# Dynamic Blocking Problems for a Model of Fire Propagation

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**Abstract**

This paper contains a survey of recent work on a class of dynamic blocking problems. The basic model consists of a differential inclusion describing the growth of a set in the plane. To restrain its expansion, it is assumed that barriers can be constructed, in real time. Here the issues of major interests are: (i) whether the growth of the set can be eventually blocked, and (ii) what is the optimal location of the barriers, minimizing a cost criterion.

After introducing the basic definitions and concepts, the paper reviews various results on the existence or non-existence of blocking strategies. A theorem on the existence of an optimal strategy is then recalled, together with various necessary conditions for optimality. Sufficient conditions for optimality and a numerical algorithm for the computation of optimal barriers are also discussed, together with several open problems.

## 1 Introduction

Consider a set in the plane, which expands as time increases. Assume that its growth can be restrained by constructing barriers, in real time. In this setting, a natural problem is whether one can completely block the growth of the set, by surrounding it with barriers. In addition, given a cost criterion, it is also of interest to determine the optimal location of these barriers.

Dynamic blocking problems of this kind were first considered in [5], motivated by the optimal control of wild fires. Let $R(t) \subset I\!R^2$ denote the region burned by the fire at time $t$. To restrict its growth, assume that a barrier can be constructed, along a one-dimensional curve. We shall denote by $\gamma(t) \subset I\!R^2$ the portion of this barrier constructed within time $t$. In the case of a forest fire, one may think of a thin strip of land which is either soaked with water poured from an airplane or a helicopter, or cleared from all vegetation using a bulldozer, or sprayed with fire extinguisher by a team of firemen. In any case, this will prevent the fire from crossing that particular strip of land. In connection with this model, it is then natural to ask whether it is possible to stop the fire, and what is the best strategy to achieve this goal, minimizing the total value of the burned region.

Aim of this paper is to survey the main concepts and results in the theory of dynamic blocking problems, developed in [5, 6, 7, 9, 10, 11, 12, 15, 23], and discuss various open questions.

After a precise description of the mathematical model, in Section 2 we introduce an equivalent way to formulate both the blocking problem and the optimization problem. In Section 3 we recall the main results on the existence or non-existence of blocking strategies. The remainder of the paper is concerned with optimal strategies. The basic existence theorem [7] is presented in Section 4. Section 5 reviews the classification of arcs in an optimal barrier and the necessary conditions for optimality proved in [5, 11, 23], while Section 6 describes a recent result concerning sufficient conditions for optimality. A numerical algorithm for computing the optimal barrier, introduced in [9], is here presented in Section 7. Finally, in Section 8 we discuss several remaining open problems.

## 1.1 A model for fire propagation

Several models for fire propagation have been proposed in the literature, see for example [17, 18, 19, 21]. In our basic model, the set $R(t) \subset I\!\!R^2$ burned by the fire up to time $t$ is described as the reachable set for a differential inclusion. More precisely, consider the Cauchy problem

$$\dot{x} \in F(x), \qquad\qquad x(0) \in R_0, \tag{1.1}$$

where the upper dot denotes a derivative w.r.t. time. Here $R_0$ describes the region where the fire is initially burning at time $t = 0$, while $F(x)$ is a set of propagation velocities.

If barriers are not present, for each $t \geq 0$ the set $R(t)$ reached by the fire is defined as

$$R(t) \doteq \Big\{x(t)\,;\;\; x(\cdot) \text{ absolutely continuous}\,,\;\; x(0) \in R_0\,,\;\; \dot{x}(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0,t]\Big\}. \tag{1.2}$$

We shall always assume that the initial set $R_0 \subset I\!\!R^2$ is nonempty and bounded. Moreover, we assume that $F : I\!\!R^2 \mapsto I\!\!R^2$ is a Lipschitz continuous multifunction whose values $F(x)$ are compact, convex sets containing the origin. Clearly, this implies

$$R(t_1) \subseteq R(t_2) \qquad\qquad \text{whenever} \quad t_1 < t_2\,. \tag{1.3}$$

According to (1.2), the propagation speed of a fire front in the normal direction is computed by

$$h(x) = \max_{v \in F(x)} \langle \mathbf{n}(x),\, v\rangle\,. \tag{1.4}$$

Here $x$ is any point along the boundary $\partial R(t)$ of the set burned up to time $t$, while $\mathbf{n}(x)$ denotes the unit outer normal vector to the boundary, at the point $x$. By $\langle \cdot, \cdot \rangle$ we denote the Euclidean inner product in $I\!\!R^2$.

An alternative way to describe this same model of fire propagation relies on the solution of a Hamilton-Jacobi (H-J) equation [17]. For each $x \in I\!\!R^2$, call

$$T(x) \doteq \inf\ \{t \geq 0\,; \quad x \in R(t)\} \tag{1.5}$$

the minimum time taken by the fire to reach the point $x$, starting from the initial set $R_0$. The function $T(\cdot)$ can now be computed by solving the nonlinear PDE

$$H(x, \nabla T(x)) = 0\,, \qquad\qquad H(x,p) \doteq \max_{v \in F(x)} \langle p,\, v\rangle - 1\,, \tag{1.6}$$

