On the Intersection of a Clarke cone with a Boltyanskii Cone

Alberto Bressan

Department of Mathematics, Penn State University, University Park, Pa. 16802 U.S.A.

e-mail: bressan@math.psu.edu

Abstract. We provide an example of two closed sets $S_1, S_2 \subset \mathbb{R}^4$ such that $S_1 \cap S_2 = \{0\}$. Yet, at the origin, a Boltyanskii tangent cone C_1 to S_1 and the Clarke tangent cone C_2 to S_2 are strongly transversal. This settles a question originally proposed by H. Sussmann.

1 - Introduction

Let $t \mapsto x^*(t)$ and $t \mapsto u^*(t)$ be respectively a trajectory and an optimal control for the Mayer problem

$$\begin{cases} \text{minimize} & \varphi\big(x(T)\big) \\ \text{subject to} & \dot x(t) = f\big(t,\, x(t),\, u(t)\big) \\ \text{and} & x(0) = \bar x\,, \qquad \xi(T) \in S\,. \end{cases}$$

Here $x \in \mathbb{R}^m$ is the state variable, while $u \in U \subseteq \mathbb{R}^m$ is the control variable.

A set of necessary conditions for optimality of the pair (x^*, u^*) is provided by the celebrated Pontryagin Maximum Principle (PMP), which has been extensively discussed in the literature on the mathematical theory of control [4]. In essence, these necessary conditions can be traced to a separation property.

On one hand we have a set S_1 of "reachable points", i.e. points $x\big(T, u\big)$ that can be reached at the terminal time T by means of admissible controls $t \mapsto u(t) \in U$. On the other hand, we can consider a set S_2 of "profitable points", i.e. points that satisfy the terminal constraint $x \in S$ and achieve a lower cost: $\varphi(x) \leq \varphi\big(x^*(T)\big)$, with equality holding only if $x = x^*(T)$.

Necessary conditions for optimality are typically obtained by constructing a tangent cone C_1 to the set S_1 at the terminal point $x^*(T)$, and a tangent cone C_2 to S_2 still at $x^*(T)$. If these cones are transversal, under suitable assumptions one can conclude that the intersection $S_1 \cap S_2$ is non-trivial, i.e. it contains points other than $x^*(T)$. Hence, the pair (x^*, u^*) is not optimal.

Reversing the argument, the optimality of the trajectory-control pair (x^*, u^*) implies that the tangent cones C_1 and C_2 are weakly separated. This provides an alternative way to state the PMP. For a deeper discussion of the PMP and for various extensions of this optimality principle we refer to the papers [1–3] and [5–9].

As previously remarked, results on intersections of tangent cones play a crucial role in deriving necessary optimality conditions. In this direction, a question posed by H. Sussmann (see Conjecture 3.6.4 in [7]) is the following:

Let $n \geq 1$ and let S_1, S_2 be closed subsets of $\mathbb{R}^n$ such that $0 \in S_1 \cap S_2$. Assume that

- C_1 *is a Boltyanskii approximating cone to* S_1 *at* 0*;*
- C_2 *is the Clarke tangent cone to* S_2 *at* 0*;*
- C_1 *and* C_2 *are strongly transversal.*

Does this imply that 0 *belongs to the closure of* $(S_1 \cap S_2) \setminus \{0\}$ *?*

The positive results, in dimension $n \leq 3$, follow from standard topological arguments.

Aim of the present paper is to prove that this question has a negative answer, in every space dimension $n \geq 4$. In Section 2 we review the basic definitions, and discuss an example in space dimension $n = 3$. Our main counterexample is then given in Section 3, where we construct two sets $S_1, S_2 \subset R^4$. At the origin, the Clarke tangent cone to S_1 and a Boltyanskii tangent cone to S_2 are strongly transversal. Yet, the two sets have trivial intersection, namely $S_1 \cap S_2 = \{0\}$.

Based on the present counterexample, the forthcoming paper by H. Sussmann [10] exhibits an optimal control problem and an optimal trajectory for which the usual conclusions of the PMP are not true, if the Clarke tangent cone to the terminal set is used instead of a Boltyanskii approximating cone.

2 - Preliminary analysis

For reader's convenience, we first recall some basic definitions.

Definition 1. A nonempty set $C \subseteq \mathbb{R}^n$ is a *cone* if

$$\text{whenever} \quad v \in C \quad \text{and} \quad r \geq 0 \quad \text{it follows that} \quad rv \in C\,.$$

