

# Discrete Bidding Strategies for a Random Incoming Order

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## Abstract

This paper is concerned with a model of a one-sided limit order book, viewed as a noncooperative game for  $n$  players. Agents offer various quantities of an asset at different prices, ranging over a finite set  $\Omega_\nu = \{(i/\nu)\bar{P}; i = 1, \dots, \nu\}$ , competing to fulfill an incoming order, whose size  $X$  is not known a priori. Players can have different payoff functions, reflecting different beliefs about the fundamental value of the asset and probability distribution of the random variable  $X$ .

For a wide class of random variables, we prove that the optimal pricing strategies for each seller form a compact and convex set. By a fixed point argument, this yields the existence of a Nash equilibrium for the bidding game.

As  $\nu \rightarrow \infty$ , we show that the discrete Nash equilibria converge to an equilibrium solution for a bidding game where prices range continuously over the whole interval  $[0, \bar{P}]$ .

**Keywords:** measure-valued optimization, optimality conditions, optimal pricing strategy, bidding game, Nash equilibrium, limit order book.

## 1 Introduction

This paper is concerned with a bidding game for  $n$  players. In the basic setting, we assume that an external buyer asks for a random amount of  $X > 0$  of shares of a certain asset. This amount will be bought at the lowest available price, as long as this price does not exceed a given upper bound  $\bar{P}$ . One or more sellers offer various quantities of this asset at different prices, competing to fulfill the incoming order, whose size is not known a priori.

Having observed the prices asked by his competitors, each seller must determine an optimal pricing strategy, maximizing his expected payoff. Clearly, when other sellers are present, asking a higher price for the asset increases the expected profit achieved by the sale, but reduces the probability that the asset will actually be sold. This same model was recently

studied in [6], assuming that the prices asked by the bidders could range over the continuum of real numbers. In such case, under natural hypotheses the authors proved that the non-cooperative game admits a unique Nash equilibrium, where all players (except one, at most) adopt price distributions which are absolutely continuous w.r.t. Lebesgue measure.

In the present paper we analyze the more realistic case where the price distributions take a discrete set of values, ranging within the finite set

$$\Omega_\nu = \left\{ p_k \doteq \frac{k}{\nu} \bar{P}; \quad k \in \{1, \dots, \nu\} \right\},$$

for some integer  $\nu > 1$ . In our model we assume that the  $i$ -th player owns an amount  $\kappa_i > 0$  of asset. He can put all of it on sale at a given price, or offer different portions at different prices. In general, his strategy will thus be described by a vector

$$\mu_i = (\mu_{i1}, \dots, \mu_{i\nu}), \quad \mu_{i\ell} \geq 0, \quad \sum_{\ell=1}^{\nu} \mu_{i\ell} = \kappa_i, \quad (1.1)$$

where  $\mu_{ik}$  denotes the amount offered for sale by the  $i$ -th player at the price  $p_k$ .

The paper has two main goals, namely (i) to study the Nash equilibrium, when prices range in a discrete set of values, and (ii) to show that, as  $\nu \rightarrow \infty$ , these discrete equilibrium solutions converge to the solution of a Nash equilibrium where prices range on a continuum interval of values  $[0, \bar{P}]$ .

A major difference between the discrete and continuum case is how bids are prioritized. When prices range over all continuum values, as proved in [6], in a Nash equilibrium no two players offer a positive amount of asset exactly at the same price. Hence, if a buying order of size  $X$  arrives, the amounts sold by the various players are uniquely determined by the prices at which the sellers post their offers. However, in the discrete case, it may well happen that two or more players offer positive amounts of asset for sale at exactly the same price  $p_k$ . In this case, one needs to specify in which order the bids of the various players will be executed. As modeling assumption, we assume the following.

**(H1)** Players are given a fixed ranking. Hence, if they all put assets on sale at any given price  $p_\ell$ , the buyer will start by taking assets from the first player, then from the second, and so on, until his order is fulfilled.

There are further differences between the models studied in [6] and in the present paper. In [6], all players assigned the same probability distribution to the random variable  $X$ , and had the same payoff function. Here we consider the far more general case of heterogeneous players, holding different beliefs about the fundamental value of the assets put on sale and on the random variable  $X$  describing the size of the incoming order.

The techniques used in the analysis are also different. In [6] the existence of a Nash equilibrium was proved by solving a system of ODEs providing necessary conditions, and checking that the solution actually provided the unique Nash equilibrium to the bidding game. The present paper, on the other hand, is based on the analysis of the best reply map. Under suitable assumptions, we prove that this map is upper semicontinuous with compact convex values, hence by Kakutani's theorem it has a fixed point.

The remainder of the paper is organized as follows. Section 2 collects the main definitions and assumptions of our model. In particular, in (2.7)-(2.8) we introduce three classes of

random variables, which we call of **Type A<sub>+</sub>**, **Type A<sub>0</sub>**, and **Type B**, depending on whether the function  $\psi(s) = Prob.(\{X > s\})$  is log-convex, log-linear, or log-concave. In Sections 3 and 4 we study the optimization problem faced by each single player, determining necessary conditions for optimality and studying the topological properties of the set of solutions. We show that for random variables of Type A<sub>+</sub> the optimal strategy is always unique, while for random variables of Type A<sub>0</sub> there is a compact, convex set of optimal strategies. Finally, in the case of random variables of Type B, the set of optimal strategies may not be convex (in fact, not even connected).

Section 5 contains our main results on the existence or non-existence of a Nash equilibria for discrete bidding strategies. If each random variable  $X_i$  is of type A<sub>+</sub> or A<sub>0</sub>, then we prove that a Nash equilibrium exists. On the other hand, if all random variables  $X_i$  are of type B, then there cannot be any Nash equilibrium.

**Remark 1.** In [6] the existence result was proved by solving a system of ODEs and explicitly constructing the optimal strategies for each player. That approach also yielded the uniqueness of the Nash equilibrium. Here we use a more standard topological technique, applying Kakutani's fixed point theorem to a family of discrete approximation. By its nature, this approach is more general but does not yield information about the uniqueness of the Nash equilibrium.

In Section 6 we consider the limit as  $\nu \rightarrow \infty$ , so that the mesh size approaches zero. By possibly choosing a subsequence, we prove that a sequence of discrete Nash equilibria converges to a continuum Nash equilibrium, where strategies take values within the real line. In particular, this analysis allows us to extend the existence theorem proved in [6] to the case of heterogeneous players, with different payoff functions.

Several recent papers have been devoted to the modeling of the limit order book. In some of these models [3, 7, 13] the prices vary in a discrete set, while in other models [1, 10, 12, 14] the prices are allowed to range in a continuum set. The main goal of these models is to study an optimization problem for an agent who wants to execute market orders or post limit orders, or both (as in [9]), maximizing a prescribed utility function.

The main feature of our model is that we consider the limit order book as the outcome of a noncooperative **game**, where several different players are trying to maximize their (possibly different) utility functions. In this aspect, our model is similar to the model by Roşu [15]. A major difference lies in the fact that in our model the utility functions of the players are determined by the probability distributions for the size of the incoming market order, and not by the expected waiting time of the order execution. Nash equilibria for some models of the limit order book have been recently studied also in [4].

In addition to the classical paper [11], for an introduction to non-cooperative games and Nash equilibria we refer to [5, 8, 16, 17].

## 2 Discrete bidding strategies

Consider a bidding game for  $n$  players, competing to fulfill a incoming order whose size is not a priori known. The optimization problem faced by the  $i$ -th player can be formulated in terms of the following data.

(i) A non-negative random variable  $X_i$ , whose distribution is described by

$$\psi_i(s) = \text{Prob.}\{X_i \geq s\}.$$

This probability distribution reflects the beliefs of the  $i$ -th agent about the size of the incoming order.

(ii) A constant  $\bar{p}_i > 0$  describing the fundamental value of the asset, according to the  $i$ -th player. This means that, for the  $i$ -th player, it is indifferent to keep the asset or to sell it at the unit price  $\bar{p}_i$ .

(iii) The total amount  $\kappa_i > 0$  of assets put on sale the  $i$ -th player.

(iv) A nondecreasing function  $\Phi_i : \Omega_\nu \mapsto \mathbb{R}_+$ . Here  $\Phi_i(p_k)$  describes the total amount of assets put on sale by all the other agents at prices  $\leq p_k$ , having priority over the bid of the  $i$ -th player at price  $p_k$ .

As remarked at (1.1), the  $i$ -th player must optimally choose a vector  $\mu_i = (\mu_{i1}, \dots, \mu_{i\nu})$ , where  $\mu_{ik}$  denotes the amount of asset put on sale at price  $p_k = k\bar{P}/\nu$ . It will be convenient to reformulate this problem, using the variable  $\beta \in [0, \kappa_i]$  to label a particular asset owned by the  $i$ -th agent. By a **pricing strategy** for the  $i$ -th player we mean any function

$$\phi_i : [0, \kappa_i] \mapsto \Omega_\nu = \left\{ \frac{j\bar{P}}{\nu}, \quad j = 1, 2, \dots, \nu \right\}. \quad (2.2)$$

which is (i) nondecreasing, (ii) left continuous, and (iii) right continuous at  $\beta = 0$ . Here  $\phi_i(\beta)$  is the price at which the  $\beta$ -asset is offered for sale.

