Hyperbolic Systems of Conservation Laws

I - Basic Concepts

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- Hyperbolic conservation laws: definitions, examples
- Some explicit solutions
- Nonlinear effects: loss of regularity and wave interactions
- Definition of weak solutions
- Shock waves: Rankine-Hugoniot equations
- Non-uniqueness of weak solutions, admissibility conditions


Available on: http://www.math.psu.edu/bressan/LectureNotes.html
The Scalar Conservation Law

\[ u_t + f(u)_x = 0 \quad \text{(conservation law)} \]

- \( t = \) time variable
- \( x = \) space variable
- \( u = \) conserved quantity
- \( f(u) = \) flux

\[ u_t + f'(u)u_x = 0 \quad \text{(quasilinear, first order PDE)} \]
Derivation of the equation

\[ \frac{d}{dt} \int_a^b u(t, x) \, dx = f(u(t, a)) - f(u(t, b)) = [\text{inflow at } a] - [\text{outflow at } b] \]

\[ \int_a^b u_t(t, x) \, dx = - \int_a^b f(u(t, x))_x \, dx \]

\[ \int_a^b \left[ u_t(t, x) + f(u(t, x))_x \right] \, dx = 0 \quad \text{for all } a < b \]

\[ u_t + f(u)_x = 0 \]
Example: Traffic Flow

\[ \rho = \text{density of cars} \]

\[ \rho = \text{traffic density} = \text{number of cars per unit length} \]

\[ \frac{d}{dt} \int_a^b \rho(t, x) \, dx = [\text{flux of cars entering at } a] - [\text{flux of cars exiting at } b] \]

\text{flux} = [\text{number of cars crossing a point } x \text{ per unit time}] = [\text{density}] \times [\text{velocity}]
A conservation law for the traffic density  
(Lighthill & Witham 1955, Richards, 1956)

\[ \rho = \text{traffic density} \]

\[ v = v(\rho) = \text{velocity of cars} \quad \text{(is a decreasing function of the traffic density} \ \rho) \]

\[ f = \rho v(\rho) = \text{flux} \]

\[ \rho_t + \left[ \rho v(\rho) \right]_x = 0 \quad (LWR) \]
Systems of Conservation Laws

\[
\begin{align*}
\frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \ldots, u_n) &= 0, \\
&\quad \cdots \\
\frac{\partial}{\partial t} u_n + \frac{\partial}{\partial x} f_n(u_1, \ldots, u_n) &= 0.
\end{align*}
\]

\[u_t + f(u)_x = 0\]

\[u = (u_1, \ldots, u_n) \in \mathbb{R}^n \quad \text{conserved quantities}\]

\[f = (f_1, \ldots, f_n) : \mathbb{R}^n \mapsto \mathbb{R}^n \quad \text{fluxes}\]
Hyperbolic Systems

\[ u_t + f(u)_x = 0 \quad u = u(t, x) \in \mathbb{R}^n \]

\[ u_t + A(u)u_x = 0 \quad A(u) = Df(u) \quad A_{ij} = \frac{\partial f_i}{\partial u_j} \]

The system is **strictly hyperbolic** if each \( n \times n \) matrix \( A(u) \) has real distinct eigenvalues

\[ \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) \]

\[ \implies \text{there exist bases of} \]
right eigenvectors \( r_1(u), \ldots, r_n(u) \) (column vectors)

left eigenvectors \( l_1(u), \ldots, l_n(u) \) (row vectors)

\[ Ar_i = \lambda_i r_i \quad l_i A = \lambda_i l_i \]
Isentropic gas dynamics (in Lagrangian coordinates)

- $\rho =$ density of the gas
- $v = \rho^{-1} =$ specific volume
- $u =$ velocity
- $p = p(v) =$ pressure \quad ($p(v) = kv^{-\gamma}$, with $1 \leq \gamma \leq 3$)

\[
\begin{align*}
\begin{cases}
    v_t - u_x &= 0 \\
    u_t + p(v)_x &= 0
\end{cases}
\end{align*}
\]
\[
\begin{aligned}
\begin{cases}
\quad v_t - u_x &= 0 \\
\quad u_t + p(v)_x &= 0
\end{cases}
\end{aligned}
\]

