On Discontinuous Differential Equations

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1 - Introduction.

Consider the Cauchy problem for an ordinary differential equation

$$\dot{x} = g(t, x), \qquad x(0) = \bar{x}, \qquad t \in [0, T]. \tag{1.1}$$

When g is continuous, the local existence of solutions is provided by Peano's theorem. Several existence and uniqueness results are known also in the case of a discontinuous right hand side [7]. We recall here the classical theorem of Carathéodory [8]:

Theorem A. *Let* $g : [0, T] \times I\!R^n \mapsto I\!R^n$ *be a bounded function.*

(i) If the map $t \mapsto g(t, x)$ *is measurable for each* x *and the map* $x \mapsto g(t, x)$ *is continuous for each* t*, then the Cauchy problem (1.1) has at least one solution.*

(ii) If the map $t \mapsto g(t, x)$ *is measurable for each* x *and the map* $x \mapsto g(t, x)$ *is Lipschitz continuous for each* t*, with a uniform Lipschitz constant, then the Cauchy problem (1.1) has a unique solution, depending Lipschitz continuously on the initial data* $\bar{x}$*.*

By a solution of (1.1) we mean an absolutely continuous function $x : [0,T] \mapsto I\!R^n$ such that

$$x(t) = \bar{x} + \int_0^t g\big(t, x(t)\big)\, dt \qquad \text{for all} \quad t \in [0,T]. \tag{1.2}$$

More recent results rely on the notions of directional continuity and of bounded directional variation of a vector field. More precisely, given a closed convex cone $\Gamma \subset I\!R^m$, we say that a (possibly discontinuous) map $\phi : I\!R^m \mapsto I\!R^n$ is *directionally continuous* if at each point $p \in I\!R^m$ one has

$$\lim_{p' \to p,\ p' - p \in \Gamma} \phi(p') \ = \ \phi(p). \tag{1.3}$$

We say that the map ϕ has *bounded directional variation* if

$$\sup \left\{ \sum_{i=1}^{N} \big|\phi(p_i) - \phi(p_{i-1})\big| \ ; \qquad N \geq 1, \quad p_i - p_{i-1} \in \Gamma \ \text{ for every } i \right\} \ < \ \infty. \tag{1.4}$$

The following existence and uniqueness results were proved in [1, 9] and in [2], respectively.

Theorem B. *Let $\big|g(t,x)\big| \leq L < M$ for all t, x.*

(i) Assume that g is directionally continuous in the direction of the cone

$$\Gamma \doteq \big\{(t,x); \quad |x| \leq Mt\big\} \ \subset I\!R^{n+1}. \tag{1.5}$$

Then the Cauchy problem (1.1) admits at least one solution.

(ii) Assume that g has bounded directional variation in the direction of the cone Γ in (1.5). Then the Cauchy problem (1.1) has a unique solution, depending Lipschitz continuously on the initial data $\bar{x}$.

Further uniqueness results can be found in [3, 4]. Concerning existence, see also the interesting work [10].

Aim of the present paper is to prove two theorems on the existence and the uniqueness of solutions to the autonomous Cauchy problem

$$\dot{x} = f(x), \qquad x(0) = \bar{x} \in I\!R^m, \qquad t \in [0,T], \tag{1.6}$$

extending the classical results of Carathéodory. As a preliminary, we observe that the equation (1.1) can be rewritten as an autonomous problem on $I\!R^{n+1}$, introducing the variable $x_0 = t$ and the vector field $f(x_0, x) = \big(1,\ g(x_0, x)\big)$. Under the assumptions of Theorem A, the function f can jump across the hyperplanes of the form $x_0 =$ constant. These hyperplanes are certainly transversal to f. Namely, taking the inner product of f with their normal vector, one trivially has

$$f \cdot \mathbf{n} = \big(1,\ g(x_0, x)\big) \cdot (1,\ 0) \equiv 1.$$

The next theorem shows that this transversality condition is indeed the key ingredient toward the existence of solutions.

Theorem 1. *Assume that f in (1.6) has the form*

$$f(x) \doteq F\Big(g_1\big(\tau_1(x),\ x\big),\ \ldots\ ,\ g_N\big(\tau_N(x),\ x\big)\Big), \tag{1.7}$$

where:

(i) Each map $\tau_i : \mathbb{R}^m \mapsto \mathbb{R}$ is continuously differentiable. Each $g_i : \mathbb{R} \times \mathbb{R}^m \mapsto \mathbb{R}$ is a Carathéodory function, i.e. measurable in t and continuous in x. Moreover, $F : \mathbb{R}^N \mapsto \mathbb{R}^m$ is continuous.

(ii) For some compact set $K \subset \mathbb{R}^m$, at every point x there holds:

$$f(x) \in K, \qquad \nabla\tau_i(x) \cdot z > 0 \quad \textit{for every} \quad z \in K. \tag{1.8}$$

Then the Cauchy problem (1.6) has at least one solution.

Remark 1. The assumption (ii) can be easily recognized as a transversality condition. Indeed, by $(1.8)_1$ every trajectory of (1.6) satisfies the differential inclusion $\dot{x} \in K$. Hence by $(1.8)_2$ this trajectory must cross transversally any hypersurface of the form $\tau_i(x) = \text{constant}$. According to the definition (1.7), these are the surfaces across which f can jump.

