Hyperbolic Systems of Conservation Laws

III - The Cauchy Problem

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The Cauchy problem for a hyperbolic system of conservation laws

\[ u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x) \quad (CP) \]

Assume: \( \bar{u} \in L^1(\mathbb{R}; \mathbb{R}^n) \) has small total variation

\[
\text{Tot.Var.}\{u\} = \sup \left\{ \sum_{i} |u(x_i) - u(x_{i-1})| ; \quad x_0 < x_1 < \cdots < x_N \right\}
\]

Construct a function \( u = u(t, x) \) such that

- \( \text{Tot.Var.}\{u(t, \cdot)\} \leq C \) for all \( t \geq 0 \)
- the map \( t \mapsto u(t, \cdot) \) is continuous with values in \( L^1(\mathbb{R}) \)
- For every test function \( \varphi \in C^1_c(\mathbb{R}^2) \), there holds

\[
\int_0^t \int_{-\infty}^{+\infty} \left[ u \varphi_t + f(u) \varphi_x \right] dx dt + \int_{-\infty}^{+\infty} \bar{u}(x) \varphi(0, x) dx = 0
\]
Global solution to the Cauchy problem
(for initial data with small total variation)

\[ u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x) \]  \hfill (CP)

**Theorem (J. Glimm, 1965)**

Assume:
- system is strictly hyperbolic
- each characteristic field is either linearly degenerate or genuinely nonlinear

Then there exists a constant \( \delta > 0 \) such that, if the initial data satisfies

\[ \bar{u} \in L^1(\mathbb{R}; \mathbb{R}^n), \quad \text{Tot.Var.}(\bar{u}) \leq \delta, \]

then (CP) has an entropy admissible weak solution \( u = u(t, x) \) defined for all \( t \geq 0 \).
Building block: the Riemann Problem

\[ u_t + f(u)_x = 0 \quad u(0, x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases} \]

Self-similar solution: \( u(t, x) = U \left( \frac{x}{t} \right) \)

Main technique: construct approximate solutions by piecing together solutions of Riemann problems
Constructing approximate solutions by piecing together solutions of Riemann problems

- approximate the initial data $\bar{u} \in L^1$ with a piecewise constant function $\bar{v}$
- solve a Riemann problem at each point where $\bar{v}$ has a jump
The Glimm scheme

- fix a spatial mesh $\Delta x$, choose $\Delta t < \frac{\Delta x}{2 \cdot [\text{max wave speed}]}$

  $\implies$ waves from different Riemann problems will not interact before time $\Delta t$

- at time $\Delta t$, approximate again the solution with a piecewise constant function, solve each Riemann problem

- repeat the construction at times $2\Delta t, 3\Delta t, \ldots$
Main issues: Letting $\Delta x, \Delta t \to 0$, make sure that

- The total variation of approximate solutions remains small
- The approximation errors generated at all time steps do not pile up
Front tracking approximations

- approximate the initial data $\bar{u} \in L^1$ with a piecewise constant function $\bar{v}$
- at each point where $\bar{v}$ has a jump, construct a piecewise constant approximate solution to the Riemann problem
- at the first time $t_1$ where two fronts interact, construct a piecewise constant approximate solution to the Riemann problem
- repeat the construction at all times $t_2, t_3, \ldots$ where two fronts interact
Main issues: make sure that

- The total variation of approximate solutions remains small
- The total number of wave fronts does not become infinite
Piecewise constant approximate solution to a Riemann problem

replace centered rarefaction waves with piecewise constant rarefaction fans
Interaction estimates (waves of different families)

GOAL: estimate the strengths of the waves in the solution of a Riemann problem, depending on the strengths of the two interacting waves $\sigma'$, $\sigma''$

Incoming: an $i$-wave of strength $\sigma'$ and a $j$-wave of strength $\sigma''$