Quasilinear form:

\[
\begin{pmatrix}
\, v_t \\
\, u_t \\
\end{pmatrix} + \begin{pmatrix}
\, 0 & -1 \\
\, p'(v) & 0 \\
\end{pmatrix} \begin{pmatrix}
\, v_x \\
\, u_x \\
\end{pmatrix} = \begin{pmatrix}
\, 0 \\
\, 0 \\
\end{pmatrix}
\]

The eigenvalues of the Jacobian matrix \( A \doteq Df \) are

\[
\lambda_1 = -\sqrt{-p'(v)}, \quad \lambda_2 = \sqrt{-p'(v)}
\]

\( p'(v) < 0 \implies \) the system is strictly hyperbolic
Euler equations of gas dynamics (1755)

\[
\begin{align*}
    \rho_t + (\rho v)_x &= 0 \quad \text{(conservation of mass)} \\
    (\rho v)_t + (\rho v^2 + p)_x &= 0 \quad \text{(conservation of momentum)} \\
    (\rho E)_t + (\rho E v + p v)_x &= 0 \quad \text{(conservation of energy)}
\end{align*}
\]

\[\rho = \text{mass density}\]

\[v = \text{velocity}\]

\[E = e + v^2/2 = \text{energy density per unit mass (internal + kinetic)}\]

\[p = p(\rho, e) \quad \text{constitutive relation}\]
The scalar equation with linear flux

\[
\begin{align*}
  u_t + f(u)_x &= 0 & f(u) &= \lambda u + c \\
  u_t + \lambda u_x &= 0 & u(0, x) &= \phi(x)
\end{align*}
\]

Explicit solution: \( u(t, x) = \phi(x - \lambda t) \)

**traveling wave** with speed \( f'(u) = \lambda \)
A linear hyperbolic system

\[ u_t + Au_x = 0 \quad u(0, x) = \phi(x) \]

\[ \lambda_1 < \cdots < \lambda_n \quad \text{eigenvalues} \quad r_1, \ldots, r_n \quad \text{eigenvectors} \]

Explicit solution: linear superposition of travelling waves

\[ u(t, x) = \sum_{i=1}^{n} \phi_i(x - \lambda_i t) r_i \]
\[ u_t + A u_x = 0 \]

\[
u(t, x) = \sum_i \phi_i(x - \lambda_i t) r_i \quad \text{is a solution}
\]

\[
u_t = \sum_i \phi'_i(x - \lambda_i t)(-\lambda_i) r_i = -\sum_i \phi'_i(x - \lambda_i t) A r_i
\]

\[
A u_x = A \left( \sum_i \phi'_i(x - \lambda_i t) r_i \right) = \sum_i \phi'_i(x - \lambda_i t) A r_i
\]
Example: D’Alembert solution

Linear wave equation: $u_{tt} - c^2 u_{xx} = 0$ \quad (u_{xt} - u_{tx} = 0)

\[
\begin{align*}
\begin{cases}
  u_1 = u_t \\
  u_2 = u_x
\end{cases}
\implies
\begin{cases}
  u_{1,t} - c^2 u_{2,x} = 0 \\
  u_{2,t} - u_{1,x} = 0
\end{cases}
\end{align*}
\]

\[
\begin{pmatrix}
  u_{1,t} \\
  u_{2,t}
\end{pmatrix} +
\begin{pmatrix}
  0 & -c^2 \\
  -1 & 0
\end{pmatrix}
\begin{pmatrix}
  u_{1,x} \\
  u_{2,x}
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  0
\end{pmatrix}
\]

Wave speeds = eigenvalues = $\pm c$

\[u(t, x) = \Phi_1(x - ct) + \Phi_2(x + ct)\]
Nonlinear effects: nonconstant eigenvalues

\[ u_t + A(u)u_x = 0 \]

