A Dynamic Model of the Limit Order Book

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Abstract

We consider an equilibrium model of the Limit Order Book in a stock market, where a large number of competing agents post "buy" or "sell" orders. For the "one-shot" game, it is shown that the two sides of the LOB are determined by the distribution of the random size of the incoming order, and by the maximum price accepted by external buyers (or the minimum price accepted by external sellers). We then consider an iterated game, where more agents come to the market, posting both market orders and limit orders. Equilibrium strategies are found by backward induction, in terms of a value function which depends on the current sizes of the two portions of the LOB. The existence of a unique Nash equilibrium is proved under a natural assumption, namely: the probability that the external order is so large that it wipes out the entire LOB should be sufficiently small.

1 The Nash equilibrium for a one-shot game

A bidding game related to a continuum model of a one-sided Limit Order Book (LOB) was recently considered in [5, 6, 7, 8], proving the existence and uniqueness of a Nash equilibrium and determining the optimal strategies for the various agents. In the basic model, it is assumed that an external buyer asks for a random amount $X > 0$ of a given asset. This amount will be bought at the lowest available price, as long as this price does not exceed some (random) upper bound $\overline{P}$. Several sellers offer various quantities of this same asset at different prices, competing to fulfill the incoming order, whose size is not known a priori.

Having observed the prices asked by his competitors, each agent must determine an optimal strategy, maximizing his expected payoff. Because of the presence of the other sellers and of the upper bound $\overline{P}$ on the acceptable price, asking a higher price for an asset reduces the probability of selling it. As a consequence, a unique shape of the LOB is determined, which represents a Nash equilibrium between the various agents.

The models considered in [5, 6, 7, 8] all have the form of a "one shot game". All players' payoffs are completely determined as soon as one single incoming order is received.

The present paper has two main goals. First, we study a two-sided LOB, where the incoming order can be either a buy or a sell order. In this situation, a large number of agents, each holding a small amount of cash and stock, must determine a pricing strategy, including sales offers as well as bid prices at which they propose to buy new stock. The two sides of the LOB are correlated by the assumption that external agents will agree to the transaction only if the price is sufficiently close to the mean bid-ask price. In this setting, we prove two results showing that the shape of the two-sided LOB can be uniquely determined, depending on (i) the total amount of stocks that agents put on sale or bid to buy on the LOB, and (ii) the distribution of the random variables X, Y describing the sizes of the incoming buy or sell orders.

As a second extension, we consider a time-dependent problem involving sequence of N incoming orders $X_1, \ldots, X_N$. Each can be either a buy order or a sell order. The random variables X_j, $j = 1, \ldots, N$, describing the amount of stock that the external agents want to buy (or sell), are assumed to be mutually independent. Again, we seek conditions ensuring that, at each time step $i = 1, \ldots, N$, the shape of the two-sided LOB can be uniquely determined, by backward induction. We point out a major difference between the "one-shot" game and the dynamic model involving multiple time steps. Namely, in a game involving one single external order, the payoff for a player holding an amount c of cash and an amount s of stock is

$$J = c + p_0 s,$$

where p_0 is an underlying fundamental value of the stock, known to all agents posting bids on the LOB. On the other hand, at an intermediate time i, this expected payoff will have the more general form

$$J_i = c \cdot V_i^C(x,y) + s \cdot V_i^S(x,y).$$

Here $V_i^C(x,y)$ and $V_i^S(x,y)$ denote the expected payoffs to an agent that holds a unit amount of cash or stock at the i-th time step, assuming that the sizes of the "sell" and "buy" portions of the LOB at that time are x, y respectively. This reflects the fact that, during the time periods $i+1, \ldots, N$, an agent can achieve some additional profits by repeatedly buying and selling stock on the LOB at favorable prices. As already shown in [8], these expected profits strongly depend on the size of the LOB. As the total amount of bids posted on the LOB increases, there is a stronger competition among agents, and hence a smaller expected profit for each one of them.

The article is organized as follows. In Section 2 we study a two-sided one-shot game, in the case where the maximum price accepted by an external buyer, and the minimum price accepted by an external seller, are deterministic functions of the mean bid-ask price. In Theorem 1 the existence of a unique shape for the LOB is proved under one main assumption. Namely, the probability that the incoming order is very large, wiping out the entire "buy" or "sell" portion of the order book, should be sufficiently small.

In Section 3 we study the more general case where the maximum or minimum prices acceptable to external agents are random as well. Under suitable assumptions on the distributions of these random variables, Theorem 2 provides the existence of a unique shape for the two-sided LOB. Remarkably, no assumption on the size "buy" or "sell" portion of the order book is here needed.

A detailed description the evolution model for the two-sided LOB is given in Section 4. Finally, in Section 5 we derive conditions for the existence of a unique shape for the evolution of the two-sided LOB, together with a priori bounds on the value functions V_i^C, V_i^S.

The present models are meant to capture some features of the Limit Order Book. In particular: (i) its shape, depending on the distribution of the random external orders, (ii) the expected profit achieved by agents posting limit orders, depending on the total size of the LOB, and hence on the competition among these agents. On the other hand, in our present model the incoming orders are regarded as independent random variables, which do not carry information about the fundamental value of the stock. The issue of how to extract information from the size and frequency of incoming orders will be a topic for a future work.

There is a large and growing literature modeling different aspects of the Limit Order Book [2, 9, 10, 11, 12, 13, 15, 16, 18, 19]. In particular, the spreading of information and the price impact and of a large external order have been studied in [1, 3, 4]. For a survey, we refer to [14] or [17].

2 The two-sided LOB for the one-shot game

We consider a continuum model of the Limit Order Book, described by a density function $\phi = \phi(s)$, as in Fig. 1, right. Calling p_0 the "fundamental value" of the stock, known to all agents posting bids on the LOB, the function ϕ will describe sell orders posted on the LOB for prices $p > p_0$, and buy orders for $p < p_0$. In other words, for $p_0 < p_1 < p_2$, the integral

$$\int_{p_1}^{p_2} \phi(s)\, ds \tag{2.1}$$

gives the total amount of stock that the agents offer for sale at price $p \in [p_1, p_2]$. On the other hand, for $p_1 < p_2 < p_0$, the integral (2.1) gives the total amount of stock that the agents are willing to buy at price $p \in [p_1, p_2]$. The minimum ask price (i.e., the lowest price at which some agent offers to sell stock) is denoted by

$$p_A \doteq \inf\left\{p > p_0\,; \quad \int_{p_0}^{p} \phi(s)\, ds > 0\right\}, \tag{2.2}$$

while the maximum bid price (i.e., the highest price at which some agent offers to buy stock) is denoted by

$$p_B \doteq \sup\left\{p < p_0\,; \quad \int_{p}^{p_0} \phi(s)\, ds > 0\right\}. \tag{2.3}$$

Throughout the following, we denote the **mean bid-ask price** as

$$\overline{p} \doteq \frac{p_A + p_B}{2}\,. \tag{2.4}$$

In a basic model, one can assume that the maximum price that an external buyer is willing to pay (or the minimum price that an external seller is willing to accept) is a given multiple of the mean price $\overline{p}$. In this case, external agents will buy stock only at a price $p \leq (1+\delta)\overline{p}$.

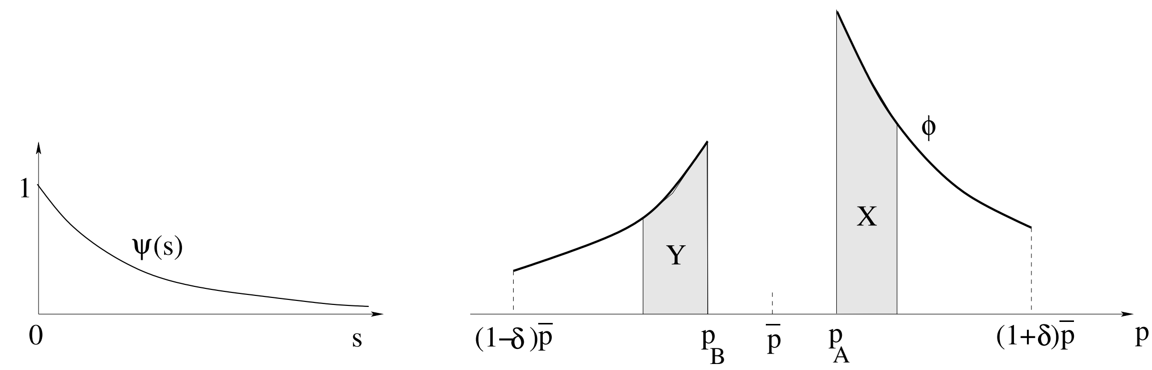

Figure 1: Left: a distribution function for the random variable X, describing the size of the external order. Right: a possible shape of the limit order book. If the external order is a buy order with size $X > 0$, all the stocks in the shaded region on the right (with area $= X$), will be sold. If the external order is a sell order for an amount $Y > 0$ of stocks, all the buy orders in the shaded region on the left (with area $= Y$), will be executed.

Similarly, an external agent will agree to sell his stock only at a price $p \geq (1-\delta)\overline{p}$. Here $\delta > 0$ is a small constant, given a priori.

A more general assumption, considered in [8] for a one-side LOB, is that the maximum price acceptable to external agents is random. We assume here that an external buyer will agree to the transaction only at a price $p \leq Q^b \cdot \overline{p}$, where $Q^b \geq 1$ is a random variable. Similarly, an external seller will agree to the transaction only at a price $p \geq Q^s \cdot \overline{p}$, where Q^s is another random variable, ranging in $[0,1]$.

In general, an external order is thus executed as follows (Fig. 1, right).

CASE 1: a buy order of size X. In this case the external buyer will take all stocks whose price ranges in the interval $[p_0,\, p(X)]$, where

$$p(X) \;=\; \sup\left\{ p > p_0\,;\;\; p \leq Q^b \cdot \overline{p},\;\;\; \int_{p_0}^{p} \phi(s)\,ds \;\leq\; X \right\}.$$

CASE 2: a sell order of size Y. In this case the external seller will fulfill all the bids whose price ranges in the interval $[p(Y),\, p_0]$, where

$$p(Y) \;=\; \inf\left\{ p < p_0\,;\;\; p \geq Q^s \cdot \overline{p},\;\;\; \int_{p}^{p_0} \phi(s)\,ds \;\leq\; Y \right\}.$$

Assume that, **after** the external order has been executed, the payoff for any player holding an amount c in cash and s in stock is given by

$$J \;=\; c + s\,p_0\,. \tag{2.5}$$

The following analysis will show that, given the distribution of the random variables X, Y, the shape of the limit order book is entirely determined by p_0 and the quantities

$$\bar{x} \;\doteq\; \int_{p_0}^{+\infty} \phi(p)\,dp\,, \qquad\qquad \bar{y} \;\doteq\; \int_{0}^{p_0} \phi(p)\,dp, \tag{2.6}$$

respectively the total amount of stock in the "sell" and in the "buy" portion of the LOB.

The heart of the argument goes as follows. First, for a given mean price $\overline{p}$, we show that the both the "sell" and the "buy" portions of the LOB are uniquely determined. In particular, the minimum ask price p_A and the maximum bid price p_B are uniquely determined as functions of $\overline{p}$. In the case where

$$\frac{1}{2}\left(\frac{d}{d\overline{p}}\,p_A + \frac{d}{d\overline{p}}\,p_B\right) \;<\; 1\,, \tag{2.7}$$

the map $\overline{p} \mapsto \frac{p_A+p_B}{2}$ is a strict contraction, hence it has a unique fixed point. This will provide the unique shape of the LOB.

