Control Theory: a Brief Tutorial

Alberto Bressan

Department of Mathematics, Penn State University

bressan@math.psu.edu
ODE’s and control systems

\[
\dot{x}(t) = \frac{d}{dt}x(t)
\]

\[
\dot{x} = f(x)
\]  

(ODE)

\[
\dot{x} = f(x, u(t)), \quad u(t) \in U
\]  

(control system)

\[
\dot{x} \in F(x) = \{f(t, u); \ u \in U\}
\]  

(differential inclusion)
Example 1 - boat on a river

\( x(t) = \) position of a boat on a river

\( \nu(x) \) velocity of the water

\( M = \) maximum speed of the boat relative to the water

\[
\dot{x} = f(x, u(t)) = \nu(x) + u(t) \quad u \in U = \{ \omega \in \mathbb{R}^2, |\omega| \leq M \} \quad (CS)
\]

\[
\dot{x} \in F(x) = \{ \nu(x) + \omega ; \ |\omega| \leq M \} \quad (DI)
\]
Example 2 - cart on a rail

\[ x(t) = \text{position of the cart} \]
\[ y(t) = \text{velocity of the cart} \]
\[ u(t) = \text{force pushing or pulling the cart (control function)} \]

\[ m\ddot{x} = u(t), \quad m = \text{mass of the cart} \]

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \frac{1}{m}u(t)
\end{align*}
\]

\[ u(t) \in [-1, 1] \]
Example 3 - fishery management

\[ x(t) = \text{amount of fish in a lake, at time } t \]

\[ M = \text{maximum population supported by the habitat} \]

\[ u(t) = \text{harvesting effort (control function)} \]

\[ \dot{x} = \alpha x(M - x) - xu, \quad u(t) \in [0, u^{max}] \]
Example 4 - systems with scalar control entering linearly

\[ \dot{x} = f(x) + g(x) u \quad u \in [-1, 1] \]

\[ \dot{x} \in F(x) = \{ f(x) + g(x) u ; \quad u \in [-1, 1] \} \]
Example 5 - car steering

Barycenter of the car: \( B = (x, y) \in \mathbb{R}^2 \)  
Orientation angle: \( \theta \)

The driver controls the motion of the car by acting on the gas pedal and on the steering wheel.

Control function has two components:

\[
\begin{align*}
&u(t) \in [-m, M], \\
&\alpha(t) \in [-\bar{\alpha}, \bar{\alpha}].
\end{align*}
\]

\[
\begin{cases}
\dot{x} = u \cos \theta \\
\dot{y} = u \sin \theta \\
\dot{\theta} = \alpha u
\end{cases}
\]
Open-loop controls

\[ \dot{x} = f(x, u) \]

If \( u = u(t) \) is assigned as a function of time, we say that \( u \) is an open-loop control.

**Theorem**

Assume that the function \( f(x, u) \) is differentiable w.r.t. \( x \). Then for every (possibly discontinuous) control function \( u(t) \) the Cauchy problem

\[ \dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0 \]

has a unique solution.
If $u = u(x)$ is assigned as a function of the state variable $x$, we say that $u$ is a **closed-loop (or feedback) control**.

**Theorem**

Assume that the function $f(x, u)$ is differentiable w.r.t. both $x$ and $u$, and that the feedback control function $u(x)$ is differentiable w.r.t. $x$.

Then the Cauchy problem

$$\dot{x}(t) = f(x(t), u(x(t))), \quad x(t_0) = x_0$$

has a unique solution.
Designing a control function

\[ \dot{x} = f(x, u), \quad u(t) \in U \]

Possible goals:

- Reach a target in minimum time
- Construct a feedback control function \( u = u(x) \) which stabilizes the system at the origin.
- Construct an open-loop control \( u(t) \) which is optimal for a given cost criterion.
Two strategies for crossing a river by boat
Problem: construct a feedback control \( u(x) \in U \) such that all trajectories of the ODE

\[
\dot{x} = f(x, u(x))
\]

(which start sufficiently close to the origin) satisfy

**asymptotic stability:** \( \lim_{t \to +\infty} x(t) = 0 \)
Asymptotic stabilization by a feedback control

\[ \dot{x} = f(x, u(x)) = g(x) \]

\[ x = (x_1, \ldots, x_n), \quad u = (u_1, \ldots, u_m), \quad f = (f_1, \ldots, f_n) \]

**Theorem**

Consider the ODE on \( \mathbb{R}^n \)

\[ \dot{x} = g(x), \quad g(0) = 0 \]