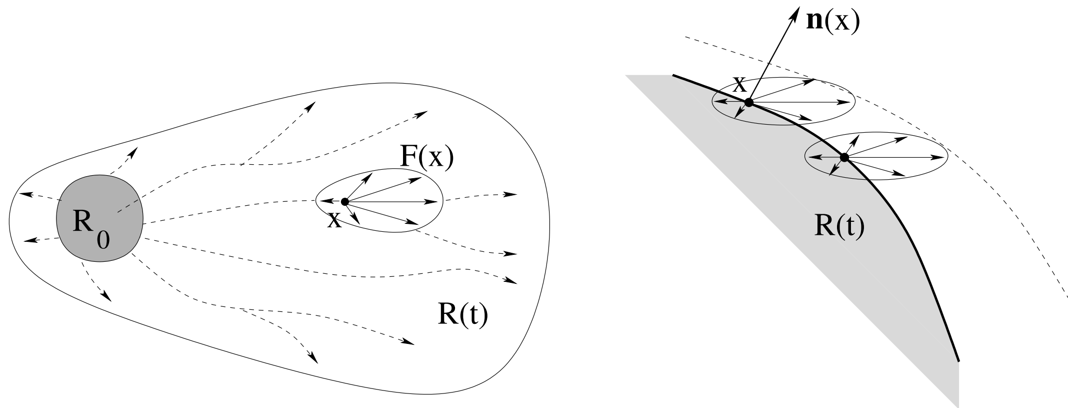


Figure 1: Left: the region $R(t)$ burned by the fire at time $t > 0$ is described as the set reached by trajectories of the differential inclusion (1.1). Right: according to this model, the fire front propagates in the normal direction with speed given by (1.4).

with boundary data

$$T(x) \;=\; 0 \qquad\qquad \text{for} \quad x \in R_0 \,. \tag{1.7}$$

The level set $\{x\,;\;\; T(x) = t\}$ describes the position of the fire front at time $t > 0$. We remark that, in general, the solution of (1.6)-(1.7) may not be smooth. In this case, the H-J equation (1.6) must be suitably interpreted in a viscosity sense [4].

While the representation based on differential inclusions is useful for theoretical analysis, the H-J equation leads to more efficient computational algorithms, based on the level set method [17, 20].

### 1.2 Barriers

We assume that the spreading of the fire can be controlled by constructing barriers, in real time. Intuitively, we think of a barrier as a curve (or a family of curves) in the plane, which the fire cannot cross. Since the wall is constructed in real time, simultaneously with fire propagation, a restriction on its length must be imposed. Calling $\gamma(t)$ the portion of the curve constructed up to time $t$, if $\sigma$ is the speed at which the wall is constructed we thus have the constraint

$$[\text{length of } \;\gamma(t)] \;\leq\; \sigma t \qquad\qquad \text{for every} \;\; t \geq 0 \,. \tag{1.8}$$

A more general situation can be considered. Indeed, the construction of the barrier can be faster in certain places than others. For example, if water is sprayed by an helicopter on top of the fire, this operation can be carried out more quickly in areas close to a lake or a water reservoir. To model this fact, we consider a continuous, strictly positive function $\psi : I\!\!R^2 \mapsto I\!\!R_+$. Calling $\gamma(t) \subset I\!\!R^2$ the portion of the wall constructed within time $t \geq 0$, we make the following assumptions:

**(H1)** For every $0 \leq t_1 \leq t_2$ one has $\gamma(t_1) \subseteq \gamma(t_2)$ .

**(H2)** Each $\gamma(t)$ is a rectifiable curve whose length satisfies

$$\int_{\gamma(t)} \psi \, dm_1 \;\leq\; t \qquad\qquad \text{for every} \;\; t \geq 0\,. \tag{1.9}$$

The first assumption states that, after being constructed a barrier cannot be destroyed, or moved to another place. For a precise mathematical definition of *rectifiable set* we refer to [1]. Roughly speaking, a rectifiable set is the most general type of one-dimensional set for which a concept of *length* can be introduced. In the integral formula (1.9), $m_1$ denotes the one-dimensional Hausdorff measure, normalized so that $m_1(\Gamma)$ yields the usual length of a smooth curve $\Gamma$. Notice that $1/\psi(x)$ is the speed at which the wall can be constructed, at the location $x$. In particular, if $\psi(x) \equiv \sigma^{-1}$ is constant, then the constraint (1.9) reduces to (1.8). In the above model, we assume that the construction speed $1/\psi(x)$ depends only on the spatial location. It would be of interest to consider a model where this speed depends also on time. This more general case remains yet to be studied.

**Remark 1.** In general, the curve $\gamma(t)$ need not be connected. For example, it may be the union of two separate barriers, produced by two teams of firemen working independently at different locations.

A blocking strategy $t \mapsto \gamma(t)$ satisfying (H1)-(H2) will be called an **admissible strategy**. In addition, we say that the strategy $\gamma$ is **complete** if it satisfies

**(H3)** For every $t \geq 0$ there holds

$$\int_{\gamma(t)} \psi \, dm_1 \;=\; t\,, \qquad\qquad \gamma(t) \;=\; \bigcap_{s>t} \gamma(s)\,. \tag{1.10}$$

Moreover, if $\gamma(t)$ has positive upper density at a point $x$, i.e. if

$$\limsup_{r\to 0+} \frac{m_1\Big(B(x,r)\cap\gamma(t)\Big)}{r} > 0\,,$$

then $x \in \gamma(t)$.

Here $B(x,r)$ denotes the open ball centered at $x$ with radius $r$. As proved in [7], for every admissible strategy $t \mapsto \gamma(t)$ one can construct a second admissible strategy $t \mapsto \tilde\gamma(t) \supseteq \gamma(t)$, which is complete.

**Remark 2.** The assumption (H3) provides some weak regularity for the sets $\gamma(t)$. In general, one cannot require that these sets be closed, because the closure of a set $\Gamma$ can have much bigger total length. For example, consider a rectifiable set $\Gamma \subset I\!R^2$ containing all points with rational coordinates, and having total length $m_1(\Gamma) \leq 1$. In this case, the closure $\overline{\Gamma}$ is the entire plane, with one-dimensional measure $m_1(\overline{\Gamma}) = m_1(I\!R^2) = \infty$.

When a barrier is being constructed, the set reached by the fire is reduced. This leads to the definition of the new reachable set

$$R^\gamma(t) \doteq \Big\{x(t)\,;\ \ x(\cdot)\ \text{ absolutely continuous}\,,\ \ x(0)\in R_0\,,\\ \dot x(\tau)\in F(x(\tau))\ \text{ for a.e. } \tau\in[0,t]\,,\qquad x(\tau)\notin\gamma(\tau)\ \text{ for all } \tau\in[0,t]\Big\}\,. \tag{1.11}$$

According to (1.11), at any time $\tau$ the fire cannot cross the portion $\gamma(\tau)$ of the wall which is already in place. Clearly, the burned set will thus be smaller: $R^\gamma(t)\subseteq R(t)$ for every $t\geq 0$.