Given a vector  $(\mu_{i1}, \dots, \mu_{i\nu})$  as in (1.1), since all the assets  $\beta \in [0, \kappa_i]$  are equivalent, any function  $\phi_i : [0, \kappa_i] \mapsto \Omega_\nu$  such that

$$\text{meas}\left(\{\beta \in [0, \kappa_i]; \phi_i(\beta) = p_k\}\right) = \mu_{ik}, \quad k = 1, \dots, \nu, \quad (2.3)$$

would yield the same expected payoff. By requiring that  $\phi$  satisfies the properties (i)–(iii), we single out a unique such function, namely

$$\phi_i(\beta) = p_k \quad \text{if and only if} \quad \sum_{j=1}^{k-1} \mu_{ij} < \beta \leq \sum_{j=1}^k \mu_{ij}. \quad (2.4)$$

The expected payoff for the  $i$ -th player is then computed by

$$J_i(\phi_i, \Phi_i) \doteq \int_0^{\kappa_i} (\phi_i(\beta) - \bar{p}_i) \cdot \psi_i\left(\beta + \Phi_i(\phi_i(\beta))\right) d\beta. \quad (2.5)$$

Notice that  $\phi_i(\beta) - \bar{p}_i$  is the profit achieved by selling the asset  $\beta$  at price  $\phi_i(\beta)$ , while  $\psi_i(\beta + \Phi_i(\phi_i(\beta)))$  is the probability that this particular asset will be actually sold.

We say that  $\phi_i^*$  is an **optimal pricing strategy** if

$$J_i(\phi_i, \Phi_i) \leq J_i(\phi_i^*, \Phi_i)$$

for every other admissible strategy  $\phi : [0, \kappa_i] \mapsto \Omega_\nu$ .

Next, consider  $n$  agents offering for sale quantities  $\kappa_1, \dots, \kappa_n$  of the same asset. Let  $\mu_{ik}$  be the amount put on sale at price  $p_k$  by the  $i$ -th agent. Recalling the prioritizing rule (H1), we define the functions

$$\Phi_i(p_\ell) \doteq \sum_{k < \ell, j \neq i} \mu_{jk} + \sum_{j < i} \mu_{j\ell}. \quad (2.6)$$

We say that an  $n$ -tuple of strategies  $(\phi_1^*, \dots, \phi_n^*)$  is a **Nash equilibrium** if, defining the corresponding functions  $\Phi_i^*$  as in (2.6), one has

$$J_i(\phi_i, \Phi_i^*) \leq J_i(\phi_i^*, \Phi_i^*)$$

for every  $i = 1, \dots, n$  and any other pricing strategy  $\phi_i : [0, \kappa_i] \mapsto \Omega_\nu$ .

As indicated by the analysis in [6], the existence of a Nash equilibrium strongly depends on the properties of the random variables  $X_i$ . Throughout the following, we assume

**(H2)** For  $i = 1, \dots, n$ , the maps  $s \mapsto \psi_i(s)$  are twice continuously differentiable and satisfy

$$\psi_i(0) = 1, \quad \psi_i(+\infty) = 0, \quad \psi_i'(s) < 0 \quad \text{for all } s > 0.$$

Motivated by [6], we shall consider three main classes of random variables, depending on whether the function

$$\psi(s) = \text{Prob.}\{X \geq s\}. \quad (2.7)$$

is log-convex, log-linear, or log-concave.

**Definition.** We say that the random variable  $X$  in (2.7) is

$$\begin{aligned} \text{of type } \mathbf{A}_+ & \quad \text{if } (\ln \psi(s))'' > 0 \quad \text{for all } s > 0, \\ \text{of type } \mathbf{A}_0 & \quad \text{if } (\ln \psi(s))'' = 0 \quad \text{for all } s > 0, \\ \text{of type } \mathbf{B} & \quad \text{if } (\ln \psi(s))'' < 0 \quad \text{for all } s > 0. \end{aligned} \quad (2.8)$$

We observe that  $\psi$  is of type  $A_0$  if and only if

$$\psi(s) = e^{-\lambda s} \quad \text{for some } \lambda > 0.$$

On the other hand, when  $\alpha > 0$  the probability distribution

$$\psi(s) = \frac{1}{(1+s)^\alpha}$$

is of type  $A_+$ , while

$$\psi(s) = e^{-s^2}$$

yields a probability distribution of type B. Roughly speaking, a probability distribution is of type  $A_+$  if its density decays slower than a negative exponential, and of type B if its density decays faster than a negative exponential.

### 3 Necessary conditions for an optimal bidding strategy

Let a positive, nondecreasing function  $\Phi_i : [0, \bar{P}] \mapsto \mathbb{R}_+$  be given, and consider the optimization problem for the  $i$ -th player, who wishes to maximize the expected payoff  $J_i(\phi, \Phi_i)$  in (2.5) over all admissible strategies  $\phi : [0, \kappa_i] \mapsto \Omega_\nu$ . As remarked at (2.4)-(2.3), the set of admissible strategies can be identified with the set  $\mathcal{S}_i$  of  $\nu$ -tuples  $\mu_i = (\mu_{i1}, \dots, \mu_{i\nu})$ , where  $\mu_{ik}$  is the amount of asset that the  $i$ -th player puts on sale at price  $p_k$ .

In this section we derive necessary conditions for optimality. In the case of two equal players, these will provide an explicit formula describing the discrete Nash equilibrium. We recall that the functions  $\phi_i$  are assumed to be nondecreasing and left continuous, therefore at each point  $\beta$  the left and right limits satisfy  $\phi(\beta-) = \phi(\beta) \leq \phi(\beta+)$ .

**Proposition 1 (optimality conditions).** *Given a positive, nondecreasing function  $\Phi_i$  let  $\phi_i : [0, \kappa] \rightarrow \Omega_\nu$  be a best reply for the  $i$ -th player. If  $\bar{\beta} \in ]0, \kappa_i[$  is a point of jump of  $\phi$ , so that  $\phi_i(\bar{\beta}+) > \phi_i(\bar{\beta})$ , then*

$$(\phi_i(\bar{\beta}+) - \bar{p}_i) \cdot \psi_i(\bar{\beta} + \Phi_i(\phi_i(\bar{\beta}+))) = (\phi_i(\bar{\beta}) - \bar{p}_i) \cdot \psi_i(\bar{\beta} + \Phi_i(\phi_i(\bar{\beta}))). \quad (3.9)$$

**Proof.** Consider the following perturbation:

$$\phi^{\varepsilon-} \doteq \begin{cases} \phi_i(\bar{\beta}+) & \text{if } \beta \in [\bar{\beta} - \varepsilon, \bar{\beta}], \\ \phi_i(\beta) & \text{otherwise.} \end{cases}$$

A direct computation yields

$$\begin{aligned} J_i(\phi^{\varepsilon-}) &= \left( \int_0^{\bar{\beta}-\varepsilon} + \int_{\bar{\beta}}^{\kappa} \right) (\phi_i(\beta) - \bar{p}_i) \cdot \psi_i(\beta + \Phi_i(\phi_i(\beta))) d\beta \\ &\quad + \int_{\bar{\beta}-\varepsilon}^{\bar{\beta}} (\phi_i(\bar{\beta}+) - \bar{p}_i) \cdot \psi_i(\beta + \Phi_i(\phi_i(\bar{\beta}+))) d\beta. \end{aligned}$$

Since  $\phi_i$  is optimal, we conclude

$$0 \geq \frac{d}{d\varepsilon} J_i(\phi^{\varepsilon-}) \Big|_{\varepsilon=0} = -(\phi_i(\bar{\beta}) - \bar{p}_i) \cdot \psi_i(\bar{\beta} + \Phi_i(\phi_i(\bar{\beta}))) + (\phi_i(\bar{\beta}+) - \bar{p}_i) \cdot \psi_i(\bar{\beta} + \Phi_i(\phi_i(\bar{\beta}+))).$$

Similarly, by considering the perturbation:

$$\phi^{\varepsilon+} \doteq \begin{cases} \phi_i(\bar{\beta}) & \text{if } \beta \in [\bar{\beta}, \bar{\beta} + \varepsilon], \\ \phi_i(\beta) & \text{otherwise,} \end{cases}$$

we obtain the converse inequality. Hence (3.9) holds.  $\square$

By a similar perturbation argument, one obtains:

**Proposition 2.** *Given a positive, nondecreasing function  $\Phi_i$ , let  $\mu_i = (\mu_{i1}, \dots, \mu_{i\nu})$  be a best reply for the  $i$ -th player.*

If  $\mu_{i\ell} > 0$  for some  $\ell > 1$ , then

$$(p_\ell - \bar{p}_i) \cdot \psi_i \left( \sum_{k=1}^{\ell-1} \mu_{ik} + \Phi_i(p_\ell) \right) \geq (p_{\ell-1} - \bar{p}_i) \cdot \psi_i \left( \sum_{k=1}^{\ell-1} \mu_{ik} + \Phi_i(p_{\ell-1}) \right). \quad (3.10)$$

Moreover, if  $\mu_{i\ell} > 0$  for some  $\ell < \nu$ , then

$$(p_\ell - \bar{p}_i) \cdot \psi_i \left( \sum_{k=1}^{\ell} \mu_{ik} + \Phi_i(p_\ell) \right) \geq (p_{\ell+1} - \bar{p}_i) \cdot \psi_i \left( \sum_{k=1}^{\ell} \mu_{ik} + \Phi_i(p_{\ell+1}) \right). \quad (3.11)$$