Definition 2. Let $S \subseteq \mathbb{R}^n$. The *Clarke tangent cone* to a point $x \in S$ is the set of all vectors $v \in \mathbb{R}^n$ such that the following holds. For every sequence $x_k \to x$ with $x_k \in S$ for all $k \geq 1$, there exist a sequence $v_k \to v$ such that

$$\liminf_{h \to 0+} \frac{d\big(x_k + hv_k\,,\ S\big)}{h} = 0\,.$$

Definition 3. Given a set $S \subseteq \mathbb{R}^n$ and a point $x \in S$, we say that a closed convex cone $C \subseteq \mathbb{R}^n$ is a *Boltyanskii approximating cone* to S at x if there exists a continuous map $F : C \mapsto S$ such that

$$\lim_{v \to 0,\, v \in C} \frac{F(v) - x - v}{|v|} = 0\,.$$

Definition 4. Two convex cones $C_1, C_2 \subseteq \mathbb{R}^n$ are *strongly transversal* if $C_1 \cap C_2 \neq \{0\}$ and moreover

$$C_1 - C_2 \doteq \big\{v_1 - v_2\,;\quad v_1 \in C_1\,,\quad v_2 \in C_2\big\} = \mathbb{R}^n\,.$$

Before giving a counterexample in dimension $n = 4$, it is useful to see what can happen in dimension $n = 3$.

Consider $I\!R^3$ with coordinates t, x, y, Let $\phi : I\!R \mapsto [0,1]$ be a $\mathcal{C}^\infty$ function such that

$$\begin{array}{ll} \phi(\xi) > 0 & \text{if } \ 0 < \xi < 1\,, \\ \phi(\xi) = 0 & \text{if } \ \xi \leq 0 \ \text{ or } \ \xi \geq 1\,. \end{array} \tag{1}$$

Consider a sequence of times t_m decreasing to zero, say $t_m = m^{-1}$. Moreover, choose a positive sequence ε_m decreasing to zero and a sequence of integers $N_m \to \infty$ such that

$$\frac{\varepsilon_m}{t_m} \to 0\,, \qquad\qquad \frac{\varepsilon_m}{t_{m-1} - t_m} \to 0\,, \tag{2}$$

$$t_{m-1} - t_m < \frac{t_m}{2N_m}\,. \tag{3}$$

We wish to construct two sets S_1, S_2 whose tangent cones at the origin will be

$$C_1 = \big\{(t,x,y)\,;\ \ t \geq 0, \ \ x = 0\,, \ \ |y| \leq t\,\big\}\,,$$

$$C_2 = \big\{(t,x,y)\,;\ \ t \geq 0, \ \ y = 0 \ \ |x| \leq t\big\}\,.$$

To define S_2 we proceed as follows. For each $m \geq 1$ we divide the segment $[-t_m,\ t_m]$ into N_m equal parts. This is achieved by setting

$$x_{m,k} = -t_m + \frac{2kt_m}{N_m}\,,$$

so that

$$-t_m = x_{m,0} < x_{m,1} < \cdots < x_{m,N_m} = t_m\,.$$

For $t \in [t_m,\ t_{m-1}]$ and $j = 1, \ldots, N_m$, define the spatial interval

$$I_{m,j}(t) \doteq \big[x_{m,j-1} - (t - t_m), \ \ x_{m,j} + (t - t_m)\big]\,.$$

Notice that each of these closed intervals become wider as time increases. However, if the time interval $[t_m,\ t_{m-1}]$ is sufficiently short, so that (3) holds, then disjoint intervals will not overlap for $t \in [t_m,\ t_{m-1}]$. More precisely

$$I_{m,j}(t) \cap I_{m,\ell}(t) = \emptyset \qquad\qquad \text{whenever } \ \ell \geq j+2\,.$$

As S_2 we take the set

$$S_2 \doteq \bigcup_{m\geq 1} \Bigg[\bigg\{(t,x,y)\,;\ \ t \in [t_m,\ t_{m-1}]\,, \ \ x \in I_{m,j}\,, \ \ y = \varepsilon_m \phi\Big(\frac{t - t_m}{t_{m-1} - t_m}\Big) \ \ j \ \text{ even}\bigg\}$$
$$\cup \bigg\{(t,x,y)\,;\ \ t \in [t_m,\ t_{m-1}]\,, \ \ x \in I_{m,j}\,, \ \ y = -\varepsilon_m \phi\Big(\frac{t - t_m}{t_{m-1} - t_m}\Big) \ \ j \ \text{ odd}\bigg\} \Bigg]\,.$$

If the numbers ε_m converge to zero fast enough, so that (2) holds, then the Clarke tangent cone to S_2 to the origin is precisely C_2.