To guarantee the existence of solutions, some kind of transversality condition is necessary, as shown by obvious counterexample

$$\dot{x} = \begin{cases} 1 & \text{if} \quad x < 0, \\ -1 & \text{if} \quad x \geq 0, \end{cases} \qquad x(0) = 0.$$

Our second result is concerned with the uniqueness and continuous dependence of solutions.

Theorem 2. *Assume that f in (1.6) has the form*

$$f(x) \doteq g\big(\tau(x),\ x\big), \tag{1.9}$$

where:

(i) The function $\tau : \mathbb{R}^m \mapsto \mathbb{R}$ is continuously differentiable. The map $g : \mathbb{R} \times \mathbb{R}^m \mapsto \mathbb{R}^m$ is measurable w.r.t. t and uniformly Lipschitz continuous w.r.t. x.

(ii) There exists a compact set K such that

$$f(x) \in K, \qquad \nabla\tau(x) \cdot z > 0 \quad \textit{for every} \quad x \in \mathbb{R}^m, \quad z \in K. \tag{1.10}$$

Moreover, the gradient $\nabla\tau$ has bounded directional variation w.r.t. the cone $\Gamma \doteq \big\{\lambda z;\ \ \lambda \geq 0,\ z \in K\big\}$.

Then the Cauchy problem (1.6) has a unique solution, depending on the initial data $\bar{x}$ in a locally Lipschitz continuous way.

Remark 2. In the case where $m = n + 1$, $x = (x_0, \ldots, x_n)$ and $\tau(x) \equiv x_0$, the above theorem reduces to part (ii) of Theorem A. Roughly speaking, in Carathéodory's theorem one allows jumps across the hyperplanes $x_0 =$ constant. On the other hand, in Theorem 2 we allow jumps across the hypersurfaces $\tau(x) =$ constant, provided that these surfaces are transversal to the vector field f and the direction of their tangent planes does not wiggle too much.

The reader should also notice that in Theorem 2 the assumption of bounded directional variation is placed on the gradient $\nabla\tau$. This situation is quite different from part (ii) of Theorem B, where one assumes that the vector field f itself has bounded directional variation.

Remark 3. In Theorems 1 and 2, the scalar functions τ, τ_i were assumed to be $\mathcal{C}^1$. This assumption simplifies some technical aspects of the proofs, but may likely be relaxed. We conjecture that the same results hold if τ, τ_i are only assumed Lipschitz continuous, and the conditions (1.8), (1.10) are duly reformulated in terms of Clarke generalized gradients [5].

Remark 4. If in Theorem 2 we drop the key assumption that the directional variation of $\nabla\tau$ be bounded, then the uniqueness of solutions may fail. This will be illustrated by an example in the last section of this paper. On the other hand, the uniqueness result stated in [4] allows f to have discontinuities along a set of lines whose slopes have unbounded directional variation. However, the validity of this theorem relies on the very special structure of f, linked to the solution of a scalar conservation law.

2 - Proof of Theorem 1.

It is not restrictive to assume that $\bar{x} = 0$. Define the Picard operator $u \mapsto \mathcal{P}u$

$$(\mathcal{P}u)(t) \doteq \int_0^t F\Big(g_1\big(\tau_1(u(s)), u(s)\big), \cdots, g_N\big(\tau_N(u(s)), u(s)\big)\Big)\, ds. \tag{2.1}$$

We will prove that this operator is continuous on the compact set

$$\mathcal{U} \doteq \left\{ u : [0,T] \mapsto I\!R^m; \quad u(0) = 0, \quad \frac{u(t) - u(s)}{t - s} \in K \quad \text{for all} \quad t > s \right\}. \tag{2.2}$$

Let $\varepsilon > 0$ be given. Applying the theorem of Scorza-Dragoni [11] to each map g_i, $i = 1, \ldots, N$, we obtain the existence of a closed set J_i with

$$\mathrm{meas}(I\!R \setminus J_i) \le \varepsilon, \tag{2.3}$$

such that g_i is continuous restricted to the set $J_i \times I\!R^m$. Define the closed set

$$A \doteq \{x \in I\!R^m; \quad \tau_i(x) \in J_i \ \text{ for every } i = 1, \ldots, N\}.$$

By (1.7), the composed map f is continuous restricted to A. Using the extension theorem of Dugundji [6, p. 188] we now construct a continuous map $\tilde{f} : I\!R^m \mapsto K$ such that $\tilde{f} = f$ on A.

Call $K = \max_{z \in K} |z|$. By (2.2), every function $u \in \mathcal{U}$ thus takes values inside the closed ball

$$X \doteq \{x \in I\!R^m; \quad |x| \le T \cdot |K|\}.$$

By $(1.8)_2$ and the continuity of the gradients $\nabla\tau_i$, there exists a strictly positive δ_0 such that

$$\nabla\tau_i(x) \cdot z \ge \delta_0 > 0 \qquad \text{for all } x \in X, \ z \in K, \tag{2.4}$$

because the sets X, K are compact. As a consequence, for each $u \in \mathcal{U}$ the maps $t \mapsto \tau_i\big(u(t)\big)$ from $[0, T]$ into $I\!R$ are strictly increasing. Namely,

$$\frac{d\tau_i\big(u(t)\big)}{dt} \;=\; \nabla\tau_i \cdot \dot{u} \;\ge\; \delta_0 \;>\; 0. \tag{2.5}$$

For a fixed $u \in \mathcal{U}$, call $I_u \subset [0, T]$ the set of times t such that $u(t) \notin A$, i.e.