The alternative description of the reachable sets based on the PDE (1.6) can also be implemented in this more general case, in the presence of barriers. A characterization of the minimum time function $T(\cdot)$ as the solution to a Hamilton-Jacobi equation with obstacles was recently proved in [15].

## 1.3 Blocking and optimization problems

In the above setting, two natural problems arise. The first one is concerned with dynamic control, the second with optimization.

**(BP1) Blocking Problem.** Given a multifunction $F$, a construction speed $1/\psi$ and a bounded initial set $R_0$, decide whether there exists an admissible strategy $t\mapsto\gamma(t)$ such that the corresponding reachable sets $R^\gamma(t)$ remain uniformly bounded for all $t\geq 0$.

In other words, we ask whether it possible to construct a barrier $\gamma(\cdot)$ such that

$$R^\gamma(t)\ \subseteq\ B_r\qquad\qquad \text{for all } t>0 \tag{1.12}$$

for some fixed ball $B_r$ centered at the origin with radius $r$. If this is the case, we say that a blocking strategy exists. Here one should keep in mind that the barrier must be constructed in real time, simultaneously with the advancement of the fire front. (Fig. 2). Clearly, a blocking strategy can exist only if the construction speed of the barrier is sufficiently fast, compared with the speed at which fire propagates.

To describe an optimization problem, one needs to introduce a cost functional. This should take into account:

- The value of the area burned by the fire.

- The cost of building the barrier.

Following [5], we consider two continuous, non-negative functions $\alpha,\beta:\mathbb{R}^2\mapsto\mathbb{R}_+$ and define the functional

$$J(\gamma)\ =\ \int_{R^\gamma_\infty}\alpha\,dm_2+\int_{\gamma_\infty}\beta\,dm_1\,. \tag{1.13}$$

Here $m_2$ denotes the two-dimensional Lebesgue measure, while $m_1$ is the one-dimensional Hausdorff measure. Moreover, the domains of integration $R^\gamma_\infty$, $\gamma_\infty$ are defined respectively as

$$R^\gamma_\infty\ \doteq\ \bigcup_{t\geq 0}R^\gamma(t)\,,\qquad\qquad \gamma_\infty\ \doteq\ \bigcup_{t\geq 0}\gamma(t)\,. \tag{1.14}$$

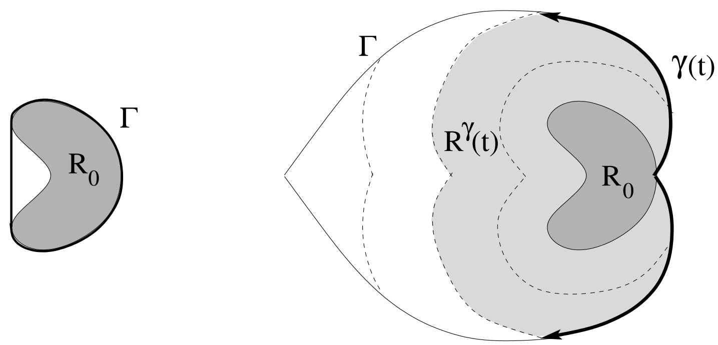


Figure 2: Left: a "static" blocking problem. Here $\Gamma$ is the curve of minimum length that entirely surrounds the set $R_0$. Right: a "dynamic" blocking problem, where the barrier $\Gamma$ is constructed at the same time as the set $R^\gamma$ burned by the fire expands. Here the thick curve $\gamma(t)$ denotes the portion of the barrier constructed up to time $t$, while the shaded region $R^\gamma(t)$ denotes the set burned by the fire, at time $t$.

In our model, $\alpha(x)$ is the value of a unit area of land at the point $x$, while $\beta(x)$ is the cost of building a unit length of wall at the point $x$. The set $R^\gamma_\infty$ describes the entire region burned by the fire, while $\gamma_\infty$ is the entire barrier. The first integral on the right hand side of (1.13) thus accounts for the value of the land burned by the fire, while the second integral yields the total cost of constructing the barrier. This leads to

**(OP1) Optimization Problem.** Find an admissible strategy $t \mapsto \gamma(t)$ for which the corresponding functional $J(\gamma)$ at (1.13) attains its minimum value.

In the remainder of this paper we discuss several recent results and open questions, in connection with the above blocking and optimization problems. In particular, the following issues are of interest:

- existence or non-existence of blocking strategies,

- existence of optimal strategies,

- necessary conditions for optimality,

- sufficient conditions for optimality,

- regularity of optimal barriers,

- numerical computation of optimal barriers.

For future reference, we list a set of assumptions which will be used in the remainder of the paper.

**(A1)** The initial set $R_0$ is nonempty, open and bounded. Its boundary satisfies $m_2(\partial R_0) = 0$.

**(A2)** The multifunction $F$ is Lipschitz continuous w.r.t. the Hausdorff distance. For each $x \in I\!\!R^2$ the set $F(x)$ is nonempty, closed and convex and contains the origin in its interior.

**(A3)** For every $x \in I\!R^2$ one has $\alpha(x) \geq 0$, $\beta(x) \geq \beta_0 > 0$, and $\psi(x) \geq \psi_0 > 0$. Moreover, $\alpha$ is locally integrable, while $\beta$ and $\psi$ are both lower semicontinuous.

We recall that the Hausdorff distance between two compact sets $X, Y$ is defined as

$$d_H(X,Y) \doteq \max\Big\{ \max_{x\in X} d(x,Y) \,, \max_{y\in Y} d(y,X)\Big\},$$

where

$$d(x,Y) \doteq \inf_{y\in Y} d(x,y)$$

and $d(x,y) \doteq |x-y|$ is the Euclidean distance on $I\!R^2$. The multifunction $F$ is Lipschitz continuous if there exists a constant $L$ such that

$$d_H(F(x),\, F(y)) \leq L\cdot d(x,y)\,,$$

for every couple of points $x, y$. For the basic theory of multifunctions and differential inclusions we refer to [3].