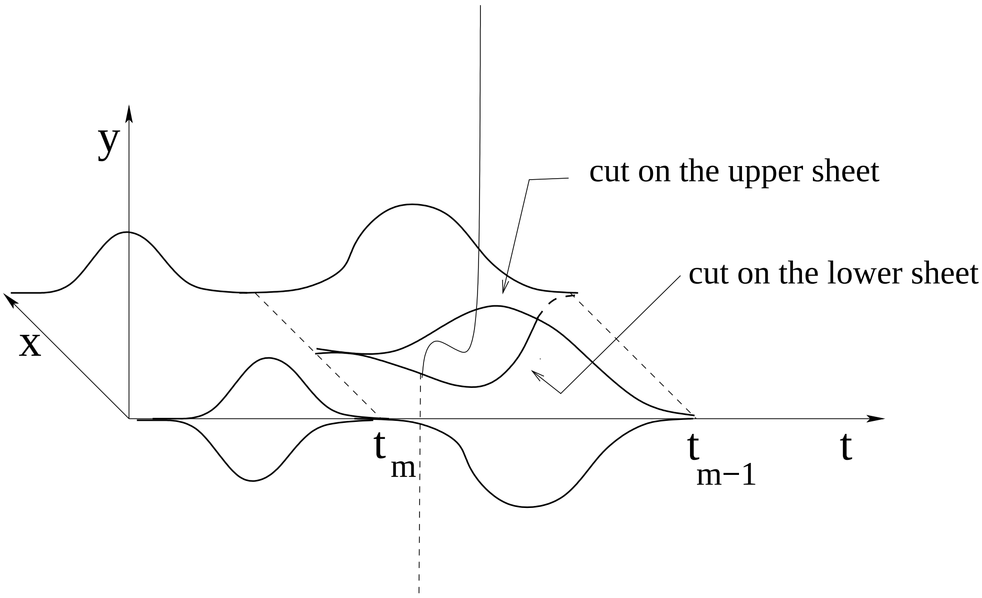

Figure 1: For $t \neq t_m$, a nearly vertical thread can pass through both sheets.

We now consider whether is it possible to slightly deform the cone C_1, by means of a map

$$\varphi : (t, 0, y) \mapsto \big(t,\ x(t, y), y\big)$$

with

$$\big|x(t, y)\big| = o\big(|t|\big)\,,$$

so that the image

$$S_1 \doteq \varphi(C_1)$$

intersects S_2 only at the origin.

Of course this is not possible. Indeed, at each time $t = t_m$ there must be an intersection. However, we notice that we could choose φ so that intersections occur only in arbitrarily small neighborhoods of the points $(t_m, 0, 0)$. Indeed, consider a time $\tau \notin \{t_1, t_2, t_3 \ldots\}$, say with $t_m < \tau < t_{m-1}$. The intersection of S_2 with the plane $\{t = \tau\}$ then consists of two sets of segments, at heights

$$y = \pm\varepsilon_m\, \phi\Big(\frac{\tau - t_m}{t_{m-1} - t_m}\Big)$$

One can thus deform a vertical line on the same plane so that no intersection occurs (fig. 1).

The construction performed in the previous example amounted to the creation of "pockets", which could be used by a vertical thread in order to cross both sheets without intersection. In dimension $n = 3$ this can be only partially successful, since the crossings at times $t = t_m$ $(m \geq 1)$ cannot be avoided.

Performing a similar construction in dimension $n = 4$, however, one can completely remove any intersection (except at the origin), thus answering on the negative side the question posed by H. Sussmann. The crucial difference between dimension $n = 3$ and $n = 4$ is illustrated in the following remarks, which we regard as evident:

Remark 1. Fix $\delta > \delta' > 0$ and $h_1 < h_2$. On the x-y plane, consider two countable sets of disjoint segments, all of length $2\delta'$:

$$I_k^1 \doteq \Big\{(x,y)\,;\quad y = h_1\,,\quad \big|x-(2k)\delta\big| \le \delta'\Big\}\,,$$

$$I_k^2 \doteq \Big\{(x,y)\,;\quad y = h_2\,,\quad \big|x-(2k+1)\delta\big| \le \delta'\Big\}\,,$$

If $x = f(y)$, $x = \tilde f(y)$ are two continuous functions whose graphs do not intersect any of the segments I_k^i, in general it is not possible to construct a homotopy preserving this property (fig. 2). In other words, there exists no continuous map $F : [0,1] \times I\!\!R \mapsto I\!\!R$ such that

$$F(0,x) = f(x)\,, \qquad\qquad F(1,x) = \tilde f(x)$$

and the graph of F does not intersect any of the segments I_k^i.

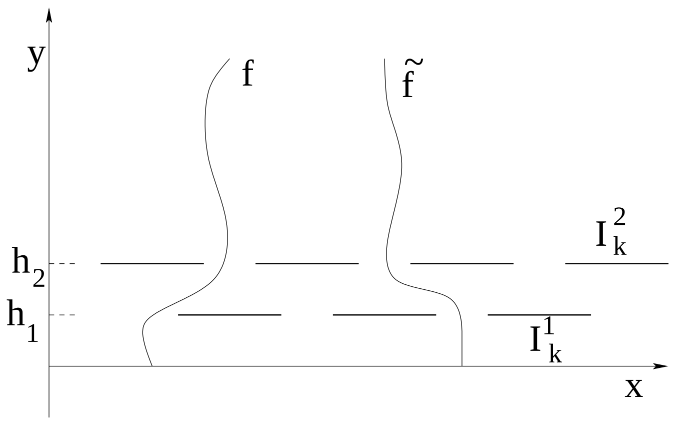

Figure 2: The functions $x = f(y)$ and $x = \tilde f(y)$ are not homotopic (avoiding the segments).