**Proof.** Setting  $\mu_{i0} = 0$ , we can rewrite the optimization problem as follows:

$$\begin{aligned} \text{maximize : } J(\mu_i) &\doteq \sum_{k=1}^{\nu} \int_{\mu_{i1} + \dots + \mu_{i,k-1}}^{\mu_{i1} + \dots + \mu_{ik}} (p_k - \bar{p}) \cdot \psi(\beta + \Phi_i(p_k)) d\beta, \\ \text{subject to : } \sum_{k=1}^{\nu} \mu_{ik} &= \kappa_i, \quad \mu_{ik} \geq 0, \quad \text{for all } k = 1, \dots, \nu. \end{aligned} \quad (3.12)$$

If  $\mu_{i\ell} > 0$  for some  $\ell > 1$ , consider the perturbed strategy  $\mu_i^\varepsilon = (\mu_{i1}^\varepsilon, \mu_{i2}^\varepsilon, \dots, \mu_{i\nu}^\varepsilon)$  defined by

$$\begin{cases} \mu_{i\ell-1}^\varepsilon = \mu_{i\ell-1} + \varepsilon, \\ \mu_{i\ell}^\varepsilon = \mu_{i\ell} - \varepsilon, \\ \mu_{ik}^\varepsilon = \mu_{ik} \end{cases} \quad \text{if } k \notin \{\ell-1, \ell\}.$$

This is clearly admissible, as long as  $0 \leq \varepsilon \leq \mu_{i\ell}$ . Since  $\mu_i$  is a best reply, we must have  $J(\mu_i) \geq J(\mu_i^\varepsilon)$  for all  $\varepsilon > 0$ , hence

$$\left. \frac{d}{d\varepsilon} J(\mu_i^\varepsilon) \right|_{\varepsilon=0} \leq 0. \quad (3.13)$$

Replacing  $\mu_{ik}$  with  $\mu_{ik}^\varepsilon$  in (3.12) and differentiating w.r.t.  $\varepsilon$ , from (3.13) we obtain (3.10). The inequality (3.11) is proved in an entirely similar way.  $\square$

In the particular case where the random incoming order is exponentially distributed, we obtain:

**Corollary 1.** Assume that  $\psi_i(s) = e^{-\lambda_i s}$ , and let  $(\mu_{i1}, \dots, \mu_{i\nu})$  be an optimal pricing strategy for the  $i$ -th player. Then

$$\begin{aligned} \mu_{i\ell} > 0 &\implies \Phi_i(p_\ell) - \Phi_i(p_{\ell-1}) \leq \frac{1}{\lambda_i} \ln \left( \frac{p_\ell - \bar{p}_i}{p_{\ell-1} - \bar{p}_i} \right), \\ \mu_{i,\ell-1} > 0 &\implies \Phi_i(p_\ell) - \Phi_i(p_{\ell-1}) \geq \frac{1}{\lambda_i} \ln \left( \frac{p_\ell - \bar{p}_i}{p_{\ell-1} - \bar{p}_i} \right). \end{aligned}$$

## 4 Properties of the set of best replies

In this section, given a positive, nondecreasing function  $\Phi_i : \Omega_\nu \mapsto \mathbb{R}$ , we analyze the set of best replies for the  $i$ -th player. These are left-continuous, nondecreasing functions  $\phi_i : [0, \kappa_i] \mapsto \Omega_\nu = \{p_1, p_2, \dots, p_\nu = \bar{P}\}$  as in (2.2), which maximize the expected payoff (2.5).

To simplify our notation, given  $0 < \bar{p} < \bar{P}$ ,  $\kappa > 0$ , and a positive nondecreasing function  $\Phi : \Omega_\nu \mapsto \mathbb{R}_+$ , we consider the optimization problem

$$\text{maximize: } J(\phi, \Phi) \doteq \int_0^\kappa (\phi(\beta) - \bar{p}) \psi(\beta + \Phi(\phi(\beta))) d\beta. \quad (4.14)$$

The maximum is sought among all nondecreasing functions

$$\phi : [0, \kappa] \mapsto \Omega_\nu.$$

As in (2.4), setting

$$\mu_\ell \doteq \text{meas}(\{\beta; \phi(\beta) = p_\ell\}),$$

the set of admissible strategies can be identified with the set of vectors

$$\mathcal{S} \doteq \left\{ \mu = (\mu_1, \dots, \mu_\nu); \quad \mu_\ell \geq 0, \quad \sum_{\ell=1}^\nu \mu_\ell = \kappa \right\},$$

Clearly  $\mathcal{S}$  is a compact, convex subset of  $\mathbb{R}^\nu$ . Setting  $\mu_0 = 0$ , a direct computation yields

$$J(\phi, \Phi) = J(\mu, \Phi) \doteq \sum_{k=1}^\nu \int_{\mu_1 + \dots + \mu_{k-1}}^{\mu_1 + \dots + \mu_k} (p_k - \bar{p}) \cdot \psi(\beta + \Phi(p_k)) d\beta. \quad (4.15)$$

From (4.15) it is clear that the map  $\mu \mapsto J(\mu, \Phi)$  is continuous. Hence it attains a maximum on the compact set  $\mathcal{S}$ . More precisely, the set  $\mathcal{S}_{max}$  of vectors  $(\mu_1, \dots, \mu_\nu)$  where the maximum is attained is a nonempty, compact subset of  $\mathcal{S}$ . Aim of this section is to study the geometry of this set  $\mathcal{S}_{max}$ .

**Lemma 1.** *If the random variable  $X$  is of type  $A_+$ , then the optimization problem (4.14) admits a unique optimal solution.*

**Proof. 1.** Let  $\phi_1, \phi_2 : [0, \kappa] \mapsto \{p_1, \dots, p_\nu\}$  be two optimal strategies. If  $\phi_1 \neq \phi_2$ , then (by possibly permuting the indices 1,2) we can find a maximal open interval  $]a, b[ \subseteq [0, \kappa]$  such that

$$\phi_1(\beta) < \phi_2(\beta) \quad \text{for all } a < \beta < b. \quad (4.16)$$

We claim that, for  $\theta = a$  and for  $\theta = b$ , the above implies

$$\int_0^\theta (\phi_1(\beta) - \bar{p}) \psi(\beta + \Phi(\phi_1(\beta))) d\beta = \int_0^\theta (\phi_2(\beta) - \bar{p}) \psi(\beta + \Phi(\phi_2(\beta))) d\beta. \quad (4.17)$$

Indeed, assume that on the contrary

$$\int_0^a (\phi_1(\beta) - \bar{p}) \psi(\beta + \Phi(\phi_1(\beta))) d\beta > \int_0^a (\phi_2(\beta) - \bar{p}) \psi(\beta + \Phi(\phi_2(\beta))) d\beta. \quad (4.18)$$



Observe that the function

$$\tilde{\phi}(\beta) = \begin{cases} \phi_1(\beta) & \text{if } \beta < a, \\ \phi_2(\beta) & \text{if } \beta > a, \end{cases}$$

is nondecreasing, because we are assuming that  $]a, b[$  is maximal among all open intervals satisfying (4.16). If (4.18) holds, then  $\tilde{\phi}$  yields an expected payoff strictly greater than both  $\phi_1$  and  $\phi_2$ , against the assumption. The other cases are ruled out by a similar argument.

**2.** If (4.16) holds, then for every  $\theta \in [a, b]$  the interpolated strategy

$$\phi^\theta(\beta) = \begin{cases} \phi_1(\beta) & \text{if } \beta < \theta, \\ \phi_2(\beta) & \text{if } \beta > \theta, \end{cases}$$

is also admissible. We now define  $J(\theta) \doteq J(\phi^\theta, \Phi)$ . We claim that

$$J \in \mathcal{C}([a, b]) \cap \mathcal{C}^1(]a, b[). \quad (4.19)$$

Indeed,

$$J(\theta) = \int_0^\theta (\phi_1(\beta) - \bar{p}) \cdot \psi(\beta + \Phi(\phi_1(\beta))) d\beta + \int_\theta^\kappa (\phi_2(\beta) - \bar{p}) \cdot \psi(\beta + \Phi(\phi_2(\beta))) d\beta,$$

and the continuity of the map  $\theta \mapsto J(\theta)$  is clear. To prove continuous differentiability, we compute

$$\frac{d}{d\theta} J(\theta) = (\phi_1(\theta) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_1(\theta))) - (\phi_2(\theta) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_2(\theta))). \quad (4.20)$$

At points where both  $\phi_1$  and  $\phi_2$  are constant, the continuity of the right hand side of (4.20) is trivial. Next, let  $\theta$  be one of the finitely many points where  $\phi_1$  has an upward jump. Since  $\phi_1$  is optimal, the necessary conditions (3.9) yield

$$(\phi_1(\theta+) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_1(\theta+))) = (\phi_1(\theta-) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_1(\theta-))).$$

The same equality holds at points where  $\phi_2$  jumps. Hence the map  $\theta \mapsto J(\theta)$  is continuously differentiable.

**3.** Next, consider an interior point  $\theta^* \in ]a, b[$  where  $\frac{d}{d\theta} J(\theta) = 0$ . We claim that  $J$  attains a strict local maximum at  $\theta^*$ .