Eigenvectors depend on \( u \) \( \implies \) waves change shape

\[ \implies \text{possible blow up of the gradient } u_x \]
Constructing solutions by the method of characteristics

\[
\begin{align*}
  u_t + f'(u)u_x &= 0, & u(0, x) &= \bar{u}(x) \\
  \frac{dx(t)}{dt} &= f'(u) & x(0) &= x_0 \\
  \frac{du(t, x(t))}{dt} &= 0 & u(0, x_0) &= \bar{u}(x_0)
\end{align*}
\]

\[
u_t + f'(u)u_x = \frac{d}{dt} u(t, x(t)) = 0
\]

\[
\begin{align*}
  u(x_0) + t \cdot f'(\bar{u}(x_0)) &= \bar{u}(x_0) \\
  \text{for all } t \geq 0
\end{align*}
\]
Loss of regularity

\[ u_t + f'(u)u_x = 0 \]

\[ (u_x)_t + f'(u)(u_x)_x + f''(u)(u_x)^2 = 0 \]

\[ \frac{d}{dt} u_x(t, x(t)) = -f''(u)(u_x)^2 \quad \text{[along a characteristic} \quad \dot{x} = f'(u)\]

If \( f \) nonlinear \( \Rightarrow f''(u) \neq 0 \quad \Rightarrow u_x \) can blow up in finite time

Global solutions can be found only in a space of discontinuous functions
Nonlinear effects: nonconstant eigenvectors

$$u_t + A(u)u_x = 0$$

eigenvectors depend on $u \implies$ nontrivial wave interactions
Smooth solutions - evolution of wave components

\[ u_t = - A(u)u_x \]

\[ \lambda_i(u) = i\text{-th eigenvalue} \quad l_i(u), r_i(u) = i\text{-th eigenvectors of } A(u) \]

\[ u^i_x = l_i \cdot u_x = [i\text{-th component of } u_x] = [\text{density of } i\text{-waves in } u] \]

\[ u_x = \sum_{i=1}^{n} u^i_x r_i(u) \quad u_t = - \sum_{i=1}^{n} \lambda_i(u) u^i_x r_i(u) \]

differentiate first equation w.r.t. \( t \), second one w.r.t. \( x \)

\[ \implies \text{ evolution equation for scalar components } u^i_x \]

\[ (u^i_x)_t + (\lambda_i u^i_x)_x = \sum_{j>k} (\lambda_j - \lambda_k) \left( l_i \cdot [r_j, r_k] \right) u^j_x u^k_x \]
source terms: \( (\lambda_j - \lambda_k) \left( l_i \cdot [r_j, r_k] \right) u^j_x u^k_x \)

= amount of \( i \)-waves produced by the interaction of \( j \)-waves with \( k \)-waves

\[
\lambda_j - \lambda_k = \text{[difference in speed]}
\]
\[
= \text{[rate at which } j \text{-waves and } k \text{-waves cross each other]}
\]

\[
u^j_x u^k_x = \text{[density of } j \text{-waves]} \times \text{[density of } k \text{-waves]}
\]

\[
[r_j, r_k] = (Dr_k) r_j - (Dr_j) r_k \quad \text{(Lie bracket)}
\]
\[
= \text{[directional derivative of } r_k \text{ in the direction of } r_j]
\]
\[
- \text{[directional derivative of } r_j \text{ in the direction of } r_k]}
\]

\[
l_i \cdot [r_j, r_k] = i\text{-th component of the Lie bracket } [r_j, r_k] \text{ along the basis of eigenvectors } \{r_1, \ldots, r_n\}
\]
Nonlinear wave interactions

\[ (u^i_x)_t + (\lambda_i u^i_x)_x = \sum_{j > k} (\lambda_j - \lambda_k) \left( l_i \cdot [r_j, r_k] \right) u^j_x u^k_x \]
Discontinuous solutions

conservation equation: \[ u_t + f(u)_x = 0 \]

\[ \int\int (u_t + f(u)_x) \phi \, dx \, dt = 0 \quad \text{for all } \phi \]

Integrating by parts:

\[ -\int\int \{u\phi_t + f(u)\phi_x\} \, dx \, dt = 0 \quad \text{for all } \phi \in C^1_c \]
Weak solutions

**Definition.** A function \( u = u(t, x) \) is a **weak solution** to the conservation law

\[
    u_t + f(u)_x = 0
\]

if \( u, f(u) \in L^1_{loc} \) and

\[
    \iint \{ u \phi_t + f(u) \phi_x \} \, dx \, dt = 0 \quad \text{for all} \quad \phi \in C^1_c
\]