Outgoing: waves of strengths $\sigma_1, \ldots, \sigma_n$. Then

$$|\sigma_i - \sigma'| + |\sigma_j - \sigma''| + \sum_{k \neq i,j} |\sigma_k| = \mathcal{O}(1) \cdot |\sigma' \sigma''|$$
\[ |\sigma_i - \sigma'| + |\sigma_j - \sigma''| + \sum_{k \neq i,j} |\sigma_k| = O(1) \cdot |\sigma' \sigma''| \]

**Proof.** Consider the \( C^2 \) function \( \Phi : \mathbb{R}^2 \mapsto \mathbb{R}^n \)

\[ \Phi(\sigma', \sigma'') = (\sigma_1, \ldots, \sigma_j - \sigma'', \ldots, \sigma_i - \sigma', \ldots, \sigma_n) \]

The identity \( \Phi(0, \sigma'') = \Phi(\sigma', 0) = (0, 0, \ldots, 0) \) implies

\[ \left| \Phi(\sigma', \sigma'') \right| = O(1) \cdot |\sigma' \sigma''| \]
Interaction estimates (waves of the same family)

Incoming: two \( i \)-waves of strengths \( \sigma' \) and \( \sigma'' \)

Outgoing: waves of strengths \( \sigma_1, \ldots, \sigma_n \). Then

\[
|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| = \mathcal{O}(1) \cdot |\sigma' \sigma''|
\]
Glimm functionals (controlling the total variation)

Total strength of waves:

\[ V(t) = \sum_{\alpha} |\sigma_{\alpha}| \]

\[ \text{Tot.Var.}\{u(t, \cdot)\} \leq V(t) \]

Wave interaction potential:

\[ Q(t) = \sum_{(\alpha, \beta) \in A} |\sigma_{\alpha} \sigma_{\beta}| \]

\[ A \ni \text{couples of approaching wave fronts} \]
Definition. Two wave-fronts, located at \(x_\alpha < x_\beta\), of families \(k_\alpha, k_\beta \in \{1, \ldots, n\}\) are \textbf{approaching} if

- either \(k_\alpha > k_\beta\)
- or else \(k_\alpha = k_\beta\) and at least one of the fronts is a shock

\[\implies\] the two fronts may interact in the future
Changes in $V$, $Q$ at time $\tau$ when the fronts $\sigma_\alpha, \sigma_\beta$ interact:

$$\Delta V(\tau) = O(1) \cdot |\sigma_\alpha \sigma_\beta|$$

$$\Delta Q(\tau) = -|\sigma_\alpha \sigma_\beta| + O(1) \cdot V(\tau-) |\sigma_\alpha \sigma_\beta|$$
Bounding the total variation

\[ \Delta V(\tau) \leq C_0 |\sigma_\alpha \sigma_\beta| \]
\[ \Delta Q(\tau) \leq -|\sigma_\alpha \sigma_\beta| + C_0 V(\tau^-)|\sigma_\alpha \sigma_\beta| \]

Claim: the map \( t \mapsto V(t) + 2C_0 Q(t) \) is nonincreasing, as long as \( V \) remains small. Indeed:

\[ \Delta V(\tau) + 2C_0 \Delta Q(\tau) \leq C_0 |\sigma_\alpha \sigma_\beta| - 2C_0 |\sigma_\alpha \sigma_\beta| + 2C_0^2 V(\tau^-)|\sigma_\alpha \sigma_\beta| \leq 0 \]

as long as \( V(\tau) < (2C_0)^{-1} \)

Total variation initially small \( \Rightarrow \) global BV bounds

\[ \text{Tot.Var.}\{u(t, \cdot)\} \leq V(t) \leq V(t) + C_0 Q(t) \leq V(0) + C_0 Q(0) \leq (2C_0)^{-1} \]

\( \Rightarrow \) front tracking approximations can be constructed for all \( t \geq 0 \)
Keeping finite the number of wave fronts

At each interaction point, the **Accurate Riemann Solver** yields a solution, possibly introducing several new fronts.

