Extremal Solutions of Differential Inclusions via Baire Category: a Dual Approach

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July 14, 2013

Abstract

Let F be a continuous multifunction on $I\!R^n$ with compact convex values. For any vector w, let $F^w(x) \subseteq F(x)$ be the subset of points which maximize the inner product with w. Call W the set of all continuous functions $w : [0,T] \mapsto I\!R^n$ with the following property: all solutions to the Cauchy problem $\dot x(t) \in F^{w(t)}(x(t))$, $x(0) = 0$, are also solutions to $\dot x(t) \in extF(x(t))$. We prove that W is residual in $\mathcal{C}([0,T];\, I\!R^n)$.

1 Introduction

For $t \in [0,T]$, consider the Cauchy problem

$$\dot x(t) \in F\big(x(t)\big)\,, \qquad\qquad x(0) \;=\; 0\,, \tag{1.1}$$

where $x \mapsto F(x) \subset I\!R^n$ is a bounded, Hausdorff continuous multifunction with compact convex values. Call $\mathcal{F} \subset \mathcal{C}\big([0,T];\, I\!R^n\big)$ the set of Carathéodory solutions of (1.1). Moreover, call $\mathcal{F}^{ext}$ the set of trajectories of

$$\dot x(t) \;\in\; ext\,F(x(t))\,, \qquad\qquad x(0) \;=\; 0\,, \tag{1.2}$$

where the time derivative takes values within the set of extreme points of $F(x)$.

If F is Lipschitz continuous w.r.t. the Hausdorff distance, it is well known that the set of extremal solutions $\mathcal{F}^{ext}$ is a residual subset of $\mathcal{F}$, i.e. it contains the intersection of countably many open dense subsets. By an application of Baire's theorem, this implies that the set $\mathcal{F}^{ext}$ is nonempty and every solution of (1.1) can be uniformly approximated by solutions of (1.2).

In this paper we wish to develop an alternative approach, still based on Baire category but from a dual point of view. For every $w \in I\!R^n$, consider the compact, convex subset of vectors in $F(x)$ which maximize the inner product with w:

$$F^w(x) \;\doteq\; \Big\{ y \in F(x)\,;\;\; \langle y, w\rangle \;=\; \max_{y' \in F(x)} \langle y', w\rangle \Big\}. \tag{1.3}$$

For each continuous path $t \mapsto w(t)$, the multifunction $F^w(t,x) \doteq F^{w(t)}(x)$ is upper semicontinuous with compact, convex values. Hence the Cauchy problem

$$\dot{x}(t) \in F^{w(t)}\big(x(t)\big), \qquad x(0) = 0, \tag{1.4}$$

has a non-empty, compact set of solutions $\mathcal{F}^w \subset \mathcal{C}([0,T];\, I\!R^n)$. Our main result shows that, for a generic function $w \in \mathcal{C}([0,T];\, I\!R^n)$ all solutions of (1.4) are also solutions of (1.2).

Theorem 1. *Let F be a bounded, Hausdorff continuous multifunction on $I\!R^n$, with compact convex values. Then the set $W \doteq \Big\{w(\cdot)\,;\ \ \mathcal{F}^w \subseteq \mathcal{F}^{ext}\Big\}$ is a residual subset of $\mathcal{C}([0,T];,I\!R^n)$.*

Incidentally, this yields yet another proof of the classical theorem of Filippov [15].

Corollary 1. *Let G be a bounded, Hausdorff continuous multifunction on $I\!R^n$, with compact values. Then, the Cauchy problem*

$$\dot{x}(t) \in G\big(x(t)\big) \qquad t \in [0,T], \qquad x(0) = 0, \tag{1.5}$$

has at least one Carathéodory solution.

Indeed, taking the convex hulls $F(x) \doteq coG(x)$, one obtains a multifunction F satisfying the assumptions of Theorem 1. Since $extF(x) \subseteq G(x)$, every solution of (1.2) is also a solution of (1.5). An application of Theorem 1 yields the result. □

The use of Baire category methods in the analysis of differential inclusions started with the seminal paper by Cellina [6], and was initially developed by De Blasi and Pianigiani [10, 11, 12, 13], providing new existence results for non-convex valued differential inclusions in Banach spaces. A proof of Filippov's theorem [15] based on Baire category arguments was first achieved in [3]. As shown by Plis' counterexample [17], when the multifunction F is continuous but not Lipschitz, the set $\mathcal{F}^{ext}$ of solutions to (1.2) need not be dense on the set $\mathcal{F}$ of all solutions to (1.1). However, by means of a "set valued" extension of Baire's theorem, where points are replaced by compact sets, the following result was proved in [3].