Assume that, at \( x = 0 \), the \( n \times n \) Jacobian matrix \( Dg = (\partial g_i / \partial x_j) \) has all eigenvalues with strictly negative real part. Then the origin is an asymptotically stable equilibrium.
A sufficient condition for asymptotic feedback stabilization

\[ \dot{x} = f(x, u(x)) = g(x) \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \]

Assume \( f(0, 0) = 0 \). At \( x = 0 \), consider

- the \( n \times n \) matrix \( A = (A_{ij}) = (\frac{\partial f_i}{\partial x_j}) \)
- the \( n \times m \) matrix \( B = (B_{ik}) = (\frac{\partial f_i}{\partial u_k}) \)
- the \( m \times n \) matrix \( C = (C_{kj}) = (\frac{\partial u_k}{\partial x_j}) \)

The Jacobian matrix of \( g \) at the origin is given by

\[ (Dg)_{ij} = \left( A_{ij} + \sum_k B_{ik} C_{kj} \right) = A + BC \]

Here the matrices \( A, B \) depend on \( f \), while the matrix \( C \) can be chosen as we like.

If we can find an \( m \times n \) matrix \( C \) such that \( A + BC \) has all of its eigenvalues with negative real part, then the feedback control \( u(x) = Cx \) is asymptotically stabilizing.
Optimal control problems

\[ \dot{x} = f(x, u), \quad u(t) \in U, \quad x(0) = x_0, \quad t \in [0, T] \]

**Goal:** Choose a control \( u(t) \in U \) such that the corresponding trajectory maximizes the payoff

\[ J = \psi(x(T)) - \int_0^T L(x(t), u(t)) \, dt \]

\[ = \text{[terminal payoff]} - \text{[running cost]} \]
Removing the running cost

\[
\begin{align*}
\text{maximize:} & \quad \psi(x(T)) - \int_0^T L(x(t), u(t)) \, dt \\
\text{subject to:} & \quad \dot{x} = f(x, u), \quad x(0) = \bar{x} \in \mathbb{R}^n, \quad u(t) \in U.
\end{align*}
\]

Adding the scalar equation

\[\dot{x}_{n+1} = L(x(t), u(t)), \quad x_{n+1}(0) = 0\]

We obtain a problem in Mayer form (no running cost):

\[
\text{maximize:} \quad \tilde{\psi}\left(x_1(T), \ldots, x_{n+1}(T)\right) = \psi(x(T)) - x_{n+1}(T)
\]
Consider the problem

\[
\begin{align*}
\text{maximize:} & \quad \psi(x(T)) \\
\text{subject to:} & \quad \dot{x} = f(x, u), \quad x(0) = x_0, \quad u(t) \in U.
\end{align*}
\]

Assume that for every \( x \) the set of possible velocities

\[
F(x) = \{ f(x, u) ; \ u \in U \}
\]

is closed, bounded, and convex.

Then an optimal (open-loop) control \( u : [0, T] \to U \) exists.
Existence of optimal controls (with dynamics linear w.r.t. $u$)

Consider the problem

maximize: $\psi(x(T)) - \int_0^T L(x(t), u(t)) \, dt$

subject to: $\dot{x} = f(x) + g(x)u$, $x(0) = x_0$, $u(t) \in [a, b]$.

Assume that the cost function $L$ is convex in $u$, for every fixed $x$.

Then an optimal (open-loop) control $u : [0, T] \mapsto U$ exists.
Example (non-existence of an optimal control)

minimize: $x_2(T)$

subject to: \[
\begin{align*}
\dot{x}_1 &= u \\
\dot{x}_2 &= x_1^2
\end{align*}
\] 
\[
\begin{align*}
x_1(0) &= 0 \\
x_2(0) &= 0 \\
u(t) &\in \{-1, 1\}
\end{align*}
\]

The above equations imply

\[
x_1(t) = \int_0^t u(s) \, ds,
\]
\[
x_2(T) = \int_0^T \left( \int_0^t u(s) \, ds \right)^2 \, dt \geq 0
\]
Taking highly oscillatory control functions

\[
    u_n(t) = \begin{cases} 
        +1 & \text{if } t \in (k-1)\frac{T}{n}, k\frac{T}{n]}, \text{ } k \text{ even} \\
        -1 & \text{if } t \in (k-1)\frac{T}{n}, k\frac{T}{n]}, \text{ } k \text{ odd}
    \end{cases}
\]

the corresponding solutions satisfy

\[
    x_{1,n}(t) \to 0, \quad x_{2,n}(t) \to 0 \text{ uniformly for } t \in [0, T]
\]