$$I_u \doteq \Big\{t \in [0, T]; \quad \tau_i\big(u(t)\big) \notin J_i \ \text{ for some } i = 1, \cdots, N\Big\}.$$

Because of (2.3) and (2.5), the measure of I_u satisfies

$$\text{meas}(I_u) \le N\varepsilon/\delta_0. \tag{2.6}$$

To prove the continuity of $\mathcal{P}$, call $\widetilde{\mathcal{P}}$ the Picard operator corresponding to the function $\tilde{f}$, i.e.

$$(\widetilde{\mathcal{P}}u)(t) \doteq \int_0^t \tilde{f}\big(u(s)\big)\, ds.$$

Clearly, $\widetilde{\mathcal{P}}$ is continuous, hence for any fixed $u \in \mathcal{U}$ there exists a $\delta > 0$, such that

$$\big\|\widetilde{\mathcal{P}}v - \widetilde{\mathcal{P}}u\big\| \le \varepsilon \qquad \text{whenever } \ v \in \mathcal{U}, \ \ \|v - u\| \le \delta. \tag{2.7}$$

We now observe that the difference between the Picard operators $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ is small. Indeed, for every $v \in \mathcal{U}$, (2.6) implies

$$\big\|\widetilde{\mathcal{P}}v - \mathcal{P}v\big\| \;\le\; \sup_{x \in X} \big|f(x) - \tilde{f}(x)\big| \cdot \text{meas}(I_v) \;\le\; 2|K| \cdot N\varepsilon/\delta_0. \tag{2.8}$$

Together, (2.7) and (2.8) yield

$$\big\|\mathcal{P}v - \mathcal{P}u\big\| \le \big\|\mathcal{P}v - \widetilde{\mathcal{P}}v\big\| + \big\|\widetilde{\mathcal{P}}v - \widetilde{\mathcal{P}}u\big\| + \big\|\widetilde{\mathcal{P}}u - \mathcal{P}u\big\| \le \varepsilon + 4|K|N\varepsilon/\delta_0, \tag{2.9}$$

for every $v \in \mathcal{U}$ with $\|v-u\| \le \delta$. Since $\varepsilon > 0$ in (2.9) was arbitrary, this shows that the Picard operator $u \mapsto Pu$ is continuous, mapping the compact set $\mathcal{U}$ into itself. By applying the Schauder fixed point theorem we thus obtain the existence of a solution to the Cauchy problem (1.6).

3 - Proof of Theorem 2.

For any given $\bar{x} \in \mathbb{R}^m$, the existence of a solution follows from Theorem 1. The main part of the proof consists in showing that, given a radius $R > 0$ and any two solutions

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big), \qquad x(0) = x_0, \\ \dot{y}(t) &= f\big(y(t)\big), \qquad y(0) = y_0, \end{aligned} \tag{3.1}$$

with $|x_0|, |y_0| \le R$, one has the estimate

$$\big|x(t) - y(t)\big| \le C_R \cdot |x_0 - y_0| \qquad t \in [0, T], \tag{3.2}$$

for some constant C_R depending only on f and R. The uniqueness of solutions is an obvious consequence of (3.2). The proof is given in four steps.

STEP 1. We first study the case where f, in addition to the assumptions (i) and (ii) in Theorem 2, is piecewise smooth. More precisely, we assume that f has the form

$$f(x) = g_k(x) \qquad \text{if} \quad \tau_k \le \tau(x) < \tau_{k+1}, \tag{3.3}$$

for some increasing sequence of times $\{\tau_k;\ k \in \mathbb{Z}\}$. Here the functions g_k have uniformly bounded $\mathcal{C}^1$ norm, say with

$$\sup_{x,k} \big|g_k(x)\big| \le C_0, \qquad \sup_{x,k} \big|D_x g_k(x)\big| \le C_1 \tag{3.4}$$

for some constants C_0, C_1. Under these additional regularity assumptions, the uniqueness of solutions of (1.6) is clear. Our aim is to derive the uniform estimate (3.2) by studying the evolution of infinitesimal tangent vectors.

Consider a one-parameter family of solutions

$$\dot{x}^\varepsilon(t) = f\big(x^\varepsilon(t)\big), \qquad x^\varepsilon(0) = x_0^\varepsilon, \tag{3.5}$$

regarded as small perturbations of a reference solution $x^0(\cdot) = x(\cdot)$. Define the first order tangent vector

$$\mathbf{v}(t) = \lim_{\varepsilon \to 0+} \frac{x^\varepsilon(t) - x(t)}{\varepsilon}. \tag{3.6}$$

Call t_k, $k \in \mathbb{Z}$, the times where the reference solution $x(\cdot)$ crosses the hypersurfaces $\tau(x) = \tau_k$. By (1.10), all these crossings are transversal. According to the standard theory of piecewise smooth differential equations [7, 8], if the limit (3.6) exists at time $t = 0$, then the tangent vector $\mathbf{v}$ is well defined for all $t \in [0, T]$, $t \neq t_k$, $k \in \mathbb{Z}$. The time evolution of $\mathbf{v}$ is governed by the linear equation

$$\dot{\mathbf{v}}(t) = D_x g_k(x(t)) \cdot \mathbf{v}(t) \qquad \text{for} \quad t \in]t_k,\ t_{k+1}[\,, \tag{3.7}$$

together with impulses at the crossing times t_k. To describe the linear impulse at time t_k, call

$$\mathbf{n}_k \doteq \frac{\nabla\tau\big(x(t_k)\big)}{\big|\nabla\tau\big(x(t_k)\big)\big|}$$

the unit normal vector to the surface $\tau = \tau_k$ at the point $x(t_k)$. Moreover, define (fig. 1)

$$f_k \doteq \lim_{t \to t_k^+} f\big(x(t)\big) = g_k\big(x(t_k)\big), \qquad \tilde{f}_k \doteq \lim_{t \to t_{k+1}^-} f\big(x(t)\big) = g_k\big(x(t_{k+1})\big), \tag{3.8}$$

$$\mathbf{v}_k \doteq \lim_{t \to t_k^+} \mathbf{v}(t), \qquad \tilde{\mathbf{v}}_k \doteq \lim_{t \to t_{k+1}^-} \mathbf{v}(t). \tag{3.9}$$