Theorem 2. *Let F be a bounded, Hausdorff continuous multifunction on $I\!R^n$, with compact convex values. Then, for every continuous selection $f(x) \in F(x)$, at least one of the solutions to the Cauchy problem*

$$\dot{x}(t) = f(x(t)) \qquad t \in [0,T], \qquad x(0) = 0,$$

can be uniformly approximated by solutions of (1.2).

Relying on Baire category arguments, an extension of the classical bang-bang theorem for control systems was obtained in [5]. Applications of these ideas to partial differential inclusions and variational problems have been developed in [4, 9]. A spectacular application of the Baire category method was recently given by De Lellis and Székelyhidi [14], constructing a large number of weak solutions to the Euler equations of inviscid, incompressible fluids. In particular, their approach also yields solutions with compact support in t-x space.

Surveys of the Baire category method in connection with differential inclusions can be found in [7, 16, 19]. For a comprehensive introduction to theory of multifunctions and differential inclusions, we refer to the classical monograph [1].

2 Proof of the main theorem

This section is devoted to the proof of Theorem 1.

We first observe that in the one-dimensional case the result is trivial. Indeed, if $F(x) \subset I\!\!R$ then

$$W \doteq \Big\{ w(\cdot)\,;\;\; \mathcal{F}^w \subseteq \mathcal{F}^{ext} \Big\} \supseteq \Big\{ w(\cdot)\,;\;\; w(t) \neq 0 \qquad \text{for a.e. } t \in [0,T] \Big\}, \tag{2.1}$$

and the set on the right hand side of (2.1) is clearly residual in $\mathcal{C}([0,T];\, I\!\!R)$. Throughout the following, we thus consider the problem in dimension $n \geq 2$.

For any compact convex set $K \subset I\!\!R^n$, define the Choquet function

$$\Phi(K,y) \;\doteq\; \sup \left\{ \int_0^1 |f(\xi) - y|^2 \, d\xi\,; \quad f : [0,1] \mapsto K, \quad \int_0^1 f(\xi)\, d\xi = y \right\}, \tag{2.2}$$

with the understanding that $\Phi(K,y) = -\infty$ if $y \notin K$. Notice that $\Phi(K,y)$ can be regarded as the maximum variance among all random variables supported inside K with mean y. As proved in [2], the function Φ is upper semicontinuous w.r.t. both arguments, and for each K the map $y \mapsto \Phi(K,y)$ is concave down.

A solution $x(\cdot)$ of (1.1) lies in $\mathcal{F}^{ext}$ if and only if

$$\int_0^T \Phi(F(x(t)),\, \dot{x}(t))\, dt \;=\; 0\,.$$

To prove the theorem we need to show that the set

$$W \;\doteq\; \Big\{ w\,; \quad \int_0^T \Phi(F(x(t)),\, \dot{x}(t))\, dt \;=\; 0 \qquad \text{for every} \quad x \in \mathcal{F}^w \Big\}$$

is of second Baire category in $\mathcal{C}([0,T];\, I\!\!R^n)$. This is the case if, for every $\varepsilon > 0$, the set

$$W^\varepsilon \;\doteq\; \Big\{ w\,; \quad \int_0^T \Phi(F(x(t)),\, \dot{x}(t))\, dt \;<\; \varepsilon \qquad \text{for every} \quad x \in \mathcal{F}^w \Big\}$$

is open and dense in $\mathcal{C}([0,T];\, I\!\!R^n)$. The proof will be given in several steps.

1. We show that W^ε is open. Consider a sequence of continuous functions $w_k \in \mathcal{C}([0,T])$, converging to w uniformly on $[0,T]$. Assume that $w_k \notin W^\varepsilon$ for every k. Then there exists a solution $x_k \in \mathcal{F}^{w_k}$ such that

$$\int_0^T \Phi(F(x_k(t)),\, \dot{x}_k(t))\, dt \;\geq\; \varepsilon\,.$$

By taking a subsequence, we can assume the uniform convergence $x_k \to x$ for some Lipschitz continuous function $x(\cdot) \in \mathcal{F}^w$. By the upper semicontinuity and concavity of Φ we conclude that

$$\int_0^T \Phi(F(x(t)),\, \dot{x}(t))\, dt \;\geq\; \varepsilon\,.$$

as well. Hence the complement of W^ε is closed.