## 2 An equivalent formulation

In its original formulation [5], an admissible strategy was defined as a set-valued map $t \mapsto \gamma(t) \subset I\!R^2$. Indeed, for each $t \geq 0$ one needs to describe the portion of the wall constructed within time $t$. The subsequent paper [9] showed that the both the blocking problem and the optimization problem can be reformulated in a simpler way, where an admissible strategy is determined by one single rectifiable set $\Gamma \subset I\!R^2$. This approach is particularly useful for the numerical computation of optimal strategies. We review here the main ideas.

Consider a rectifiable set $\Gamma \subset I\!R^2$ which is **complete**, in the sense that it contains all of its points of positive upper density:

$$\limsup_{r\to 0+} \frac{m_1\Big(B(x,r)\cap\Gamma\Big)}{r} > 0 \qquad \Longrightarrow \qquad x\in\Gamma\,.$$

The set reached at time $t$ by trajectories of the differential inclusion (1.1) without crossing $\Gamma$ is then defined as

$$\begin{aligned} R^\Gamma(t) \doteq \Big\{x(t)\,;\ \ x(\cdot) \text{ absolutely continuous}\,,\ \ x(0)\in R_0\,, \\ \dot x(\tau)\in F(x(\tau)) \text{ for a.e. } \tau\in[0,t]\,,\quad x(\tau)\notin\Gamma \text{ for all } \tau\in[0,t]\Big\}. \end{aligned} \tag{2.1}$$

Throughout the following, $\overline{S}$ will denote the closure of a set $S$. We say that the rectifiable set $\Gamma$ is **admissible** in connection with the differential inclusion (1.1) and the bound on the construction speed (1.9) if

$$\int_{\Gamma\cap\overline{R^\Gamma(t)}} \psi\, dm_1 \leq t \qquad \text{for all } t\geq 0\,. \tag{2.2}$$

Of course, this means that the strategy

$$t \mapsto \gamma(t) \doteq \Gamma\cap\overline{R^\Gamma(t)} \tag{2.3}$$

is admissible according to (1.9).

In analogy with (1.14), we denote by

$$R^\Gamma_\infty \doteq \bigcup_{t\geq 0} R^\Gamma(t) \tag{2.4}$$

the entire region burned by the fire. Both the blocking problem (BP1) and the optimization problem (OP1) can now be reformulated in a simpler way, involving one single barrier $\Gamma$.

**(BP2) Blocking Problem:** Find an admissible rectifiable set $\Gamma$ such that the corresponding region $R^\Gamma_\infty$ is bounded.

**(OP2) Optimization Problem:** Find an admissible rectifiable set $\Gamma \subset I\!R^2$ such that the cost

$$J(\Gamma) = \int_{R^\Gamma_\infty} \alpha\, dm_2 + \int_\Gamma \beta\, dm_1 \tag{2.5}$$

attains the minimum possible value.

As proved in [9], under the assumptions (A1)–(A3) the two formulations are equivalent. In particular, if $t \mapsto \gamma(t)$ is a complete, optimal strategy for (OP1), then the rectifiable set

$$\Gamma \doteq \left(\bigcup_{t\geq 0} \gamma(t)\right) \setminus \left(\bigcup_{t\geq 0} R^\gamma(t)\right) \tag{2.6}$$

is admissible and provides an optimal solution to the minimization problem (OP2). Viceversa, if the set $\Gamma$ provides an optimal solution to (OP2), then the strategy $\gamma(\cdot)$ in (2.3) is optimal for (OP1).

**Remark 3.** For each $t \geq 0$, the set $\gamma(t)$ in (2.3) is the part of the wall $\Gamma$ touched by the fire at time $t$. This is the portion that actually needs to be put in place within time $t$, in order to restrain the fire. The remaining portion $\Gamma \setminus \gamma(t)$ can be constructed at a later time. On the other hand, given a strategy $\gamma(\cdot)$, the set $\Gamma$ in (2.6) consists of the "useful" part of all walls constructed by $\gamma$. Portions of a wall, which are constructed in a region already reached by the fire, are clearly useless.

**Remark 4.** By the assumption (A2), the fire propagates with positive speed in every direction. Hence, for a given initial domain $R_0$, the set $R^\Gamma_\infty$ in (2.4) burned by the fire can be characterized as the union of all connected components of $I\!R^2 \setminus \Gamma$ which intersect $R_0$.

Given a barrier $\Gamma$, we can define the minimum time function as

$$T^\Gamma(x) \doteq \inf \left\{t\,;\quad x \in \overline{R^\Gamma(t)}\right\}. \tag{2.7}$$

A characterization of this function as a solution of a H-J equation with obstacles was recently provided by De Lellis and Robyr in [15]. Before stating their result, we recall that a map $u : I\!R^2 \mapsto I\!R$ is in the space SBV of Special functions with Bounded Variation if

- its distributional derivative $Du$ is a measure,
- decomposing this measure $Du = \nabla u + D^{jump}u + D^{Cantor}u$ as the sum of an absolutely continuous part (w.r.t. Lebesgue measure), a jump part, and a Cantor part, the last component vanishes: $D^{Cantor}u \equiv 0$.

A family $\mathcal{S}^\Gamma$ of subsolutions to the H-J equation (1.6) with $\Gamma$ as an obstacle can now be defined as follows. For any $t \geq 0$, we shall denote by $u \wedge t$ the truncated function

$$(u \wedge t)(x) \doteq \min\{u(x), t\}. \tag{2.8}$$

**Definition.** A function $u : I\!R^2 \mapsto [0, \infty]$ is in the set $\mathcal{S}^\Gamma$ if

(i) For every $t \geq 0$ one has $(u \wedge t) \in SBV$, and the set of jump points $J_{u\wedge t} \subseteq \text{Supp}(D^{jump}u)$ is contained inside the barrier. Namely, $m_1(J_{u\wedge t} \setminus \Gamma) = 0$.