Remark 2. Fix $\delta > \delta' > 0$ and $h_1 < h_2 < h_3 < h_4$. In x-y-z space, consider four countable sets of disjoint squares, parallel to the x-y plane, located at four different heights $z = h_i$. These will all have side length $2\delta'$ and centers at points in the lattice $\delta\, Z\!\!\!Z^2$.

$$
\begin{aligned}
Q^1_{jk} &\doteq \Big\{(x,y,z)\,;\quad z = h_1\,,\quad \big|x-(2j)\delta\big| \le \delta'\,,\quad \big|y-(2k)\delta\big| \le \delta'\Big\}\,,\\
Q^2_{jk} &\doteq \Big\{(x,y,z)\,;\quad z = h_2\,,\quad \big|x-(2j+1)\delta\big| \le \delta'\,,\quad \big|y-(2k)\delta\big| \le \delta'\Big\}\,,\\
Q^3_{jk} &\doteq \Big\{(x,y,z)\,;\quad z = h_3\,,\quad \big|x-(2j)\delta\big| \le \delta'\,,\quad \big|y-(2k+1)\delta\big| \le \delta'\Big\}\,,\\
Q^4_{jk} &\doteq \Big\{(x,y,z)\,;\quad z = h_4\,,\quad \big|x-(2j+1)\delta\big| \le \delta'\,,\quad \big|y-(2k+1)\delta\big| \le \delta'\Big\}\,.
\end{aligned}
$$

Let now $(x,y) = f(z)$ and $(x,y) = \tilde f(z)$ be two continuous functions, whose graphs do not intersect any of the squares Q^i_{jk}. Then they are homotopic (fig. 3), i.e. we can now construct a continuous map $F : [0,1] \times I\!\!R \mapsto I\!\!R^2$ such that

$$F(0,z) = f(z)\,, \qquad\qquad F(1,z) = \tilde f(z)$$

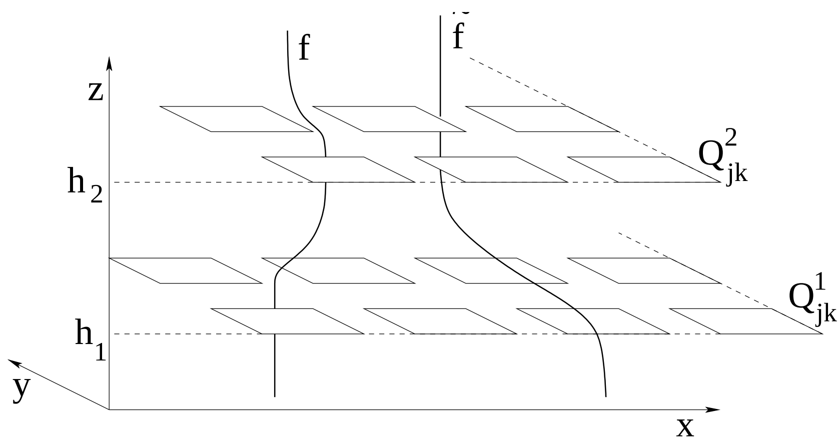

Figure 3: The functions $(x,y) = f(z)$ and $(x,y) = \tilde{f}(z)$ are homotopic (avoiding the squares).

and the graph of F does not intersect any of the squares Q^i_{jk}.

The four-dimensional counterexample

In the space $I\!R^4$ with coordinates (t,x,y,z) we will construct two closed sets S_1, S_2, having the origin as their only point in common. A Boltyanskii tangent cone to S_1 at the origin will be

$$C_1 \doteq \big\{(t,x,y,z)\,;\quad t \geq 0\,,\quad |z| \leq t\,,\quad x = y = 0\big\}\,,$$

while the Clarke tangent cone to S_2 at the origin will be

$$C_2 \doteq \big\{(t,x,y,z)\,;\quad t \geq 0\,,\quad z = 0\,,\quad |x| \leq t\,,\quad |y| \leq t\big\}\,.$$

The strong transversality of the two cones is thus clear.

We begin by constructing the set S_2. Define the decreasing sequence of times

$$t_m \doteq \frac{1}{m}\,. \tag{5}$$

Each time slice

$$S_2(\tau) \doteq S_2 \cap \{t = \tau\}$$

will be a subset of the x-y-z space $I\!R^3$ consisting of a finite union of square patches.