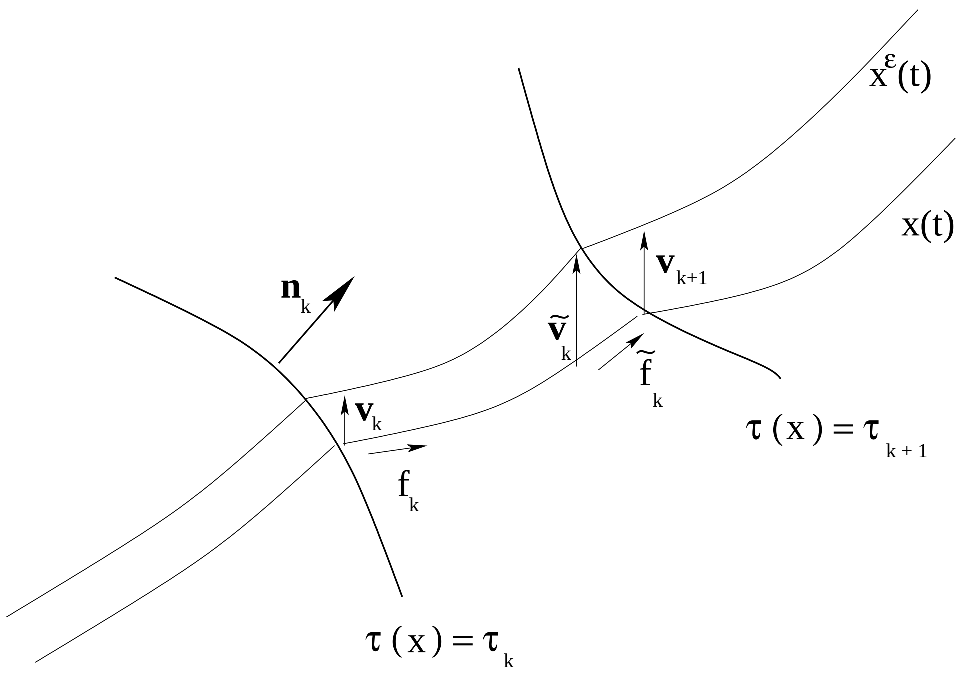

figure 1

With the above notations, an elementary computation shows that, at the crossing time t_k, the values $\mathbf{v}(t_k+) = \mathbf{v}_k$ and $\mathbf{v}(t_k-) = \tilde{\mathbf{v}}_{k-1}$ satisfy the linear relation

$$\mathbf{v}_k = \tilde{\mathbf{v}}_{k-1} + (f_k - \tilde{f}_{k-1}) \frac{\tilde{\mathbf{v}}_{k-1} \cdot \mathbf{n}_k}{\tilde{f}_{k-1} \cdot \mathbf{n}_k}. \tag{3.10}$$

Our next goal is to derive a priori bounds on the size of $\mathbf{v}$. In the following, with the Landau symbol $\mathcal{O}(1)$ we denote a quantity whose norm is uniformly bounded. The bound may depend on T, on the constants C_0, C_1 in (3.4) and $\delta_0 > 0$ in (2.4), and on the total directional variation of $\nabla\tau$, but not on the particular solution $x(\cdot)$ in (3.1).

Recalling (3.8), from (3.4) we deduce

$$\tilde{f}_k - f_k = g_k\big(x(t_{k+1})\big) - g_k\big(x(t_k)\big) = \mathcal{O}(1)(t_{k+1} - t_k). \tag{3.11}$$

Moreover, recalling (3.9), from (3.7) we deduce

$$\tilde{\mathbf{v}}_k - \mathbf{v}_k = \mathbf{v}(t_{k+1}^-) - \mathbf{v}(t_k^+) = \mathcal{O}(1)(t_{k+1} - t_k)|\mathbf{v}_k|. \tag{3.12}$$

In the following, we use the superscripts ^{N_k} and ^{T_k} to denote the components of a vector which are parallel and tangent to $\mathbf{n}_k$, respectively. More precisely, we set

$$\mathbf{v}_k^{N_k} = \mathbf{v}_k \cdot \mathbf{n}_k, \qquad\qquad \mathbf{v}_k^{T_k} = \mathbf{v}_k - \mathbf{v}_k^{N_k}\mathbf{n}_k. \tag{3.13}$$

The same notations are used for f_k. In addition, for $t \in]t_k,\ t_{k+1}[$ we define

$$w_k \doteq \frac{\mathbf{v}_k^{N_k}}{f_k^{N_k}} \qquad\qquad z_k \doteq \mathbf{v}_k^{T_k} - f_k^{T_k}\, w_k. \tag{3.14}$$