2. The remaining steps will show that W^ε is dense. Let $B \subset I\!R^n$ be the closed unit ball and let $\partial B = \{x \in I\!R^n\,;\ \ |x| = 1\}$ be its $n-1$ dimensional boundary. Two remarks are in order:

(i) Without loss of generality we can assume that $F(x) \subseteq B$ for every $x \in I\!R^n$.

(ii) In dimension $n \geq 2$, every continuous function $\tilde w : [0,T] \mapsto I\!R^n$ can be uniformly approximated by a continuous function $w : [0,T] \mapsto I\!R^n \setminus \{0\}$. Since $F^w(x) = F^{w/|w|}(x)$ depends only on the unit vector parallel to w, it suffices to show that W^ε is dense on the set of all continuous functions with values in ∂B.

As a preliminary, consider the function $\varphi : I\!R^n \times \partial B \mapsto I\!R$, defined as

$$\varphi(x,w) \;\doteq\; \max\Big\{\Phi(F(x),y)\,;\quad y \in F^w(x)\Big\}.$$

For each fixed x, it is well known (see for example [18]) that for a.e. $w \in \partial B$ the set $F^w(x)$ is a singleton and $\varphi(x,w) = 0$.

We claim that φ is upper semicontinuous. Indeed, consider a sequence $(x_k, w_k) \to (\bar x, \bar w)$. For each $k \geq 1$ choose $y_k \in F^{w_k}(x_k)$ such that $\varphi(x_k, w_k) = \Phi(F(x_k), y_k)$. Taking a subsequence we can assume $y_k \to y \in F^{\bar w}(\bar x)$. The upper semicontinuity of Φ implies

$$\varphi(\bar x, \bar w) \;\geq\; \Phi(F(\bar x), \bar y) \;\geq\; \limsup_{k\to\infty} \Phi(F(x_k), y_k) \;=\; \limsup_{k\to\infty} \varphi(x_k, w_k)\,.$$

3. For any large constant $\lambda > 0$, consider the Lipschitz continuous approximation

$$\varphi^\lambda(x,w) \;\doteq\; \max\Big\{\varphi(x',w') - \lambda|x'-x| - \lambda|w'-w|\,;\quad x', w' \in I\!R^n,\ \ |w'| = 1\Big\}. \tag{2.3}$$

This is the minimum among all continuous functions with Lipschitz constant λ which are $\geq \varphi$. Clearly, the values $\varphi^\lambda(x,w)$ decrease pointwise to $\varphi(x,w)$ as $\lambda \to \infty$.

We claim that, for every $\delta > 0$, there exists λ sufficiently large so that

$$\int_{\partial B} \varphi^\lambda(x,w)\,dw \;<\; \delta \tag{2.4}$$

for every $x \in I\!R^n$, $|x| \leq T$. Here the integration is w.r.t. the probability measure uniformly distributed over the $(n-1)$-dimensional sphere $\partial B \subset I\!R^n$.

If the claim were false, there would be a sequence of points x_k such that

$$\int_{\partial B} \varphi^k(x_k,w)\,dw \;\geq\; \delta \tag{2.5}$$

for every integer $k \geq 1$. Taking a convergent subsequence we can assume $x_k \to \bar x$. By upper semicontinuity, this would yield

$$\int_{\partial B} \varphi(\bar x,w)\,dw \;\geq\; \delta\,. \tag{2.6}$$

This yields a contradiction because $\varphi(\bar{x}, w) = 0$ for a.e. $w \in \partial B$.

4. We can now prove that W^ε is dense. Fix $\rho > 0$ and let any continuous function $z : [0, T] \mapsto \partial B$ be given.

By the previous step, choosing $\lambda > 0$ sufficiently large, we can assume that the average value of φ^λ over the $(n-1)$-dimensional spherical cap centered at z with radius ρ satisfies

$$\fint_{\{|\zeta|=1\,,|\zeta-z|<\rho\}} \varphi^\lambda(x, \zeta)\, d\zeta \;\leq\; \varepsilon \tag{2.7}$$

for every $x \in \mathbb{R}^n$ such that $|x| \leq T$, and for every $z \in \partial B$.