The total number of fronts can become infinite in finite time.

Need: a **Simplified Riemann Solver**, producing only one "non-physical" front.
The generation number of a wave front

In a front tracking approximation, to each front one can attach a generation
number \( \nu \geq 1 \)

- Fronts originating from the initial data at time \( t = 0 \) are the “primal ancestors”. Their generation number is 1
- When two fronts interact, of generations \( \nu_1, \nu_2 \), the new fronts emerging from the interaction have generation \( \nu = \max\{\nu_1, \nu_2\} + 1 \)
- When two fronts of generations \( \nu_1, \nu_2 \) belong to the same \( i \)-family and merge together, the outgoing \( i \)-front has generation \( \nu = \min\{\nu_1, \nu_2\} \)
Non-physical waves

When a sequence of front tracking approximations \((u_N)_{N \geq 1}\) is constructed, in the \(N\)-th approximate solution \(u_N\) all fronts of order \(> N\) are discarded.

They are replaced by non-physical waves, traveling with a fixed speed \(\lambda^*\).
Estimates on the total strength of waves

Define: \( V_k = \text{total strength of all wave fronts of generation } k \)

\[
\begin{align*}
V_1 & \approx \text{Tot.Var.}\{\bar{u}\} \leq \delta_0 \\
V_2 & \leq C_0 \cdot V_1^2 \\
V_3 & \leq C_0 \cdot (V_1 + V_2) V_2 \\
& \quad \ldots \\
V_k & \leq C_0 \cdot (V_1 + V_2 + \cdots + V_{k-1}) V_{k-1}
\end{align*}
\]

\[\implies V_k \leq (C \delta_0)^k \to 0\]

Discarding all wave fronts of generation \( > k \), the error is

\[O(1) \cdot (C \delta_0)^k \to 0 \quad \text{as} \quad k \to \infty\]
Alternative constructions

- For $2 \times 2$ systems, non-physical fronts can be avoided altogether. Total number of fronts remains bounded.

- Instead of keeping track of generation numbers, one could require that, if two wave-fronts of strengths $\sigma', \sigma''$ interact, then

\[
\begin{align*}
|\sigma' \sigma''| \geq \delta_n & \quad \Rightarrow \quad \text{Accurate Riemann Solver is used} \\
|\sigma' \sigma''| < \delta_n & \quad \Rightarrow \quad \text{Simplified Riemann Solver is used}
\end{align*}
\]

Then construct a sequence of approximate solutions $(u_n)_{n \geq 1}$ letting $\delta_n \to 0$

- In numerical simulations, to reduce the number of fronts one can attach non-physical fronts to other fronts.
  But this requires more careful estimates on front interactions.
A sequence of approximate solutions

\[ u_t + f(u)_x = 0 \quad u(0,x) = \bar{u}(x) \]

\((u_n)_{n \geq 1}\) sequence of approximate front tracking solutions

- initial data satisfy \( \| u_n(0, \cdot) - \bar{u} \|_{L^1} \leq \varepsilon_n \to 0 \)
- all shock fronts in \( u_n \) are entropy-admissible
- each rarefaction front in \( u_n \) has strength \( \leq \varepsilon_n \), travels with speed \( \lambda_i(u^-) \)
- at each time \( t \geq 0 \), the total strength of all non-physical fronts in \( u_n(t, \cdot) \) is \( \leq \varepsilon_n \)
A compactness theorem

Helly’s theorem

Let $u_n : [0, T] \times \mathbb{R} \to \mathbb{R}^n$ be a sequence of functions such that, for every $n \geq 1$:

- $|u_n(t, x)| \leq C$ \hspace{1cm} (uniform bound)
- $\text{Tot.Var.}\{u_n(t, \cdot)\} \leq C_0$ \hspace{1cm} (total variation bound)
- $\|u_n(t, \cdot) - u_n(s, \cdot)\|_{L^1(\mathbb{R})} \leq L|t - s|$ \hspace{1cm} (Lipschitz continuity in $L^1$)