Indeed, choose  $\delta > 0$  such that  $\phi_1, \phi_2$  are both constant on the interval  $]\theta^*, \theta^* + \delta[$ . For  $\theta \in ]\theta^*, \theta^* + \delta[$  we then have

$$\begin{aligned} \frac{d^2}{d\theta^2} J(\theta) &= \frac{d}{d\theta} \left[ (\phi_1(\theta) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_1(\theta))) - (\phi_2(\theta) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_2(\theta))) \right] \\ &= (\phi_1(\theta) - \bar{p}) \cdot \psi'(\theta + \Phi(\phi_1(\theta))) - (\phi_2(\theta) - \bar{p}) \cdot \psi'(\theta + \Phi(\phi_2(\theta))) \\ &= (\phi_1(\theta) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_1(\theta))) \left[ \frac{\psi'(\theta + \Phi(\phi_1(\theta)))}{\psi(\theta + \Phi(\phi_1(\theta)))} - \frac{\psi'(\theta + \Phi(\phi_2(\theta)))}{\psi(\theta + \Phi(\phi_2(\theta)))} \right] \\ &\quad + \left[ (\phi_1(\theta) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_1(\theta))) - (\phi_2(\theta) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_2(\theta))) \right] \frac{\psi'(\theta + \Phi(\phi_2(\theta)))}{\psi(\theta + \Phi(\phi_2(\theta)))} \\ &\doteq A(\theta) + B(\theta). \end{aligned} \quad (4.21)$$

We now observe that

$$\lim_{\theta \rightarrow \theta^*+} A(\theta) < 0 \quad (4.22)$$

because  $X$  is of type  $A_+$  and hence  $(\ln \psi)'' > 0$ . On the other hand, by (4.20)

$$\begin{aligned} \lim_{\theta \rightarrow \theta^*+} B(\theta) &= \lim_{\theta \rightarrow \theta^*+} \left[ (\phi_1(\theta) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_1(\theta))) - (\phi_2(\theta) - \bar{p}) \cdot \psi(\theta + \Phi(\phi_2(\theta))) \right] \\ &= \lim_{\theta \rightarrow \theta^*+} \frac{d}{d\theta} J(\theta) = 0. \end{aligned} \quad (4.23)$$

From (4.22)-(4.23) it follows that  $\frac{d^2}{d\theta^2} J(\theta) < 0$  for all  $\theta \in ]\theta^*, \theta^* + \varepsilon]$ , with  $\varepsilon > 0$  small enough. An entirely similar argument shows that  $\frac{d^2}{d\theta^2} J(\theta) < 0$  also for  $\theta \in [\theta^* - \varepsilon, \theta^*[$ . Hence  $J$  attains a strict local maximum at  $\theta^*$ .

4. As a consequence of (4.17), when  $\theta = a$  and  $\theta = b$  the maximum expected payoff is achieved:

$$J(\phi^a, \Phi) = J(\phi^b, \Phi) = J(\phi_1, \Phi) = J(\phi_2, \Phi).$$

We conclude that the function  $J$  in (4.19) must achieve its global minimum on  $[a, b]$  at some interior point  $\theta^* \in ]a, b[$ . This implies  $\frac{dJ}{d\theta}(\theta^*) = 0$ . and hence, as proved in the previous step,  $J$  must attains a strict local maximum at  $\theta^*$ . We thus reach a contradiction, proving the result.  $\square$

**Lemma 2.** *Assume that the random variable  $X$  is of type  $A_0$ , with  $\psi(s) = e^{-\lambda s}$ . Then there exists a subset of prices  $\Omega_{opt} \subseteq \{p_1, \dots, p_\nu\} = \Omega_\nu$  with the following property. A pricing strategy  $\phi : [0, \kappa] \mapsto \Omega_\nu$  is optimal if and only if  $\phi(\beta) \in \Omega_{opt}$  for every  $\beta$ .*

**Proof. 1.** Let  $\phi : [0, \kappa] \mapsto \Omega_\nu$  be an optimal strategy. We claim that there exists a constant  $C$  such that

$$(p_k - \bar{p})e^{-\lambda\Phi(p_k)} = C \quad (4.24)$$

for all  $k \in \{1, \dots, \nu\}$  such that

$$\mu_k \doteq \text{meas}\left(\{\beta; \phi(\beta) = p_k\}\right) > 0. \quad (4.25)$$

Indeed, if there is only one price  $p_k$  such that  $\mu_k > 0$ , then the conclusion is trivial. Next, assume that  $\phi$  takes values in the subset  $\Omega \doteq \{p_{j_1}, p_{j_2}, \dots, p_{j_m}\} \subseteq \Omega_\nu$ , for some indices  $j_1 < j_2 < \dots < j_m$ . For  $1 < \ell \leq m$  and  $|\varepsilon|$  small enough, consider the perturbed strategy  $\phi^\varepsilon$  where the amount  $\mu_k^\varepsilon$  offered at price  $p_k$  is

$$\mu_k^\varepsilon = \begin{cases} \mu_k & \text{if } k \neq j_{\ell-1}, \quad k \neq j_\ell, \\ \mu_k - \varepsilon & \text{if } k = j_{\ell-1}, \\ \mu_k + \varepsilon & \text{if } k = j_\ell. \end{cases}$$

By optimality we must have

$$0 = \frac{d}{d\varepsilon} J(\phi^\varepsilon, \Phi) \Big|_{\varepsilon=0} = \lambda(p_{j_\ell} - \bar{p})e^{-\lambda_i\Phi(p_{j_\ell})} - \lambda(p_{j_{\ell-1}} - \bar{p})e^{-\lambda\Phi(p_{j_{\ell-1}})}.$$

By induction on  $\ell = 2, 3, \dots, m$ , we conclude that all quantities in (4.24), for  $\mu_k > 0$ , coincide.

2. If  $\tilde{\phi}$  is any other pricing strategy taking values in the same set  $\{p_{j_1}, p_{j_2}, \dots, p_{j_m}\}$ , then  $\tilde{\phi}$  is optimal as well. Indeed, thanks to (4.24) we obtain

$$\begin{aligned} J(\tilde{\phi}, \Phi) &= \sum_{k=1}^{\nu} \int_{\{\beta; \tilde{\phi}(\beta)=p_k\}} (p_k - \bar{p}) e^{-\lambda(\beta + \Phi(p_k))} d\beta \\ &= C \int_0^{\kappa} e^{-\lambda\beta} d\beta = \frac{C}{\lambda} (1 - e^{-\lambda\kappa}) = J(\phi, \Phi). \end{aligned}$$

3. Let now  $\phi, \phi' : [0, \kappa] \mapsto \Omega_{\nu}$  be any two optimal pricing strategies, whose values range over the subsets

$$\Omega \doteq \{p_{j_1}, p_{j_2}, \dots, p_{j_m}\}, \quad \Omega' \doteq \{p_{j'_1}, p_{j'_2}, \dots, p_{j'_{m'}}\},$$

respectively. Since  $\phi$  is optimal, by (4.24) there exists a constant  $C$  such that

$$(p_{j(\ell)} - \bar{p}) e^{-\lambda\Phi(p_{j(\ell)})} = C \quad \ell = 1, \dots, m.$$

Similarly, since  $\phi'$  is optimal, there exists a constant  $C'$  such that

$$(p_{j'_\ell} - \bar{p}) e^{-\lambda\Phi(p_{j'_\ell})} = C' \quad \ell = 1, \dots, m'.$$

Being both optimal,  $\phi$  and  $\phi'$  achieve the same payoff. Hence  $C = C'$  and any strategy supported on  $\Omega \cup \Omega'$  is also optimal.

We can now define the set  $\Omega_{opt}$  as the set of all prices that lie in the range of some optimal pricing strategy. By the previous analysis, any pricing strategy  $\phi$  taking values inside  $\Omega_{opt}$  is optimal.  $\square$

**Lemma 3.** *Let the random variable  $X$  be of type  $B$ , and let  $\phi : [0, \kappa] \mapsto \Omega_{\nu}$  be an optimal strategy. Then  $\phi$  is constant.*

**Proof.** By contradiction, assume that  $\phi$  has a jump at  $\beta^* \in ]0, \kappa[$ . Then the necessary conditions (3.9) hold. For  $|\varepsilon|$  sufficiently small, define

$$\begin{aligned} \phi^\varepsilon(\beta) &= \begin{cases} \phi(\beta^* +) & \text{if } \beta \in [\beta^* + \varepsilon, \beta^*] \\ \phi(\beta) & \text{otherwise} \end{cases} & \text{for } \varepsilon < 0, \\ \phi^\varepsilon(\beta) &= \begin{cases} \phi(\beta^* -) & \text{if } \beta \in [\beta^*, \beta^* + \varepsilon] \\ \phi(\beta) & \text{otherwise} \end{cases} & \text{for } \varepsilon > 0. \end{aligned}$$

By optimality we have

$$\left. \frac{d}{d\varepsilon} J(\phi^\varepsilon, \Phi) \right|_{\varepsilon=0} = 0. \quad (4.26)$$