For sake of clarity, we first study the case where the maximum price accepted by external buyers and the minimum price accepted by external sellers are

$$p_{max} \;=\; (1+\delta_1)\overline{p}, \qquad\qquad p_{min} \;=\; (1-\delta_2)\overline{p}, \tag{2.8}$$

respectively, for some $\delta_1, \delta_2 > 0$. Afterwards, we shall consider the general case where these prices are random.

(I) - Computing the "sell" portion of the LOB.

In the case of a buy order with random size X, let

$$\text{Prob.}\big\{X > s\big\} \;=\; \Psi(s), \qquad\qquad s \geq 0, \tag{2.9}$$

be the distribution of this random variable, and assume

(A1) *The map $s \mapsto \Psi(s)$ is continuously differentiable and satisfies*

$$\Psi(0) \;=\; 1, \qquad \Psi(+\infty) \;=\; 0, \qquad\qquad \Psi'(s) \;<\; 0 \quad \textit{for all} \;\; s \in [0, +\infty[\,. \tag{2.10}$$

For $p > p_0$ we call

$$U(p) \;\doteq\; \int_{p_0}^{p} \phi(s)\,ds \;=\; [\text{amount of stock offered for sale at price} \leq p]. \tag{2.11}$$

By (2.9), the probability that a stock offered at price $p \leq (1+\delta_1)\overline{p}$ will be sold is

$$Prob.\big\{X > U(p)\big\} \;=\; \Psi(U(p)). \tag{2.12}$$

The assumption that the LOB represent an equilibrium implies that the expected profit from a unit amount of stock put on sale is a constant. On the support of U' (i.e. on the set of prices at which some stock is offered for sale), by (2.5) and (2.12) it follows

$$\Psi(U(p))\cdot(p-p_0) \;=\; C, \tag{2.13}$$

for some constant C independent of p. Differentiating (2.13) w.r.t. p we obtain an ODE for U, namely

$$U'(p) \;=\; -\,\frac{\Psi(U(p))}{\Psi'(U(p))}\cdot\frac{1}{p-p_0}\,. \tag{2.14}$$

Observe that by (A1) we have $\Psi' < 0$. hence the right hand side of (2.14) is non-negative for $p > p_0$.

As in (2.6), let $\bar{x}$ be the total amount of stock offered for sale in the LOB. Then the ODE (3.8) should be solved with the terminal condition

$$U\big((1+\delta_1)\overline{p})\big) \;=\; \bar{x}\,. \tag{2.15}$$

Since

$$p_A \;=\; \inf\Big\{p > p_0\;;\;\; U(p) > 0\Big\} \tag{2.16}$$

and, according to (2.11), the function U is absolutely continuous, we have $U(p_A) = 0$ and $\Psi(U(p_A)) = 1$. Hence, the constant C in (2.13) can be computed equivalently as

$$C \;=\; (p_A - p_0) \;=\; \Psi(\bar{x})\cdot\big((1+\delta_1)\overline{p} - p_0\big). \tag{2.17}$$

This yields

$$p_A \;=\; (1+\delta_1)\Psi(\bar{x})\overline{p} + \big(1-\Psi(\bar{x})\big)p_0, \tag{2.18}$$

$$\frac{d}{d\overline{p}}\,p_A \;=\; (1+\delta_1)\Psi(\bar{x}). \tag{2.19}$$

(II) - Computing the "buy" portion of the LOB.

Next, consider the "buy" portion of the LOB. In the case of an external sell order of random size Y, let the distribution of this random variable be

$$\text{Prob.}\big\{Y > s\big\} \;=\; \Phi(s) \qquad\qquad s \geq 0\,. \tag{2.20}$$

We assume that the map $s \mapsto \Phi(s)$ satisfies the same conditions as in **(A1)**. Given a mean bid-ask price $\overline{p}$, the external agent will agree to the transaction only as long as the price ranges within an interval $[(1-\delta_2)\overline{p},\,\overline{p}]$.

In analogy with (2.11), for $p < p_0$ we call

$$U(p) \;\doteq\; \int_p^{p_0} \phi(s)\,ds \;=\; [\text{amount of stock that agents bid to buy at price} \geq p]. \tag{2.21}$$

The expected profit from a unit amount of cash, bidding at price p, is

$$Prob.\;\{Y > U(p)\}\cdot\Big(\frac{p_0}{p} - 1\Big). \tag{2.22}$$

Since the expected profit in (2.22) is constant over the support of U', we have

$$\Phi(U(p))\cdot\Big(\frac{p_0}{p} - 1\Big) \;=\; C, \tag{2.23}$$

for some constant C. Differentiating (2.23) w.r.t. p we obtain an ODE for U, namely

$$U'(p) \;=\; \frac{\Phi(U(p))}{\Phi'(U(p))}\cdot\frac{p_0}{p(p_0 - p)}\,. \tag{2.24}$$

Notice that here the right hand side is negative, because $p < p_0$ while $\Phi' < 0 < \Phi$. This is consistent with the definition (2.21).

Calling $\bar{y}$ the total amount of stock for which agents post buying bids, the above ODE must be solved with the boundary condition

$$U\big((1-\delta_2)\overline{p}\big) \;=\; \bar{y}. \tag{2.25}$$

We have

$$p_B \;=\; \sup\Big\{p < p_0 \;;\; U(p) > 0\Big\}. \tag{2.26}$$

Then the constant C in (2.23) can be equivalently computed as

$$C \;=\; \frac{p_0}{p_B} - 1 \;=\; \Phi(\bar{y}) \cdot \left(\frac{p_0}{(1-\delta_2)\overline{p}} - 1\right). \tag{2.27}$$

This yields

$$p_B \;=\; \left(\frac{\Phi(\bar{y})}{(1-\delta_2)\overline{p}} + \frac{1-\Phi(\bar{y})}{p_0}\right)^{-1}. \tag{2.28}$$

Hence

$$\frac{d}{d\overline{p}}\, p_B \;=\; \frac{p_B^2}{(1-\delta_2)\overline{p}^2}\,\Phi(\bar{y})\,. \tag{2.29}$$

The previous analysis leads to

Theorem 1. *Assume that the random sizes X, Y of an external "buy" and a "sell" order have distributions given by (2.9), (2.20), respectively, and satisfy the assumptions* **(A1)**. *Moreover, assume that the external agent will agree to the transaction if the price is $\leq (1+\delta_1)\overline{p}$ in case of a buyer, and $\geq (1-\delta_2)\overline{p}$ in case of a seller, where $\overline{p}$ is the mean bid-ask price.*

Let $\bar{x}$, $\bar{y}$ be the total amount of stock for which selling bids and buying bids are posted on the LOB, respectively. Assume that

$$(1+\delta_1)\Psi(\bar{x}) + \frac{(1+\delta_2)^2}{1-\delta_2}\Phi(\bar{y}) \;<\; 2\,. \tag{2.30}$$

Then there exists a unique two-sided LOB satisfying (2.14) and (2.24).

Proof. 1. By the previous analysis, both sides of the LOB are uniquely determined as soon as the mean bid-ask price $\overline{p}$ is given, specifying that no sell order (resp. buy order) is posted in the LOB when $(1+\delta_1)\overline{p} < p_0$ (resp. $p_0 < (1-\delta_2)\overline{p}$). Recalling (2.18), (2.28), we set

$$p_A \;=\; \begin{cases} p_0 & \text{if } \ \overline{p} \leq \frac{p_0}{1+\delta_1} \\ (1+\delta_1)\Psi(\bar{x})\overline{p} + \big(1-\Psi(\bar{x})\big)p_0 & \text{if } \ \overline{p} > \frac{p_0}{1+\delta_1} \end{cases}$$

and

$$p_B \;=\; \begin{cases} \left(\frac{\Phi(\bar{y})}{(1-\delta_2)\overline{p}} + \frac{1-\Phi(\bar{y})}{p_0}\right)^{-1} & \text{if } \ \overline{p} < \frac{p_0}{1-\delta_2} \\ p_0 & \text{if } \ \overline{p} \geq \frac{p_0}{1-\delta_2}\,. \end{cases}$$

The theorem can thus be proved by showing that the continuous map

$$\overline{p} \;\mapsto\; \frac{1}{2}p_A + \frac{1}{2}p_B \;\doteq\; F(\overline{p}) \tag{2.31}$$

has a unique fixed point.

2. We claim that the function F in (2.31) maps the interval

$$I \doteq \left[\frac{p_0}{1+\delta_2}\,,\ \frac{p_0}{1-\delta_1}\right]$$

into itself. Indeed, we have

$$(1+\delta_1)\Psi(\bar x)\overline{p} + \big(1-\Psi(\bar x)\big)p_0 \;=\; p_0 + \Big[(1+\delta_1)\overline{p} - p_0\Big]\psi(\bar x) \;\in\; \Big[p_0, (1+\delta_1)\overline{p}\Big] \tag{2.32}$$

if $(1+\delta_1)\overline{p} \geq p_0$ and

$$\left(\frac{\Phi(\bar y)}{(1-\delta_2)\overline{p}} + \frac{1-\Phi(\bar y)}{p_0}\right)^{-1} \;=\; \frac{(1-\delta_2)\overline{p}p_0}{(1-\delta_2)\overline{p} + \Big[p_0 - (1-\delta_2)\overline{p}\Big]\Phi(\bar y)} \;\in\; \Big[(1-\delta_2)\overline{p}, p_0\Big] \tag{2.33}$$

if $p_0 \geq (1-\delta_2)\overline{p}$. Hence, combining (2.31), (2.32) and (2.33), we obtain for $\overline{p} \in I$

$$\frac{p_0}{1+\delta_2} \;\leq\; \frac{1}{2}\Big[p_0 + (1-\delta_2)\overline{p}\Big] \;\leq\; F(\overline{p}) \;\leq\; p_0 \;\leq\; \frac{p_0}{1-\delta_1} \quad \text{if } \overline{p} \leq \frac{p_0}{1+\delta_1}\,,$$

$$\frac{p_0}{1+\delta_2} \;\leq\; \frac{1}{2}\Big[p_0 + (1-\delta_2)\overline{p}\Big] \;\leq\; F(\overline{p}) \;\leq\; \frac{1}{2}\Big[(1+\delta_1)\overline{p} + p_0\Big] \;\leq\; \frac{p_0}{1-\delta_1} \quad \text{if } \frac{p_0}{1+\delta_1} < \overline{p} < \frac{p_0}{1-\delta_2}\,,$$

$$\frac{p_0}{1+\delta_2} \;\leq\; p_0 \;\leq\; F(\overline{p}) \;\leq\; \frac{1}{2}\Big[(1+\delta_1)\overline{p} + p_0\Big] \;\leq\; \frac{p_0}{1-\delta_1} \quad \text{if } \frac{p_0}{1-\delta_2} \leq \overline{p}\,.$$

By continuity, F has a fixed point.

3. Differentiating (2.31) w.r.t. $\overline{p}$ and using (2.19), (2.29), and (2.30), one obtains

$$\frac{d}{d\overline{p}}F(\overline{p}) \;\leq\; \frac{1}{2}\left((1+\delta_1)\Psi(\bar x) + \frac{p_B^2}{(1-\delta_2)\overline{p}^2}\Phi(\bar y)\right) \;\leq\; \frac{1}{2}\left((1+\delta_1)\Psi(\bar x) + \frac{(1+\delta_2)^2}{1-\delta_2}\Phi(\bar y)\right) \;<\; 1\,.$$

This proves that F is a strict contraction, having a unique fixed point $\overline{p} = \frac{p_A+p_B}{2}$. □

Remark 1. In the above theorem, $\Psi(\bar x)$ is the probability that an external buy order is so large that the entire "sell" portion of the LOB is wiped out, while $\Phi(\bar y)$ is the probability that the external sell order is so large that the entire "buy" portion of the LOB is wiped out. In essence, the assumption (2.30) requires that the sizes $\bar x, \bar y$ of the LOB are large enough, compared with the random sizes of external orders.