The quantities $\tilde{w}_k$ and $\tilde{z}_k$ are defined similarly. By (1.10), the quantities $|f_k^{N_k}|$ are uniformly positive. We thus have the estimates

$$|w_k| = \mathcal{O}(1)\cdot|\mathbf{v}_k|, \qquad\qquad |z_k| = \mathcal{O}(1)\cdot|\mathbf{v}_k|, \tag{3.15}$$

$$\begin{aligned} |\mathbf{v}_k| = \left|\mathbf{v}_k^{N_k}\right| + \left|\mathbf{v}_k^{T_k}\right| \ &\le\ \left|f_k^{N_k}\right|\ |w_k| + |z_k| + \left|f_k^{T_k}\right|\ |w_k| \\ = |f_k|\ |w_k| + |z_k| \ &=\ \mathcal{O}(1)\cdot\big(|w_k| + |z_k|\big). \end{aligned} \tag{3.16}$$

Bounds on the size of $\mathbf{v}$ can thus be obtained from estimates on w_k and z_k.

From (3.13) and (3.10) it follows

$$\begin{aligned} w_k &= \frac{\mathbf{v}_k\cdot\mathbf{n}_k}{f_k\cdot\mathbf{n}_k} \\ &= \frac{1}{f_k\cdot\mathbf{n}_k}\left[\tilde{\mathbf{v}}_{k-1}\cdot\mathbf{n}_k + (f_k\cdot\mathbf{n}_k - \tilde{f}_{k-1}\cdot\mathbf{n}_k)\frac{\tilde{\mathbf{v}}_{k-1}\cdot\mathbf{n}_k}{\tilde{f}_{k-1}\cdot\mathbf{n}_k}\right] \\ &= \frac{\tilde{\mathbf{v}}_{k-1}\cdot\mathbf{n}_k}{\tilde{f}_{k-1}\cdot\mathbf{n}_k} \\ &= \tilde{w}_{k-1} + \mathcal{O}(1)\cdot|\tilde{\mathbf{v}}_{k-1}|\big|\mathbf{n}_k - \mathbf{n}_{k-1}\big|. \end{aligned} \tag{3.17}$$

In addition, using (3.11) and (3.12) we deduce

$$\begin{aligned} \tilde{w}_{k-1} - w_{k-1} &= \frac{\tilde{\mathbf{v}}_{k-1}\cdot\mathbf{n}_{k-1} - \mathbf{v}_{k-1}\cdot\mathbf{n}_{k-1}}{\tilde{f}_{k-1}\cdot\mathbf{n}_{k-1}} + \left[\frac{\mathbf{v}_{k-1}\cdot\mathbf{n}_{k-1}}{\tilde{f}_{k-1}\cdot\mathbf{n}_{k-1}} - \frac{\mathbf{v}_{k-1}\cdot\mathbf{n}_{k-1}}{f_{k-1}\cdot\mathbf{n}_{k-1}}\right] \\ &= \mathcal{O}(1)(t_k - t_{k-1})\ |\mathbf{v}_{k-1}|. \end{aligned} \tag{3.18}$$

Together, (3.12), (3.17) and (3.18) yield

$$w_k = w_{k-1} + \mathcal{O}(1) \cdot \Big\{|\mathbf{n}_k - \mathbf{n}_{k-1}| + |t_k - t_{k-1}|\Big\} \left|\mathbf{v}_{k-1}\right|. \tag{3.19}$$

Similar estimates can be obtained for the component z_k, namely

$$\begin{aligned} z_k - \tilde z_{k-1} &= \Big[\mathbf{v}_k^{T_k} - f_k^{T_k} w_k\Big] - \Big[\tilde{\mathbf{v}}_{k-1}^{T_{k-1}} - \tilde f_{k-1}^{T_{k-1}} \tilde w_{k-1}\Big] \\ &= \Big[\tilde{\mathbf{v}}_{k-1}^{T_k} + \Big(f_k^{T_k} - \tilde f_{k-1}^{T_k}\Big) \tilde w_{k-1} - f_k^{T_k} w_k\Big] - \Big[\tilde{\mathbf{v}}_{k-1}^{T_{k-1}} - \tilde f_{k-1}^{T_{k-1}} \tilde w_{k-1}\Big] \\ &= \Big(\tilde{\mathbf{v}}_{k-1}^{T_k} - \tilde{\mathbf{v}}_{k-1}^{T_{k-1}}\Big) + f_k^{T_k}\left(\tilde w_{k-1} - w_k\right) + \tilde w_{k-1}\Big(\tilde f_{k-1}^{T_{k-1}} - \tilde f_{k-1}^{T_k}\Big) \\ &= \mathcal{O}(1)\left|\mathbf{n}_k - \mathbf{n}_{k-1}\right| \ \left|\tilde{\mathbf{v}}_{k-1}\right|, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \tilde z_{k-1} - z_{k-1} &= \Big(\tilde{\mathbf{v}}_{k-1}^{T_{k-1}} - \mathbf{v}_{k-1}^{T_{k-1}}\Big) + \Big(f_{k-1}^{T_{k-1}} - \tilde f_{k-1}^{T_{k-1}}\Big) w_{k-1} + \tilde f_{k-1}^{T_{k-1}}\left(w_{k-1} - \tilde w_{k-1}\right) \\ &= \mathcal{O}(1)\left|t_k - t_{k-1}\right| \ \left|\mathbf{v}_{k-1}\right|. \end{aligned} \tag{3.21}$$