(ii) $u = 0$ on $R_0$ and $H(x, \nabla u(x)) \leq 0$ for a.e. $x$.

As proved in [15], the function $T^\Gamma$ defined by (2.7) admits the following characterization:

**Theorem 1.** *Let the assumptions (A1)-(A2) hold. Then the minimum time function $T^\Gamma$ is the unique maximal element of $\mathcal{S}^\Gamma$.*

# 3 Existence of blocking strategies

In this section we discuss the existence or nonexistence of an admissible strategy $\gamma(\cdot)$ which restrains the fire within a uniformly bounded region.

We recall that a blocking problem is specified by assigning

- the multifunction $x \mapsto F(x)$ describing the propagation velocity of the fire,

- the set $R_0$, describing the initial location of the fire,

- the funtion $x \mapsto \psi(x)$ determining the speed $\sigma = 1/\psi$ at which the barrier can be constructed.

For different data, a simple but useful comparison result holds.

**Lemma 1.** *Consider two blocking problems, the first with data $(F, R_0, \psi)$, the second with data $(\widetilde{F}, \widetilde{R}_0, \tilde{\psi})$. Assume that $R_0 \subseteq \widetilde{R}_0$ and*

$$F(x) \subseteq \widetilde{F}(x), \qquad \psi(x) \leq \tilde{\psi}(x) \qquad \textit{for all } x \in I\!R^2. \tag{3.1}$$

*If a blocking strategy for the second problem exists, then the the first problem admits a blocking strategy as well.*

Indeed, let $t \mapsto \tilde{\gamma}(t)$ be an admissible strategy for the second problem, such that the corresponding reachable sets satisfy $\widetilde{R}^{\tilde{\gamma}}(t) \subseteq B_r$ for some fixed radius $r$ and all $t \geq 0$. Since

$$\int_{\tilde{\gamma}(t)} \psi \, dm_1 \;\leq\; \int_{\tilde{\gamma}(t)} \tilde{\psi} \, dm_1 \;\leq\; t\,,$$

the strategy $\tilde{\gamma}(\cdot)$ is admissible for the first problem as well. Call $R^{\tilde{\gamma}}$, $\widetilde{R}^{\tilde{\gamma}}$ the corresponding sets reached by the fire in the first and second problem, respectively. By the assumptions,

$$R^{\tilde{\gamma}}(t) \;\subseteq\; \widetilde{R}^{\tilde{\gamma}}(t) \;\subseteq B_r \qquad\qquad \text{for all } t \geq 0.$$

□

### 3.1 The isotropic case

We begin by discussing the isotropic case, where the fire propagates with unit speed in all directions, while the barrier is constructed at a constant speed $\sigma > 0$. In other words, we assume that $F(x) = \overline{B}_1$ is the closed disc centered at the origin with radius 1, and the constraint (1.8) holds.

We observe that, in this isotropic case, the family of solutions of (1.1) is invariant under rotations and translations. It is also invariant under a group of rescaling transformations. Namely, consider an initial set $R_0$ and an admissible strategy $t \mapsto \gamma(t)$, and let $R^{\gamma}(t)$ be the corresponding reachable sets, defined at (1.11). Given any $\lambda > 0$, define the rescaled barriers

$$\tilde{\gamma}(t) \;\doteq\; \lambda\, \gamma(t/\lambda)$$

and the initial set $\widetilde{R}_0 \doteq \lambda R_0$. It is now easy to check that the blocking strategy $\tilde{\gamma}(\cdot)$ is also admissible, and the corresponding sets reached by the fire are given by

$$R^{\tilde{\gamma}}(t) \;=\; \lambda\, R^{\gamma}(t/\lambda)\,. \tag{3.2}$$

Combining (3.2) with the comparison result stated in Lemma 1, one can prove that the solvability of the blocking problem depends only on the speed $\sigma$, and not on the initial set $R_0$.

**Lemma 2.** *Let a construction speed $\sigma > 0$ be given. If there exists a (nonempty) bounded open set $R_0^*$ for which the blocking problem can be solved, then a blocking strategy exists for every bounded set $R_0$.*

Indeed, by a rescaling followed by a translation, every bounded set $R_0$ can be mapped into a subset of $R_0^*$. The result thus follows from Lemma 1. □

To understand the solvability of the blocking problem in the isotropic case, it thus suffices to study the case where

$$F(x) \;\equiv\; \overline{B}_1\,, \qquad\qquad R_0 \;=\; B_1 \qquad\qquad \psi(x) \;=\; \frac{1}{\sigma}\,, \tag{3.3}$$

where $B_1$ is the open disc centered at the origin with unit radius. By a comparison argument it is clear that there must be a constant $\sigma^* > 0$ such that

- for $\sigma > \sigma^*$ a blocking strategy exists,

- for $\sigma < \sigma^*$ blocking strategy does not exists.

At the present date, the exact value of this constant $\sigma^*$ is not known. The analysis in [5, 10] has shown that $\sigma^* \in [1, 2]$. We recall the main results in this direction.

**Theorem 2.** *Let (3.3) hold. If $\sigma > 2$, a blocking strategy exists.*

**Proof.** Assuming $\sigma > 2$, consider the positive constant $\lambda \doteq \left(\frac{\sigma^2}{4} - 1\right)^{-1/2}$, so that $\sigma = 2\sqrt{1+\lambda^2}/\lambda$. Using polar coordinates $r, \theta$, let $\Gamma$ be the closed curve consisting of two arcs of logarithmic spirals:

$$\Gamma \doteq \Big\{(r\cos\theta\,,\ r\sin\theta)\ ;\quad r = e^{\lambda|\theta|}\,,\quad \theta \in [-\pi,\,\pi]\Big\}.$$

As in (2.3), let

$$\gamma(t) \doteq \Gamma \cap \overline{R^\Gamma(t)} = \Big\{(r\cos\theta\,,\ r\sin\theta)\ ;\quad r = e^{\lambda|\theta|}\,,\quad \theta \in [-\pi,\,\pi]\,,\quad r \leq 1+t\Big\}$$

be the portion of this barrier reached by the fire within time $t$. By the choice of $\lambda$, an elementary computation yields $m_1(\gamma(t)) = \sigma t$, showing that $\gamma(\cdot)$ is an admissible strategy. This strategy solves the blocking problem, because the reachable sets $R^\gamma(t)$ are all contained inside the region bounded by the closed curve $\Gamma$. □

On the other hand, proving the non-existence of a blocking strategy when $\sigma$ is small is a far more difficult task. In this direction, two entirely different arguments were developed in [5] and [10].