Example 1. In the case where the random incoming orders X, Y have exponential distribution, say $\Psi(s) = e^{-\gamma s}$, $\Phi(s) = e^{-\beta s}$, the equations determining the shape of the LOB take a particularly simple form. Indeed, the ODE (2.14) determining the "sell" part of the LOB becomes

$$U'(p) \;=\; \frac{1}{\gamma}\cdot\frac{1}{p-p_0}\,. \tag{2.34}$$

On the other hand, the ODE (2.24) determining the "buy" part of the LOB becomes

$$U'(p) \;=\; -\frac{1}{\beta}\cdot\frac{p_0}{p(p_0-p)}\,. \tag{2.35}$$

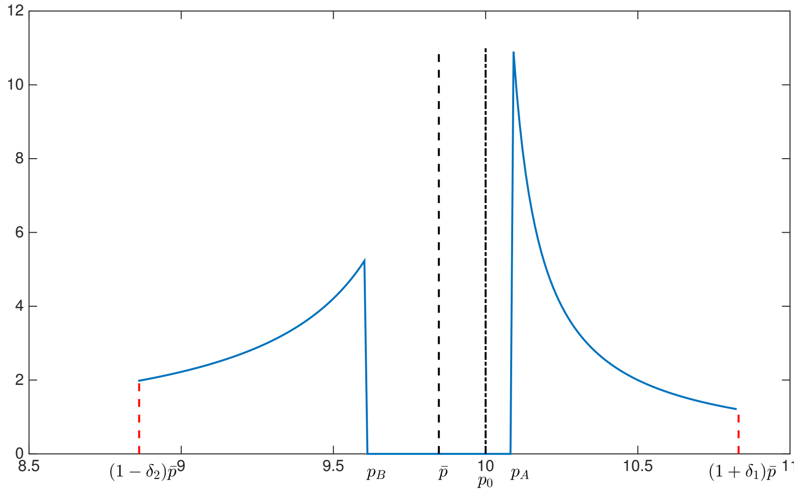

Figure 2: A plot of the density function ϕ, with data as in (2.39). In this case, solving (2.36)–(2.38) we find $p_A = 10.0831$, $p_B = 9.6097$, $\bar{p} = 9.8464$.

Let $\bar{x}$, $\bar{y}$ be the total amounts of stock on the "sell" and "buy" portions of the LOB, and let δ_1, δ_2 be given, as in (2.8).

The density function ϕ in (2.1), describing the two sides of the LOB, is here determined by

$$\phi(s) \;=\; \begin{cases} \dfrac{1}{\gamma(s-p_0)} & \text{if} \quad s\in\big[p_A\,,\ (1+\delta_1)\bar{p}\big], \\[2mm] \dfrac{p_0}{\beta s(p_0-s)} & \text{if} \quad s\in\big[(1-\delta_2)\bar{p}\,,\ p_B\big], \\[2mm] 0 & \text{otherwise.} \end{cases}$$

The constants $\bar{p}, p_A, p_B$ are implicitly determined by the three equations

$$\bar{x} \;=\; \int_{p_A}^{(1+\delta_1)\bar{p}} \phi(s)\,ds \;=\; \frac{1}{\gamma}\ln\frac{(1+\delta_1)\bar{p}-p_0}{p_A-p_0}\,, \tag{2.36}$$

$$\bar{y} \;=\; \int_{(1-\delta_2)\bar{p}}^{p_B} \phi(s)\,ds \;=\; \frac{1}{\beta}\ln\frac{\big(p_0-(1-\delta_2)\bar{p}\big)p_B}{(1-\delta_2)\bar{p}(p_0-p_B)}\,, \tag{2.37}$$

$$\bar{p} \;=\; \frac{p_A+p_B}{2}\,. \tag{2.38}$$

Figures 2 and 3 show the density ϕ and the integral functions U in (2.11) and (2.21), in the case where

$$\delta_1 = \delta_2 = \frac{1}{10}, \qquad p_0 = 10, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \bar{x} = \bar{y} = \ln 10. \tag{2.39}$$

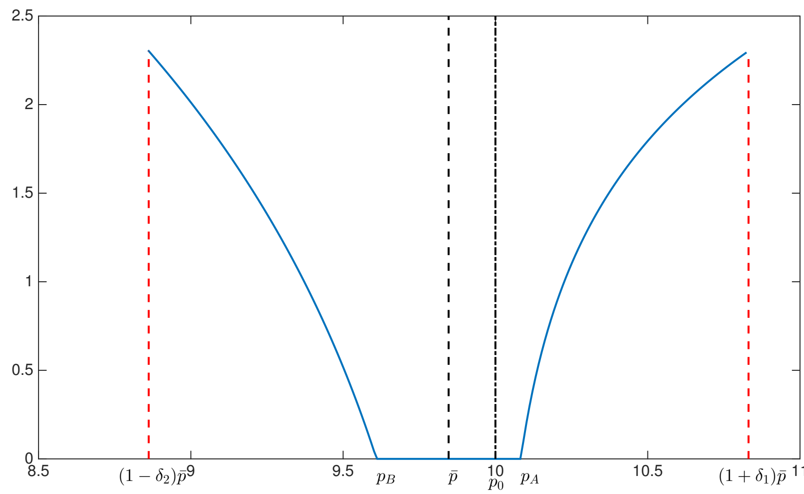

Figure 3: A plot of the functions $U(p)$ in (2.11) and (2.21), with data as in (2.39).

3 The two-sided LOB with random acceptable prices

We now consider the more general case where the maximum price $Q^b \cdot \bar p$ acceptable to a buyer and the minimum price $Q^s \cdot \bar p$ acceptable to a seller are random variables.

For example, one could let Q^b be a random variable such that

$$\text{Prob.}\{Q^b > s\} \;=\; h(s) \qquad\qquad s \geq 0. \tag{3.1}$$

Here $h(\cdot)$ is a continuous map, twice continuously differentiable on the open interval $s \in \,]1,\, 1+\delta_1[$ for some $\delta_1 \in]0,1[$, which satisfies

$$h(s) \;=\; 1 \quad s \in [0,1], \qquad h(s) \;=\; 0 \quad s \geq 1+\delta_1, \qquad h'(s) \;<\; 0 \quad s \in]1,\, 1+\delta_1[, \tag{3.2}$$

$$(\ln h(s))'' \leq 0 \qquad \text{for all } s \in]1,\, 1+\delta_1[\,. \tag{3.3}$$

A natural choice in (3.1) is

$$h(s) \;=\; \begin{cases} 1 & \text{if} \quad s \in [0,1], \\ 1 - \dfrac{s-1}{\delta_1} & \text{if} \quad s \in [1,\, 1+\delta_1], \\ 0 & \text{if} \quad s > 1+\delta_1\,. \end{cases} \tag{3.4}$$

In the following, we always assume that, after the external order has been executed, the payoff of any agent holding an amount c in cash and s in stock is given by (2.5).

(I) The "sell" portion of the LOB, with random acceptable prices.

As in (2.11), let $U(p)$ be the total amount of stock offered for sale at price $\leq p$. Assume that the maximum price accepted by an external buyer is $Q^b\,\bar p$, where Q^b is the random variable in (3.1). Moreover, assume that

$$\bar p \;\leq\; \Big(1 + \frac{1}{\gamma - 1}\Big) p_0\,, \tag{3.5}$$

where $\gamma > 1$ is defined by

$$\gamma \doteq \max\left\{\frac{1}{\delta_1}, \quad -h'(1+)\right\}. \tag{3.6}$$

The expected payoff for a seller asking a price p is

$$\Psi(U(p)) \cdot h\left(\frac{p}{\overline{p}}\right) \cdot (p - p_0) = C. \tag{3.7}$$

The assumption that the LOB represents an equilibrium implies that C is a constant independent of p. Differentiating (3.7) we thus obtain

$$U'(p) = -\frac{\Psi(U(p))}{\Psi'(U(p))} \cdot \left(\frac{1}{p - p_0} + \frac{1}{\overline{p}} \cdot \frac{h'(p/\overline{p})}{h(p/\overline{p})}\right). \tag{3.8}$$

Throughout the following we use the notation

$$a \vee b \doteq \max\{a, b\}, \qquad a \wedge b \doteq \min\{a, b\}.$$

Let $\overline{p}$ be given. For any $p \in \left]p_0 \vee \overline{p}\,,\, (1+\delta_1)\overline{p}\right[$, we define

$$\Lambda(p) \doteq \frac{1}{p - p_0} + \frac{1}{\overline{p}} \cdot \frac{h'(p/\overline{p})}{h(p/\overline{p})}. \tag{3.9}$$

Observe that $\Lambda(p) = \frac{1}{p - p_0}$ when $p < \overline{p}$. By (3.3), the map $p \mapsto \Lambda(p)$ is strictly decreasing. If $\overline{p} > p_0$, by (3.5) and (3.6) we have

$$\Lambda(\overline{p}+) = \frac{1}{\overline{p} - p_0} + \frac{1}{\overline{p}} \cdot h'(1+) \geq \frac{1}{\overline{p} - p_0} - \frac{1}{\overline{p}} \cdot \gamma = \frac{p_0\gamma - (\gamma - 1)\overline{p}}{\overline{p}(\overline{p} - p_0)} \geq 0.$$

If $\overline{p} \leq p_0$, then $\Lambda(p_0+) = +\infty$. Moreover, by (3.2)-(3.3) and Gronwall's inequality it follows

$$\lim_{s \to (1+\delta_1)-} \frac{h'(s)}{h(s)} = -\infty.$$

Hence $\Lambda(p) \to -\infty$ as $p \to (1 + \delta_1)\overline{p}-$.

By continuity and monotonicity, there exists a unique $p^\sharp \in \left[p_0 \vee \overline{p}\,,\, (1+\delta_1)\overline{p}\right[$ such that $\Lambda(p^\sharp) = 0$. It satisfies

$$\Lambda(p) > 0 \iff p \in \left]p_0, p^\sharp\right[. \tag{3.10}$$

By the definition (2.11), the derivative of U must be positive. By (2.10), (3.8) and (3.10), no sell order can be posted at a price $p > p^\sharp$.

The ODE (3.8) should be solved with terminal condition

$$U(p^\sharp) = \bar{x}, \tag{3.11}$$

where $\bar{x}$ is the total amount of stocks offered for sale on the LOB. Call

$$p_A = \inf\left\{p \in \left]p_0\,,\, p^\sharp\right[;\;\; U(p) > 0\right\} \tag{3.12}$$

the minimum ask price. This implies $U(p_A) = 0$ and hence $\Psi(U(p_A)) = 1$. The constant C in (3.7) can be computed by taking $p = p_A$, so that

$$C = h\left(\frac{p_A}{\overline{p}}\right)(p_A - p_0). \tag{3.13}$$

Lemma 1. *Assume that, in addition to (3.2)-(3.3), the function h satisfies*

$$[(\ln h)']^2(s) + (\ln h)'(s) + (s-1)\cdot(\ln h)''(s) \;\geq\; 0\,, \tag{3.14}$$

for all $s \in]1,\; 1+\delta_1[$. *Then* $0 < \frac{d}{d\overline{p}}\, p_A < 1$.