For any  $\varepsilon \in ]0, \varepsilon_0[$  sufficiently small, a similar computation as in (4.21) now yields

$$\begin{aligned}
& \frac{d^2}{d\varepsilon^2} J(\phi^\varepsilon, \Phi) \\
&= \frac{d}{d\varepsilon} \left[ (\phi(\beta^*) - \bar{p}) \cdot \psi(\beta^* + \varepsilon + \Phi(\phi(\beta^*))) - (\phi(\beta^*+) - \bar{p}) \cdot \psi(\beta^* + \varepsilon + \Phi(\phi(\beta^*+))) \right] \\
&= (\phi(\beta^*) - \bar{p}) \cdot \psi'(\beta^* + \varepsilon + \Phi(\phi(\beta^*))) - (\phi(\beta^*+) - \bar{p}) \cdot \psi'(\beta^* + \varepsilon + \Phi(\phi(\beta^*+))) \\
&= (\phi(\beta^*) - \bar{p}) \cdot \psi(\beta^* + \varepsilon + \Phi(\phi(\beta^*))) \\
&\quad \cdot \left[ \frac{\psi'(\beta^* + \varepsilon + \Phi(\phi(\beta^*)))}{\psi(\beta^* + \varepsilon + \Phi(\phi(\beta^*)))} - \frac{\psi'(\beta^* + \varepsilon + \Phi(\phi(\beta^*+)))}{\psi(\beta^* + \varepsilon + \Phi(\phi(\beta^*+)))} \right] \\
&\quad + \left[ (\phi(\beta^*) - \bar{p}) \cdot \psi(\beta^* + \varepsilon + \Phi(\phi(\beta^*))) - (\phi(\beta^*+) - \bar{p}) \cdot \psi(\beta^* + \varepsilon + \Phi(\phi(\beta^*+))) \right] \\
&\quad \cdot \frac{\psi'(\beta^* + \varepsilon + \Phi(\phi(\beta^*+)))}{\psi(\beta^* + \varepsilon + \Phi(\phi(\beta^*+)))} \\
&\doteq A(\varepsilon) + B(\varepsilon).
\end{aligned}$$

A similar argument as in (4.22)-(4.23), but using the fact that  $X$  is of type B and hence  $(\ln \psi(s))'' < 0$  for all  $s > 0$ , we now obtain

$$\lim_{\varepsilon \rightarrow 0^+} A(\varepsilon) > 0, \quad \lim_{\varepsilon \rightarrow 0^+} B(\varepsilon) = 0.$$

Therefore, for all  $\varepsilon \in ]0, \varepsilon_0]$  sufficiently small,

$$\frac{d^2}{d\varepsilon^2} J(\phi^\varepsilon, \Phi) > 0.$$

Together with (4.26), this proves that  $\phi$  cannot be an optimal strategy.  $\square$

Assuming that the variable  $X$  is of type B, the following example shows that, for any  $\kappa > 0$ , one can construct a piecewise constant  $\Phi$  with exactly one jump such that the optimization problem (4.14) has exactly two solutions. In particular, the solution set is not convex.

**Example 1.** Assume  $(\ln \psi(s))'' < 0$  for all  $s > 0$ , and let

$$\Phi(p) = \begin{cases} 0 & \text{if } p < p_k, \\ \alpha & \text{if } p \geq p_k. \end{cases} \quad (4.27)$$

Since by the previous Lemma any optimal pricing strategy must be constant, the only two optimal candidates are  $\phi(\beta) \equiv p_{k-1}$ , or  $\phi(\beta) \equiv \bar{P}$ . The corresponding payoffs are

$$J(p_{k-1}) = (p_{k-1} - \bar{p}) \int_0^\kappa \psi(\beta) d\beta, \quad J(\bar{P}) = (\bar{P} - \bar{p}) \int_\alpha^{\alpha+\kappa} \psi(\beta) d\beta.$$

Consider the function

$$f(\alpha) \doteq \int_{\alpha}^{\alpha+\kappa} \psi(\beta) d\beta.$$

It is easy to see that there exists a solution  $\alpha^*$  to the equation  $f(\alpha) = \frac{p_{k-1} - \bar{p}}{\bar{P} - \bar{p}} f(0)$ . This follows from the fact that  $f$  is continuously differentiable,  $f' < 0$ , and  $\lim_{\alpha \rightarrow \infty} f(\alpha) = 0$ . Hence, by choosing  $\alpha = \alpha^*$  in (4.27), we see that  $\phi(\beta) \equiv p_{k-1}$  and  $\phi(\beta) \equiv \bar{P}$  are both optimal, and this shows that the set of best replies is not connected, thus not convex.

## 5 Existence of a discrete Nash equilibrium

In this section we prove the existence of a Nash equilibrium, when each probability distribution  $\psi_i$  is either of type  $A_0$  or of type  $A_+$ .

**Theorem 1.** *Consider a discrete pricing game for  $n$  players, with strategies given by (2.2) and payoffs as in (2.5), (2.6). Assume that the selling priorities are determined by (H1) and that each probability distribution  $\psi_i$  is either of type  $A_0$  or  $A_+$ . Then the game admits a Nash equilibrium.*

**Proof. 1.** The set of admissible strategies for the  $i$ -th player can be identified with the compact convex set

$$\mathcal{S}_i \doteq \left\{ \mu_i = (\mu_{i1}, \dots, \mu_{i\nu}); \mu_{ij} \geq 0, \sum_{j=1}^{\nu} \mu_{ij} = \kappa_i \right\}.$$

On the cartesian product  $\mathcal{S} \doteq \mathcal{S}_1 \times \dots \times \mathcal{S}_n$  consider the multifunction

$$R : (\mu_1, \mu_2, \dots, \mu_n) \mapsto R_1 \times R_2 \times \dots \times R_n, \quad (5.28)$$

where  $R_i \subseteq \mathcal{S}_i$  is the set of all best replies for player  $i$  to the strategies  $(\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \mu_n)$  adopted by the other  $n - 1$  players.

**2.** By a standard argument we now show that the multifunction  $R$  in (5.28) has closed graph. The map

$$(\mu_1, \dots, \mu_n) \mapsto J_i(\mu_1, \dots, \mu_n)$$

defined as in (2.5)-(2.6), describing the payoff to the  $i$ -th player, is a continuous function on  $\mathcal{S}_1 \times \dots \times \mathcal{S}_n$ . Since each set  $\mathcal{S}_i$  is compact, the map

$$J_i^{max}(\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n) \doteq \max_{\eta_i \in \mathcal{S}_i} J_i(\mu_1, \dots, \mu_{i-1}, \eta_i, \mu_{i+1}, \dots, \mu_n)$$

is also a continuous function of its  $n - 1$  arguments. Therefore, the set

$$Graph(R) = \left\{ (\mu_1, \dots, \mu_n, \eta_1, \dots, \eta_n) ; \right.$$

$$\left. J_i(\mu_1, \dots, \mu_{i-1}, \eta_i, \mu_{i+1}, \dots, \mu_n) = J_i^{max}(\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n) \text{ for all } i = 1, \dots, n \right\}$$

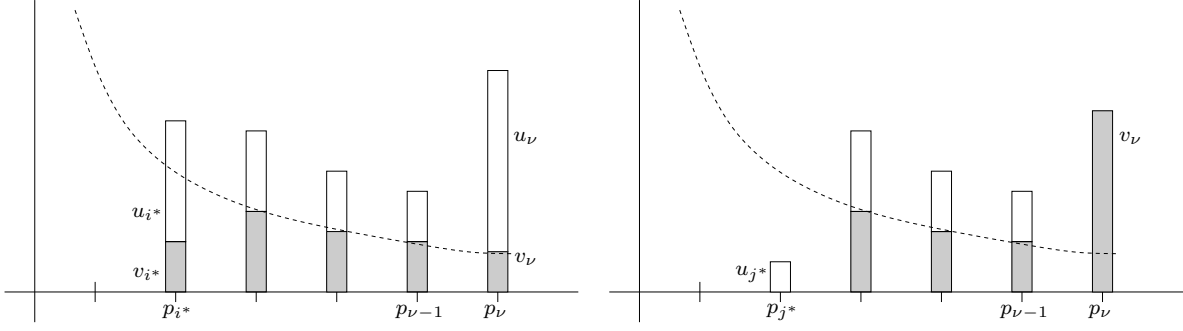


Figure 1: Two examples of Nash equilibria for two players with the same payoff function, selling different amounts of asset. Here  $\bar{p}_1 = \bar{p}_2 = 0$ ,  $\psi_1(s) = \psi_2(s) = e^{-s}$ .

is closed.

**3.** Having a closed graph, the multifunction  $R$  is upper semicontinuous [2]. By Lemma 2 in the previous section, if  $\psi_i$  is of type  $A_0$  then each set  $R_i$  of best replies has the form

$$R_i \doteq \left\{ \mu_i = (\mu_{i1}, \dots, \mu_{i\nu}); \mu_{ij} \geq 0, \sum_{j=1}^{\nu} \mu_{ij} = \kappa_i, \mu_{ij} = 0 \text{ if } p_j \notin \Omega_{opt} \right\}.$$

for some subset  $\Omega_{opt} \subseteq \{p_1, \dots, p_\nu\}$ . In particular, each set  $R_i$  is compact and convex.

On the other hand, if  $\psi_i$  is of type  $A_+$ , then by Lemma 1 each  $R_i$  reduces to a single point.