Therefore,

$$z_k = z_{k-1} + \mathcal{O}(1) \cdot \Big\{|\mathbf{n}_k - \mathbf{n}_{k-1}| + |t_k - t_{k-1}|\Big\} \left|\mathbf{v}_{k-1}\right|. \tag{3.22}$$

Introducing the scalar quantity $y_k \doteq |w_k| + |z_k|$, from (3.19), (3.22) and (3.16) we deduce

$$y_k \le \Big(1 + \mathcal{O}(1) \cdot \{|\mathbf{n}_k - \mathbf{n}_{k-1}| + |t_k - t_{k-1}|\}\Big) y_{k-1}.$$

By induction on k, for any integers $p < q$ we obtain

$$y_q \le \exp\left\{C' \cdot \sum_{k=p+1}^{q} \big(|\mathbf{n}_k - \mathbf{n}_{k-1}| + |t_k - t_{k-1}|\big)\right\} y_p, \tag{3.23}$$

for some constant C'. Recalling the assumption on the directional variation of $\nabla\tau$, we now have

$$\sum_k |t_k - t_{k-1}| \le T, \qquad\qquad \sum_k |\mathbf{n}_k - \mathbf{n}_{k-1}| \le \text{const.} \tag{3.24}$$

From (3.23) and (3.24), since by (3.15)-(3.16) the quantities y_k are uniformly equivalent to the corresponding norms $\big|\mathbf{v}(t_k)\big|$, we finally obtain the estimate

$$\big|\mathbf{v}(t)\big| \le C_R \cdot \big|\mathbf{v}(s)\big| \qquad\qquad \text{for every} \ \ 0 \le s < t \le T, \tag{3.25}$$

for some constant C_R. Observe that C_R may depend on R through the quantity

$$\delta_0 \doteq \min\Big\{\nabla\tau(x) \cdot z \ ; \qquad z \in K, \ \ |x| \le R + T|K|\Big\}. \tag{3.26}$$

STEP 2. Relying on the uniform bounds (3.25) on tangent vectors, it is easy to derive the estimate (3.2) in the piecewise smooth case. Indeed, let the initial data x_0, y_0 be given. We then construct a one-parameter family of solutions $x^\theta : [0,T] \mapsto I\!R^m$, satisfying

$$\dot x^\theta(t) = f\big(x^\theta(t)\big), \qquad x^\theta(0) = \theta y_0 + (1-\theta)x_0, \qquad \theta \in [0,1].$$

Defining the tangent vectors

$$\mathbf{v}^\theta(t) \doteq \lim_{\varepsilon\to 0} \frac{x^{\theta+\varepsilon}(t) - x^\theta(t)}{\varepsilon},$$

for all $t \in [0,T]$, from (3.25) it follows

$$\big|y(t) - x(t)\big| \le \int_0^1 \left|\frac{d}{d\theta} x^\theta(t)\right| d\theta = \int_0^1 \big|\mathbf{v}^\theta(t)\big|\, d\theta \le C_R \int_0^1 \big|\mathbf{v}^\theta(0)\big|\, d\theta = C_R |y_0 - x_0|, \tag{3.27}$$

proving (3.2). In this piecewise smooth case, the evolution equation in (1.6) thus generates a uniformly Lipschitz continuous flow. In the following, to denote the unique solution of the Cauchy problem (1.6), we shall use the semigroup notation

$$x(t) = S_t \bar x.$$

We recall that, if $w : [0,T] \mapsto I\!R^m$ is any Lipschitz function, one has the error estimate

$$\big\|w(t) - S_t w(0)\big\| \le L \cdot \int_0^T \left(\liminf_{h\to 0+} \frac{\big|w(t+h) - S_h w(t)\big|}{h} \right) dt, \qquad t \in [0,T], \tag{3.28}$$

where L is the Lipschitz constant of the semigroup. In particular, if w solves the perturbed equation

$$\dot w(t) = f\big(w(t)\big) + e(t), \tag{3.29}$$

and satisfies the bounds $\big|w(t)\big| \le R + t|K|$, from (3.27) and (3.28) we deduce

$$\big|w(t) - S_t w(0)\big| \le C_R \cdot \int_0^t \big|e(s)\big|\, ds \qquad t \in [0,T]. \tag{3.30}$$

STEP 3. The general case will be treated using an approximation procedure. Fix any radius R arbitrarily large, and define

$$\mathcal{U} \doteq \left\{ u : [0,T] \mapsto I\!R^m; \quad \big|u(0)\big| \le R, \quad \frac{u(t) - u(s)}{t-s} \in K \quad \text{for all} \quad t > s \right\}. \tag{3.31}$$

Lemma 1. *Let f, g, τ be as in Theorem 2, and let R and $\varepsilon > 0$ be given. Then there exists a piecewise smooth function $\hat f : I\!R^m \mapsto K$ of the form*

$$\hat f(x) = \hat g_k(x) \qquad \text{if} \quad \tau_k \le \tau(x) < \tau_{k+1}, \tag{3.32}$$

with the following properties. Each $\hat g_k$ is smooth, and its Lipschitz constant satisfies

$$Lip(\hat g_k) \leq \sup_{t\in I\!\!R} Lip\big(g(t,\cdot)\big). \tag{3.33}$$

Moreover, the Picard operators determined by f and $\hat f$ are close, namely

$$\sup_{u\in\mathcal{U}} \int_0^T \Big|\hat f\big(u(t)\big) - f\big(u(t)\big)\Big|\, dt \leq \varepsilon. \tag{3.34}$$