Proof. For any fixed $\overline{p}$, we have

$$-\ln\Psi(\bar{x}) \;=\; \int_{p_A}^{p^\sharp} -\frac{\Psi'}{\Psi}(U(p))\cdot U'(p)\,dp \;=\; \int_{p_A}^{p^\sharp} \Lambda(p)\,dp\,. \tag{3.15}$$

Differentiating (3.15) w.r.t. $\overline{p}$ and recalling that $\Lambda(p^\sharp)=0$, we obtain

$$0 \;=\; \int_{p_A}^{p^\sharp} \frac{\partial}{\partial\overline{p}}\Lambda(p)\,dp - \frac{d}{d\overline{p}}\,p_A\cdot\Lambda(p_A)\,. \tag{3.16}$$

The assumptions (3.2), (3.3) and the identity (3.16) imply that $0 < \frac{d}{d\overline{p}}\,p_A$. Moreover

$$\frac{d}{d\overline{p}}\,p_A \;=\; \Big[\int_{p_A}^{p^\sharp}\frac{\partial}{\partial\overline{p}}\Lambda(p)\,dp\Big]\Big/\Lambda(p_A) \;\leq\; \Big[\int_{p_A}^{p^\sharp}\frac{\partial}{\partial\overline{p}}\Lambda(p)\,dp\Big]\Big/\Lambda(p_A\vee\overline{p}) \tag{3.17}$$

$$=\; \Big[\int_{p_A\vee\overline{p}}^{p^\sharp}\frac{\partial}{\partial\overline{p}}\Lambda(p)\,dp\Big]\Big/\Big[\int_{p_A\vee\overline{p}}^{p^\sharp}-\frac{\partial}{\partial p}\Lambda(p)\,dp\Big]\,. \tag{3.18}$$

Fix $p \in]p_A\vee\overline{p}, p^\sharp[$. Since

$$0 \;<\; \Lambda(p) \;=\; \frac{1}{p-p_0} + \frac{1}{\overline{p}}\cdot\frac{h'}{h}\Big(\frac{p}{\overline{p}}\Big)\,,$$

one has

$$\frac{\partial}{\partial p}\,\Lambda(p) \;=\; -\,\frac{1}{(p-p_0)^2} + \frac{1}{\overline{p}^2}\cdot\Big(\frac{h'}{h}\Big)'\Big(\frac{p}{\overline{p}}\Big) \;<\; \frac{1}{\overline{p}^2}\cdot\Big[-\Big(\frac{h'}{h}\Big)^2\Big(\frac{p}{\overline{p}}\Big) + \Big(\frac{h'}{h}\Big)'\Big(\frac{p}{\overline{p}}\Big)\Big]\,. \tag{3.19}$$

By (3.19) and the assumption (3.14) we obtain

$$\frac{\partial}{\partial p}\,\Lambda(p) \;<\; \frac{1}{\overline{p}^2}\cdot\Big[\frac{h'}{h}\Big(\frac{p}{\overline{p}}\Big) + \frac{p}{\overline{p}}\cdot\Big(\frac{h'}{h}\Big)'\Big(\frac{p}{\overline{p}}\Big)\Big] \;=\; -\,\frac{\partial}{\partial\overline{p}}\,\Lambda(p) \tag{3.20}$$

for every $p \in]p_A, p^\sharp[$. Therefore, by (3.17) and (3.20) we have $\frac{d}{d\overline{p}}\,p_A < 1$. □

Example 2. In the case where the random variable Q^b is uniformly distributed over the interval $[1, 1+\delta_1]$, i.e. the map h is given by (3.4), the condition (3.14) is satisfied whenever $\delta_1 \leq 1$. Indeed, one can compute

$$p^\sharp \;=\; \frac{1}{2}\Big(p_0 + (1+\delta_1)\overline{p}\Big)\,.$$

However, notice that we cannot have $\frac{d}{d\overline{p}}\,p_A < 1$ if $\delta_1 > 1$ and the total amount $\bar{x}$ of stock put on sale on the LOB is very small.

On the other hand, for any $0 < \delta_1 \leq \mu$, all the assumptions in Lemma 1 are satisfied by taking

$$h(s) \;=\; \begin{cases} 1 & \text{if} \quad s \in [0,1], \\ \left(\dfrac{1+\delta_1 - s}{\delta_1}\right)^{\mu} & \text{if} \quad s \in [1, 1+\delta_1], \\ 0 & \text{if} \quad s > 1+\delta_1\,. \end{cases} \tag{3.21}$$

(II) The "buy" portion of the LOB, with random acceptable prices.

Given a mean bid-ask price $\overline{p}$, we assume that the external agent will agree to the transaction only as long as the price ranges within an interval $[Q^s\,\overline{p},\,\overline{p}]$, where $0 < Q^s < 1$ is an independent random variable. Let

$$\text{Prob.}\big\{Q^s < s\big\} \;=\; g(s), \qquad\qquad s \geq 0, \tag{3.22}$$

and assume that the map $g(\cdot)$ is continuous, $\mathcal{C}^2$ on some interval $]1-\delta_2, 1[$, with $0 < \delta_2 < 1/3$, and satisfies

$$g(s) \;=\; 0 \quad s \in [0, 1-\delta_2], \qquad g(s) \;=\; 1 \quad s \geq 1, \qquad g'(s) \;>\; 0 \quad s \in\,]1-\delta_2,\, 1[\,, \tag{3.23}$$

$$(\ln g(s))'' \leq 0 \qquad \text{for all } s \in\,]1-\delta_2, 1[\,. \tag{3.24}$$

Furthermore, we assume that $\overline{p}$ is such that

$$\overline{p} \;\geq\; p_0\Big(1 - \frac{1}{\sigma}\Big)\,, \tag{3.25}$$

where $\sigma > 2$ is defined by

$$\sigma \;\doteq\; \max\left\{\frac{2(1-\delta_2)}{1-2\delta_2}\,,\; g'(1-)\right\}. \tag{3.26}$$

In particular, we have

$$(1-\delta_2)\,\overline{p} \;\geq\; \frac{p_0}{2}\,. \tag{3.27}$$

As in (2.21), we denote by $U(p)$ the total amount of stock that agents are offering to buy at price $> p$.

The expected profit from a unit amount of cash bidding at a price p is

$$\textit{Prob.}\ \{X > U(p)\} \cdot \textit{Prob.}\ \Big\{p > Q^s\,\overline{p}\Big\} \cdot \Big(\frac{p_0}{p} - 1\Big). \tag{3.28}$$

Since the expected profit in (3.28) is constant over the support of U', we have

$$\Phi(U(p)) \cdot g\Big(\frac{p}{\overline{p}}\Big) \cdot \Big(\frac{p_0}{p} - 1\Big) \;=\; C, \tag{3.29}$$

for some constant C. Differentiating (3.29) w.r.t. p we obtain an ODE for U, namely

$$U'(p) \;=\; \frac{\Phi(U(p))}{\Phi'(U(p))} \cdot \left(\frac{p_0}{p(p_0-p)} - \frac{1}{\overline{p}} \cdot \frac{g'(p/\overline{p})}{g(p/\overline{p})}\right). \tag{3.30}$$

For every $p \in \left]\,(1-\delta_2)\overline{p}\,,\ p_0 \wedge \overline{p}\,\right[\setminus \{\overline{p}\}$, define

$$\Lambda(p) \;=\; \frac{p_0}{p(p_0-p)} - \frac{1}{\overline{p}} \cdot \frac{g'(p/\overline{p})}{g(p/\overline{p})}\,. \tag{3.31}$$

Observe that, under the assumptions (3.24) and (3.27), the map $p \mapsto \Lambda(p)$ is strictly increasing. If $\overline{p} < p_0$, by (3.25) and (3.26) we have

$$\Lambda(\overline{p}-) \;=\; \frac{p_0}{\overline{p}(p_0-\overline{p})} - \frac{1}{\overline{p}} \cdot g'(1-) \;\geq\; \frac{p_0}{\overline{p}(p_0-\overline{p})} - \frac{1}{\overline{p}} \cdot \sigma \;=\; \frac{p_0(1-\sigma)+\sigma\overline{p}}{\overline{p}(p_0-\overline{p})} \;\geq\; 0\,.$$

If $\overline{p} \geq p_0$, then $\Lambda(p_0-) = +\infty$. Moreover, observe that (3.23)-(3.24) and Gronwall's lemma imply that $\lim_{s\to(1-\delta_2)+} \frac{g'(s)}{g(s)} = +\infty$. Hence $\Lambda(p) \to -\infty$ as $p \to (1-\delta_2)\,\overline{p}+$.

By continuity and monotonicity, there exists a unique $p^\flat \in \left]\,(1-\delta_2)\overline{p},\ p_0 \wedge \overline{p}\,\right]$ such that $\Lambda(p^\flat) = 0$. One has

$$\Lambda(p) \;>\; 0 \qquad \Longleftrightarrow \qquad p \in \left]\,p^\flat, p_0\,\right[\,. \tag{3.32}$$

By the definition (2.21), the derivative of U must be negative. By **(A1)**, (3.30) and (3.32), no buy order can be posted at a price $p < p^\flat$.

The ODE (3.30) must be solved with terminal condition

$$U(p^\flat) \;=\; \bar{y}\,, \tag{3.33}$$

where $\bar{y}$ is the total amount of stocks for which bids are posted in the LOB. Call

$$p_B \;=\; \sup\Big\{p \in \left]\,p^\flat\,,\, p_0\,\right[\,;\ \ U(p) > 0\Big\} \tag{3.34}$$

the maximum bid price. By **(A1)**, we have that $U(p_B) = 0$. In this setting, the expected profit in (3.29) from a unit amount of cash can be computed by taking $p = p_B$, namely

$$C \;=\; g\Big(\frac{p_B}{\overline{p}}\Big) \cdot \Big(\frac{p_0}{p_B} - 1\Big)\,. \tag{3.35}$$

Lemma 2. *Assume that the function g in (3.22) satisfies*

$$\frac{1}{s^2} + (\ln g)''(s) - \frac{1}{4}[(\ln g)']^2(s) \;\leq\; (\ln g)'(s) + s\,(\ln g)''(s) \;<\; 0 \qquad \textit{for all } s \in]1-\delta_2, 1[\,. \tag{3.36}$$

Then $0 < \frac{d}{d\overline{p}}\, p_B < 1$.