Applying Kakutani's fixed point theorem [2, 5], we obtain an  $n$ -tuple of admissible strategies  $(\mu_1^*, \dots, \mu_n^*) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_n$  such that  $\mu_i^* \in R_i(\mu_1^*, \dots, \mu_{i-1}^*, \mu_{i+1}^*, \dots, \mu_n^*)$  for every  $i = 1, \dots, n$ . This provides a Nash equilibrium to the discrete pricing game.  $\square$

**Example 2 (discrete Nash equilibrium for two players).** Consider a bidding game for two players with the same payoff functional. More precisely, assume that in (2.5) one has  $\bar{p}_1 = \bar{p}_2 = \bar{p}$ ,  $\psi_1(s) = \psi_2(s) = e^{-s}$ . Assume that Player 1 has selling priority, in case both players ask the same price. For notational convenience, we write  $v_\ell \doteq \mu_{1\ell}$ ,  $u_\ell \doteq \mu_{2\ell}$ . Define the quantities

$$\bar{v}_k \doteq \ln \frac{p_k - \bar{p}}{p_{k-1} - \bar{p}}, \quad \bar{u}_k \doteq \ln \frac{p_{k+1} - \bar{p}}{p_k - \bar{p}},$$

$$i^* \doteq \sup \left\{ i; \sum_{k=i}^n \bar{v}_k \geq \kappa_2 \right\}, \quad j^* \doteq \sup \left\{ j; \sum_{k=j}^{n-1} \bar{u}_k \geq \kappa_1 \right\}.$$

Two cases can arise, depending on the total amounts  $\kappa_1, \kappa_2$  of assets put on sale by the two players.

CASE 1:  $\kappa_1 \geq \sum_{k=i^*}^{\nu-1} \bar{u}_k$ . Then a Nash equilibrium is given by:

$$u_k = \begin{cases} 0 & k \leq i^* - 1 \\ \bar{u}_k & i^* \leq k \leq \nu - 1 \\ \kappa_1 - \sum_{\ell=i^*}^{\nu-1} \bar{u}_\ell & k = \nu \end{cases} \quad v_k = \begin{cases} 0 & k \leq i^* - 1 \\ \kappa_2 - \sum_{\ell=i^*+1}^{\nu} \bar{v}_\ell & k = i^* \\ \bar{v}_k & i^* + 1 \leq k \leq \nu. \end{cases}$$

Indeed, the following optimality conditions for Player 2 are satisfied:

- (i) Any pricing strategy  $\phi_2$  taking values within the set  $\{p_{i^*}, \dots, p_\nu\}$  yields the same payoff for Player 2;
- (ii) Any strategy  $\phi_2$  taking values on a set which is not a subset of  $\{p_{i^*}, \dots, p_\nu\}$  yields a strictly lower payoff.

To verify (i) consider any pricing strategy  $\phi_2 : [0, \kappa_2] \mapsto \{p_1, p_2, \dots, p_\nu\}$ , taking values inside the set  $\{p_{i^*}, \dots, p_\nu\}$ . We have

$$\begin{aligned} J(\phi_2) &= \sum_{k=i^*}^{\nu} \int_{\phi_2^{-1}(p_k)} (p_k - \bar{p}) e^{-\beta - v_{i^*} - \ln \frac{p_k - \bar{p}}{p_{i^*} - \bar{p}}} d\beta \\ &= (p_{i^*} - \bar{p}) e^{-v_{i^*}} \sum_{k=i^*}^{\nu} \int_{\phi_2^{-1}(p_k)} e^{-\beta} d\beta = (p_{i^*} - \bar{p}) e^{-v_{i^*}} (1 - e^{-\kappa_2}). \end{aligned}$$

By the definition of  $i^*$  it follows

$$J(\phi_2) > (p_{i^*} - \bar{p}) e^{-\ln \frac{p_{i^*} - \bar{p}}{p_{i^*-1} - \bar{p}}} (1 - e^{-\kappa_2}) = (p_{i^*-1} - \bar{p}) (1 - e^{-\kappa_2}),$$

which is the payoff corresponding to the constant pricing strategy  $\phi_2(\beta) \equiv p_{i^*-1}$ .

Consider now another strategy,  $\phi_2^\varepsilon(\beta)$  where Player 2 sells an amount  $\varepsilon > 0$  of shares at price  $p_{i^*-1}$ . The corresponding expected payoff is

$$\begin{aligned} J(\phi_2^\varepsilon) &= \int_0^\varepsilon (p_{i^*-1} - \bar{p}) e^{-\beta} d\beta + (p_{i^*} - \bar{p}) e^{-v_{i^*}} \int_\varepsilon^{\kappa_2} e^{-\beta} d\beta \\ &= (p_{i^*-1} - \bar{p}) (1 - e^{-\varepsilon}) + (p_{i^*} - \bar{p}) e^{-v_{i^*}} (e^{-\varepsilon} - e^{-\kappa_2 + \varepsilon}). \end{aligned}$$

A simple computation shows that

$$\begin{aligned} \frac{d}{d\varepsilon} J(\phi_2^\varepsilon) &= e^{-\varepsilon} [(p_{i^*-1} - \bar{p}) - (p_{i^*} - \bar{p}) e^{-v_{i^*}} (1 + e^{-\kappa_2 + 2\varepsilon})] \\ &< - (p_{i^*-1} - \bar{p}) e^{-\kappa_2 + \varepsilon} < 0 \quad \text{for all } \varepsilon > 0, \end{aligned}$$

where we assume that  $\nu$  is large enough so that  $p_{i^*-1} > \bar{p}$  and used the fact that  $v_{i^*} < \bar{v}_{i^*}$ . We conclude that

$$J(\phi_2^\varepsilon) < J(\phi_2), \quad \text{for all } \varepsilon > 0$$

proving (ii).

With similar arguments, we can verify the same optimality conditions for Player 1. If  $\phi_1(\beta)$  is any pricing strategy taking values in  $\{p_{i^*}, \dots, p_\nu\}$ , then

$$J(\phi_1) = \sum_{k=i^*}^{\nu} \int_{\phi_1^{-1}(p_k)} (p_k - \bar{p}) e^{-\beta - \ln \frac{p_k - \bar{p}}{p_{i^*} - \bar{p}}} d\beta = (p_{i^*} - \bar{p})(1 - e^{-\kappa_1})$$

and it is clearly not optimal to sell at lower prices, since Player 1 has the priority.

CASE 2:  $\kappa_1 \leq \sum_{k=i^*}^{\nu-1} \bar{u}_k$ . Then a Nash equilibrium is given by:

$$u_k = \begin{cases} 0 & k \leq j^* - 1 \\ \kappa_1 - \sum_{\ell=j^*+1}^{\nu-1} \bar{u}_\ell & k = j^* \\ \bar{u}_k & j^* + 1 \leq k \leq \nu - 1 \\ 0 & k = \nu \end{cases} \quad v_k = \begin{cases} 0 & k \leq j^* \\ \bar{v}_k & j^* + 1 \leq k \leq \nu - 1 \\ \kappa_2 - \sum_{\ell=j^*+1}^{\nu-1} \bar{v}_\ell & k = \nu. \end{cases}$$

Again, we can show that any strategy  $\phi_1$  yields the same payoff to the first player if it takes values in the set  $\{p_{j^*+1}, \dots, p_\nu\}$  and that the payoff for Player 2 is the same for any pricing strategy supported on  $\{p_{j^*}, \dots, p_{\nu-1}\}$ . Indeed, we have

$$J(\phi_1) = \sum_{k=j^*+1}^{\nu} (p_k - \bar{p}) \int_{\phi_2^{-1}(p_k)} e^{-\beta - u_{j^*} - \ln \frac{p_k - \bar{p}}{p_{j^*} - \bar{p}}} d\beta = (p_{j^*+1} - \bar{p})e^{-u_{j^*}}(1 - e^{-\kappa_1}),$$

$$J(\phi_2) = (p_{j^*} - \bar{p}) \int_0^{u_{j^*}} e^{-\beta} d\beta + \sum_{k=j^*+1}^{\nu-1} (p_k - \bar{p}) \int_{\phi_2^{-1}(p_k)} e^{-\beta - \ln \frac{p_k - \bar{p}}{p_{j^*} - \bar{p}}} d\beta = (p_{j^*} - \bar{p})(1 - e^{-\kappa_2}).$$

In Theorem 1, the assumption that every probability distribution is of type  $A_+$  or  $A_0$  was crucial. We now give another example showing that, if each probability distribution is of type B then a Nash equilibrium in general does not exist.

**Example 3.** Consider a discrete bidding game for two players, putting on sale  $\kappa_1, \kappa_2 > 0$  amounts of shares, and let  $\bar{p}_1 \leq \bar{p}_2 < \bar{P}$  be given. Assume that  $(\ln \psi_i(s))'' > 0$  for  $i = 1, 2$  and all  $s > 0$ . As usual, we assume that Player 1 has priority over Player 2. By Lemma 3, every optimal strategy  $\phi_i$  is constant, hence any Nash equilibrium  $(\phi_1^*, \phi_2^*)$  has the form

$$\phi_1^*(\beta) \equiv p_{j_1} = \frac{j_1 \bar{P}}{\nu}, \quad \phi_2^*(\beta) \equiv p_{j_2} = \frac{j_2 \bar{P}}{\nu},$$

for some  $j_1, j_2 \in \{1, \dots, \nu\}$ . Clearly  $p_{j_1} > \bar{p}_1$  and  $p_{j_2} > \bar{p}_2 \geq \bar{p}_1$ , otherwise one of the payoffs would be negative.