In addition to the function ϕ in (1), let $\tilde{\phi} : I\!R \mapsto [0,1]$ be a $\mathcal{C}^\infty$ function such that

$$\begin{cases} \tilde{\phi}(\xi) > 0 & \text{if} \quad -1 < \xi < 1\,, \\ \tilde{\phi}(\xi) = 0 & \text{if} \quad \xi \leq -1 \quad \text{or} \quad \xi \geq 1\,. \end{cases} \tag{6}$$

Moreover, define the function

$$\delta(t,x) = t^6 \cdot \tilde{\phi}\Big(\frac{x}{t^{4/3}} - 2\cos\frac{\pi}{t}\Big)\,. \tag{7}$$

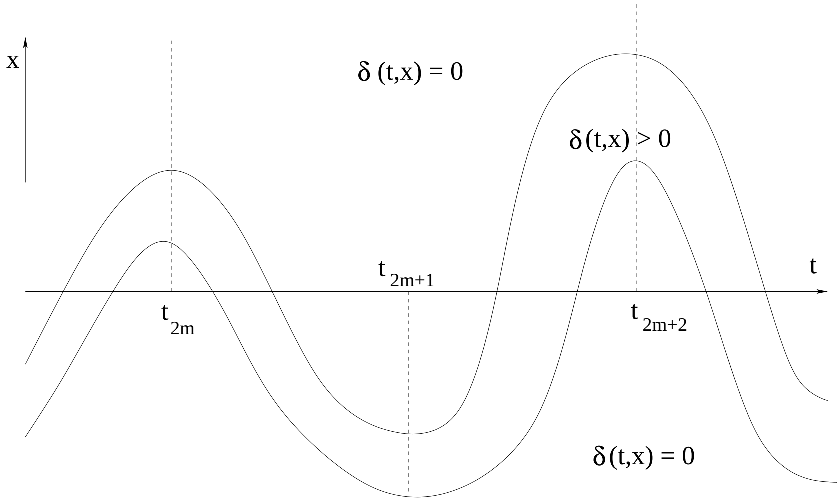

Figure 4: the region in the t-x plane where the sheets are separated.

This will determine the separation in height between the four layers of squares. Notice that

$$\delta > 0 \qquad \Longleftrightarrow \qquad \left| x - 2t^{4/3} \cos \frac{\pi}{t} \right| < t^{4/3}. \tag{8}$$

Moreover

$$\cos \frac{\pi}{t_m} = \cos m\pi = \begin{cases} 1 & \text{if} \quad m \text{ is even,} \\ -1 & \text{if} \quad m \text{ is odd.} \end{cases}$$

The region where $\delta > 0$ is depicted in fig. 4.

Before writing down lengthy formulas, we explain the underlying idea of in plain words.

(Knitting problem) *Consider four sheets of fabric spread over a horizontal table, the x-y plane (see fig. 5). At a given time t, the heights of the four sheets are given by the smooth functions*

$$z = i \cdot \delta(t, x) \qquad \qquad i = 0, 1, 2, 3.$$

where $\delta(t, x)$ is as in (7). On each of these sheets we single out finitely many square patches: $A_{ij}(t)$ on the first sheet, $B_{i,j}(t)$ on the second, etc... Any two distinct A_{ij} do not intersect, and the same is true for the B_{ij}, C_{ij}, and D_{ij}.

We now want to pass an (almost vertical) thread through the four sheets of cloth, moving the thread continuously in time, without ever touching any of the square patches A_{ij}, B_{ij}, C_{ij}, and D_{ij}. The crossing cannot take place on regions where the four sheets stick together (i.e. where $\delta(t, x) = 0$), because the vertical projections of all these patches covers the x-y plane. However, we can easily pass our thread through the four sheets, as long as the crossing takes place over a region of the x-y plane where the four sheets are separated, i.e. where $\delta(t, x) > 0$.

We now specify details. At every time t, the projection of the set S_2^τ on the x-y plane is exactly the square $\big\{ |x| \leq t, \ |y| \leq t \big\}$. In x-y-z space, the time slice $S_2(\tau)$ consists of finitely many squares. The ones having center at points with $x > 0$ will be treated separately from the ones having centers at points with $x < 0$.

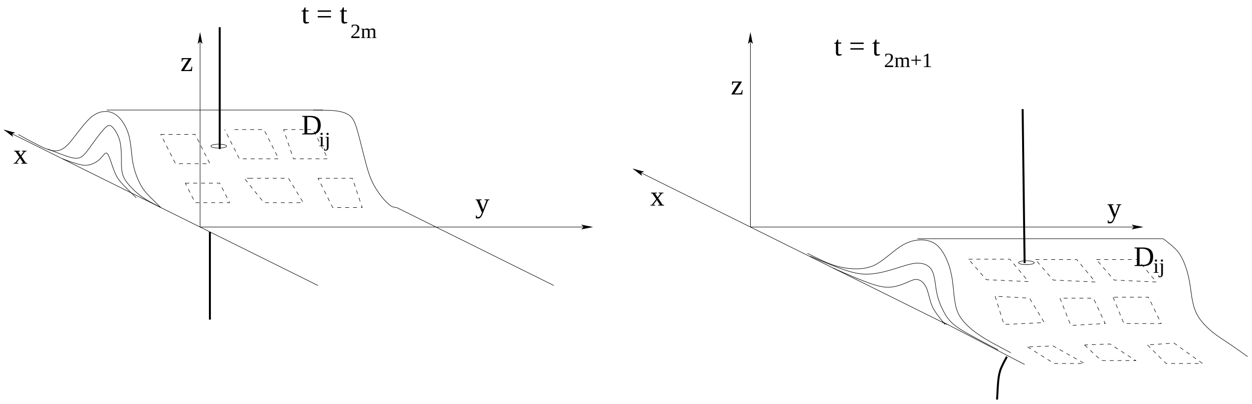

Figure 5: A thread crossing four sheets of fabric, without ever touching the square patches on them.