Proof. For any $\overline{p}$, we have

$$-\ln \Phi(\bar{y}) \;=\; \int_{p^\flat}^{p_B} \frac{\Phi'}{\Phi}(U(p)) \cdot U'(p)\, dp \;=\; \int_{p^\flat}^{p_B} \Lambda(p)\, dp\,. \tag{3.37}$$

Differentiating (3.37) w.r.t. $\overline{p}$ and recalling that $\Lambda(p^\flat) = 0$, we obtain

$$0 \;=\; \int_{p^\flat}^{p_B} \frac{\partial}{\partial \overline{p}} \Lambda(p)\, dp + \frac{d}{d\overline{p}}\, p_B \cdot \Lambda(p_B)\,. \tag{3.38}$$

The second inequality in (3.36) and (3.38) imply $0 < \frac{d}{d\overline{p}}\, p_B$. Moreover,

$$\frac{d}{d\overline{p}}\, p_B \;=\; \Big[\int_{p^\flat}^{p_B} -\frac{\partial}{\partial \overline{p}}\Lambda(p)\,dp\Big]\Big/\Lambda(p_B) \;\leq\; \Big[\int_{p^\flat}^{p_B} -\frac{\partial}{\partial \overline{p}}\Lambda(p)\,dp\Big]\Big/\Lambda(\overline{p}\wedge p_B) \tag{3.39}$$

$$= \Big[\int_{p^\flat}^{\overline{p}\wedge p_B} -\frac{\partial}{\partial \overline{p}}\Lambda(p)\,dp\Big]\Big/\Big[\int_{p^\flat}^{\overline{p}\wedge p_B} \frac{\partial}{\partial p}\Lambda(p)\,dp\Big]\,. \tag{3.40}$$

Fix $p \in \,]p^\flat\,,\,\overline{p}\wedge p_B[\,$. Since

$$0 \;<\; \Lambda(p) \;=\; \frac{p_0}{p(p_0-p)} - \frac{1}{\overline{p}}\cdot\frac{g'}{g}\Big(\frac{p}{\overline{p}}\Big)\,,$$

and by (3.27), one has

$$\begin{aligned}
\frac{\partial}{\partial p}\Lambda(p) \;&=\; -\frac{1}{p^2} + \frac{1}{(p_0-p)^2} - \frac{1}{\overline{p}^2}\cdot\Big(\frac{g'}{g}\Big)'\Big(\frac{p}{\overline{p}}\Big)\\
&>\; -\frac{1}{p^2} + \frac{1}{p_0^2}\cdot\frac{p^2}{\overline{p}^2}\Big(\frac{g'}{g}\Big)^2\Big(\frac{p}{\overline{p}}\Big) - \frac{1}{\overline{p}^2}\cdot\Big(\frac{g'}{g}\Big)'\Big(\frac{p}{\overline{p}}\Big)\\
&>\; -\frac{1}{p^2} + \frac{1}{4\overline{p}^2}\Big(\frac{g'}{g}\Big)^2\Big(\frac{p}{\overline{p}}\Big) - \frac{1}{\overline{p}^2}\cdot\Big(\frac{g'}{g}\Big)'\Big(\frac{p}{\overline{p}}\Big)\,.
\end{aligned} \tag{3.41}$$

The inequality (3.41) and the assumption (3.36) yield

$$\frac{\partial}{\partial p}\Lambda(p) \;>\; -\frac{1}{\overline{p}^2}\cdot\left[\frac{g'(p/\overline{p})}{g(p/\overline{p})} + \frac{p}{\overline{p}}\cdot\Big(\frac{g'}{g}\Big)'\Big(\frac{p}{\overline{p}}\Big)\right] \;=\; -\frac{\partial}{\partial \overline{p}}\Lambda(p) \tag{3.42}$$

for every $p \in]p^\flat, p_B[$. Therefore, by (3.39) and (3.42) we have $\frac{d}{d\overline{p}}\, p_B < 1$. □

Example 3. Consider a random variable Q^s which is uniformly distributed over the interval $[1-\delta_2, 1]$, so that g is given by

$$g(s) \;=\; \begin{cases} 0 & \text{if} \quad s\in[0, 1-\delta_2],\\ \delta_2^{-1}(s-1+\delta_2) & \text{if} \quad s\in[1-\delta_2, 1],\\ 1 & \text{if} \quad s>1\,. \end{cases} \tag{3.43}$$

Then the condition (3.36) is satisfied.

Based on the previous analysis, we can now prove

Theorem 2. *Assume that the random sizes X, Y of an external "buy" and a "sell" order have distributions given by (2.9), (2.20), respectively, and satisfy the assumptions* **(A1)**. *Moreover, assume that the external agent will agree to the transaction if the price is $\leq Q^b\overline{p}$ in case of a buyer, and $\geq Q^s\overline{p}$ in case of a seller, where $\overline{p}$ is the mean bid-ask price, Q^b is a random variable in (3.1) satisfying (3.2), (3.3), (3.14), and Q^s is a random variable in (3.22) satisfying (3.23), (3.24) and (3.36).*

Then for any given sizes $\bar{x}, \bar{y} > 0$ of the "sell" and of the "buy" portions of the LOB, the mean bid-ask price $\overline{p}$ and the two-sided LOB are uniquely determined.

Proof. 1. For any choice of the mean price $\overline{p}$, the minimum ask price p_A and the maximum bid price p_B are uniquely determined by solving the Cauchy problem (3.8), (3.11), and the Cauchy problem (3.30), (3.33), respectively.

Let γ and σ as in (3.6) and (3.26). Consider the interval

$$I \doteq \Big[(1-\frac{1}{\sigma})p_0\,,\ (1+\frac{1}{\gamma-1})p_0\Big] \tag{3.44}$$

and define the map

$$\bar{p} \mapsto F(\bar{p}) = \frac{p_A+p_B}{2}\,,$$

where p_A and p_B were defined at (3.12) and (3.34), respectively. The proof will be achieved by showing that F maps I into itself and has a unique fixed point.

2. If $-h'(1+)\geq \frac{1}{\delta_1}$, then $\gamma=-h'(1+)$ and

$$F(\overline{p}) = \frac{p_B+p_A}{2} \leq \frac{p_0+p^\sharp}{2}\,. \tag{3.45}$$

As in (3.10), here $p^\sharp=p^\sharp(\overline{p})$ is the unique point where the map $p\mapsto \Lambda(p)$ in (3.9) vanishes. By (3.3) we have

$$0 = \Lambda(p^\sharp) = \frac{1}{p^\sharp-p_0}+\frac{1}{\overline{p}}\cdot\frac{h'(p^\sharp/\overline{p})}{h(p^\sharp/\overline{p})} \leq \frac{1}{p^\sharp-p_0}+\frac{1}{\overline{p}}\cdot h'(1+)\,.$$

This yields

$$p^\sharp \leq p_0+\frac{\overline{p}}{-h'(1+)}\,. \tag{3.46}$$

Combining (3.45) and (3.46), we have

$$F(\overline{p}) \leq p_0+\frac{\overline{p}}{-2h'(1+)} \leq p_0\Big(1+\frac{1}{2(\gamma-1)}\Big)\,.$$

If $-h'(1+)\leq\frac{1}{\delta_1}$, then $\gamma=\frac{1}{\delta_1}$ and

$$F(\overline{p}) = \frac{p_B+p_A}{2} \leq \frac{p_0+(1+\delta_1)\overline{p}}{2} \leq p_0\Big(1+\frac{\delta_1}{2}+\frac{1+\delta_1}{2(\gamma-1)}\Big) = p_0\Big(1+\frac{1}{\gamma-1}\Big)\,. \tag{3.47}$$

3. Next, we prove that $F(\overline{p})\geq p_0\big(1-\frac{1}{\sigma}\big)$ for all $\overline{p}\in I$. If $g'(1-)\geq\frac{2(1-\delta_2)}{1-2\delta_2}$, then $\sigma=g'(1-)$ and

$$F(\overline{p}) = \frac{p_B+p_A}{2} \geq \frac{p^\flat+p_0}{2}\,. \tag{3.48}$$

As in (3.32), let $p^\flat = p^\flat(\overline{p})$ be the point where the function Λ in (3.31) vanishes. Since $\frac{d}{d\overline{p}}p^\flat>0$, it will be sufficient to check that

$$p^\flat \geq p_0\big(1-\frac{2}{\sigma}\big) \tag{3.49}$$

in the case where $\overline{p}<p_0$.

By (3.24), we have

$$0 \;=\; \Lambda(p^\flat) \;=\; \frac{p_0}{p^\flat(p_0-p^\flat)} - \frac{1}{\overline{p}}\cdot\frac{g'(p^\flat/\overline{p})}{g(p^\flat/\overline{p})} \;\leq\; \frac{p_0}{p^\flat(p_0-p^\flat)} - \frac{1}{\overline{p}}\cdot g'(1-)\,. \tag{3.50}$$

Moreover, by (3.50), (3.27) and the definition of $p^\flat$, we obtain

$$p_0 - p^\flat \;\leq\; \frac{p_0}{\sigma}\frac{\overline{p}}{p^\flat} \;<\; \frac{p_0}{\sigma p^\flat}p_0 \;\leq\; \frac{p_0}{\sigma p^\flat}2(1-\delta_2)\overline{p} \;<\; \frac{2p_0}{\sigma}\,. \tag{3.51}$$

Hence (3.49) holds. Combining (3.48) and (3.49) one obtains

$$F(\overline{p}) \;\geq\; p_0\Big(1-\frac{1}{\sigma}\Big)\,.$$

In the remaining case where $g'(1-) < \frac{2(1-\delta_2)}{1-2\delta_2}$, one has $\sigma = \frac{2(1-\delta_2)}{1-2\delta_2}$ and

$$p_B \;>\; (1-\delta_2)\overline{p} \;\geq\; p_0\Big(1-\delta_2-\frac{1-\delta_2}{\sigma}\Big) \;=\; \frac{p_0}{2}\,.$$

Therefore

$$F(\overline{p}) \;=\; \frac{p_B+p_A}{2} \;\geq\; \frac{3}{4}p_0 \;\geq\; \frac{p_0}{2(1-\delta_2)} \;=\; p_0\Big(1-\frac{1}{\sigma}\Big)\,,$$

since $0 < \delta_2 < 1/3$.

4. By the previous two steps, F maps the closed interval I in (3.44) into itself. Hence it has a fixed point. By Lemmas 1 and 2 we have $0 < \frac{d}{d\bar{p}}F(\bar{p}) < 1$. Hence the map F is a strict contraction, with a unique fixed point. □

4 The dynamic model

We now consider a repeated game, including a sequence of N random incoming orders $X_1, \ldots,$ X_N. Assume that the X_i are independent, identically distributed random variables. In addition, at each time t_i, $i = 1, 2, \ldots, N-1$, agents can post on the LOB new sell or buy orders.

If $\overline{p} = \frac{p_A+p_B}{2}$ is the mean bid-ask price, we assume that external buyers and external sellers will agree to the transaction if the price is $\leq (1+\delta_1)\overline{p}$, and $\geq (1-\delta_2)\overline{p}$, respectively.

The state variable. At each time t_i, the state is described by two positive variables: (x_i, y_i), where

- x_i is the total amount of stock in the "sell" portion of the LOB, at time t_i,
- y_i is the total amount of stock in the "buy" portion of the LOB, at time t_i.

The evolution equation. At each time t_i, an external buy order of random size X_i, or a sell order of size Y_i will arrive. After this order is executed, the corresponding part of the LOB shrinks in size, while the other portion remains unchanged. More precisely, using the notation $a_+ \doteq \max\{a, 0\}$, the new sizes are

$$\begin{cases} \tilde{x}_i &= (x_i - X_i)_+ \,, \\ \tilde{y}_i &= y_i \,, \end{cases} \qquad\qquad \begin{cases} \tilde{x}_i &= x_i, \\ \tilde{y}_i &= (y_i - Y_i)_+ \,, \end{cases}$$

in case of a buy order or a sell order, respectively.

To account for the fact that agents can post new sell or buy orders on the LOB (or remove some of the old ones), we consider a transition probability density $f(x, y;\, \tilde{x}, \tilde{y})$. Here

$$\text{Prob.}\Big\{x_{i+1} \le \xi,, \quad y_{i+1} \le \eta \;\Big|\; \tilde{x}_i = \tilde{x}\,, \quad \tilde{y}_i = \tilde{y}\Big\} \;=\; \int_0^\xi \int_0^\eta f(x, y;\, \tilde{x}, \tilde{y})\, dxdy\,. \tag{4.1}$$

If one assumes that limit orders are never removed (unless they are executed), then one has the implication

$$x < \tilde{x} \quad \text{or} \quad y < \tilde{y} \qquad \Longrightarrow \qquad f(x, y;\, \tilde{x}, \tilde{y}) \;=\; 0.$$

Let $P^s \in [0, 1]$ be the probability that at time t_i a "sell" order arrives, and let $P^b = 1 - P^s$ be the probability that at time t_i a "buy" order arrives. Here P^s and P^b are fixed constants. Then the sizes of the "buy" and "sell" portions of the LOB are described by a Markov process.