If  $\nu$  is large enough, then

$$p_\nu \int_{\kappa_2}^{\kappa_2 + \kappa_1} \psi_1(\beta) d\beta < p_{\nu-1} \int_0^{\kappa_1} \psi_1(\beta) d\beta, \quad p_\nu \int_{\kappa_1}^{\kappa_1 + \kappa_2} \psi_2(\beta) d\beta < p_{\nu-1} \int_0^{\kappa_2} \psi_2(\beta) d\beta. \quad (5.29)$$

We now observe that



- if  $j_1 = j_2 < \nu$ , then the strategy  $\phi_2 \equiv p_\nu$  yields a strictly higher expected payoff for Player 2;
- if  $j_1 = j_2 = \nu$ , then by (5.29) the strategy  $\phi_2 \equiv p_{j_2-1}$  yields a strictly higher expected payoff for Player 2;
- if  $j_1 < j_2$ , then the strategy  $\phi_1 \equiv p_{j_2}$  yields a strictly higher expected payoff for Player 1;
- if  $j_2 < j_1 < \nu$ , then the strategy  $\phi_1 \equiv p_\nu$  yields a strictly higher expected payoff for Player 1;
- if  $j_2 + 1 < j_1$ , then the strategy  $\phi_2 \equiv p_{j_2+1}$  yields a strictly higher expected payoff for Player 2;
- if  $j_2 = \nu - 1$  and  $j_1 = \nu$ , then by (5.29) the strategy  $\phi_2 \equiv p_{\nu-1}$  yields a strictly higher expected payoff for Player 1.

Since the above cases exhaust all possibilities, conclude that a Nash equilibrium cannot exist.

## 6 Convergence of discrete approximations

In this section we let  $\nu \rightarrow \infty$ , so that the mesh size  $\bar{P}/\nu$  approaches zero. We show that any weak limit of discrete Nash equilibria provides a Nash equilibrium for a bidding game where prices are allowed to range continuously over the reals.

**Theorem 2.** *Let  $\kappa_1, \dots, \kappa_n > 0$  be given. Assume that, for every  $\nu \geq 1$ , the  $n$ -tuple of strategies  $(\phi_1^\nu, \dots, \phi_n^\nu)$  provides a Nash equilibrium to the discrete bidding game in (2.5), (2.6). By selecting an infinite subset of indices  $I \subseteq \mathbb{N}$ , one can achieve the pointwise convergence*

$$\lim_{\nu \in I, \nu \rightarrow \infty} \phi_i^\nu(\beta) = \phi_i^*(\beta) \quad \beta \in [0, \kappa_i], \quad i = 1, \dots, n, \quad (6.30)$$

for some nondecreasing functions  $\phi_i^* : [0, \kappa_i] \mapsto [0, \bar{P}]$ . The  $n$ -tuple  $(\phi_1^*, \dots, \phi_n^*)$  provides a Nash equilibrium to the bidding game, with prices ranging continuously over the reals.

**Proof. 1.** By possibly choosing a subsequence (corresponding to an infinite subset of indices  $I \subseteq \mathbb{N}$ ), since all functions  $\phi_i^\nu$  are nondecreasing, by Helly's compactness theorem we can assume the pointwise convergence in (6.30), for some nondecreasing functions  $\phi_1^*, \dots, \phi_n^*$ .

**2.** We claim that each limit strategy  $\phi_i^*$  is optimal for the  $i$ -th player, in reply to the left continuous function

$$\Phi_i^-(p) = \sum_{j \neq i} \text{meas}(\{\beta \in [0, \kappa_j]; \phi_j^*(\beta) < p\}). \quad (6.31)$$

Indeed, let

$$\Phi_i^\nu(p) = \sum_{j \neq i} \text{meas}(\{\beta \in [0, \kappa_j]; \phi_j^\nu(\beta) < p\}).$$

Since  $\Phi_i^-(p)$  is nondecreasing and left continuous, hence lower semicontinuous, for every  $\beta \in [0, \kappa_i]$  the pointwise convergence  $\phi_i^\nu(\beta) \rightarrow \phi_i^*(\beta)$  yields

$$\Phi_i^-(\phi_i^*(\beta)) \leq \liminf_{\nu \rightarrow \infty} \Phi_i^\nu(\phi_i^\nu(\beta)).$$

Therefore

$$\limsup_{\nu \rightarrow \infty} \psi_i(\beta + \Phi_i^\nu(\phi_i^\nu(\beta))) \leq \psi_i(\beta + \Phi_i^*(\phi_i^*(\beta))).$$

By Fatou's lemma, we conclude

$$\begin{aligned} J_i(\phi_i^*, \Phi_i^-) &= \int_0^{\kappa_i} (\phi_i^*(\beta) - \bar{p}_i) \cdot \psi_i(\beta + \Phi_i^-(\phi_i^*(\beta))) d\beta \\ &\geq \limsup_{\nu \rightarrow \infty} \int_0^{\kappa_i} (\phi_i^\nu(\beta) - \bar{p}_i) \cdot \psi_i(\beta + \Phi_i^\nu(\phi_i^\nu(\beta))) d\beta = \limsup_{\nu \rightarrow \infty} J_i(\phi_i^\nu, \Phi_i^\nu). \end{aligned} \quad (6.32)$$

Next, let  $\phi_i : [0, \kappa_i] \mapsto [0, \bar{P}]$  be any admissible strategy for the  $i$ -th player and let  $\varepsilon > 0$  be given. For each  $\nu$  we can construct a (unique) discrete-valued strategy  $\varphi_i^\nu : [0, \kappa_i] \mapsto \Omega_\nu$  such that

$$\phi_i(\beta) - \frac{\bar{P}}{\nu} < \varphi_i^\nu(\beta) \leq \phi_i(\beta) \quad \text{for all } \beta \in [0, \kappa_i].$$

This strategy satisfies

$$J_i(\varphi_i^\nu, \Phi_i^-) \geq J_i(\phi_i, \Phi_i^-) - \kappa_i \frac{\bar{P}}{\nu}. \quad (6.33)$$

Indeed

$$\begin{aligned} J_i(\varphi_i^\nu, \Phi_i^-) &> \int_0^{\kappa_i} \left( \phi_i(\beta) - \frac{\bar{P}}{\nu} - \bar{p}_i \right) \psi_i(\beta + \Phi_i^-(\varphi_i^\nu(\beta))) d\beta \\ &= \int_0^{\kappa_i} (\phi_i(\beta) - \bar{p}_i) \psi_i(\beta + \Phi_i^-(\varphi_i^\nu(\beta))) d\beta - \kappa_i \frac{\bar{P}}{\nu} \\ &\geq J_i(\phi_i, \Phi_i^-) - \kappa_i \frac{\bar{P}}{\nu}. \end{aligned}$$

For all  $\nu \geq 1$  sufficiently large we have

$$J_i(\phi_i, \Phi_i^-) \leq J_i(\varphi_i^\nu, \Phi_i^-) + \varepsilon \leq J_i(\varphi_i^\nu, \Phi_i^\nu) + 2\varepsilon \leq J_i(\phi_i^\nu, \Phi_i^\nu) + 2\varepsilon. \quad (6.34)$$

The first inequality in (6.34) is an immediate consequence of (6.33). The second follows from the lower semicontinuity of  $\Phi_i^-$  and the pointwise relation  $\Phi_i^-(p) \leq \liminf_{\nu \rightarrow \infty} \Phi_i^\nu(p)$ . The third inequality follows from the optimality of  $\phi_i^\nu$  in response to  $\Phi_i^\nu$ . Together, (6.34) and (6.32) yield

$$J_i(\phi_i, \Phi_i^-) \leq \limsup_{\nu \rightarrow \infty} J_i(\phi_i^\nu, \Phi_i^\nu) + 2\varepsilon \leq J_i(\phi_i^*, \Phi_i^-) + 2\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we conclude that  $\phi_i^*$  is an optimal reply to  $\Phi_i^-$ .

**3.** Our next goal is to prove that, for each  $i = 1, \dots, n$ , the limit function  $\Phi_i^-$  in (6.31) is Lipschitz continuous on  $[0, \bar{P}]$ . Define

$$\delta \doteq \min_j \psi_j(\kappa_1 + \dots + \kappa_n) \cdot (\bar{P} - \bar{p}_j) > 0. \quad (6.35)$$

We claim that, for every  $i \in \{1, \dots, n\}$ , it is never optimal for the  $i$ -th player to put anything on sale at price  $p < \bar{p}_i + \delta$ . Indeed, by putting the same asset for sale at price  $\bar{P}$ , his expected gain (per unit of asset) would be

$$\geq \psi_j(\kappa_1 + \dots + \kappa_n) \cdot (\bar{P} - \bar{p}_i) \geq \delta > (p - \bar{p}_i).$$

This proves our claim.