**Definition 2.** \( u = u(t, x) \) is a **weak solution** to the Cauchy problem

\[
    u_t + f(u)_x = 0 \quad \text{for all} \quad \phi \in C^1_c
\]

if \( u, f(u) \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}) \) and

\[
    \int_0^{+\infty} \int_{-\infty}^{+\infty} \{ u \phi_t + f(u) \phi_x \} \, dx \, dt + \int_{-\infty}^{+\infty} \bar{u}(x) \phi(0, x) \, dx = 0 \quad \text{for all} \quad \phi \in C^1_c(\mathbb{R}^2)
\]
Limit of weak solutions

\[ u_t + f(u)_x = 0 \]

Assume: \( u_n \) is a weak solution for each \( n \geq 1 \), and

\[ u_n \to u, \quad f(u_n) \to f(u) \quad \text{in} \quad L^1_{loc} \]

then

\[
\int \int \left\{ u \phi_t + f(u) \phi_x \right\} \, dx \, dt = \lim_{n \to \infty} \int \int \left\{ u_n \phi_t + f(u_n) \phi_x \right\} \, dx \, dt = 0
\]

for all \( \phi \in C^1_c \)

Hence \( u \) is also a weak solution \( \quad \text{(convergence of derivatives is not needed)} \)
Shock solutions

\[ u_t + f(u)_x = 0 \]

\[ u(t, x) = \begin{cases} 
  u^- & \text{if } x < \lambda t \\
  u^+ & \text{if } x > \lambda t 
\end{cases} \]

is a weak solution if and only if it satisfies

**Rankine - Hugoniot equations**

\[ \lambda \cdot [u^+ - u^-] = f(u^+) - f(u^-) \] (RH)

[speed of the shock] × [jump in the state] = [jump in the flux]
Derivation of the Rankine-Hugoniot Equations

\[ 0 = \int \int \left\{ u \phi_t + f(u) \phi_x \right\} \, dx \, dt = \int \int_{\Omega^+ \cup \Omega^-} \text{div} \left( u \phi, f(u) \phi \right) \, dx \, dt \]

\[ = \int_{\partial \Omega^+} n^+ \cdot v^+ \, ds + \int_{\partial \Omega^-} n^- \cdot v^- \, ds \]

\[ = \int \left[ \lambda(u^+ - u^-) - (f(u^+) - f(u^-)) \right] \phi(t, \lambda t) \, dt \]

\[ v \triangleq \left( u \phi, f(u) \phi \right) \]

\[ n^+ = \frac{(\lambda, -1)}{\sqrt{\lambda^2 + 1}} \]

\[ ds = \sqrt{\lambda^2 + 1} \, dt \]
Alternative formulation:

\[ \lambda (u^+ - u^-) = f(u^+) - f(u^-) = \int_0^1 Df(\theta u^+ + (1 - \theta)u^-) \cdot (u^+ - u^-) \, d\theta \]

\[ = A(u^+, u^-) \cdot (u^+ - u^-) \]

\[ A(u, v) \equiv \int_0^1 Df(\theta u + (1 - \theta)v) \, d\theta = \text{[averaged Jacobian matrix]} \]

The Rankine-Hugoniot conditions hold if and only if

\[ \lambda(u^+ - u^-) = A(u^+, u^-)(u^+ - u^-) \]

- The jump \( u^+ - u^- \) is an eigenvector of the averaged matrix \( A(u^+, u^-) \)
- The speed \( \lambda \) coincides with the corresponding eigenvalue
The Rankine-Hugoniot condition for the scalar conservation law \( u_t + f(u)_x = 0 \)

\[
\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{u^+ - u^-} \int_{u^-}^{u^+} f'(s) \, ds
\]

[speed of the shock] = [slope of secant line through \( u^-, u^+ \) on the graph of \( f \)]

= [average of the characteristic speeds between \( u^- \) and \( u^+ \)]
A piecewise smooth function \( u = u(t, x) \) with jumps along the curves

\[
t \mapsto x_i(t), \quad t \in [a_i, b_i], \quad i = 1, \ldots, n
\]
is a weak solution to the system of conservation laws \( u_t + f(u)_x = 0 \) if and only if