The value functions. Consider any point (x, y) in state space. For any $i = 1, 2, \ldots, N$, we denote by

$$V_i^C(x, y), \qquad\qquad V_i^S(x, y), \tag{4.2}$$

the maximum expected payoffs that an agent can achieve at the terminal time t_N, provided that at time t_i

- the two portions of the LOB have sizes x, y, and
- the agent owns a unit of cash, or a unit of stock, respectively

We wish to describe the evolution of the LOB, in terms of the following data:

- The random variables X, Y, describing the size of the external (buy or sell) orders.
- The transition probability density $f(\cdot, \cdot;\, x, y)$, describing the new limit orders posted in the LOB.
- The terminal value $\bar{\beta}$ of a unit of stock.

This should be solved by backward induction, computing the value functions V_i^C, V_i^S for $i = N, N-1, \ldots, 2, 1$. The terminal conditions imply that at the final time $t = t_N$ one has

$$V_N^C(x, y) \;\equiv\; 1, \qquad\qquad V_N^S(x, y) \;\equiv\; \bar{\beta}. \tag{4.3}$$

Assume that the value functions V_{i+1}^C, V_{i+1}^S are known. At time t_i, let the "sell" and "buy" portions of the LOB have sizes (x_i, y_i). To compute $V_i^C(x_i, y_i)$ we proceed as follows. First, assume that at time t_i a buying order arrives, of random size X_i. The portion $\widetilde{X}_i = \min\{X_i, x_i\}$ of this order will be executed. The expected payoffs, for an agent holding a unit of stock or a unit of cash at time t_{i+1}, are thus computed as

$$E^{X_i}\left[\int V^S_{i+1}(x,y)\, f\big(x,y;\, (x_i - X_i)_+\,, y_i\big)\, dxdy\right], \tag{4.4}$$

$$E^{X_i}\left[\int V^C_{i+1}(x,y)\, f\big(x,y;\, (x_i - X_i)_+\,, y_i\big)\, dxdy\right]. \tag{4.5}$$

Next, assume that at time t_i a sell order arrives, of random size Y_i. The portion $\widetilde{Y}_i = \min\{Y_i, y_i\}$ of this order will be executed. The expected payoffs, for an agent holding a unit of stock or a unit of cash at time t_{i+1}, are then computed as

$$E^{Y_i}\left[\int V^S_{i+1}(x,y)\, f\big(x,y;\, x_i, (y_i - Y_i)_+\big)\, dxdy\right], \tag{4.6}$$

$$E^{Y_i}\left[\int V^C_{i+1}(x,y)\, f\big(x,y;\, x_i, (y_i - Y_i)_+\big)\, dxdy\right]. \tag{4.7}$$

5 Dynamic evolution of the LOB

Assume that the values of a unit of cash $V^C = V^C_{i+1}(\xi,\eta)$ and the value of a unit of stock $V^S = V^S_{i+1}(\xi,\eta)$ at time $t = t_{i+1}$ are known, depending on the sizes (ξ,η) of the two parts of the LOB at time t_{i+1}. Moreover, let x, y be the sizes of the "sell" and "buy" portions of the LOB at time t_i. We wish to find the shape of the LOB at time t_i.

5.1 The "sell" portion of the LOB.

As in (2.9), let the random variable X describe the size of the incoming "buy" order. Moreover, let $U(p)$ be the amount of stock offered for sale at price $\leq p$, as in (2.11). Then the expected payoff by putting a unit of stock on sale at price p is

$$\begin{aligned} & p\cdot \text{Prob.}\{X > U(p)\}\cdot E\Big[V^C \;\Big|\; X > U(p)\Big] + \text{Prob.}\{X < U(p)\}\cdot E\Big[V^S \;\Big|\; X < U(p)\Big] \\ = \;& p\cdot \int_{U(p)}^{\infty}\left(\iint V^C(\xi,\eta) f(\xi,\eta;\,(x-s)_+, y)\, d\xi d\eta\right)\cdot\big(-\Psi'(s)\big)ds \\ & \quad + \int_0^{U(p)}\left(\iint V^S(\xi,\eta) f(\xi,\eta;\,(x-s)_+, y)\, d\xi d\eta\right)\cdot\big(-\Psi'(s)\big)ds\,. \end{aligned} \tag{5.1}$$

Notice that in the case where $V^C \equiv \alpha$ and $V^S \equiv \beta$ are constant, the quantity in (5.1) reduces to

$$p\,\Psi(U(p))\cdot\alpha + (1 - \Psi(U(p))\cdot\beta\,.$$

Assuming that the LOB is a Nash equilibrium, we deduce that the quantity in (5.1) is constant on the support of U' (i.e., it is constant on the set of all prices at which some stock is actually offered for sale). Differentiating the right hand side of (5.1) w.r.t. p, one obtains

$$
\begin{aligned}
0 \;=\; & \int_{U(p)}^{\infty} \left(\iint V^C(\xi,\eta) f\big(\xi,\eta;\,(x-s)_+,y\big)\,d\xi d\eta\right)\cdot\big(-\Psi'(s)\big)ds \\
& + U'(p)\,\Psi'(U(p))\cdot\iint \Big(pV^C(\xi,\eta)-V^S(\xi,\eta)\Big) f\big(\xi,\eta;\,(x-U(p))_+,y\big)\,d\xi d\eta\,.
\end{aligned}
\tag{5.2}
$$

Notice again that, in the case where $V^C \equiv \alpha$ and $V^S \equiv \beta$, the above equation reduces to

$$
\Psi(U(p)) + U'(p)\Psi'(U(p))\Big(p-\frac{\beta}{\alpha}\Big) \;=\; 0,
$$

which yields (2.14), with $p_0 = \beta/\alpha$.

We regard (5.2) as an ODE for the function $U(p)$, where the right hand side depends on p, x, y and on the functions V^C, V^S. This must be solved with boundary condition

$$
U\big((1+\delta_1)\overline{p}\big) \;=\; x\,. \tag{5.3}
$$

5.2 The "buy" portion of the LOB.

As in (2.20), let Y be the random size of the incoming "sell"order. Moreover, let $U(p)$ be the total amount of stock that agents bid to buy at price $\geq p$, as in (2.21). Then the expected payoff for an agent who offers to buy a unit of stock at price p is

$$
\begin{aligned}
& \text{Prob.}\{Y<U(p)\}\cdot E\Big[V^C\;\Big|\;Y<U(p)\Big] + \frac{1}{p}\,\text{Prob.}\{Y>U(p)\}\cdot E\Big[V^S\;\Big|\;Y>U(p)\Big] \\
= \;& \int_0^{U(p)} \left(\iint V^C(\xi,\eta) f(\xi,\eta;\,x,(y-s)_+)\,d\xi d\eta\right)\cdot\big(-\Phi'(s)\big)ds \\
& + \frac{1}{p}\int_{U(p)}^{+\infty} \left(\iint V^S(\xi,\eta)(\xi,\eta;\,x,(y-s)_+)\,d\xi d\eta\right)\cdot\big(-\Phi'(s)\big)ds\,.
\end{aligned}
\tag{5.4}
$$

Assuming that the LOB is a Nash equilibrium, we deduce that the quantity in (5.4) is constant on the support of U' (i.e., it is constant on the set of all prices at which some agent is bidding to buy the stock). Differentiating the right hand side of (5.4) w.r.t. p, one obtains

$$
\begin{aligned}
0 \;=\; & -\frac{1}{p^2}\int_{U(p)}^{\infty} \left(\iint V^S(\xi,\eta) f(\xi,\eta;\,x,(y-s)_+)\,d\xi d\eta\right)\cdot\big(-\Phi'(s)\big)ds \\
& - U'(p)\,\Phi'(U(p))\cdot\iint \Big(V^C(\xi,\eta)-\frac{1}{p}V^S(\xi,\eta)\Big) f(\xi,\eta;\,x,(y-U(p))_+)\,d\xi d\eta\,.
\end{aligned}
\tag{5.5}
$$

In the special case where $V^C \equiv \alpha$ and $V^S \equiv \beta$, the above equation reduces to

$$
-\frac{\beta}{p^2}\Phi(U(p)) - U'(p)\Phi'(U(p))\left(\alpha-\frac{\beta}{p}\right) \;=\; 0,
$$

which yields (2.24), with $p_0 = \beta/\alpha$.

We regard (5.5) as an ODE for the function $U(p)$, where the right hand side depends on p, x, y, and on the functions V^C, V^S. This must be solved with boundary condition

$$
U((1-\delta_2)\overline{p}) \;=\; y\,. \tag{5.6}
$$

5.3 Existence of the two-sided LOB.

Let the mean bid-ask price $\overline{p}$ be given.

- By solving the Cauchy problem (5.2)-(5.3), we obtain the function $U(p)$ = amount of stock which agents offer for sale at price $\leq p$. Given the total amount x of stock offered for sale, the minimum ask price is then determined by the implicit equation

$$U(p_A) \;=\; 0\,. \tag{5.7}$$

- By solving the Cauchy problem (5.5)-(5.6), we obtain the function $U(p)$ = amount of stock which agents offer to buy at price $\geq p$. Given the total amount y of stock which agents bid to buy, the maximum bid price is then determined by the implicit equation

$$U(p_B) \;=\; 0\,. \tag{5.8}$$

To establish the existence and uniqueness of the two-sided LOB, we need to show that, under suitable assumptions, the map

$$\overline{p} \;\mapsto\; \frac{p_A(\overline{p}) + p_B(\overline{p})}{2} \tag{5.9}$$

is a strict contraction, hence it has a unique fixed point. As in the proof of Theorem 1, the heart of the matter is to estimate the partial derivatives $\partial p_A/\partial\overline{p}$ and $\partial p_B/\partial\overline{p}$.

To fix the ideas, assume we have a priori bounds

$$V^C_{min} \;\leq\; V^C(\xi,\eta) \;\leq\; V^C_{max}\,, \qquad\qquad V^S_{min} \;\leq\; V^S(\xi,\eta) \;\leq\; V^S_{max} \tag{5.10}$$

for all $\xi \geq 0$, $\eta \geq 0$. In connection with (5.2), these imply

$$\begin{aligned} U'(p) \;&=\; \frac{1}{-\Psi'(U(p))}\cdot\frac{\displaystyle\int_{U(p)}^{+\infty}\left(\iint V^C(\xi,\eta) f\big(\xi,\eta;\,(x-s)_+,y\big)\,d\xi d\eta\right)\cdot\big(-\Psi'(s)\big)ds}{\displaystyle\iint \Big(pV^C(\xi,\eta) - V^S(\xi,\eta)\Big) f\big(\xi,\eta;\,(x-U(p))_+,y\big)\,d\xi d\eta} \\ &\geq\; \frac{V^C_{min}}{pV^C_{max} - V^S_{min}}\cdot\frac{\Psi(U(p))}{-\Psi'(U(p))}\,. \end{aligned} \tag{5.11}$$

Notice that the right hand side of (5.11) approaches $+\infty$ as p decreases to V^S_{min}/V^C_{max}.