Then there exists a convergent subsequence: $u_{nk} \to u$ in $L^1_{loc}([0, T] \times \mathbb{R})$
Existence of a convergent subsequence of front-tracking approximations

\[ \text{Tot.Var.} \{ u_n(t, \cdot) \} \leq C \]

\[ \| u_n(t, \cdot) - u_n(s, \cdot) \|_{L^1} \leq |t - s| \cdot [\text{total strength of all wave fronts}] \cdot [\text{maximum speed}] \leq L \cdot |t - s| \]

Helly’s compactness theorem \( \implies \) a subsequence converges: \( u_{n_k} \to u \) in \( L^1_{loc} \)
Claim: $u = \lim_{n \to \infty} u_n$ is a weak solution

$$\int \int \{ \phi_t u + \phi_x f(u) \} \, dx \, dt = 0 \quad \phi \in C^1_c ([0, \infty[ \times \mathbb{R})$$

Need to show:

$$\lim_{n \to \infty} \int \int \{ \phi_t u_n + \phi_x f(u_n) \} \, dx \, dt = 0$$
Assume $\phi(t, x) = 0$ outside the strip $[0, T] \times \mathbb{R}$. Define

$$\Delta u_n(t, x_\alpha) \doteq u_n(t, x_\alpha^+) - u_n(t, x_\alpha^-)$$

$$\Delta f(u_n(t, x_\alpha)) \doteq f(u_n(t, x_\alpha^+)) - f(u_n(t, x_\alpha^-))$$

$$\Phi_n \doteq (\phi \cdot u_n, \phi \cdot f(u_n)).$$

Use the divergence theorem on each polygonal domain $\Gamma_j$ where $u_n$ is constant:

$$\sum_j \iint_{\Gamma_j} \text{div} \Phi_n(t, x) \, dx \, dt = \sum_j \int_{\partial \Gamma_j} \Phi_n \cdot n \, d\sigma$$
\[
\int_0^T \int_{-\infty}^{\infty} \left\{ \phi_t(t,x)u_n(t,x) + \phi_x(t,x)f(u_n(t,x)) \right\} \, dx \, dt = \sum_j \int_{\partial \Gamma_j} \Phi_n \cdot n \, d\sigma
\]

\[
= \int_0^T \sum_{\alpha \in S \cup R \cup N \cup P} \left[ \dot{x}_\alpha(t) \cdot \Delta u_n(t,x_\alpha) - \Delta f(u_n(t,x_\alpha)) \right] \phi(t,x_\alpha(t)) \, dt
\]

Jump along \( x_\alpha \) satisfies the Rankine-Hugoniot equations

\[
\Rightarrow \quad \dot{x}_\alpha(t) \cdot \Delta u_n(t,x_\alpha) - \Delta f(u_n(t,x_\alpha)) = 0
\]
By how much do the R-H equations fail in a front tracking approximate solution?

- $\alpha \in S$ (shock front) \implies \dot{x}_\alpha(t) \cdot \Delta u_n(t, x_\alpha) - \Delta f(u_n(t, x_\alpha)) = 0$

- $\alpha \in NP$ (non-physical front) \implies \dot{x}_\alpha(t) \cdot \Delta u_n(t, x_\alpha) - \Delta f(u_n(t, x_\alpha)) = O(\sigma_\alpha)$

- $\alpha \in R$ (rarefaction front) \implies \dot{x}_\alpha(t) \cdot \Delta u_n(t, x_\alpha) - \Delta f(u_n(t, x_\alpha)) = O(\sigma_\alpha^2)$