(i) Squares with centers with $x > 0$. To define these squares, assume $t_{2m+1} \leq \tau \leq t_{2m-1}$. Divide the upper half-square

$$Q^+_{2m+1} \doteq \Big\{(x,y)\,;0 \leq x \leq t_{2m+1}\,, \quad -t_{2m+1} \leq y \leq t_{2m+1}\Big\}$$

into $2N^2_{2m+1}$ equal smaller squares, each with side of length t_{2m+1}/N_{2m+1}. The centers of these squares are located at the points

$$(x_i, y_j) \doteq \left(\frac{(i-\frac{1}{2})t_{2m+1}}{N_{2m+1}}\,,\ \frac{(j-\frac{1}{2})t_{2m+1}}{N_{2m+1}}\right) \qquad -N_{2m+1} < i \leq N_{2m+1}\,, \quad 1 \leq j \leq N_{2m+1}\,.$$

We divide these squares in four classes, depending on whether i, j are even or odd. For $\tau \in [t_{2m+1},\ t_{2m-1}]$ we then define

$$A_{i,j}(\tau) = \Big\{(\tau,x,y,z)\,;\ |x-x_i| \leq \frac{t_{2m+1}}{2N_{2m+1}} + (t-t_{2m+1})\,,\ \ |y-y_j| \leq \frac{t_{2m+1}}{2N_{2m+1}} + (t-t_{2m+1})\,,\ \ z=0\Big\}$$

$$B_{i,j}(\tau) = \Big\{(\tau,x,y,z)\,;\ |x-x_i| \leq \frac{t_{2m+1}}{2N_{2m+1}} + (t-t_{2m+1})\,,\ \ |y-y_j| \leq \frac{t_{2m+1}}{2N_{2m+1}} + (t-t_{2m+1})\,,\ \ z=\delta(\tau,x)\Big\}$$

$$C_{i,j}(\tau) = \Big\{(\tau,x,y,z)\,;\ |x-x_i| \leq \frac{t_{2m+1}}{2N_{2m+1}} + (t-t_{2m+1})\,,\ \ |y-y_j| \leq \frac{t_{2m+1}}{2N_{2m+1}} + (t-t_{2m+1})\,,\ \ z=2\delta(\tau,x)\Big\}$$

$$D_{i,j}(\tau) = \Big\{(\tau,x,y,z)\,;\ |x-x_i| \leq \frac{t_{2m+1}}{2N_{2m+1}} + (t-t_{2m+1})\,,\ \ |y-y_j| \leq \frac{t_{2m+1}}{2N_{2m+1}} + (t-t_{2m+1})\,,\ \ z=3\delta(\tau,x)\Big\}.$$

And finally:

$$S_2^{(\text{upper})}(\tau) \doteq \left(\bigcup_{\substack{i \text{ even} \\ j \text{ even}}} A_{ij}(\tau)\right) \cup \left(\bigcup_{\substack{i \text{ odd} \\ j \text{ even}}} B_{ij}(\tau)\right) \cup \left(\bigcup_{\substack{i \text{ even} \\ j \text{ odd}}} C_{ij}(\tau)\right) \cup \left(\bigcup_{\substack{i \text{ odd} \\ j \text{ odd}}} D_{ij}(\tau)\right).$$

The lower portion of the set $S_2(\tau)$ is defined similarly, except that in this case we assume $\tau \in [t_{2m},\ t_{2m-2}]$. The lower squares are obtained dividing the half square

$$Q^-_{2m} \doteq \Big\{(x,y)\,; -t_{2m} \leq x \leq 0\,, \quad -t_{2m} \leq y \leq t_{2m}\Big\}$$

into $2N^2_{2m}$ equal smaller squares, each with side of length t_{2m}/N_{2m}. Call (x_i, y_j) the centers of these squares. These will not be confused with the previous ones because now $i < 0$. We define

$$A_{i,j}(\tau) = \Big\{(\tau,x,y,z)\,;\ \ |x-x_i| \leq \frac{t_{2m}}{2N_{2m}} + (t-t_{2m})\,,\ \ |y-y_j| \leq \frac{t_{2m}}{2N_{2m}} + (t-t_{2m})\,,\ \ z=0\Big\}$$

$$B_{i,j}(\tau) = \Big\{(\tau,x,y,z)\,;\ \ |x-x_i| \leq \frac{t_{2m}}{2N_{2m}} + (t-t_{2m})\,,\ \ |y-y_j| \leq \frac{t_{2m}}{2N_{2m}} + (t-t_{2m})\,,\ \ z=\delta(\tau,x)\Big\}$$