**Theorem 3.** *Let (3.3) hold. If $\sigma \leq 1$, a blocking strategy does not exist.*

With reference to Figure 3, this result can be motivated by the following intuitive argument. Assume that, for a construction speed $\sigma > 0$, a blocking strategy exists. Let $\Gamma$ be the entire barrier constructed by this strategy, and let $\bar{x} \in \Gamma$ be the point where the "last brick" of the wall $\Gamma$ is placed (see Fig. 3). Calling $T^\Gamma(\cdot)$ the minimum time function in (2.7), we must have

$$T^\Gamma(\bar{x}) = \sup_{x\in\Gamma} T^\Gamma(x) \geq \frac{m_1(\Gamma)}{\sigma}. \tag{3.4}$$

Indeed, the right hand side measures the total time needed to construct the barrier $\Gamma$. If the fire reaches the point $\bar{x}$ before this barrier is completed, it will spill outside, hence the reachable sets $R^\Gamma(t)$ will not remain within the bounded set enclosed by $\Gamma$.

We now estimate the left hand side of (3.4), i.e. the time needed by the fire to reach the point $\bar{x}$. Let $\gamma_0$ be a shortest path joining a point $x_0 \in R_0$ with $\bar{x}$ without crossing $\Gamma$. By prolonging this path backwards, we can find a point $x_1 \in \Gamma$ and a path $\gamma_1 \supset \gamma_0$, such that $\gamma_1$ is the shortest path joining $x_1$ with $\bar{x}$ without crossing $\Gamma$. An estimate on the length of $\gamma_1$ can be obtained as follows. Starting from $x_1$, move along $\Gamma$ either clockwise or counterclockwise until the point $\bar{x}$ is reached. This yields two curves, say $\gamma_2$ and $\gamma_3$. The length of these curves satisfies

$$m_1(\gamma_2) + m_2(\gamma_3) \ \leq \ 2m_1(\Gamma)\,. \tag{3.5}$$

Hence

$$T^\Gamma(\bar{x}) \ = \ m_1(\gamma_0) \ < \ m_1(\gamma_1) \ \leq \ \min\{m_1(\gamma_2), m_1(\gamma_3)\} \ \leq \ m_1(\Gamma)\,. \tag{3.6}$$

If $\sigma \leq 1$, the inequality (3.6) is in contradiction with (3.4), hence a blocking strategy cannot exists. For a rigorous proof based on these ideas we refer to [10].

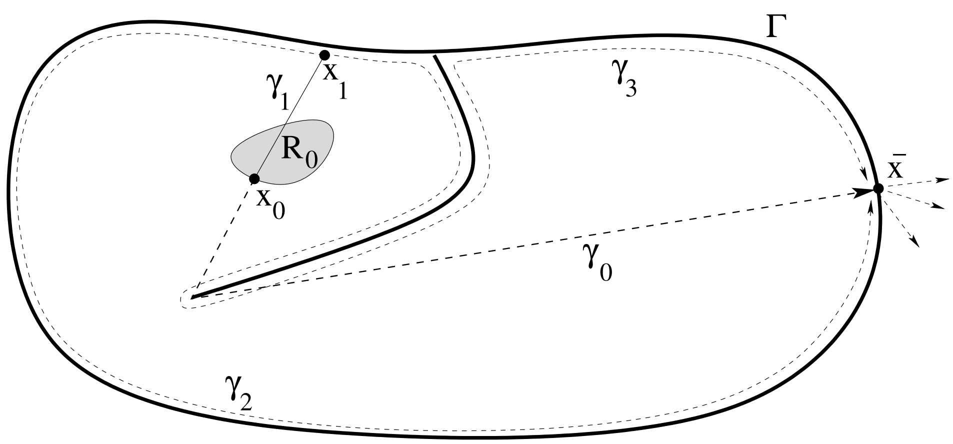


Figure 3: If the barrier is constructed at speed $\sigma \leq 1$, then the fire reaches the point $\bar{x}$ and spills outside before the barrier is completed. Hence no blocking strategy can exist.

While the blocking problem on the entire plane is not yet fully understood, a sharp result is available in the case where fire propagation is restricted to a half plane $R^+_2 \doteq \{(x_1, x_2)\,;\ \ x_2 \geq 0\}$. This models a situation where the presence of a river, or a lake, provides a natural barrier to fire propagation. In this case, the definition of reachable sets in (1.11) must be modified, requiring that all trajectories remain in the half plane $R^2_+$:

$$\begin{aligned} R^\gamma(t) \doteq \ \Big\{ & x(t)\,;\ \ x(\cdot) \ \text{ absolutely continuous}\,, \ \ x(0) \in R_0\,, \\ & \dot{x}(\tau) \in F(x(\tau)) \ \text{ for a.e. } \tau \in [0,t]\,, \qquad x(\tau) \in I\!\!R^2_+ \setminus \gamma(\tau) \ \text{ for all } \tau \in [0,t] \Big\}\,. \end{aligned} \tag{3.7}$$

The following result was proved in [10].

**Theorem 4.** *Let (3.3) hold. Then, restricted to the half plane, for every bounded initial set* $R_0 \subset I\!\!R^2_+$ *a blocking strategy exists if and only if* $\sigma > 1$.