Indeed: $u^+ = R_i(\sigma_\alpha)(u^-) = u^- + \sigma_\alpha r_i(u^-) + O(\sigma_\alpha^2), \quad \dot{x}_\alpha = \lambda_i(u^-)$

\[
f(u^+) - f(u^-) = Df(u^-)(u^+ - u^-) + O(|u^+ - u^-|^2)
\]

\[
= Df(u^-) \sigma_\alpha r_i(u^-) + O(\sigma_\alpha^2)
\]

\[
= \lambda_i(u^-) \sigma_\alpha r_i(u^-) + O(\sigma_\alpha^2)
\]

\[
= \dot{x}_\alpha(u^+ - u^-) + O(\sigma_\alpha^2)
\]
Error estimate for front tracking approximations

\[
\left| \int_0^T \int_{-\infty}^{\infty} \left\{ \phi_t(t, x) u_n(t, x) + \phi_x(t, x) f(u_n(t, x)) \right\} \, dx \, dt \right|
\leq \int_0^T \left| \sum_{\alpha \in S \cup R \cup N} \left[ \dot{x}_\alpha(t) \cdot \Delta u_n(t, x_\alpha) - \Delta f(u_n(t, x_\alpha)) \right] \phi(t, x_\alpha(t)) \right| \, dt
\leq \| \phi \|_{L^\infty} \cdot \left\{ \mathcal{O}(1) \cdot \sum_{\alpha \in R} |\sigma_\alpha^2| + \mathcal{O}(1) \cdot \sum_{\alpha \in N} |\sigma_\alpha| \right\}
\leq \| \phi \|_{L^\infty} \cdot \left\{ \mathcal{O}(1) \cdot \varepsilon_n \sum_{\alpha \in R} |\sigma_\alpha| + \mathcal{O}(1) \cdot \varepsilon_n \right\}
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]

Hence the limit \( u_n \rightarrow u \) yields a weak solution
Entropy admissibility

\[ u_t + f(u)_x = 0 \]

\[
\left\{ \begin{array}{ll}
\eta(u) & \text{convex entropy} \\
q(u) & \text{entropy flux}
\end{array} \right.
\]

For a wave-front tracking approximate solution \( u_n \), one has

- \( \alpha \in S \) (shock front) \( \implies \dot{x}_\alpha(t) \cdot \Delta \eta(u_n)(t, x_\alpha) - \Delta q(u_n)(t, x_\alpha) \geq 0 \)

- \( \alpha \in NP \) (non-physical front) \( \implies \dot{x}_\alpha(t) \cdot \Delta \eta(u_n)(t, x_\alpha) - \Delta q(u_n)(t, x_\alpha) = O(\sigma_\alpha) \)

- \( \alpha \in R \) (rarefaction front) \( \implies \dot{x}_\alpha(t) \cdot \Delta \eta(u_n)(t, x_\alpha) - \Delta q(u_n)(t, x_\alpha) = O(\sigma_\alpha^2) \)
Entropy estimate for front tracking approximations

For a test function \( \phi \in C^1_c, \phi \geq 0 \), one has

\[
\begin{align*}
\int_0^T \int_{-\infty}^\infty \left\{ \phi_t(t, x) \eta(u_n(t, x)) + \phi_x(t, x) q(u_n(t, x)) \right\} \, dx \, dt \\
eq \int_0^T \sum_{\alpha \in S \cup R \cup N^P} \left[ \dot{x}_\alpha(t) \cdot \Delta \eta(u_n(t, x_\alpha)) - \Delta q(u_n(t, x_\alpha)) \right] \phi(t, x_\alpha(t)) \, dt \\
\geq - \|\phi\|_{L^\infty} \cdot \left\{ \mathcal{O}(1) \cdot \sum_{\alpha \in R} |\sigma_\alpha^2| + \mathcal{O}(1) \cdot \sum_{\alpha \in N^P} |\sigma_\alpha| \right\} \\
\geq - \|\phi\|_{L^\infty} \cdot \left\{ \mathcal{O}(1) \cdot \varepsilon_n \sum_{\alpha \in R} |\sigma_\alpha| + \mathcal{O}(1) \cdot \varepsilon_n \right\} \\
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\end{align*}
\]