$$C_{i,j}(\tau) = \Big\{(\tau,x,y,z)\,;\ \ |x-x_i| \leq \frac{t_{2m+1}}{2N_{2m}} + (t-t_{2m})\,,\ \ |y-y_j| \leq \frac{t_{2m}}{2N_{2m}} + (t-t_{2m})\,,\ \ z=2\delta(\tau,x)\Big\}$$

$$D_{i,j}(\tau) = \Big\{(\tau,x,y,z)\,;\ \ |x-x_i| \leq \frac{t_{2m}}{2N_{2m}} + (t-t_{2m})\,,\ \ |y-y_j| \leq \frac{t_{2m}}{2N_{2m}} + (t-t_{2m})\,,\ \ z=3\delta(\tau,x)\Big\}.$$

Then we set

$$S_2^{(\text{lower})}(\tau) \doteq \left(\bigcup_{\substack{i \text{ even} \\ j \text{ even}}} \widetilde{A}_{ij}(\tau)\right) \cup \left(\bigcup_{\substack{i \text{ odd} \\ j \text{ even}}} \widetilde{B}_{ij}(\tau)\right) \cup \left(\bigcup_{\substack{i \text{ even} \\ j \text{ odd}}} \widetilde{C}_{ij}(\tau)\right) \cup \left(\bigcup_{\substack{i \text{ odd} \\ j \text{ odd}}} \widetilde{D}_{ij}(\tau)\right).$$

Finally, for every $\tau > 0$ we define

$$S_2(\tau) \doteq S_2^{(\text{upper})}(\tau) \cup S_2^{(\text{lower})}(\tau)\,,$$

$$S_2 \doteq \{0\} \cup \bigcup_{\tau>0} S_2(\tau)\,.$$

Next, we claim that there exists a continuous function $\varphi : C_1 \mapsto I\!R^2$, say $(x,y) = \varphi(t,0,0,z)$ such that

$$\varphi(t,0,0,z) = o\big(|t|\big) \qquad\qquad \text{as } t \to 0\,,\quad (t,0,0,z) \in C_1\,,$$

and moreover the graph

$$S_1 \doteq \Big\{(t,x,y,z)\,;\ \ (x,y) = \varphi(t,0,0,z)\,,\quad (t,0,0,z) \in C_1\Big\}$$

intersects the set S_2 only at the origin.

Intuitively, at each time $\tau > 0$ the one-dimensional set $S_1^\tau \doteq S_1 \cap \{t = \tau\}$ gives the position of the thread, which should not touch any of the square patches $A_{ij}(\tau)$, $B_{ij}(\tau)$, $C_{ij}(\tau)$, $D_{ij}(\tau)$.

In view of Remark 2, such a continuous function φ exists provided that:

• The width of the region where $\delta(\tau,\cdot)>0$ is much larger than the sides of the single pathes $A_{ij}(\tau)$, $B_{ij}(\tau)$, $C_{ij}(\tau)$, $D_{ij}(\tau)$.

• Square patches on the same sheet remain disjoint, i.e.

$$\text{[length of a side]} < \text{[distance between two centers]}$$

Since patches grow in time at unit rate, this last condition is a consequence of the inequalities

$$t_{2m-1}-t_{2m+1}<\frac{t_{2m-1}}{5N_{2m-1}}\,,\qquad t_{2m-2}-t_{2m}<\frac{t_{2m}}{5N_{2m}}\qquad (m\geq 1). \tag{9}$$

Finally, in order that the set S_2 have C_2 as Clarke tangent cone at the origin, we need

$$\frac{\partial}{\partial t}\,\delta(t,x)\to 0\,,\qquad \frac{\partial}{\partial x}\,\delta(t,x)\to 0\,,\qquad\qquad \text{as}\quad t\to 0. \tag{10}$$

Moreover, in order that the set S_1 have C_1 as a Boltyanskii tangent cone at the origin, we need

• The maximum size of the patches $A_{ij}(t)$, $B_{ij}(t)$, $C_{ij}(t)$, $D_{ij}(t)$ is $o(t)$ as $t\to 0$. Moreover,

$$\sup\,\big\{|x|\,;\quad \delta(t,x)>0\big\}=o(t)\qquad \text{as } t\to 0.$$

It is not difficult to make choices such that all the above conditions are satisfied. Indeed:

$$t_{m-2}-t_m=\mathcal{O}(t_m^2)=o(t_m^{2-\varepsilon}).$$

Therefore, if we choose N_m so that

$$\text{[size of patches at time } t_m] \approx \frac{t_m}{N_m}\approx t_m^{3/2}$$

the conditions (9) will certainly hold. To fix the ideas, we will choose

$$\begin{cases} N_m \doteq m^{1/2} & \text{if } m \text{ is even}\,,\\ N_m \doteq 9m^{1/2} & \text{if } m \text{ is odd}\,. \end{cases} \tag{11}$$

By these choices, as $t\to 0$:

• The sizes of the patches are $\mathcal{O}(t^{3/2})$

• The region where $\delta(t,\cdot)>0$, where the crossing can occur, is at a distance $O(t^{4/3})$ from the x-axis.