### 3.2 The non-isotropic case

In a realistic situation, the fire propagates in various directions at different speeds. This happens, for example, if there is wind blowing in a preferred direction. This case is modeled by replacing the unit disc $B_1$ with more general velocity sets $F(x) \subset I\!\!R^2$. Sufficient conditions for the existence of a blocking strategy for the problem (1.1), (1.8) were derived in [6].

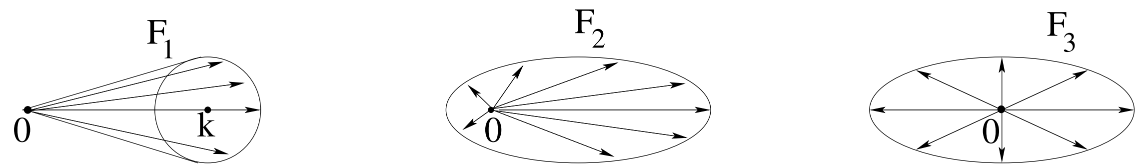


Figure 4: The velocity sets $F_1$ and $F_2$ satisfy the assumptions (3.8), while the set $F_3$ does not.

**Theorem 5.** *Assume that the velocity sets in (1.1) are independent of $x$ and have the form*

$$F(x) \equiv F = \Big\{(r\cos\theta\,,\ r\sin\theta)\,;\quad 0\le r\le \rho(\theta)\,,\quad \theta\in[-\pi,\,\pi]\Big\},$$

*where the function $\rho:[-\pi,\,\pi]\mapsto I\!R_+$ satisfies*

$$\rho(-\theta)=\rho(\theta)\,,\qquad\qquad 0\le\rho(\theta')\le\rho(\theta)\qquad\qquad \text{for all}\ \ 0\le\theta\le\theta'\le\pi\,. \tag{3.8}$$

*If the wall construction speed satisfies*

$$\sigma\ >\ [\text{vertical width of } F]\ =\ 2\max_{\theta\in[0,\pi]}\ \rho(\theta)\sin\theta$$

*then, for every bounded initial set $R_0$, a blocking strategy exists.*

Notice that the above result reduces to Theorem 3 in the case where $F=\overline{B}_1$ is the closed unit disc. Further results on existence or non-existence of blocking strategies can be obtained by comparison with the isotropic case, using Lemma 2.

# 4 Existence of optimal strategies

Consider the dynamic blocking problem with dynamics described by the differential inclusion (1.1) and by the admissibility condition (1.9). Let the cost functional be described by (1.13). The following result on the existence of optimal blocking strategies was proved in [7].

**Theorem 6.** *Let the assumptions (A1)–(A3) hold. If there exists an admissible strategy such that $J(\gamma)<\infty$, then the optimization problem (OP1)admits an optimal solution.*

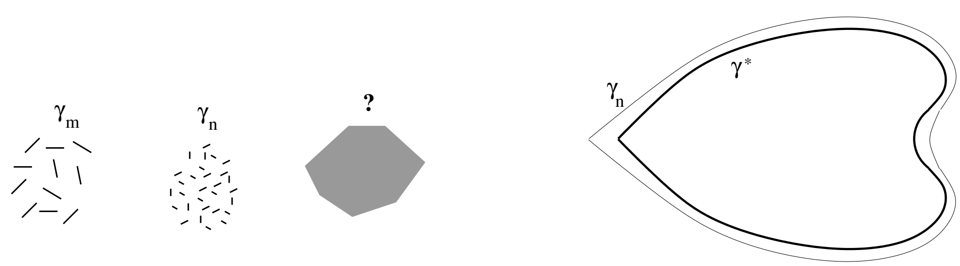


Figure 5: Left: the limit of a sequence of bounded rectifiable sets can only be interpreted in measure sense, and may not yield a rectifiable set. Right: the limit of a sequence of *connected* rectifiable sets is a rectifiable set.

The proof given in [7] relies on the direct method of the Calculus of Variations. Consider a minimizing sequence $\gamma_n(\cdot)$ of admissible strategies, such that

$$J(\gamma_n) \ \to \ \inf_{\gamma \in \mathcal{A}} J(\gamma)\,,$$

where the infimum is taken over the family of all admissible blocking strategies. An optimal strategy $\gamma^*(\cdot)$ is then obtained by taking a suitable limit of the $\gamma_n(\cdot)$.

One should be aware, however, that no regularity is known a priori about the rectifiable sets $\gamma_n(t)$. Hence (see Fig. 3) as $n \to \infty$ there is no guarantee that these sets will converge to a rectifiable set $\gamma^*(t)$. A more careful argument requires several steps. The key idea is to split each set $\gamma_n(t)$ into connected components of decreasing length, and take limits componentwise.

**1.** By possibly enlarging the sets $\gamma_n(t)$, is not restrictive to assume that the $\gamma_n(\cdot)$ are complete strategies, i.e., for every $t \geq 0$ one has

$$\left\{ x \in I\!\!R^2\,; \quad \limsup_{r \downarrow 0} \frac{m_1(B(x,r) \cap \gamma_n(t))}{r} \ > \ 0 \right\} \ \subseteq \ \gamma_n(t) \ = \ \bigcap_{s>t} \gamma_n(s).$$

**2.** For each rational time $\tau$, let the connected components of $\gamma_n(\tau)$ be ordered according to decreasing length, so that

$$\gamma_n(\tau) \ = \ \gamma_{n,1}(\tau) \cup \gamma_{n,2}(\tau) \cup \gamma_{n,3}(\tau) \cup \cdots$$

with

$$\ell_{n,1}(\tau) \ \geq \ \ell_{n,2}(\tau) \ \geq \ \ell_{n,3}(\tau) \ \geq \ \cdots\,, \qquad\qquad \ell_{n,i}(\tau) \ \doteq \ m_1(\gamma_{n,i}(\tau))\,.$$

Notice that, by completeness, each connected component $\gamma_{n,i}(\tau)$ must be closed. Taking a subsequence, as $n \to \infty$ we can assume that, for every rational time $\tau \geq 0$,

$$\ell_{n,i}(\tau) \ \to \ \ell_i(\tau) \qquad\qquad d_H(\gamma_{n,i}(\tau)\,,\ \gamma_i(\tau)) \ \to \ 0\,.$$

Here $d_H(\cdot,\cdot)$ denotes the Hausdorff distance between compact sets. We then define $\gamma(\tau) \doteq \bigcup_{\ell_i(\tau)>0} \gamma_i(\tau)$. Finally, the optimal strategy $\gamma^*(\cdot)$ is defined as the completion of $\gamma(\cdot)$.