However, there is no optimal control \( u : [0, T] \mapsto \{-1, 1\} \) which achieves \( x_2(T) = 0 \), indeed

\[
    x_2(T) = \int_0^T \left( \int_0^t u(s) \, ds \right)^2 \, dt = 0
\]

only if \( u(t) \equiv 0 \). But such a control is not admissible
Finding the optimal control

maximize: \( \psi(x(T)) \)

subject to: \( \dot{x} = f(x, u), \quad x(0) = x_0, \quad u(t) \in U \)

Let \( u^*(t) \) be an optimal control and let \( x^*(t) \) be the optimal trajectory. Derive necessary conditions for their optimality.
Preliminary: perturbed solutions of an ODE

\[ \dot{x}(t) = g(t, x(t)) \]  \hspace{1cm} (ODE)

Let \( x^*(t) \) be a solution, and consider a family of perturbed solutions

\[ x_\varepsilon(t) = x^*(t) + \varepsilon v(t) + O(\varepsilon^2) \]

How does the “first order perturbation” \( v(t) \) evolve in time?
An example

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1, \quad T = \pi \]

\[ v(0) = (-1, 0), \quad v(t) = (-\cos t, \sin t) \]
A linearized equation for the evolution of tangent vectors

\[ \dot{x}(t) = g(t, x(t)) \quad (ODE) \]

\[ x_\varepsilon(t) = x^*(t) + \varepsilon v(t) + O(\varepsilon^2) \quad (\dagger) \]

Insert (\dagger) in (ODE), and use a Taylor approximation:

\[ \dot{x}_\varepsilon(t) = g(t, x_\varepsilon(t)) \]

\[ \dot{x}^*(t) + \varepsilon \dot{v}(t) + O(\varepsilon^2) = g(t, x^*(t) + \varepsilon v(t) + O(\varepsilon^2)) \]

\[ = g(t, x^*(t)) + \frac{\partial g}{\partial x}(t, x^*(t)) \cdot \varepsilon v(t) + O(\varepsilon^2) \]

\[ \implies \dot{v}(t) = A(t)v(t), \quad A(t) = \frac{\partial g}{\partial x}(t, x^*(t)) \]
The adjoint linear system

\[ p = (p_1, \ldots, p_n), \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad A(t) \text{ is an } n \times n \text{ matrix} \]

**Lemma**

Let \( v(t) \) and \( p(t) \) be any solutions to the linear ODEs

\[
\dot{v}(t) = A(t)v(t), \quad \dot{p}(t) = -p(t)A(t)
\]

Then the product \( p(t)v(t) = \sum_i p_i v_i \) is constant.

\[
\frac{d}{dt}(pv) = \dot{p}v + p\dot{v} = (-pA)v + p(Av) = 0
\]
Deriving necessary conditions

maximize the terminal payoff: \( \psi(x(T)) \)
subject to: \( \dot{x} = f(x, u), \quad x(0) = x_0, \quad u(t) \in U. \)

\( u^*(t) = \text{optimal control}, \quad x^*(t) = \text{optimal trajectory}. \)

No matter how we change the control \( u^*(\cdot) \), the terminal payoff cannot be increased.
Needle variations

Choose an arbitrary time $\tau \in ]0, T]$ and control value $\omega \in U$.

**Needle variation:**

$$u_\varepsilon(t) = \begin{cases} 
\omega & \text{if } t \in [\tau - \varepsilon, \tau], \\
u^*(t) & \text{otherwise.}
\end{cases}$$

**Perturbed trajectory:**

$$x_\varepsilon(t) = \begin{cases} 
x^*(t) & \text{if } t \leq \tau - \varepsilon, \\
x^*(t) + \varepsilon v(t) + O(\varepsilon^2) & \text{if } t \geq \tau
\end{cases}$$
Computing the perturbed trajectory

At time $\tau$:  \[ x_\varepsilon(\tau) = x^*(\tau) + \varepsilon \left[ f(x^*(\tau), \omega) - f(x^*(\tau), u^*(\tau)) \right] + \mathcal{O}(\varepsilon^2) \]

On the interval $t \in [\tau, T]$: \[ x_\varepsilon(t) = x^*(t) + \varepsilon v(t) + \mathcal{O}(\varepsilon^2), \]

\[ \begin{cases} 
\dot{v}(t) = A(t)v(t), \\
 v(\tau) = f(x^*(\tau), \omega) - f(x^*(\tau), u^*(\tau)), 
\end{cases} \]

\[ A(t) = \frac{\partial f}{\partial x}(x^*(t), u^*(t)) \]
A family of necessary conditions