- outside the shock curves it satisfies \( u_t + Df(u)u_x = 0 \)
- along each shock curve, the left and right limits satisfy

\[
\dot{x}_i(t) \left[ u^+(t, x_i(t)) - u^-(t, x_i(t)) \right] = f \left( u^+(t, x_i(t)) \right) - f \left( u^-(t, x_i(t)) \right)
\]
Non-uniqueness of weak solutions

Burgers’ equation: \[ u_t + \left( \frac{u^2}{2} \right)_x = 0 \]

Flux: \[ f(u) = \left( \frac{u^2}{2} \right) \]

Speed of a shock joining the states \( u^- \), \( u^+ \)

\[ \lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{(u^+)^2 - (u^-)^2}{2(u^+ - u^-)} = \frac{u^+ + u^-}{2} \]
A Cauchy problem with multiple solutions

\[ u_t + (u^2/2)_x = 0, \quad u(0, x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases} \]

Each \( \alpha \in [0, 1] \) yields a weak solution

\[ u_\alpha(t, x) = \begin{cases} 
0 & \text{if } x < \alpha t/2 \\
\alpha & \text{if } \alpha t/2 \leq x < (1 + \alpha)t/2 \\
1 & \text{if } x \geq (1 + \alpha)t/2 
\end{cases} \]
Admissibility conditions on shocks

\[ u_t + f(u)_x = 0 \]

- solutions should be stable w.r.t. small perturbations
- solutions should be limits of suitable approximations (vanishing viscosity, relaxation, etc.)
- any convex entropy should not increase
Perturb the shock by inserting an intermediate state $u^* \in [u^-, u^+]$.

Initial shock is stable $\iff$ [speed of jump behind] $\geq$ [speed of jump ahead]
Initial shock is stable $\iff$ 

$speed of jump behind \geq speed of jump ahead$

\[
\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*}
\]
speed of a shock = slope of a secant line to the graph of $f$

Stability conditions:

- when $u^- < u^+$ the graph of $f$ should remain above the secant line
- when $u^- > u^+$, the graph of $f$ should remain below the secant line
Admissibility Condition (P. Lax)

A shock connecting the states $u^-, u^+$, travelling with speed $\lambda = \frac{f(u^+)-f(u^-)}{u^+-u^-}$ is admissible if

$$f'(u^-) \geq \lambda \geq f'(u^+)$$

- Geometric meaning: characteristics flow toward the shock from both sides

[Diagrams showing the admissibility condition and geometric meaning]
Vanishing viscosity

A “good” solution to the system of conservation laws

\[ u_t + f(u)_x = 0 \]

should be obtained as limit of solutions to vanishing viscosity approximations

\[ u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \quad \text{as} \quad \varepsilon \to 0 \]

**Convergence results:**

**Scalar conservation laws, several space dimensions:**


**Hyperbolic systems, one space dimension:**

Viscous shock profiles

\[ u_t + f(u)_x = 0 \]

shock solution: \[ u(t, x) = \begin{cases} u^- & \text{if } x < \lambda t \\ u^+ & \text{if } x > \lambda t \end{cases} \]

**Question:** is \( u \) a limit of viscous wave profiles?

\[ u^\varepsilon(t, x) = \phi^\varepsilon(x - \lambda t), \quad \varepsilon \to 0 \]

\[ u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \]
Viscous traveling profiles

Seek: a solution to

\[ u_t + f(u)_x = u_{xx} \]

of the form

\[ u(t, x) = \phi(x - \lambda t) \]

with

\[
\lim_{s \to -\infty} \phi(s) = u^-, \quad \lim_{s \to +\infty} \phi(s) = u^+ \]

\[-\lambda \phi' + [f(\phi)]' = \phi''\]
\[-\lambda \phi' + [f(\phi)]' = \phi''\]

\[\phi' = f(\phi) - \lambda \phi + C\]

\[\phi'(s) \to 0 \text{ as } s \to \pm \infty, \quad \text{and} \quad \phi \to u^- \quad \text{or} \quad \phi \to u^+\]