For each $t \in [0, T]$, choose a set of points

$$p_1(t), \ldots, p_\nu(t) \;\in\; \{|\zeta| = 1\,, |\zeta - z(t)| < \rho\}$$

which is "nearly uniformly distributed on the spherical cap centered at $z(t)$ with radius ρ", in the sense that

$$\left| \fint_{\{|\zeta|=1,\,|\zeta-z(t)|<\rho\}} f d\zeta \;-\; \frac{1}{\nu}\sum_{i=1}^{\nu} f(p_i(t)) \right| \;\leq\; \varepsilon \tag{2.8}$$

for every Lipschitz continuous function f with Lipschitz constant λ. It is clear that such a set can be found, and that its cardinality ν is independent of $z(t)$. Indeed, assume that the points $p_1(0), \ldots, p_\nu(0)$ in the spherical cap centered at $z(0)$ have been chosen, and satisfy (2.8) at time $t = 0$. Then for any $t \in [0, T]$ we can simply define

$$p_i(t) \;\doteq\; \Theta(t) p_i(0), \qquad\qquad i = 1, \ldots, \nu,$$

where $\Theta(t)$ is a rotation in $\mathbb{R}^n$, mapping $z(0)$ to $z(t)$.

Given a large integer N, divide the interval $[0, T]$ into N equal parts, inserting the times

$$0 \;=\; t_0 \;<\; t_1 \;<\; \cdots \;<\; t_N \;=\; T\,, \qquad\qquad t_i \;=\; \frac{iT}{N}\,.$$

Each subinterval $[t_{i-1}, t_i]$ will be further subdivided into ν equal parts. Let $t \mapsto z_N(t)$ be a piecewise constant function with jumps at the points $t_{i,j} = \frac{iT}{N} + \frac{jT}{N\nu}$, such that

$$z_N(t) \;=\; p_j(t_i) \qquad\qquad t \in [t_{i,j-1},\; t_{i,j}]\,.$$

Since $z : [0, T] \mapsto \partial B$ is continuous, for all N sufficiently large we have

$$\sup_{t\in[0,T]} |z_N(t) - z(t)| \;<\; 2\rho\,. \tag{2.9}$$

Next, consider any trajectory $t \mapsto x(t)$ with $x(0) = 0$ and with Lipschitz constant 1. Using

(2.7)-(2.8) and recalling that φ^λ has Lipschitz constant λ, we obtain

$$
\begin{aligned}
\int_0^T \varphi^\lambda(x(t), z_N(t))\,dt \;&=\; \sum_{i=1}^N \left(\frac{1}{\nu} \sum_{j=1}^\nu \int_{t_{i,j-1}}^{t_{i,j}} \varphi^\lambda(x(t), p_j(t_i))\,dt \right) \\
&\leq\; \sum_{i=1}^N \left(\frac{1}{\nu} \sum_{j=1}^\nu \int_{t_{i,j-1}}^{t_{i,j}} \Big[\varphi^\lambda(x(t_i), p_j(t_i)) + \lambda |x(t) - x(t_i)| \Big]\,dt \right) \\
&\leq\; \frac{\lambda T^2}{N} + \sum_{i=1}^N \frac{T}{N} \fint_{\{|\zeta|=1,\, |\zeta - z(t_i)| < \rho\}} \varphi^\lambda(x(t_i), \zeta)\,d\zeta + T\varepsilon \\
&\leq\; \frac{\lambda T^2}{N} + T\varepsilon + T\varepsilon \;\leq\; 3T\varepsilon
\end{aligned}
\tag{2.10}
$$

for N sufficiently large.