$u^*$ is optimal $\quad \Longrightarrow \quad \frac{d}{d\varepsilon} \psi(x_\varepsilon(T)) \bigg|_{\varepsilon=0} = \nabla \psi(x^*(T)) \cdot v(T) \leq 0$
Let the row vector $p(t)$ be the solution to

$$
\dot{p}(t) = - p(t)A(t), \quad p(T) = \nabla \psi(x^*(T))
$$

$$
A(t) = \frac{\partial f}{\partial x}(x^*(t), u^*(t))
$$

Since $v(t)$ satisfies $\dot{v}(t) = A(t)v(t)$, the product $p(t)v(t)$ is constant in time. Hence

$$
p(\tau)v(\tau) = p(T)v(T) = \nabla \psi(x^*(T)) \cdot v(T) \leq 0
$$

For every $\tau \in ]0, T]$ and $\omega \in U$, we thus have

$$
p(\tau)v(\tau) = p(\tau)[f(x^*(\tau), \omega) - f(x^*(\tau), u^*(\tau))] \leq 0
$$
Geometric interpretation of the Pontryagin Maximum Principle

For every \( \tau \in ]0, T] \), the inequality

\[
p(\tau) \left[ f(x^*(\tau), \omega) - f(x^*(\tau), u^*(\tau)) \right] \leq 0 \quad \text{for all } \omega \in U
\]

implies

\[
p(\tau) \cdot \dot{x}^*(\tau) = p(\tau) \cdot f(x^*(\tau), u^*(\tau)) = \max_{\omega \in U} \left\{ p(\tau) \cdot f(x^*(\tau), \omega) \right\} \quad (PMP)
\]

For every time \( \tau \in ]0, T] \), the speed \( \dot{x}^*(\tau) \) corresponding to the optimal control \( u^*(\tau) \) is the one maximizing the product with \( p(\tau) \).
Statement of the Pontryagin Maximum Principle

maximize the terminal payoff: $\psi(x(T))$

subject to: $\dot{x} = f(x, u), \quad x(0) = x_0, \quad u(t) \in U$.

**Theorem**

Let $t \mapsto u^*(t)$ be an optimal control and $t \mapsto x^*(t)$ be the corresponding optimal trajectory.

Let the row vector $t \mapsto p(t)$ be the solution to the linear adjoint system

$$\dot{p}(t) = -p(t)A(t), \quad A_{ij}(t) = \frac{\partial f_i}{\partial x_j}(x^*(t), u^*(t))$$

with terminal condition $p(T) = \nabla \psi(x^*(T))$.

Then, at every time $\tau \in [0, T]$ where $u^*(\cdot)$ is continuous, one has

$$p(\tau) \cdot f(x^*(\tau), u^*(\tau)) = \max_{\omega \in U} \left\{ p(\tau) \cdot f(x^*(\tau), \omega) \right\}$$
Computing the Optimal Control

STEP 1: solve the pointwise maximization problem, obtaining the optimal control $u^*$ as a function of $p, x$, i.e.

$$u^*(x, p) = \arg\max_{\omega \in U} \{p \cdot f(x, \omega)\}$$  \hspace{1cm} (1)

STEP 2: solve the two-point boundary value problem

$$\begin{align*}
\dot{x} &= f(x, u^*(x, p)) \\
\dot{p} &= -p \cdot \frac{\partial}{\partial x} f(x, u^*(x, p))
\end{align*}$$

$$\begin{align*}
x(0) &= x_0 \\
p(T) &= \nabla\psi(x(T))
\end{align*} \hspace{1cm} (2)$$

- In general, the function $u^* = u^*(p, x)$ in (1) is highly nonlinear. It may be multivalued or discontinuous.
- The two-point boundary value problem (2) can be solved by a shooting method: Guess an initial value $p(0) = p_0$ and solve the corresponding Cauchy problem. Try to adjust the value of $p_0$ so that the terminal values $x(T), p(T)$ satisfy the given conditions.
- The dual variables $p = (p_1, \ldots, p_n)$ can be interpreted as shadow prices
Example: a linear pendulum

\[ q(t) = \text{position of a linearized pendulum, controlled by an external force with magnitude } u(t) \in [-1, 1]. \]

\[ \ddot{q}(t) + q(t) = u(t), \quad q(0) = \dot{q}(0) = 0, \quad u(t) \in [-1, 1] \]

We wish to maximize the terminal displacement \( q(T) \).
\[ \ddot{q}(t) + q(t) = u(t), \quad q(0) = \dot{q}(0) = 0, \quad u(t) \in [-1, 1] \]

Equivalent control system: \( x_1 = q, \ x_2 = \dot{q} \)

\[
\begin{cases}
\dot{x}_1 = f_1(x_1, x_2, u) = x_2 \\
\dot{x}_2 = f_2(x_1, x_2, u) = u - x_1 
\end{cases}
\]

**Goal:** maximize \( \psi(x(T)) = x_1(T) \)

Let \( u^*(t) \) be an optimal control, and let \( x^*(t) \) be the optimal trajectory.

