W in (5.11) will be a lower solution, restricted to the domain Ω in (5.18). Observing that $U \leq W$ on the parabolic boundary (5.19) of Ω, we conclude that $V(t,x) \leq U(t,x)$ for all $(t,x) \in \Omega$.

5. Since $\xi > 0$ and $\eta > 0$ imply $I(\xi) > 0$ and $J(\eta) > 0$, the solutions to (5.17), and (5.21) satisfy

$$\xi(t) \to 0, \qquad \eta(t) \to 0 \qquad \text{as} \quad t \to \infty.$$

Hence $V(t,\cdot)$ and $W(t,\cdot)$ both approach $U^\sharp$ as $t\to\infty$. Since the inequalities (5.9) hold for every time $t\geq 0$ and every $p\in[p_0+\delta\,,~p_{max}-\delta]$, we obtain the uniform convergence $U(t,\cdot)\to U^\sharp$.

6. The $\mathbf{L}^1$ convergence

$$\|u(t,\cdot)-u^\sharp\|_{\mathbf{L}^1}~\to~0 \tag{5.22}$$ L1c

is now proved by means of Oleinik's estimates. Indeed, recalling that the flux function in (5.3) is a strictly concave function of u, we have an estimate of the form

$$u(t,p_2)-u(t,p_1)~\geq~-C(p_2-p_1)\qquad \hbox{for all } p_0-\delta<p_1<p_2<p_{max}+\delta\,,\quad t\geq 1. \tag{5.23}$$ OE

In particular, for $t\geq 1$ the total variation of $u(t,\cdot)$ is uniformly bounded. As a consequence, the uniform convergence $U(t,\cdot)\to U^\sharp$ implies the $\mathbf{L}^1$ convergence (5.22). □

5.2 Two or more large players.

We consider here the case of n players, with amounts $0<\kappa_1\leq\cdots\leq\kappa_n$ of asset to put on sale. Let

$$U_i(p)~=~\int_{p_0}^p u_i(x)\,dx \tag{5.24}$$ Uid2

be the total amount of asset put on sale at price $<p$ by Player i, and let $U(p)\doteq\sum_{i=1}^n U_i(p)$. At the initial time $t=0$, we assume that the supports of $u_1,\ldots,u_n$ are all contained in a compact subset of $\,]p_0,p_{max}[\,$.

Consider a situation where each player gradually modifies the prices asked for his assets, in reply to the strategies adopted by all other players. This can be modeled by the system of conservation laws

$$u_{i,t}+\left\{e^{-\lambda U(p)}\Big[h(p)-\lambda(p-p_0)\,h(p)\sum_{j\neq i}u_j(p)+(p-p_0)\,h'(p)\Big]u_i(p)\right\}_p~=~0\,, \tag{5.25}$$ gfn

with $i=1,\ldots,n$. We think of (5.25) as a system of n gradient flows, in connection with the functionals J_i in (3.2) describing the expected payoffs of the various players.

The next example shows that, for general initial data, the solution may not converge to a Nash equilibrium.

Example 2. Assume that the initial data

$$u_i(0,p)~=~\bar u_i(p),\qquad\qquad i=1,\ldots,n, \tag{5.26}$$ id5

are smooth and have disjoint supports (as in Fig. 7, left). Then, as long as the supports of the functions $u_i(t,\cdot)$ remain disjoint, the system (5.25) is equivalent to

$$u_{i,t}+\Big\{e^{-\lambda U(p)}\Big[h(p)+(p-p_0)\,h'(p)\Big]u_i(p)\Big\}_p~=~0\,. \tag{5.27}$$ gf22

In this case, all densities u_i satisfy the same linear transport equation. Hence, for every $t>0$, the solutions $u_i(t,\cdot)$ remain smooth and with disjoint supports. We now observe that every solution to the ODE

$$\dot p~=~h(p)+(p-p_0)h'(p),\qquad\qquad p(0)\in\,]p_0,p_{max}[\,, \tag{5.28}$$ pode