4. Given an integer  $\nu > 2\bar{P}/\delta$ , let  $(\mu_1^\nu, \dots, \mu_n^\nu)$  be a discrete Nash equilibrium corresponding to the mesh size  $\frac{\bar{P}}{\nu}$ . Fix some  $k < \nu$  and assume that some agent is selling at price  $p_k$ , i.e.

$$\sum_{i=1}^n \mu_{ik}^\nu > 0.$$

Let  $i^*$  be the player with lowest priority among those selling at price  $p_k$ . Then, by optimality (considering the first share he is selling at price  $p_k$ ) we obtain

$$(p_k - \bar{p}_{i^*}) \cdot \psi \left( \sum_{\ell=1}^{k-1} \sum_{i=1}^n \mu_{i\ell}^\nu + \sum_{i < i^*} \mu_{ik}^\nu \right) > (p_{k-1} - \bar{p}_{i^*}) \cdot \psi \left( \sum_{\ell=1}^{k-2} \sum_{i=1}^n \mu_{i\ell}^\nu + \sum_{i \leq i^*} \mu_{i,k-1}^\nu \right)$$

By the mean value theorem, there exists some

$$\zeta \in \left[ \sum_{\ell=1}^{k-2} \sum_{i=1}^n \mu_{i\ell}^\nu + \sum_{i \leq i^*} \mu_{i,k-1}^\nu, \sum_{\ell=1}^{k-1} \sum_{i=1}^n \mu_{i\ell}^\nu + \sum_{i < i^*} \mu_{ik}^\nu \right] \subseteq \left[ 0, \sum_{i=1}^n \kappa_i \right]$$

such that

$$\begin{aligned} & (p_k - \bar{p}_{i^*}) \cdot \left( \psi \left( \sum_{\ell=1}^{k-2} \sum_{i=1}^n \mu_{i\ell}^\nu + \sum_{i \leq i^*} \mu_{i,k-1}^\nu \right) + \psi'(\zeta) \cdot \left( \sum_{i > i^*} \mu_{i,k-1}^\nu + \sum_{i < i^*} \mu_{ik}^\nu \right) \right) \\ & > (p_{k-1} - \bar{p}_{i^*}) \cdot \psi \left( \sum_{\ell=1}^{k-2} \sum_{i=1}^n \mu_{i\ell}^\nu + \sum_{i \leq i^*} \mu_{i,k-1}^\nu \right). \end{aligned}$$

Hence

$$\left( \sum_{i > i^*} \mu_{i,k-1}^\nu + \sum_{i < i^*} \mu_{ik}^\nu \right) \leq \frac{\psi \left( \sum_{\ell=1}^{k-2} \sum_{i=1}^n \mu_{i\ell}^\nu + \sum_{i \leq i^*} \mu_{i,k-1}^\nu \right)}{-\psi'(\zeta)} \cdot \frac{p_k - p_{k-1}}{p_k - \bar{p}_{i^*}} \leq \frac{1}{c_0} \cdot \frac{\bar{P}/\nu}{\delta}, \quad (6.36)$$

with  $c_0 \doteq \inf\{-\psi'(\zeta); \zeta \in [0, \sum \kappa_i]\} > 0$ , and  $\delta$  as in (6.35).

In order to estimate  $\mu_{i^*k}^\nu$ , we observe that somebody with higher priority than  $i^*$  must be selling at price  $p_{k+1}$ , otherwise Player  $i^*$  would achieve a larger expected payoff by becoming the first seller at price  $p_{k+1}$ . Call  $j^* < i^*$  the first seller at price  $p_{k+1}$ . Then, by optimality, we have

$$(p_{k+1} - \bar{p}_{j^*}) \cdot \psi \left( \sum_{\ell=1}^k \sum_{i=1}^n \mu_{i\ell}^\nu \right) > (p_k - \bar{p}_{j^*}) \cdot \psi \left( \sum_{\ell=1}^{k-1} \sum_{i=1}^n \mu_{i\ell}^\nu + \sum_{i \leq j^*} \mu_{ik}^\nu \right).$$

As before, this yields the bound

$$\sum_{i>j^*} \mu_{ik}^\nu \leq \frac{\bar{P}}{\nu c_0 \delta}. \quad (6.37)$$

Combining (6.36) and (6.37), and recalling that  $j^* < i^*$ , we obtain

$$\sum_{i=1}^n \mu_{ik}^\nu \leq \frac{2\bar{P}}{\nu c_0 \delta}.$$

Finally, given any  $0 < a < b < \bar{P}$ , we have

$$\begin{aligned} \sum_{i=1}^n \text{meas}\left(\{\beta ; a < \phi_i^\nu(\beta) < b\}\right) &= \sum_{a < p_k < b} \sum_{i=1}^n \mu_{ik}^\nu \\ &\leq \left(\frac{b-a}{\bar{P}/\nu} + 1\right) \cdot \sup_{a < p_k < b} \sum_{i=1}^n \mu_{ik}^\nu \leq \frac{2}{c_0 \delta} \left(b - a + \frac{\bar{P}}{\nu}\right). \end{aligned} \quad (6.38)$$

Hence, by letting  $\nu \rightarrow \infty$  we see that, for any  $j$ ,

$$\text{meas}\left(\{\beta ; \phi_j^*(\beta) \in ]a, b[ \}\right) \leq \frac{2}{c_0 \delta} (b - a).$$

This implies that every function  $\Phi_i$  is Lipschitz continuous on  $[0, \bar{P}[$ . In particular,  $\Phi_i(p) = \Phi_i^-(p)$  for  $p < \bar{P}$ .

**5.** We claim that, by possibly shrinking the countable set  $I \in \mathcal{I}$  (i.e., by taking a further subsequence), in the limit  $(\phi_1^*, \dots, \phi_n^*)$  at most one of the players puts a positive amount of assets for sale at price  $\bar{P}$ .

For each  $\nu \geq 1$ , let  $(\mu_1^\nu, \dots, \mu_n^\nu)$  be a corresponding discrete Nash equilibrium. Let  $\iota(\nu) \in \{1, \dots, n\}$  denote the player with lowest priority, among those who are selling something at price  $\bar{P}$ :

$$\iota(\nu) \doteq \max\{i ; \mu_{i\nu}^\nu > 0\}.$$

By possibly choosing a further subsequence, we can assume that

$$\iota(\nu) = \iota^* \quad \text{for all } \nu.$$

If  $\iota^* = 1$ , there is nothing to prove, since only one player is selling at the highest price. If  $\iota^* > 1$ , we use the optimality conditions (applied to the first share that player  $\iota^*$  sells at price  $\bar{P}$ ) to bound the amount of shares put on sale at  $\bar{P}$  by the remaining players:

$$(\bar{P} - \bar{p}_{\iota^*}) \cdot \psi\left(\sum_{i=1}^n \kappa_i - \mu_{\iota^* \nu}^\nu\right) > (p_{\nu-1} - \bar{p}_{\iota^*}) \cdot \psi\left(\sum_{i=1}^n \kappa_i - \sum_{i=1}^{\iota^*} \mu_{i\nu}^\nu - \sum_{i>\iota^*} \mu_{i, \nu-1}^\nu\right)$$

which, as before, yields

$$\sum_{i=1}^{\iota^*-1} \mu_{i, \nu}^\nu \leq \frac{\bar{P}}{\nu c_0 \delta}. \quad (6.39)$$

As  $\nu \rightarrow \infty$ , the right hand side of (6.39) approaches zero. Combining (6.38) with (6.39), for every  $\varepsilon > 0$  and  $i \neq i^*$  we obtain

$$\text{meas}\left(\{\beta \in [0, \kappa_i]; \phi_i^*(\beta) = \bar{P}\}\right) \leq \limsup_{\nu \rightarrow \infty} \text{meas}\left(\{\beta \in [0, \kappa_i]; \phi_i^\nu(\beta) \in [\bar{P}-\varepsilon, \bar{P}]\}\right) < \frac{2\varepsilon}{c_0\delta}.$$

Since  $\varepsilon > 0$  was arbitrary, for  $i \neq i^*$  one has

$$\text{meas}\left(\{\beta \in [0, \kappa_i]; \phi_i^*(\beta) = \bar{P}\}\right) = 0,$$

proving our claim.

**6.** According to the previous analysis, for each  $i \neq i^*$  the map

$$p \mapsto \text{meas}\left(\{\beta \in [0, \kappa_i]; \phi_i^*(\beta) \leq p\}\right)$$

is Lipschitz continuous. Therefore

$$J(\phi_i^*, \Phi_i^*) = J(\phi_i^*, \Phi_i^-) \quad \text{for all } i = 1, \dots, n.$$

Recalling step 2 and observing that  $\Phi_i^-(p) \leq \Phi_i^*(p)$ , for every  $i$  and every admissible strategy  $\phi_i : [0, \kappa_i]$  we have

$$J(\phi_i, \Phi_i^*) \leq J(\phi_i, \Phi_i^-) \leq J(\phi_i^*, \Phi_i^-) = J(\phi_i^*, \Phi_i^*).$$

proving that the  $n$ -tuple of pricing strategies  $(\phi_1^*, \dots, \phi_n^*)$  provides a Nash equilibrium.  $\square$

As a consequence of Theorem 2, we obtain the existence of a Nash equilibrium for a pricing game with heterogeneous players, where prices can range continuously on the interval  $[0, \bar{P}]$ .

**Corollary 2.** *Consider a pricing game for  $n$  players, with continuum strategies  $\phi_i : [0, \kappa_i] \mapsto [0, \bar{P}]$  and payoffs as in (2.5), (2.6). Assume that each probability distribution  $\psi_i$  is either of type  $A_0$  or  $A_+$ . Then the game admits a Nash equilibrium.*

**Proof.** For each  $\nu \geq 1$ , consider the game where prices range over the discrete set  $\Omega_\nu = \left\{\frac{j\bar{P}}{\nu}; j = 1, 2, \dots, \nu\right\}$ . By Theorem 1, this game has at least one Nash equilibrium, say  $(\phi_1^\nu, \dots, \phi_n^\nu)$ . According to Theorem 2, we can then choose a subsequence such that (6.30) holds and the pointwise limit  $(\phi_1^*, \dots, \phi_n^*)$  provides a Nash equilibrium to the game with prices continuously ranging over  $[0, \bar{P}]$ .  $\square$

**Remark 2.** In the case where all players have the same payoff function and assign the same probability distribution to the random incoming order  $X$ , it was proved in [6] that the Nash equilibrium in the continuum model is unique. In this case, the entire sequence of discrete Nash equilibria must converge to this unique limit.

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