The adjoint vector \( p = (p_1, p_2) \) is found by solving the linear system of ODEs

\[
\dot{p} = -p(t)A(t), \quad p(T) = \nabla \psi(x^*(T))
\]

\[
A_{ij}(t) = \frac{\partial f_i}{\partial x_j}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

\[
\psi(x_1, x_2) = x_1, \quad (p_1(T), p_2(T)) = \left( \frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2} \right)_{x=x^*(T)} = (1, 0)
\]
\[
(p_1, p_2) = -(p_1, p_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (p_1, p_2)(T) = (1, 0) \quad (3)
\]

In this special case, we can explicitly solve the adjoint equation (3) without needing to know \( x^*, u^* \), namely

\[
(p_1, p_2)(t) = (\cos(T-t), \ \sin(T-t)) \quad (4)
\]

\[
\left\{ \begin{array}{l}
\dot{x}_1 = f_1(x_1, x_2) = x_2 \\
\dot{x}_2 = f_2(x_1, x_2) = u - x_1
\end{array} \right.
\]

Given \( p = (p_1, p_2) \), the optimal control is

\[
u^*(x, p) = \arg \max_{\omega \in [-1, 1]} \{ p \cdot f(x, \omega) \} = \arg \max_{\omega \in [-1, 1]} \left\{ p_1 x_2 + p_2 (-x_1 + \omega) \right\} = \text{sign}(p_2)
\]

By (4), the optimal control is

\[
u^*(t) = \text{sign}(p_2(t)) = \text{sign}(\sin(T - t)) \quad t \in [0, T]
\]
$x_2 = \dot{q}$

- $u = 1$
- $u = -1$

$p = (p_1, p_2)$

$p_2(t) > 0 \quad \Rightarrow \quad u^*(t) = 1$

$p_2(t) < 0 \quad \Rightarrow \quad u^*(t) = -1$
Terminal constraints

maximize: $\psi(x(T))$

with dynamics: $\dot{x} = f(x, u), \quad x(0) = x_0 \in \mathbb{R}^n, \quad u(t) \in U$

and terminal constraints: $\phi_1(x(T)) = 0, \ldots, \phi_m(x(T)) = 0$
The tangent space to the target, and its perpendicular space

**Target set:** \[ S = \left\{ x \in \mathbb{R}^n ; \; \phi_1(x) = 0, \; \ldots, \; \phi_m(x) = 0 \right\} \]

Assume: the gradient vectors \( \nabla \phi_i(x^*) \) are linearly independent.

**Tangent space** to \( S \) at \( x^* \): \[ T_S = \left\{ v \in \mathbb{R}^n ; \; \nabla \phi_i(x^*) \cdot v = 0, \; i = 1, \ldots, m \right\} \]

A vector \( p \) is perpendicular to \( T_S \) if \( p \cdot v = 0 \) for all \( v \in T_S \)

This is true if and only \( p = \sum_{i=1}^m c_i \nabla \phi_i(x^*) \) for some coefficients \( c_1, \ldots, c_m \)
The cone of profitable directions

Profitable cone: \( T_{S^+} = \left\{ v \in T_S ; \; \nabla \psi(x^*) \cdot v \geq 0 \right\} \)

A vector \( p \) satisfies \( p \cdot v \geq 0 \) for all \( v \in T_{S^+} \) if and only if

\[
p = c_0 \nabla \psi(x^*) + \sum_{i=1}^{m} c_i \nabla \phi_i(x^*)
\]

for some coefficients \( c_0, c_1 \ldots, c_m \in \mathbb{R} \), with \( c_0 \geq 0 \)
A cone of feasible directions

For every choice of $\tau \in ]0, T]$, $\omega \in U$, the needle variation of the optimal control $u^*(\cdot)$ produces a shift in the terminal point of the trajectory:

$$v^{\tau, \omega}(T) = \lim_{\varepsilon \to 0^+} \frac{x_\varepsilon(T) - x^*(T)}{\varepsilon}$$