Hence we can construct the function $(x,y)=\varphi(t,z)$ such that

$$\big|\varphi(t,z)\big|=\mathcal{O}(t^{4/3}).$$

This guarantees that C_1 is indeed a Boltyanskii tangent cone to S_1 at the origin.

Finally, the factor t^6 on the right hand side of (7) guarantees that the derivatives of $\delta(\cdot,\cdot)$ w.r.t. both t and x vanish as $t \to 0$. Hence C_2 is the Clarke tangent cone to S_2. This completes the analysis.

Remark 3. Some additional words of explanation might be helpful. With reference to the figures 5 and 6, what is going on is the following. At time $t = t_m$, the four sheets are sticking together in the half-plane $x < 0$ but are separated in some region where $x > 0$ (figure 6, left).

We split the lower half plane $\{x < 0\}$ in four classes of square patches, of size length $\ell(t_{2m}) = t_{2m}/N_{2m} = \mathcal{O}(t_{2m}^{2/3})$. This patches keep sticking together for $t > t_{2m}$ until δ becomes positive.

At time $t = t_{2m-1}$, all four sheets now coincide in the upper half plane $\{x > 0\}$, but are separated somewhere in the lower half plane (figure 6, right). It is here that the thread can cross all sheets without intersecting any of the square patches. At time $t = t_{2m-1}$, sheets in the upper half plane are split into new square patches, of different size from those defined during the previous time interval $[t_{2m-1},\ t_{2m+1}]$.

Remark 4. The factor "9" in (11) is motivated by a small technical detail. On the upper and lower half planes $\{x > 0\}$ and $\{x < 0\}$ we usually have patches of different sizes, and centers on different lattice points, on one single sheet. Indeed, the sizes of upper squares are defined at times t_{2m+1}, while the sizes of lower squares are determined at times t_{2m}. In principle, on one of the sheets we may have a configuration as shown in fig. 6, left. This would prevent the thread from moving freely from the upper to the lower half-plane. This can be easily prevented by requiring that the lower squares are considerably larger that the upper ones, as in fig. 6, right.

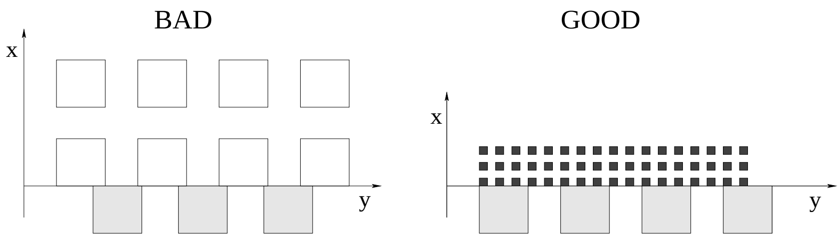

Figure 6

References

[1] F.H.Clarke, The Maximum Principle under minimal hypotheses, *S.I.A.M. J. Control Optim.* **14** (1976), 1078-1091.

[2] F.H.Clarke, *Optimization and Nonsmooth Analysis*, Wiley Interscience, New York, 1983.

[3] A.J.Krener, The higher order maximum principle and its applications to singular extremals, *S.I.A.M. J. Control Optim.* **15** (1977), 256-293.

[4] L.S.Pontryagin, V.G.Boltyanskii, R.V.Gamkrelidze and E.F.Mischenko, *The Mathematical Theory of Optimal Processes*, Wiley, New York, 1962.

[5] H.J.Sussmann, A strong version of the Maximum Principle under weak hypotheses, *Proc. 33rd IEEE Conference on Decision and Control*, Orlando, Fl. 1994, pp.1950-1956.

[6] H.J.Sussmann, A strong maximum principle for systems of differential inclusions, *Proc. 35th IEEE Conference on Decision and Control*, Kobe, Japan, IEEE publications, 1996.

[7] H.J.Sussmann, The geometrization of the maximum principle, paper written for the Proceedings of the COSY Workshop on "Control of Nonlinear and Uncertain Systems" London, February 6-8, 1998 (unpublished, available from the author upon request).

[8] H.J.Sussmann, New theories of set-valued differentials and new versions of the maximum principle of optimal control theory, in *Nonlinear Control in the Year 2000*, A.Isidori, F.Lamnabhi-Lagarrigue and W.Respondek Eds.; Springer-Verlag, 2000; pp.487-526.

[9] H.J.Sussmann, Generalized differentials, variational generators, and the maximum principle with state constraints. To appear in *Nonlinear and Optimal Control Theory*, Gianna Stefani and Paolo Nistri Eds., C.I.M.E. Lecture Notes in Mathematics, Springer-Verlag.

[10] H.J.Sussmann, On the validity of the transversality condition for different concepts of tangent cone to a set. *Proc. IEEE Conf. Decision and Control, 2006*, to appear.