Introduce the functions

$$\begin{aligned} F(U) \;&\doteq\; \int_U^{+\infty}\left(\iint V^C(\xi,\eta) f\big(\xi,\eta;\,(x-s)_+,y\big)\,d\xi d\eta\right)\cdot\big(-\Psi'(s)\big)ds\,, \\ g_C(U) \;&\doteq\; \iint V^C(\xi,\eta) f\big(\xi,\eta;\,(x-U(p))_+,y\big)\,d\xi d\eta\,, \\ g_S(U) \;&\doteq\; \iint V^S(\xi,\eta) f\big(\xi,\eta;\,(x-U(p))_+,y\big)\,d\xi d\eta. \end{aligned}$$

Then the equation in (5.11) can be written as

$$\frac{dU}{dp} \;=\; \frac{F(U)}{-\Psi'(U)}\cdot\frac{1}{p\,g_C(U) - g_S(U)}\,. \tag{5.12}$$

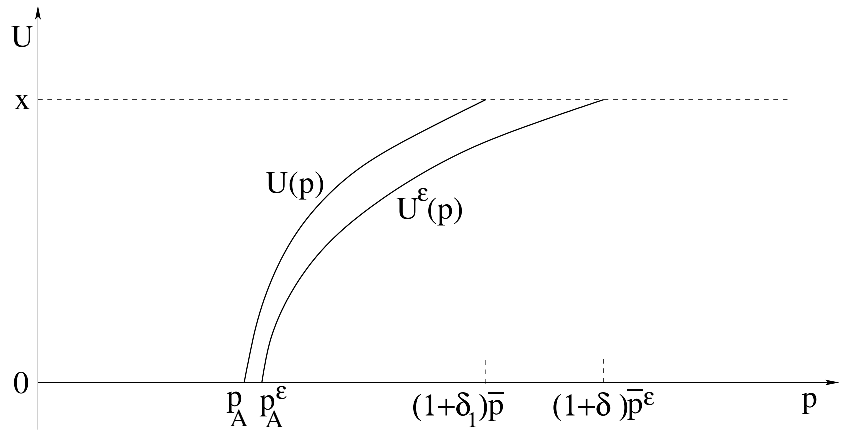

Figure 4: The ask price p_A is found by solving the Cauchy problem (5.11), (5.3), and finding the price at which $U = 0$. To estimate the rate at which p_A changes with the boundary data $\overline{p}$, it is convenient to invert the role of the variables U, p, thus obtaining the linear ODE (5.13) for $p = p(U)$. The figure shows how p_A changes when the value of $\overline{p}$ is increased.

Inverting the role of the two variables, we obtain the linear ODE

$$\frac{dp}{dU} = \frac{-\Psi'(U)}{F(U)} \cdot \big[g_C(U)p - g_S(U)\big]. \tag{5.13}$$

This should be solved with terminal data at $U = x$

$$p(x) = (1+\delta_1)\overline{p}\,. \tag{5.14}$$

The linear Cauchy problem (5.13)-(5.14) can be explicitly solved. Indeed

$$\begin{aligned} p(U) &= C_0 \exp\left\{-\int_U^x \frac{-\Psi'(w)}{F(w)} \cdot g_C(w)dw\right\} \\ &\qquad + \int_U^x \frac{-\Psi'(w)}{F(w)} \cdot g_S(w) \, \exp\left\{-\int_U^w \frac{-\Psi'(\tau)}{F(\tau)} \cdot g_C(\tau)\, d\tau\right\} dw, \end{aligned} \tag{5.15}$$

for some constant C_0. The boundary condition (5.14) yields

$$C_0 = (1+\delta_1)\overline{p}.$$

Differentiating w.r.t. $\overline{p}$ we obtain

$$\frac{\partial}{\partial \overline{p}}\, p(U) = (1+\delta_1)\exp\left\{-\int_U^x \frac{-\Psi'(w)}{F(w)}\; g_C(w)\; dw\right\}. \tag{5.16}$$

Since $p_A = p(0)$, we study the value of p at $U = 0$. Using the priori bounds (5.10) and recalling that $\ln\Psi(0) = 0$, from (5.16) we obtain

$$\begin{aligned} \frac{\partial p_A}{\partial \overline{p}} &= (1+\delta_1)\exp\left\{-\int_0^x \frac{-\Psi'(w)}{F(w)}\; g_C(w)\; dw\right\} \\ &\le (1+\delta_1)\exp\left\{-\int_0^x \frac{-\Psi'(w)}{\int_w^{+\infty} V^C_{max}(-\Psi'(s))ds}\; V^C_{min}\; dw\right\} \\ &= (1+\delta_1)\exp\left\{\frac{V^C_{min}}{V^C_{max}}\int_0^x \frac{\Psi'(w)}{\Psi(w)}\; dw\right\} \\ &= (1+\delta_1)\big(\Psi(x)\big)^{\lambda_C}, \end{aligned} \tag{5.17}$$

with $\lambda_C \doteq V^C_{min}/V^C_{max}$.

A similar analysis applies to the "buy" portion of the LOB. Indeed, (5.5) implies

$$
\begin{aligned}
U'(p) &= \frac{1}{-p^2\Phi'(U(p))}\cdot\frac{\displaystyle\int_{U(p)}^{+\infty}\left(\iint V^S(\xi,\eta)f\big(\xi,\eta;\,x,(y-s)_+\big)\,d\xi d\eta\right)\cdot\big(-\Phi'(s)\big)ds}{\displaystyle\iint\Big(V^C(\xi,\eta)-\frac{1}{p}V^S(\xi,\eta)\Big)f\big(\xi,\eta;\,x,(y-U(p))_+\big)\,d\xi d\eta}\\
&\geq \frac{V^S_{min}}{p^2V^C_{max}-pV^S_{min}}\cdot\frac{\Phi(U(p))}{-\Phi'(U(p))}\,.
\end{aligned}
\tag{5.18}
$$

Notice that the right hand side of (5.12) approaches $+\infty$ as p decreases to V^S_{min}/V^C_{max}.

Introducing the functions

$$
\begin{aligned}
G(U) &\doteq \int_U^{+\infty}\left(\iint V^S(\xi,\eta)f\big(\xi,\eta;\,x,(y-s)_+\big)\,d\xi d\eta\right)\cdot\big(-\Phi'(s)\big)ds\,,\\
\tilde g_C(U) &\doteq \iint V^C(\xi,\eta)f\big(\xi,\eta;\,x,(y-U(p))_+\big)\,d\xi d\eta\,,\\
\tilde g_S(U) &\doteq \iint V^S(\xi,\eta)f\big(\xi,\eta;\,x,(y-U(p))_+\big)\,d\xi d\eta\,,
\end{aligned}
$$

the ODE in (5.18) can be written as

$$
\frac{dU}{dp} = \frac{G(U)}{-\Phi'(U)}\cdot\frac{1}{p^2\,\tilde g_C(U)-p\,\tilde g_S(U)}\,. \tag{5.19}
$$

Inverting the role of the two variables p and U, we obtain a Bernoulli differential equation

$$
\frac{dp}{dU} = \frac{-\Phi'(U)}{G(U)}\cdot\big[p^2\,\tilde g_C(U)-p\,\tilde g_S(U)\big] \tag{5.20}
$$

with terminal data at $U=y$ given by

$$
p(y) = (1-\delta_2)\overline{p}\,. \tag{5.21}
$$

Introducing the new variable $q=1/p$, we obtain the linear Cauchy problem

$$
\frac{dq}{dU} = \frac{-\Phi'(U)}{G(U)}\cdot\big[q\,\tilde g_S(U)-\tilde g_C(U)\big],\qquad\qquad q(y) = \frac{1}{(1-\delta_2)\overline{p}}\,. \tag{5.22}
$$

An explicit computation yields

$$
\begin{aligned}
q(U) &= \frac{1}{(1-\delta_2)\overline{p}}\exp\left\{-\int_U^y\frac{-\Phi'(w)}{G(w)}\cdot\tilde g_S(w)dw\right\}\\
&\quad+\int_U^y\frac{-\Phi'(w)}{G(w)}\cdot\tilde g_C(w)\cdot\exp\left\{-\int_U^w\frac{-\Phi'(\tau)}{G(\tau)}\cdot\tilde g_S(\tau)\,d\tau\right\}dw\,,
\end{aligned}
\tag{5.23}
$$

Differentiating the solution $p(U)$ of (5.20)-(5.21) w.r.t. $\overline{p}$, by (5.23) we now obtain

$$
\frac{\partial}{\partial\overline{p}}p(U) = \frac{\partial p}{\partial q}\cdot\frac{\partial q}{\partial\overline{p}} = \frac{p^2}{(1-\delta_2)\overline{p}^2}\exp\left\{-\int_U^y\frac{-\Phi'(w)}{G(w)}\,\tilde g_S(w)\,dw\right\},
$$

Since $p_B = p(0) \leq \overline{p}$, using the a priori bounds (5.10) one obtains

$$\begin{aligned}
\frac{\partial p_B}{\partial \overline{p}} &= \frac{p_B^2}{(1-\delta_2)\,\overline{p}^2} \exp\left\{-\int_U^0 \frac{-\Phi'(w)}{G(w)}\ \tilde{g}_S(w)\ dw\right\} \\
&\leq \frac{(1+\delta_2)^2}{1-\delta_2} \exp\left\{-\int_U^0 \frac{-\Phi'(w)}{\int_w^{+\infty} V^S_{max}(-\Phi'(s))ds}\ V^S_{min}\ dw\right\} \\
&= \frac{(1+\delta_2)^2}{1-\delta_2} \big(\Phi(y)\big)^{\lambda_S},
\end{aligned} \tag{5.24}$$

with $\lambda_S \doteq V^S_{min}/V^S_{max}$. Combining the two inequalities (5.17) and (5.24), we obtain a sufficient condition for the existence of a unique mean bid-ask price $\overline{p}$.

Theorem 3. *Assume that the value functions V^C, V^S satisfy the a priori bounds (5.10). Moreover, assume that the total amount x of stock offered for sale and the total amount y that agents bid to buy are both large enough, so that*

$$(1+\delta_1)\big(\Psi(x)\big)^{\lambda_C} + \frac{(1+\delta_2)^2}{1-\delta_2}\big(\Phi(y)\big)^{\lambda_S} \ < \ 2\,, \tag{5.25}$$

with $\lambda_C \doteq V^C_{min}/V^C_{max}$, $\lambda_S \doteq V^S_{min}/V^S_{max}$.

Then the two-sided LOB has a unique equilibrium configuration.

Proof. Combining (5.17) and (5.24) with the assumption (5.25) one obtains

$$\frac{d}{d\overline{p}}\left(\frac{p_A+p_B}{2}\right) \leq \frac{1}{2}\ \left[(1+\delta_1)\big(\Psi(x)\big)^{\lambda_C} + \frac{(1+\delta_2)^2}{1-\delta_2}\big(\Phi(y)\big)^{\lambda_S}\right] \ < \ 1\,. \tag{5.26}$$

showing that the map $\overline{p} \mapsto \frac{1}{2}(p_A+p_B)$ is a strict contraction. Hence, a unique fixed point exists.

As soon as this unique mean bid-ask price $\overline{p}$ has been determined, the "sell" and the "buy" portions of the LOB are obtained by solving the Cauchy problems (5.2)-(5.3) and (5.5)-(5.6), respectively. □

Remark 3. In the above setting, $\Psi(x)$ is the probability that the external buy order is so large that it wipes out the entire "sell" portion of the LOB. Similarly, $\Phi(y)$ is the probability that the external sell order is so large that it wipes out the entire "buy" portion of the LOB. The key assumption of the theorem requires that these probabilities are sufficiently small. Notice that, if $V^C(\xi,\eta)$ and $V^S(\xi,\eta)$ are constants, then $\lambda_C = \lambda_S = 1$ and the assumption (5.25) is exactly the same as (2.30) in Theorem 1.