Let $\Gamma$ be the convex cone of all linear combinations of the vectors $v^{\tau, \omega}(T)$ with positive coefficients:

$$\Gamma \triangleq \left\{ \sum_i \theta_i v_i^{\tau_i, \omega_i}(T) ; \quad \theta_i \geq 0 \text{ for all } i \right\}$$
By combining two or more needle variations, one can perturb the terminal point of the trajectory in any direction of the form

\[ \mathbf{v} = \sum_{i} \theta_i \mathbf{v}_{i}^{\tau_i, \omega_i}(T) \quad \theta_i \geq 0. \]

Hence all directions \( \mathbf{v} \in \Gamma \) are attainable.
Necessary conditions - geometric version

$R(T) =$ reachable set at time $T$, $S =$ target set

$\Gamma =$ cone of feasible directions

$T_{S^+} =$ cone of profitable directions, which satisfy the constraints and increase the payoff function $\psi$
(PMP - geometric version)

Let $u^*(\cdot)$ be an optimal control, with trajectory $t \mapsto x^*(t) = x(t, u^*)$.
Then the cones $\Gamma$ and $T_{S^+}$ are **weakly separated**, i.e. there exists a non-zero vector $p(T)$ such that

$$p(T) \cdot v \geq 0 \quad \text{for all} \quad v \in T_{S^+}$$

$$p(T) \cdot v \leq 0 \quad \text{for all} \quad v \in \Gamma$$
(PMP - analytic version)

Let \( u^*(\cdot) \) be an optimal control, with trajectory \( t \mapsto x^*(t) = x(t, u^*) \).
Then there exists a non-zero adjoint vector \( t \mapsto p(t) \) such that

\[
p(T) = \lambda_0 \nabla \psi(x^*(T)) + \sum_{i=1}^m \lambda_i \nabla \phi_i(x^*(T)) \quad \text{with} \quad \lambda_0 \geq 0 \tag{1}
\]

\[
\dot{p}(t) = -p(t) D_x f(t, x^*(t), u^*(t)) \quad t \in [0, T] \tag{2}
\]

\[
p(\tau) \cdot f(\tau, x^*(\tau), u^*(\tau)) = \max_{\omega \in U} \left\{ p(\tau) \cdot f(\tau, x^*(\tau), \omega) \right\} \quad \text{for a.e.} \quad \tau \in [0, T] \tag{3}
\]
- In the PMP, the optimal control $u^*$ and the optimal trajectory $x^*$ do not change if we replace $p(t)$ by $c p(t)$, for any constant $c > 0$
- If $\lambda_0 > 0$, in (1) we can replace $\lambda_i$ with $\lambda_i/\lambda_0$, $i = 0, 1, \ldots, m$.

Two cases can thus arise:

- **Normal extremals**: $\lambda_0 = 1$. This is the most usual case
- **Abnormal extremals**: $\lambda_0 = 0$. In this case the function $\psi$ to be maximized does not play any role in determining the optimal control (figure above).
**Goal:** Starting at the origin with zero speed, move the cart so that at a given time $T > 0$ it comes back to the origin with maximum speed.

maximize: $\psi(x(T)) = x_2(T)$

subject to: \[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u(t)
\end{align*}
\]

$u(t) \in [-1, 1]$, \[
\begin{align*}
x_1(0) &= 0 \\
x_2(0) &= 0
\end{align*}
\]

terminal constraint: $\phi_1(x(T)) = x_1(T) = 0$
Optimality conditions

\[ p(T) = \nabla \psi(x(T)) + \lambda_1 \nabla \phi_1(x(T)) = (0, 1) + (\lambda_1, 0) \]

\[ u^* = \arg \max_{\omega \in [-1, 1]} p(t) \cdot f(x, \omega) = \arg \max_{\omega \in [-1, 1]} \{ p_1 x_2 + p_2 \omega \} = \text{sign}(p_2) \]

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \text{sign}(p_2) \\
\dot{p}_1 &= 0 \\
\dot{p}_2 &= -p_1
\end{aligned}
\]

\[
\begin{aligned}
x_1(0) &= 0 \\
p_1(0) &= 0 \\
x_2(0) &= 0 \\
p_2(0) &= 0
\end{aligned}
\]

\[
\begin{aligned}
x_1(T) &= 0 \\
p_2(T) &= 1 + (T - t) \lambda_1
\end{aligned}
\]

\[
\begin{aligned}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= \text{sign}(p_2(t)) = \begin{cases} -1 & t \in [0, \tau] \\ +1 & t \in ]\tau, T] \end{cases} \\
x_1(T) &= 0 \\
\tau &= \left(1 - \frac{1}{\sqrt{2}}\right) T
\end{aligned}
\]
$$u^*(t) = \begin{cases} 
-1 & t \in [0, \tau] \\
+1 & t \in ]\tau, T] 
\end{cases} \quad \tau = \left(1 - \frac{1}{\sqrt{2}}\right) T$$
The value function