To construct $\hat f$, we first apply the theorem of Scorza-Dragoni [11] to the Carathéodory function g and obtain a closed set J with meas$(I\!\!R\setminus J)\leq\varepsilon$, such that the restriction of g to $J\times I\!\!R^m$ is continuous. The complement of J is an open set, which can be written as a disjoint union of countably many open intervals, say $]a_\nu,\ b_\nu[\,,\ \nu\geq 1$. We then define

$$g^*(t,x) = \begin{cases} g(t,x) & \text{if}\quad t\in J,\\ \theta g(b_\nu,x)+(1-\theta)g(a_\nu,x) & \text{if}\quad t=\theta b_\nu+(1-\theta)a_\nu \quad\text{for some}\quad \nu\geq 1,\quad 0<\theta<1. \end{cases} \tag{3.35}$$

By (3.35), the function g^* is continuous in t and Lipschitz continuous in x. More precisely

$$\sup_{t\in I\!\!R} Lip\big(g^*(t,\cdot)\big) = \sup_{t\in J} Lip\big(g(t,\cdot)\big). \tag{3.36}$$

Moreover, calling $|K| = \max_{z\in K}|z|$, for any $u\in\mathcal{U}$ we have

$$\begin{aligned} \int_0^T \Big|g^*\big(\tau(u(t)),\ u(t)\big) - g\big(\tau(u(t)),\ u(t)\big)\Big|\, dt &\leq 2|K|\cdot meas\big\{t;\quad \tau(u(t))\notin J\big\}\\ &\leq 2|K|\varepsilon/\delta_0, \end{aligned} \tag{3.37}$$

where $\delta_0>0$ is the constant in (3.26). We now choose a small $\delta^*>0$ and, for each $k\in Z\!\!\!Z$, we define $\tau_k \doteq k\delta^*$ and let $\hat g_k$ be a mollification of the function $g^*(k\delta^*,\cdot)$. We then define

$$\hat g(t,x) \doteq \hat g_k(x) \qquad \text{if}\quad \tau_k\leq t<\tau_{k+1},$$

and let $\hat f$ be as in (3.32). From (3.36) it thus follows (3.33). If δ^* is sufficiently small and the mollification kernel is sufficiently close to the identity, this construction yields

$$\int_0^T \Big|g^*\big(\tau(u(t)),\ u(t)\big) - \hat g\big(\tau(u(t)),\ u(t)\big)\Big|\, dt \leq \varepsilon \qquad \text{for all } u\in\mathcal{U}. \tag{3.38}$$

Since $\varepsilon>0$ was arbitrary, (3.37) and (3.38) together yield Lemma 1.

STEP 4. We can now conclude the proof of Theorem 2. Let x,y be any two solutions, as in (3.1). To prove the estimates (3.2), let $\varepsilon>0$ be given, choose $R\doteq\max\big\{|x_0|,\ |y_0|\big\}$ and construct a function $\hat{f}$ according to Lemma 1. According to STEP 1, the semigroup $\widehat{S}$ generated by the evolution equation $\dot{x} = \hat{f}(x)$ is Lipschitz continuous, more precisely

$$\big|\widehat{S}_t x_* - \widehat{S}_t y_*\big| \leq C_R \cdot |x_* - y_*| \qquad \text{whenever } \ |x_*|,\ |y_*| \leq R, \quad t \in [0,T], \tag{3.39}$$

for some constant C_R not depending on ε. Define the quantities e_x and e_y as

$$\begin{aligned} e_x(t) &\doteq f\big(x(t)\big) - \hat{f}\big(x(t)\big), \\ e_y(t) &\doteq f\big(y(t)\big) - \hat{f}\big(y(t)\big), \end{aligned} \qquad t \in [0,T]. \tag{3.40}$$

The functions x and y are thus solutions to

$$\begin{aligned} \dot{x}(t) &= \hat{f}\big(x(t)\big) + e_x(t), \qquad & x(0) = x_0, \\ \dot{y}(t) &= \hat{f}\big(y(t)\big) + e_y(t), \qquad & y(0) = y_0. \end{aligned}$$

By (3.39) we can now use (3.30) and deduce

$$\begin{aligned} \big|y(t) - x(t)\big| &\leq \big|y(t) - \widehat{S}_t y_0\big| + \big|\widehat{S}_t y_0 - \widehat{S}_t x_0\big| + \big|\widehat{S}_t x_0 - x(t)\big| \\ &\leq C_R \cdot \int_0^t |e_y(s)|\, ds + C_R |y_0 - x_0| + C_R \cdot \int_0^t |e_x(s)|\, ds \\ &\leq 2C_R \varepsilon + C_R |y_0 - x_0| \end{aligned} \tag{3.41}$$

for every $t \in [0,T]$. Indeed, by (3.40) and (3.34) it follows

$$\int_0^T \big|e_y(s)\big|\, ds < \varepsilon, \qquad \int_0^T \big|e_x(s)\big|\, ds < \varepsilon.$$

Since ε was arbitrary, from (3.41) we deduce (3.2).

4 - A counterexample.

In Theorem 2, the assumption that $\nabla\tau$ has bounded directional variation is essential for the uniqueness of the solutions, as shown by the following counterexample.