5.4 The inductive computation of the value functions.

If the existence of a unique fixed point $\overline{p}$ is known, the value functions V^C, V^S can then be inductively computed as follows. Let $P_{buy} = P$ be the probability that the external agent is a buyer, so that $P_{sell} = (1-P)$ is the probability that the external agent is a seller.

The assumption that the LOB represents an equilibrium implies that the expected payoff for an agent holding a unit amount of stock (or a unit amount of cash) is independent of the price p he asks (or the price he bids). In particular, we can compute this payoff in the case $p = p_A$ (or $p = p_B$, respectively), where the transaction occurs with probability one.

We thus obtain the inductive relations

$$\begin{aligned} V_i^S(x,y) &= P_{buy} \cdot p_A \cdot E^{X_i}\left[\int V_{i+1}^C(\xi,\eta)\, f\big(\xi,\eta;\, (x-X_i)_+\,, y\big)\, d\xi d\eta\right] \\ &\quad + P_{sell} \cdot E^{Y_i}\left[\int V_{i+1}^S(\xi,\eta)\, f\big(\xi,\eta;\, x, (y-Y_i)_+\big)\, d\xi d\eta\right]. \end{aligned} \tag{5.27}$$

$$\begin{aligned} V_i^C(x,y) &= P_{buy} \cdot E^{X_i}\left[\int V_{i+1}^C(\xi,\eta)\, f\big(\xi,\eta;\, (x-X_i)_+\,, y\big)\, d\xi d\eta\right] \\ &\quad + P_{sell} \cdot \frac{1}{p_B} \cdot E^{Y_i}\left[\int V_{i+1}^S(\xi,\eta)\, f\big(\xi,\eta;\, x, (y-Y_i)_+\big)\, d\xi d\eta\right]. \end{aligned} \tag{5.28}$$

Notice that here p_A, p_B depend on x, y, and also on all values of the functions V^S, V^C.

In order to apply Theorem 3, and construct the value functions V_i^C, V_i^S for all $i = 1, \ldots, N$ by backward induction, we need to provide suitable upper and lower bounds.

Lemma 3. *Let V_i^S, V_i^C, $i = 1, 2, \ldots, N$ be a sequence of value functions satisfying the inductive relations (5.27)-(5.28), with*

$$V_N^C(\xi,\eta) \equiv 1\,, \qquad\qquad V_N^S(\xi,\eta) \equiv \bar\beta\,. \tag{5.29}$$

Then for all $i = 1, \ldots, N$

$$1 \;\le\; V_i^C(\xi,\eta) \;\le\; \overline{V}_i^C\,, \qquad\qquad \bar\beta \;\le\; V_i^S(\xi,\eta) \;\le\; \overline{V}_i^S\,, \tag{5.30}$$

where the upper bounds $\overline{V}_i^C$, $\overline{V}_i^S$ are defined by the following inductive relations:

$$\begin{aligned} \overline{V}_N^C &= 1\,, \qquad \overline{V}_i^C = \left[P_{buy} + P_{sell} \cdot \frac{1+\delta_2}{1-\delta_2} \cdot \frac{\overline{V}_{i+1}^S}{\bar\beta}\right] \cdot \overline{V}_{i+1}^C\,, \quad i = 1, \ldots, N-1\,, \\ \overline{V}_N^S &= \overline{\beta}\,, \qquad \overline{V}_i^S = \left[P_{buy} \cdot \frac{1+\delta_1}{1-\delta_1} \cdot \overline{V}_{i+1}^C + P_{sell}\right] \cdot \overline{V}_{i+1}^S\,, \quad i = 1, \ldots, N-1\,. \end{aligned} \tag{5.31}$$

Proof. By assumption, at the terminal time $i = N$ the value functions are constant and satisfy (5.29).

The proof will be achieved by backward induction. Assuming that V_{i+1}^C, V_{i+1}^S satisfy the bounds

$$1 \;\le\; V_{i+1}^C(\xi,\eta) \;\le\; \overline{V}_{i+1}^C\,, \qquad\qquad \bar\beta \;\le\; V_{i+1}^S(\xi,\eta) \;\le\; \overline{V}_{i+1}^S\,, \tag{5.32}$$

we will show that V_i^C, V_i^S satisfy the inequalities (5.30)-(5.31).

1. Using the functions F, g_C, g_S and G, $\tilde g_C$, $\tilde g_S$, by (5.2) and (5.5) we have

$$0 \;=\; F(U(p)) + U'(p)\Psi'(U(p)) \cdot \big(p \cdot g_C(U(p)) - g_S(U(p))\big)$$

and

$$0 \;=\; -\frac{1}{p^2}G(U(p)) - U'(p)\Phi'(U(p))\cdot\big(\tilde g_C(U(p)) - \frac{1}{p}\tilde g_S(U(p))\big)\,.$$

It follows that

$$p\cdot g_C(U(p)) - g_S(U(p)) \;\geq\; 0\,,\quad \text{for all } p\in[p_A,(1+\delta_1)\overline{p}]$$

and

$$\tilde g_C(U(p)) - \frac{1}{p}\tilde g_S(U(p)) \;\leq\; 0\,,\quad \text{for all } p\in[(1-\delta_2)\overline{p},p_B]\,.$$

In particular,

$$(1+\delta_1)\overline{p}\cdot g_C(x) \;\geq\; g_S(x) \qquad \text{and} \qquad \tilde g_C(y) \;\leq\; \frac{1}{(1-\delta_2)\overline{p}}\cdot\tilde g_S(y)\,. \tag{5.33}$$

Observe that the values in (5.27) and (5.28) can be expressed as

$$V_i^S(x,y) \;=\; P_{buy}\cdot\Big[(1+\delta_1)\overline{p}\int_x^\infty g_C(s)\big(-\Psi'(s)\big)ds + \int_0^x g_S(s)\big(-\Psi'(s)\big)ds\Big] + P_{sell}\cdot G(0). \tag{5.34}$$

$$V_i^C(x,y) \;=\; P_{buy}\cdot F(0) + P_{sell}\cdot\Big[\int_0^y \tilde g_C(s)\big(-\Phi'(s)\big)ds + \frac{1}{(1-\delta_2)\overline{p}}\int_y^\infty \tilde g_S(s)\big(-\Phi'(s)\big)ds\Big]. \tag{5.35}$$

Remarking that $g_{C,S}(s) = g_{C,S}(x)$ for every $s > x$ and $\tilde g_{C,S}(s) = \tilde g_{C,S}(y)$ for every $s > y$, and combining (5.33), (5.34) and (5.35), we obtain

$$V_i^S(x,y) \;\geq\; P_{buy}\cdot\int_0^\infty g_S(s)\big(-\Psi'(s)\big)ds + P_{sell}\cdot G(0) \;\geq\; \overline{\beta}\,,$$

$$V_i^C(x,y) \;\geq\; P_{buy}\cdot F(0) + P_{sell}\cdot\int_0^\infty \tilde g_C(s)\big(-\Phi'(s)\big)ds \;\geq\; 1\,.$$

2. Our modeling assumptions on the maximum and minimum acceptable prices yield

$$p_A \;\leq\; (1+\delta_1)\overline{p}, \qquad (1-\delta_2)\overline{p} \;\leq\; p_B\,. \tag{5.36}$$

From (5.15), (5.32) and (5.36), it follows

$$\begin{aligned} p_A \;=\; p(0) =\;& (1+\delta_1)\overline{p}\cdot\exp\Big\{-\int_0^{\bar x}\frac{-\Psi'(w)}{F(w)}\cdot g_C(w)dw\Big\} \\ &+ \int_0^{\bar x}\frac{-\Psi'(w)}{F(w)}\cdot g_S(w)\,\exp\Big\{-\int_0^w\frac{-\Psi'(\tau)}{F(\tau)}\cdot g_C(\tau)\,d\tau\Big\}\,dw \\ \geq\;& p_A\cdot\exp\Big\{-\int_0^{\bar x}\frac{-\Psi'(w)}{F(w)}\cdot g_C(w)dw\Big\} \\ &+ \frac{\bar\beta}{\bar V_{i+1}^C}\Big(1-\exp\Big\{-\int_0^{\bar x}\frac{-\Psi'(w)}{F(w)}\cdot g_C(w)dw\Big\}\Big)\,. \end{aligned} \tag{5.37}$$

Therefore, we obtain $p_A \geq \dfrac{\bar{\beta}}{\overline{V}^C_{i+1}}$.

Concerning the maximum bid price p_B, from (5.23), (5.32) and (5.36), it follows

$$\begin{aligned}\frac{1}{p_B} \;=\; q(0) \;&=\; \frac{1}{(1-\delta_2)\overline{p}} \exp\left\{-\int_0^{\bar{y}} \frac{-\Phi'(w)}{G(w)} \cdot \tilde{g}_S(w)dw\right\} \\ &\quad + \int_0^{\bar{y}} \frac{-\Phi'(w)}{G(w)} \cdot \tilde{g}_C(w) \cdot \exp\left\{-\int_0^{w} \frac{-\Phi'(\tau)}{G(\tau)} \cdot \tilde{g}_S(\tau)\; d\tau\right\} dw \\ &\geq\; \frac{1}{p_B} \exp\left\{-\int_0^{\bar{y}} \frac{-\Phi'(w)}{G(w)} \cdot \tilde{g}_S(w)dw\right\} \\ &\quad + \frac{1}{\overline{V}^S_{i+1}} \left(1 - \exp\left\{-\int_0^{\bar{y}} \frac{-\Phi'(w)}{G(w)} \cdot \tilde{g}_S(w)dw\right\}\right).\end{aligned} \tag{5.38}$$

Therefore, we have $p_B \leq \overline{V}^S_{i+1}$. It follows

$$\overline{p} \;=\; \frac{p_A + p_B}{2} \;\leq\; \frac{1}{2}(1+\delta_1)\overline{p} + \frac{1}{2}\overline{V}^S_{i+1}\,,$$

so that

$$(1+\delta_1)\overline{p} \;\leq\; \frac{1+\delta_1}{1-\delta_1}\overline{V}^S_{i+1}\,.$$

Analogously, we obtain

$$(1-\delta_2)\overline{p} \;\leq\; \frac{1-\delta_2}{1+\delta_2}\frac{\overline{\beta}}{\overline{V}^C_{i+1}}\,.$$

By (5.34)-(5.35), for any x, y it follows

$$\begin{aligned}V_i^S(x,y) \quad &\leq\; P_{buy} \cdot \left[\frac{1+\delta_1}{1-\delta_1}\overline{V}^S_{i+1}\overline{V}^C_{i+1}\Psi(x) + \overline{V}^S_{i+1}(1-\Psi(x))\right] + P_{sell} \cdot \overline{V}^S_{i+1} \\ &\leq\; \left[P_{buy} \cdot \tfrac{1+\delta_1}{1-\delta_1} \cdot \overline{V}^C_{i+1} + P_{sell}\right] \cdot \overline{V}^S_{i+1}\,,\end{aligned} \tag{5.39}$$

$$\begin{aligned}V_i^C(x,y) \quad &\leq\; P_{buy} \cdot \overline{V}^C_{i+1} + P_{sell} \cdot \left[\overline{V}^C_{i+1}(1-\Phi(y)) + \frac{1+\delta_2}{1-\delta_2}\frac{\overline{V}^C_{i+1}}{\overline{\beta}} \cdot \overline{V}^S_{i+1}\Phi(y)\right] \\ &\leq\; \left[P_{buy} + P_{sell} \cdot \tfrac{1+\delta_2}{1-\delta_2} \cdot \tfrac{\overline{V}^S_{i+1}}{\overline{\beta}}\right] \cdot \overline{V}^C_{i+1}\,.\end{aligned} \tag{5.40}$$

□

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