Consider a family of optimal control problems, with different initial data

\[ \text{minimize: } \int_{\tau}^{T} L(x(t), u(t)) \, dt + \psi(x(T)) \]

subject to: \( \dot{x} = f(x, u), \quad u(t) \in U, \quad x(\tau) = y \)

**Value function:** \( V(\tau, y) = \text{minimum cost, starting at } (\tau, y) \)
\( x^*(\cdot) = \text{optimal trajectory starting from } (\tau, y). \)

- Choose any \( \omega \in U \)
- Using the constant control \( u(t) = \omega \) on the initial interval \( t \in [\tau, \tau + \varepsilon] \), we reach a new point \( x(\tau + \varepsilon) \).
- Starting at \( x(\tau + \varepsilon) \), we use an optimal control on the remaining interval \( t \in [\tau + \varepsilon, T] \)
Cost of the perturbed control

\[
x(\tau + \varepsilon) = y + \varepsilon f(y, \omega) + o(\varepsilon)
\]

\[
\int_\tau^{\tau+\varepsilon} L(x(t), u(t)) \, dt = \varepsilon L(y, \omega) + o(\varepsilon)
\]

\[
V(\tau + \varepsilon, x(\tau + \varepsilon)) = V(\tau, y) + \varepsilon V_\tau(\tau, y) + \nabla_y V(\tau, y) \cdot \varepsilon f(y, \omega) + o(\varepsilon)
\]

\[
V(\tau, y) \leq \varepsilon L(y, \omega) + V(\tau, y) + \varepsilon V_\tau(\tau, y) + \nabla_y V(\tau, y) \cdot \varepsilon f(y, \omega) + o(\varepsilon)
\]
\[ V(\tau, y) \leq \epsilon L(y, \omega) + V(\tau, y) + \epsilon V_{\tau}(\tau, y) + \nabla_y V(\tau, y) \cdot \epsilon f(y, \omega) + o(\epsilon) \]

Dividing by \( \epsilon \) and letting \( \epsilon \to 0 \) one obtains

\[ L(y, \omega) + V_{\tau}(\tau, y) + \nabla_y V(\tau, y) \cdot f(y, \omega) \geq 0 \quad \text{for all } \omega \in U \]

Choosing \( \omega = u^*(\tau) = \text{initial value of the optimal control} \), one achieves equality:

\[ L(y, u^*(\tau)) + V_{\tau}(\tau, y) + \nabla_y V(\tau, y) \cdot f(y, u^*(\tau)) = 0 \]
The Hamilton-Jacobi PDE for the value function

\[
\min_{\omega \in U} \left\{ L(y, \omega) + V_\tau(\tau, y) + \nabla_y V(\tau, y) \cdot f(y, \omega) \right\} = 0
\]

Can be written as:

\[ V_\tau + H(y, \nabla_y V) = 0 \]

\[
H(y, \xi) \doteq \min_{\omega \in U} \left\{ L(y, \omega) + \xi \cdot f(y, \omega) \right\}
\]

Must be solved for \((\tau, y) \in [0, T] \times \mathbb{R}^n\), with terminal data

\[ V(T, y) = \psi(y) \]

If the value function is known, then the optimal feedback control is

\[ u(\tau, y) = \arg\min_{\omega \in U} \left\{ L(y, \omega) + \nabla_y V(\tau, y) \cdot f(y, \omega) \right\} \]
Example: a linear-quadratic problem

minimize: \[ \int_\tau^T \left[ a|x(t)|^2 + u^2(t) \right] dt + c|x(T)|^2 \]

subject to: \[ \dot{x} = Ax + bu, \quad x \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}, \quad x(\tau) = y \]

Here \( A \) is an \( n \times n \) matrix, \( b \in \mathbb{R}^n \).

The value function \( V = V(t, x) \) satisfies the Hamilton-Jacobi PDE

\[ V_t + \min_{\omega \in \mathbb{R}} \left\{ [a|x|^2 + \omega^2] + \nabla_x V \cdot (Ax + b\omega) \right\} = 0 \]

Solving the above minimization problem we find the optimal control

\[ u^* = \arg\min_{\omega \in \mathbb{R}} \left\{ \omega^2 + \nabla_x V \cdot b\omega \right\} = -\frac{\nabla_x V \cdot b}{2} \]