Hence the limit \( u_n \rightarrow u \) yields an entropy admissible solution.
The Glimm scheme

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]

Assume: all characteristic speeds satisfy \( \lambda_i(u) \in [0, 1] \)

This is not restrictive. If \( \lambda_i(u) \in [-M, M] \), simply change coordinates:

\[ y = x + Mt, \quad \tau = 2Mt \]
Glimm approximations

- Construct a grid in the \( t-x \) plane with step size \( \Delta t = \Delta x \)
  
  grid points: \((t_j, x_k) = (j \cdot \Delta t, k \cdot \Delta x)\)

- For each \( j \geq 0 \), \( u(t_j, \cdot) \) is piecewise constant, with jumps at the points \( x_k \).
  The Riemann problems are solved exactly, for \( t_j \leq t < t_{j+1} \)

- At time \( t_{j+1} \) the solution is again approximated by a piecewise constant function, and the procedure can repeat
Restarting procedure - by averaging?

Natural approach: take $u_{j,k} \triangleq$ average value of $u(t_j, \cdot)$ on each interval $[x_k, x_{k+1}]

- Very easy to compute (Godunov scheme)
  
  $$u_{j,k+1} = u_{j,k} + f(u_{j,k-1}) - f(u_{j,k})$$

- But the total variation can become arbitrarily large!
Choose a random sequence of numbers $\theta_1, \theta_2, \theta_3, \ldots$ uniformly distributed on $[0,1]$.

at time $t_j$, define

$$u(t_j, x) = u(t_j -, (k + \theta_j)\Delta x) \quad \text{for all } x \in [k\Delta x, (k + 1)\Delta x]$$
A deterministic scheme

In the Glimm scheme, instead of a random sequence one can use a sequence of numbers \( \theta_1, \theta_2, \theta_3, \ldots \in [0, 1] \) which is \textbf{uniformly distributed}, so that

\[
\lim_{N \to \infty} \frac{\# \{ j ; \ 1 \leq j \leq N, \ \theta_j \in [0, \lambda] \} }{N} = \lambda \quad \text{for each} \ \lambda \in [0, 1].
\]

Example:

\[ \theta_1 = 0.1, \ldots, \theta_{759} = 0.957, \ldots, \theta_{39022} = 0.22093, \ldots \]
Example: Glimm’s scheme applied to a solution containing a single shock

\[ u(t, x) = \begin{cases} 
  u^+ & \text{if } x > \lambda t \\
  u^- & \text{if } x < \lambda t 
\end{cases} \]

Fix \( T > 0 \), take \( \Delta x = \Delta t = T/N \)

\[ x(T) = \#\{j ; \ 1 \leq j \leq N, \ \theta_j \in [0, \lambda] \} \cdot \Delta t \]

\[ = \frac{\#\{j ; \ 1 \leq j \leq N, \ \theta_j \in [0, \lambda] \}}{N} \cdot T \rightarrow \lambda T \quad \text{as } N \rightarrow \infty \]
In the Glimm scheme, solutions are exact for \( t \in ]t_{j-1}, t_j[ \)

Errors are introduced by the restarting procedure

\[
\sum_{j=1}^{N} \| u(t_{j+}, \cdot) - u(t_{j-}, \cdot) \|_{L^1} \geq \sum_{j=1}^{N} |u^+ - u^-| \cdot \min\{\lambda, (1 - \lambda)\} \cdot \Delta x
\]

\[
= T \cdot |u^+ - u^-| \cdot \min\{\lambda, (1 - \lambda)\}
\]

- The sum of all the errors remains large as \( N \to \infty \)
- Convergence is achieved by a cancellation, of leading order