Recalling (2.9), we can now construct a continuous function $\tilde{z} : [0, T] \mapsto \partial B$ such that

$$
|\tilde{z}(t) - z(t)| \;\leq\; 4\rho\,, \tag{2.11}
$$

and moreover

$$
\text{meas}\Big(\{ t \in [0, T]\,; \quad \tilde{z}(t) \neq z_N(t) \} \Big) \;<\; T\varepsilon\,. \tag{2.12}
$$

Since all sets $F(x)$ are contained in the unit ball, by (2.2) it follows that $\varphi(x, w) \leq 1$ and $\varphi^\lambda(x, w) \leq 1$ for all x, w. This observation together with (2.10) and (2.12) implies

$$
\begin{aligned}
\int_0^T \varphi(x(t), \tilde{z}(t))\,dt \;&\leq\; \int_0^T \varphi^\lambda(x(t), \tilde{z}(t))\,dt \\
&\leq\; \int_0^T \varphi^\lambda(x(t), z_N(t))\,dt + \int_{\{t;\ \tilde{z}(t) \neq z_N(t)\}} \varphi^\lambda(x(t), \tilde{z}(t))\,dt \;\leq\; 4T\varepsilon
\end{aligned}
\tag{2.13}
$$

for every function $x : [0, T] \mapsto I\!R^n$ satisfying $x(0) = 0$ and with Lipschitz constant 1. In particular, this is true for every solution $x(\cdot)$ of (1.4).

The previous arguments show that the neighborhood of $z(\cdot)$ with radius 4ρ intersects the set $W^{4T\varepsilon}$. Since ρ and ε were arbitrary, this concludes the proof. □

3 Concluding remarks

Remark 1. The construction of extremal solutions developed here represents a "dual" approach, because the Baire category theorem is used within a space of functions $w(\cdot)$ taking values in the dual space $(I\!R^n)^*$. Of course, being an inner-product space, $I\!R^n$ can be identified with its dual and this difference is purely formal. However, any extension of the present results to an infinite dimensional Banach space X should refer to functions $w : [0, T] \mapsto X^*$ taking values in the dual space.

Remark 2. From Theorem 1 we immediately obtain a counterpart to Theorem 2, namely

Corollary 2. *Let F be a bounded, Hausdorff continuous multifunction on $I\!R^n$, with compact convex values. Then, for every continuous function $w : [0,T] \mapsto I\!R^n$, at least one of the solutions to the Cauchy problem*

$$\dot{x}(t) \in F^{w(t)}(x(t)) \qquad t \in [0,T], \qquad\qquad x(0) = 0\,, \tag{3.1}$$

can be uniformly approximated by solutions of (1.2).

Indeed, by Theorem 1 there exists a sequence of continuous functions $w_k \in W$, converging to w uniformly on $[0,T]$. For each $k \geq 1$, let $x_k(\cdot)$ be a solution to the Cauchy problem

$$\dot{x}(t) \in F^{w_k(t)}(x(t)) \qquad t \in [0,T], \qquad\qquad x(0) = 0\,.$$

Then $x_k(\cdot)$ is also a solution to (1.2), because $w_k \in W$. By possibly taking a subsequence, we can assume the uniform convergence $x_k \to x$, for some continuous function $x : [0,T] \mapsto I\!R^n$. By upper semicontinuity and by the convexity of the sets $F^{w(t)}(x)$, it now follows that $x(\cdot)$ provides a solution to (3.1). □

Remark 3. Our main theorem was proved within the space of all continuous functions $w : [0,T] \mapsto I\!R^n$. It would be interesting to see whether a similar result can be valid in a Banach space of functions with a given modulus of continuity. More precisely, let $\varphi : I\!R_+ \mapsto I\!R_+$ be a continuous function such that

$$\varphi(0) = 0, \qquad\qquad \varphi'(s) > 0\,, \qquad \varphi''(s) \leq 0 \qquad \text{for all } s > 0.$$

Let $\mathcal{C}^\varphi([0,T])$ be the Banach space of all functions $w : [0,T] \mapsto I\!R^n$ having modulus of continuity φ, with norm

$$\|w\|_{\mathcal{C}^\varphi} \doteq \sup_t |w(t)| + \sup_{s<t} \frac{|w(t)-w(s)|}{\varphi(t-s)}\,.$$

Assuming that the multifunction F is Lipschitz or Hölder continuous w.r.t. the Hausdorff distance, it may be of interest to understand for which functions φ the set

$$W = \Big\{ w \in \mathcal{C}^\varphi([0,T])\,; \quad \mathcal{F}^w \subseteq \mathcal{F}^{ext} \Big\}$$

is a residual subset of $\mathcal{C}^\varphi([0,T])$. In this connection, one wonders if the probabilistic approach developed in [8] in terms of Brownian paths $w(\cdot)$ can be implemented, in cases where W is a residual set in the Hölder space $C^{0,1/2}([0,T])$.

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