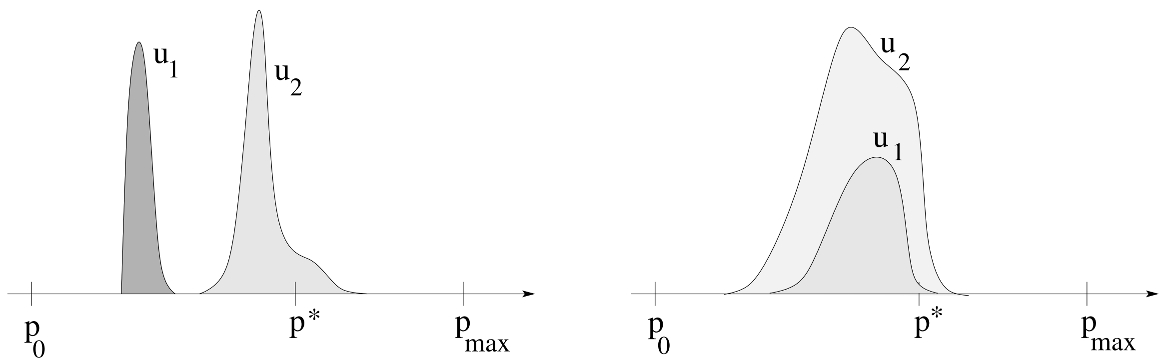

Figure 7: Left: two pricing strategies u_1, u_2 with disjoint support. For this initial data, as $t \to +\infty$ the solution to the system of conservation laws (5.29) will approach two point masses concentrated at p^*. Right: two pricing strategies satisfying the pointwise inequality $u_1 \leq u_2$. For this initial data, the solution to (5.29) converges to the unique Nash equilibrium.

converges to p^* as $t \to +\infty$. Therefore, the solutions u_i to the conservation laws (5.27) converge to Dirac masses of sizes κ_i, all located at the point p^*. When $n = 1$, this limit yields the optimal strategy for the single player. However, for $n \geq 2$, this limit is not a Nash equilibrium.

By the previous example it follows that the Nash equilibrium for n players is not dynamically stable w.r.t. small perturbations, in the topology of weak convergence of measures. Indeed, one can always perturb an equilibrium distribution in such a way that the densities $u_1, \ldots, u_n$ have disjoint supports. For such an initial data, the solution of (5.25) will not converge to the Nash equilibrium.

Next, we prove a positive result, in the case of two players and with an additional assumption on the initial data $\bar u_1\, \bar u_2$. When $n = 2$, the system (5.25) takes the form

$$\begin{cases} u_{1,t} + \Big\{ e^{-\lambda U(p)} \Big[h(p) - \lambda (p - p_0)\, u_2(p)\, h(p) + (p - p_0)\, h'(p) \Big] u_1(p) \Big\}_p \;=\; 0\,, \\[2mm] u_{2,t} + \Big\{ e^{-\lambda U(p)} \Big[h(p) - \lambda (p - p_0)\, u_1(p)\, h(p) + (p - p_0)\, h'(p) \Big] u_2(p) \Big\}_p \;=\; 0\,. \end{cases} \tag{5.29}$$

Theorem 7. *Assume that the initial data are supported on a compact subset of* $]p_0, p_{max}[$ *and satisfy* $\bar u_1 \leq \bar u_2$. *Then, as* $t \to +\infty$, *the solution of (5.29) converges to the Nash equilibrium, in the topology of weak convergence of measures.*

Proof. 1. Consider the additional variables

$$z(p) \;\doteq\; u_2(p) - u_1(p)\,, \qquad\qquad Z(p) \;=\; \int_{p_0}^{p} z(x)\, dx\,.$$

Subtracting the first equation in (5.29) from the second, one obtains the linear conservation law

$$z_t + \Big\{ e^{-\lambda U(p)} \Big[h(p) + (p - p_0)\, h'(p) \Big] z \Big\}_p \;=\; 0\,. \tag{5.30}$$

Since every solution to the ODE (5.28) approaches p^* as $t \to +\infty$, for every $\varepsilon > 0$ we can find a time $T_\varepsilon > 0$ large enough such that

$$z(t,p) \;=\; u_2(t,p) - u_1(t,p) \;=\; 0 \qquad\qquad \text{for all} \;\; p \notin [p^* - \varepsilon,\, p^* + \varepsilon]\,, \quad t > T_\varepsilon\,. \tag{5.31}$$

u01

For $t \geq T_\varepsilon$ this implies

$$Z(t,p) \;=\; \begin{cases} 0 & \text{if} \;\; p < p^* - \varepsilon\,, \\ \kappa_2 - \kappa_1 & \text{if} \;\; p > p^* + \varepsilon\,. \end{cases} \tag{5.32}$$