Consider a function $g : I\!R \times I\!R^2 \mapsto I\!R^2$ such that

$$g(t,x) = g(t) \doteq \begin{cases} (1,1), & \text{if } \ t \in \left]\dfrac{2k+1}{2k(k+1)},\ \dfrac{1}{k}\right], \\ (1,-1), & \text{if } \ t \in \left]\dfrac{1}{k+1},\ \dfrac{2k+1}{2k(k+1)}\right], \end{cases} \qquad \text{for any } k \geq 1. \tag{4.1}$$

In the plane with coordinates (x_1, x_2), define the sequences of points P_k, P'_k, Q_k and Q'_k by setting (fig. 2)

$$\begin{aligned} P_k &\doteq \left(\frac{1}{k},\ 0\right), \qquad & Q_k &\doteq \left(\frac{2k+1}{2k(k+1)},\ \frac{-1}{2k(k+1)}\right), \\ P'_k &\doteq \left(\frac{1}{k},\ \frac{1}{2k}\right), \qquad & Q'_k &\doteq \left(\frac{4k+1}{4k(k+1)},\ \frac{2k-1}{4k(k+1)}\right). \end{aligned} \tag{4.2}$$

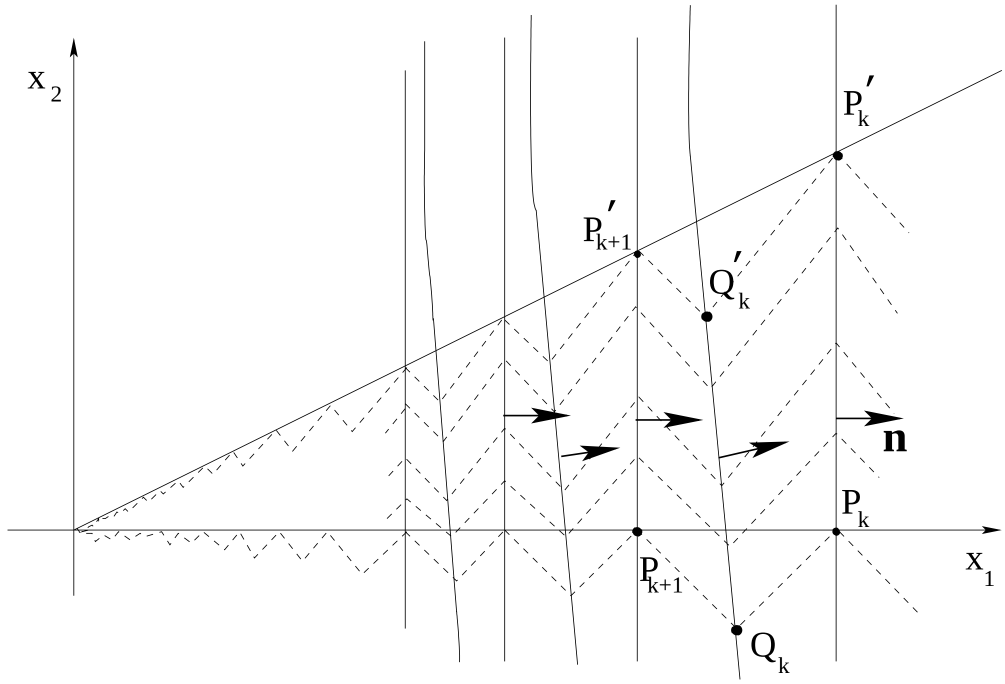

figure 2

We can now construct a $\mathcal{C}^1$ function $\tau : I\!R^2 \mapsto I\!R$ with

$$\nabla\tau(x)\cdot(1,\lambda) > 0 \qquad \nabla\tau(x)\cdot(1,-1) > 0 \qquad \text{for all } \ x \in I\!R^2 \tag{4.3}$$

and such that, for every integer $k \geq 1$,

$$\tau(x) \doteq \begin{cases} \dfrac{1}{k}, & \text{along the segment joining } P_k, P_k', \\ \dfrac{2k+1}{2k(k+1)}, & \text{along the segment joining } Q_k, Q_k'. \end{cases} \tag{4.4}$$

Letting $f(x) \doteq g\big(\tau(x)\big)$, and defining

$$K \doteq \overline{co}\big\{(1,1);(1,-1)\big\} = \big\{(1,\lambda);|\lambda|\leq 1\big\},$$

all of the assumptions in Theorem 2 are satisfied, except the one on the directional variation of $\nabla\tau$. Indeed, at all points P_k the gradient $\nabla\tau$ is parallel to the vector $(1,0)$. On the other hand, at each point Q_k this gradient is parallel to the vector $\big(1,\ 1/(2k+1)\big)$. Since $\nabla\tau$ is continuous and never vanishes, its total variation in the direction of the cone $\Gamma \doteq \big\{(x_1,x_2);\ \ |x_2| \leq x_1\big\}$ cannot be bounded.

From the definitions (4.1)-(4.4) it follows that, for each $k \geq 1$,

$$f(x) \doteq \begin{cases} (1,\ 1) & \text{on the quadrilateral with vertices } P_k, P_k', Q_k', Q_k, \\ (1,-1) & \text{on the quadrilateral with vertices } Q_k, Q_k', P_{k+1}', P_{k+1}. \end{cases} \tag{4.5}$$

One can easily check that the Cauchy problem on $\mathbb{R}^2$

$$\dot{x} = f(x), \qquad\qquad x(0) = (0,0)$$

has two distinct solutions (fig. 2). Namely, one solution passing through all the points Q_k, P_k, $k \geq 1$, and a second solution passing through all the points Q'_k, P'_k.

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