\[\implies\]

\[0 = f(u^-) - \lambda u^- + C = f(u^+) - \lambda u^+ + C\]

\[C = \lambda u^- - f(u^-) = \lambda u^+ - f(u^+)\]

\[\implies\] the Rankine-Hugoniot condition must hold
A solution to

\[ \phi' = f(\phi) - \lambda \phi - \left[ f(u^-) - \lambda u^- \right] \]

with

\[
\begin{cases} 
\phi(s) \to u^- \quad \text{as } s \to -\infty \\
\phi(s) \to u^+ \quad \text{as } s \to +\infty
\end{cases}
\]

exists provided that

- if \( u^- < u^+ \), then \( f(\phi) - \lambda \phi - \left[ f(u^-) - \lambda u^- \right] > 0 \) for all \( u^- < \phi < u^+ \)
- if \( u^+ < u^- \), then \( f(\phi) - \lambda \phi - \left[ f(u^-) - \lambda u^- \right] < 0 \) for all \( u^+ < \phi < u^- \)
Existence of a viscous shock profile

speed of a shock = slope of a secant line to the graph of \( f \)

A traveling viscous shock profile exists provided that:

- if \( u^- < u^+ \) the graph of \( f \) remains strictly above the secant line
- if \( u^- > u^+ \), the graph of \( f \) remains strictly below the secant line
A vanishing viscosity limit

Assume:

$$-\lambda \phi' + [f(\phi)]' = \phi''$$

$$\lim_{s \to -\infty} \phi(s) = u^-, \quad \lim_{s \to +\infty} \phi(s) = u^+$$

Then

$$u(t, x) = \phi(x - \lambda t) \quad \text{solves} \quad u_t + f(u)_x = u_{xx}$$

$$u^\varepsilon(t, x) = \phi \left( \frac{x - \lambda t}{\varepsilon} \right) \quad \text{solves} \quad u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}$$

Taking the limit as $\varepsilon \to 0$ one obtains the shock solution
Entropies

Given a hyperbolic system of conservation laws

$$ u_t + f(u)_x = 0 \quad u \in \mathbb{R}^n, \quad f : \mathbb{R}^n \mapsto \mathbb{R}^n \quad (HCL) $$

A scalar function $\eta(u)$ is called an entropy with entropy flux $q(u)$ if

$$ D\eta(u) \cdot Df(u) = Dq(u) $$

($n$ equations, 2 variables $\implies$ overdetermined if $n > 2$)

$\implies$ every smooth solution to (HCL) satisfies the additional conservation law

$$ \eta(u)_t + q(u)_x = 0 \quad (E) $$

Check: $$ D\eta(u) u_t + D\eta(u) Df(u) u_x = \eta(u)_t + q(u)_x = 0 $$

Note: a weak solution of (HCL) may not yield a solution of (E), if shocks are present
Example: Burger’s equation

\[ u_t + \left( \frac{u^2}{2} \right)_x = 0 \]

\[
\begin{aligned}
\text{entropy:} & \quad \eta(u) = u^2 / 2 \\
\text{entropy flux:} & \quad q(u) = u^3 / 3
\end{aligned}
\]

Check:

\[ u_t + uu_x = 0 \implies uu_t + u^2u_x = 0 \implies (u^2/2)_t + (u^3/3)_x = 0 \]
However, discontinuous weak solutions to

$$ u_t + (u^2/2)_x = 0 \quad (B) $$

are NOT weak solutions to

$$ (u^2)_t + (2u^3/3)_x = 0 \quad (E) $$

A shock of Burgers’ equation (B) with left and right states $u^-, u^+$ should travel with speed

$$ \lambda = \frac{(u^+)^2/2 - (u^-)^2/2}{u^+ - u^-} = \frac{u^+ + u^-}{2} $$

A shock for (E) with left and right states $\eta(u^+), \eta(u^-)$ should travel with speed

$$ \lambda = \frac{q(u^+) - q(u^-)}{\eta(u^+) - \eta(u^-)} = \frac{2(u^+)^3/3 - 2(u^-)^3/3}{(u^+)^2 - (u^-)^2} \neq \frac{u^+ + u^-}{2} $$
Why an entropy is not conserved by solutions with shocks?