Zt

We recall that, by assumption, $z(0,p) \geq 0$. Hence

$$u_2(t,p) - u_1(t,p) \;=\; z(t,p) \;\geq\; 0 \qquad\qquad \text{for all} \;\; t \geq 0,\;\; p \in [p_0,\, p_{max}]\,. \tag{5.33}$$

u12

2. Let U_1, U_2 be as in (5.24). By (5.33) it follows that $U_{1,p} \leq U_{2,p}$ for all t, p. Therefore U_1 satisfies

$$U_{1,t} + e^{-\lambda U} \cdot \Big[h(p) - \lambda(p - p_0)\, U_{1,p}\, h(p) + (p - p_0)\, h'(p)\Big] U_{1,p} \;\geq\; 0\,. \tag{5.34}$$

U1t

Define $U_1^\sharp$ to be the unique Lipschitz function such that

$$\begin{cases} U'(p) &=\; G(p) & \text{if} \quad p_A < p < p^*, \\ U(p) &=\; 0 & \text{if} \;\; p \leq p_A\,, \\ U(p) &=\; \kappa_1 & \text{if} \;\; p \geq p^*\,, \end{cases}$$

for some minimum asking price p_A. As in step **3** of the proof of Theorem 6, we can construct a subsolution V having the form

$$V(t,p) \;=\; U_1^\sharp(p - \xi(t)),$$

with $\xi(t)$ decreasing to zero as $t \to +\infty$. hence

$$\liminf_{t \to +\infty} \; U_1(t,p) \;\geq\; U_1^\sharp(p). \tag{5.35}$$

linf

Recalling that $U_{1,p}^\sharp(p^*) = G(p^*) = 0$, for every small $\varepsilon > 0$ we can find T_ε such that

$$U_1(t,p) \;\geq\; \kappa_1 - \varepsilon \qquad\qquad \text{for all}\; t \geq T_\varepsilon\,, \quad p \leq p^* - \varepsilon\,. \tag{5.36}$$

U1low

3. According to (5.31), for all $t > T_\varepsilon$ we have

$$U_{2,p}(t,p) \;=\; U_{1,p}(t,p) \qquad\qquad \text{for all}\; p < p^* - \varepsilon\,.$$

Terefore, restricted to the domain $[T_\varepsilon, \infty[\, \times [p_0, p^* - \varepsilon]$, the function U_1 provides a solution to

$$U_{1,t} + e^{-\lambda U(p)} \cdot \Big[h(p) - \lambda(p - p_0) h(p) U_{1,p} + (p - p_0) h'(p)\Big] U_{1,p} \;=\; 0. \tag{5.37}$$

U11t

As in step **4** of the proof of Theorem 6, we can construct a supersolution W of (5.37) having the form

$$W(t,p) \;=\; U_1^\sharp(p + \eta(t)),$$

where η satisfies

$$\dot\eta(t) \;=\; \begin{cases} -e^{-\lambda(\kappa_1+\kappa_2)} J(\eta(t)) & \text{if } \ \eta(t) > \varepsilon\,, \\ 0 & \text{if } \ \eta(t) \leq \varepsilon\,. \end{cases} \tag{5.38}$$

dotE

Letting $t \to \infty$ we thus obtain

$$\limsup_{t\to\infty}\; U_1(t,p) \;\leq\; \limsup_{t\to\infty}\; W(t,p) \;=\; U_1^\sharp(p+\varepsilon)\,.$$

Since $\varepsilon > 0$ was arbitrary, we conclude

$$\limsup_{t\to\infty} U_1(t,p) \;\leq\; U_1^\sharp(p)\,. \tag{5.39}$$

lsup

4. Together, (5.35) and (5.39) imply the pointwise convergence

$$U_1(t,p) \;\to\; U_1^\sharp(p) \qquad \text{as} \quad t\to\infty. \tag{5.40}$$

UU1

Finally, from the properties (5.32) of $Z = U_2 - U_1$ we deduce

$$\begin{cases} \lim_{t\to\infty}\; U_2(t,p) \;=\; \lim_{t\to\infty}\; U_1(t,p) \;=\; U_1^\sharp(p) & \text{if } \ p < p^*, \\[2mm] \lim_{t\to\infty}\; U_2(t,p) \;=\; \kappa_2 & \text{if } \ p > p^*. \end{cases} \tag{5.41}$$

UU2

By (5.40)-(5.41) the distribution functions U_1, U_2 converge pointwise a.e. to the Nash equilibrium distributions. This completes the proof. □

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