Hence the PDE for \( V \) takes the form

\[ V_t + \left\{ a|x|^2 + \nabla_x V \cdot Ax - \frac{|\nabla_x V \cdot b|^2}{4} \right\} = 0, \quad V(T, x) = c|x|^2 \]
Optimal control problem in infinite time horizon, with exponentially discounted cost

\[
\text{minimize: } \int_0^{+\infty} e^{-\gamma t} L(x(t), u(t)) \, dt
\]

subject to:  \( \dot{x} = f(x, u), \; u(t) \in U, \; x(0) = y \)

\textbf{Value function: } \; V(y) = \text{minimum cost, starting at } y
A PDE for the value function

\[ x^*(\cdot) = \text{optimal trajectory starting from } (\tau, y). \]

- Choose any \( \omega \in U \)
- Using the constant control \( u(t) = \omega \) on the initial interval \( t \in [0, \epsilon] \), we reach a new point \( x(\epsilon) \).
- Starting at \( x(\epsilon) \), we then use an optimal control on the remaining interval \( t \in [\epsilon, +\infty[ \).
Cost of the perturbed control

\[ x(\varepsilon) = y + \varepsilon f(y, \omega) + o(\varepsilon) \]

\[ \int_0^\varepsilon e^{-\gamma t} L(x(t), u(t)) \, dt = \varepsilon L(y, \omega) + o(\varepsilon) \]

\[ V(x(\varepsilon)) = V(y) + \nabla V(y) \cdot \varepsilon f(y, \omega) + o(\varepsilon) \]

\[ V(y) \leq \varepsilon L(y, \omega) + e^{-\gamma \varepsilon} V(x(\varepsilon)) + o(\varepsilon) \]

\[ = \varepsilon L(y, \omega) + (1 - \gamma \varepsilon) V(y) + \nabla V(y) \cdot \varepsilon f(y, \omega) + o(\varepsilon) \]
\[ V(y) \leq \varepsilon L(y, \omega) + V(y) - \gamma \varepsilon V(y) + \nabla V(y) \cdot \varepsilon f(y, \omega) + o(\varepsilon) \]

Dividing by \( \varepsilon \) and letting \( \varepsilon \to 0 \) one obtains

\[ \gamma V(y) \leq L(y, \omega) + \nabla V(y) \cdot f(y, \omega) \quad \text{for all } \omega \in U \]

Choosing \( \omega = u^*(0) = \text{initial value of the optimal control} \), one achieves equality:

\[ \gamma V(y) = L(y, u^*(0)) + \nabla V(y) \cdot f(y, u^*(0)) \]
The Hamilton-Jacobi PDE for the value function, infinite time horizon

\[
\gamma V(y) = \min_{\omega \in U} \left\{ L(y, \omega) + \nabla V(y) \cdot f(y, \omega) \right\}
\]

Can be written as:

\[
\gamma V(y) = H(y, \nabla V)
\]

\[
H(y, \xi) = \min_{\omega \in U} \left\{ L(y, \omega) + \xi \cdot f(y, \omega) \right\}
\]

Optimal feedback control:

\[
u(y) = \arg\min_{\omega \in U} \left\{ L(y, \omega) + \nabla V(y) \cdot f(y, \omega) \right\}
\]
Example - fishery management

\( x(t) = \) amount of fish, \( u(t) = \) harvesting effort,
\[ c = \text{harvesting cost}, \quad p = \text{market price of fish} \]

\[
\text{maximize: } \int_0^{+\infty} e^{-\gamma t} [p x(t) u(t) - c u(t)] \, dt
\]

subject to: \( \dot{x} = \alpha x (M - x) - xu, \quad u(t) \in [0, \kappa], \quad x(0) = y \)

The value function \( V = V(x) \) satisfies the implicit ODE

\[
\gamma V(x) = \max_{\omega \in [0, \kappa]} \left\{ p x u - c u + V' \cdot [\alpha x (M - x) - xu] \right\}
\]

Solving the maximization problem we find the optimal feedback control

\[
u^* = \arg \max_{\omega \in [0, \kappa]} [p x - c - V'(x) \cdot x] \omega \begin{cases} = 0 & \text{if } px - c - V'(x) x < 0 \\ = \kappa & \text{if } px - c - V'(x) x > 0 \\ \in [0, \kappa] & \text{if } px - c - V'(x) x = 0 \end{cases}
\]
The value function $V(x)$ is obtained by patching together solutions of two implicit ODEs

\[
\gamma V(x) = V'(x) \cdot \alpha x (M - x) \quad \text{when } u = 0
\]

\[
\gamma V(x) = (px - c)\kappa + V'(x) \cdot [\alpha x(M - x) - x\kappa] \quad \text{when } u = \kappa
\]
References