\[ u_t + \left( \frac{u^2}{2} \right)_x = 0, \quad \left( \frac{u^2}{2} \right)_t + \left( \frac{u^3}{3} \right)_x = 0 \]

area \[ = \int \int_{0 < u < u(t,x)} 1 \, du \, dx = \int u(t,x) \, dx \]

static moment w.r.t. x-axis \[ = \int \int_{0 < u < u(t,x)} u \, du \, dx = \int \frac{u^2(t,x)}{2} \, dx \]
Every point on the graph of $u(t, \cdot)$ shifts horizontally with speed $f'(u)$

- for smooth solutions, both the area and the static moment remain constant in time
- In the presence of a shock, if the area remains constant, the static moment must decrease
Entropy admissibility conditions

Assume: the hyperbolic system of conservation laws

\[ u_t + f(u)_x = 0 \]  \hspace{1cm} (HCL)

admits a \textbf{convex entropy} \( \eta(u) \) with \textbf{entropy flux} \( q(u) \), so that every smooth solution to (HCL) satisfies the additional conservation law

\[ \eta(u)_t + q(u)_x = 0 \]  \hspace{1cm} (E)

We say that a weak solution \( u = u(t, x) \) of (HCL) is \textbf{entropy admissible} if

\[ \eta(u)_t + q(u)_x \leq 0 \]

in distributional sense:

\[ \iint \{ \eta(u)\varphi_t + q(u)\varphi_x \} \ dxdt \geq 0 \quad \text{for all } \varphi \in C^1_c, \quad \varphi \geq 0 \]
Vanishing viscosity limit \(\Rightarrow\) entropy admissibility

\[
\begin{align*}
  u_t + f(u)_x &= 0, & u_t^\varepsilon + f(u^\varepsilon)_x &= \varepsilon u^\varepsilon_{xx}
\end{align*}
\]

Assume: \(u^\varepsilon \to u\) as \(\varepsilon \to 0\), \(\eta\) convex entropy, \(q\) entropy flux

\[
D\eta(u^\varepsilon)u^\varepsilon_t + D\eta(u^\varepsilon)Df(u^\varepsilon)u^\varepsilon_x = \varepsilon D\eta(u^\varepsilon)u^\varepsilon_{xx}
\]

\[
\begin{align*}
  \left[\eta(u^\varepsilon)\right]_t + \left[q(u^\varepsilon)\right]_x &= \varepsilon D\eta(u^\varepsilon)u^\varepsilon_{xx} = \varepsilon \left\{ \left[\eta(u^\varepsilon)\right]_{xx} - D^2\eta(u^\varepsilon) \cdot (u^\varepsilon_x \otimes u^\varepsilon_x) \right\}
\end{align*}
\]

\[
D^2\eta(u^\varepsilon)(u^\varepsilon_x \otimes u^\varepsilon_x) = \sum_{i,j=1}^n \frac{\partial^2 \eta(u^\varepsilon)}{\partial u_i \partial u_j} \cdot \frac{\partial u_i^\varepsilon}{\partial x} \frac{\partial u_j^\varepsilon}{\partial x} \geq 0
\]

because \(\eta\) is convex
\[
[\eta(u^\varepsilon)]_t + [q(u^\varepsilon)]_x = \varepsilon \left\{ [\eta(u^\varepsilon)]_{xx} - D^2\eta(u^\varepsilon) \cdot (u^\varepsilon_x \otimes u^\varepsilon_x) \right\}
\]

Multiply by a test function \( \varphi \geq 0, \quad \varphi \in C^\infty_c \), and integrate by parts:

\[-\iint \left\{ \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x \right\} \, dx\, dt \leq \varepsilon \iint \eta(u^\varepsilon)\varphi_{xx} \, dx\, dt.\]

If \( u^\varepsilon \to u \) in \( L^1_{loc} \) as \( \varepsilon \to 0 \), then

\[\iint \left\{ \eta(u)\varphi_t + q(u)\varphi_x \right\} \, dx\, dt \geq 0\]

proving that

\[\eta(u)_t + q(u)_x \leq 0\]

in distributional sense.