**3.** Using the lower semicontinuity of the functions $\psi, \beta$, for every $t \geq 0$ one obtains

$$\int_{\gamma^*(t)} \psi\, dm_1 \ \leq \ \liminf_{n\to\infty} \int_{\gamma_n(t)} \psi\, dm_1 \ \leq t\ , \qquad\qquad \int_{\gamma^*(t)} \beta\, dm_1 \ \leq \ \liminf_{n\to\infty} \int_{\gamma_n(t)} \beta\, dm_1\,.$$

By the first inequality, the limit strategy $\gamma^*(\cdot)$ is admissible.

**4.** The last (and most difficult) step is to prove the inequalities

$$\int_{R^{\gamma^*}(t)} \alpha\, dm_2 \ \leq \ \liminf_{n\to\infty} \int_{R^{\gamma_n}(t)} \alpha\, dm_2\,.$$

These are achieved by showing that, for every $t \geq 0$, the sets $R^{\gamma_n}(t)$ are "almost as big" as the reachable set $R^{\gamma^*}(t)$. In other words, assume that there exists a trajectory $\tau \mapsto x(\tau)$ for the fire, satisfying (1.1), and reaching a point $x(t) = \bar{x}$ without crossing the wall $\gamma^*(\tau)$ for any $\tau \in [0,t]$. Then for every $n \geq 1$ sufficiently large, there exists a trajectory $\tau \mapsto x_n(\tau)$ reaching a point $x_n(t)$ close to $\bar{x}$ without crossing the barriers $\gamma_n(\tau)$. The proof of this statement requires a careful analysis of solutions to the differential inclusion (1.1), involving both topological and mesure-theoretic arguments. A key step calls for the partition of the plane $I\!R^2$ into a checkerboard, whose squares $Q_k$ are colored either white or black depending on the length $m_1(\gamma_n(\tau) \cup Q_k)$. For all details we refer to [7].

An entirely different proof of Theorem 6, based on the analysis of the minimal time function $T^\Gamma$ in (2.7), was recently developed in [15]. We review here the main ideas.

**1.** Consider a minimizing sequence of admissible barriers $\Gamma_k$, $k \geq 1$. Let $T_k \doteq T^{\Gamma_k}$ be the corresponding minimum time functions. For every $t \geq 0$, the functions $T_k \wedge t$ are SBV functions, which admit the characterization stated in Theorem 1.

**2.** Using the Ambrosio-De Giorgi compactness theorem for SBV functions [1], one obtains a convergent subsequence $T_k \to U$ such that

(i) Recalling the notation (2.8), for every $t \geq 0$ one has $(U \wedge t) \in SBV$.

(ii) The jump set of $U \wedge t$ satisfies

$$\int_{J_{(U\wedge t)}} \psi\, dm_1 \;\leq\; \liminf_{k\to\infty} \int_{J_{(T_k\wedge t)}} \psi\, dm_1 \;\leq\; t\,. \tag{4.1}$$

**3.** By (4.1) the rectifiable set $\Gamma$, obtained by taking the completion of $J_U$, is an admissible barrier. Standard lower semicontinuity estimates now yield

$$\int_{R^\Gamma_\infty} \alpha\, dm_2 \;\leq\; \liminf_{k\to\infty} \int_{R^{\Gamma_k}_\infty} \alpha\, dm_2\,, \qquad\qquad \int_\Gamma \beta\, dm_1 \;\leq\; \liminf_{k\to\infty} \int_{\Gamma_k} \beta\, dm_1\,.$$

Hence $\Gamma$ is an optimal barrier. □

# 5 Necessary conditions for optimality

Given the minimization problem (OP2), assume that an optimal barrier $\Gamma$ exists, consisting of the union of finitely many, sufficiently regular arcs. By deriving necessary conditions for optimality, one seeks to determine these optimal arcs, as solutions to a family of ODEs together with boundary conditions.

Following a standard procedure in the Calculus of Variations, necessary conditions are obtained by the analysis of perturbations. For the present problem, however, these conditions take different forms depending on the various types of arcs. As a preliminary, one must therefore introduce a classification of optimal arcs.

Let $\Gamma$ be an admissible barrier for the differential inclusion (1.1), so that (2.2) holds. Observe that the presence of this barrier has two effects, namely: (i) it restricts the fire to the set $R^\Gamma_\infty$, consisting of all connected components of $I\!R^2 \setminus \Gamma$ which intersect the initial domain $R_0$, and (ii) within the set $R_\infty^\Gamma$, it can slow down the advancement of the fire.

This fact, illustrated in Fig. 6, can be better described as follows. Given the differential inclusion (1.1) and the barrier $\Gamma$, let $T^\Gamma(\cdot)$ be the minimum time function, defined at (2.7). Calling $T(\cdot)$ the minimum time function for the original problem (1.1) without any barrier, one clearly has

$$0 \leq T(x) \;\leq\; T^\Gamma(x) \qquad\qquad \text{for all } x \in I\!R^2.$$

We say that a point $x \in \Gamma$ belongs to the delaying portion of the barrier if, by modifying the set $\Gamma$ in an arbitrarily small neighborhood of $x$, one can change the minimal time function *somewhere else*:

**Definition.** The subset $\Gamma^d \subseteq \Gamma$ of **delaying walls** is the set of all points $x \in \Gamma$ such that, for some $\delta > 0$, the following holds. For every $\varepsilon > 0$ there exists an admissible rectifiable set $\Gamma'$ with $\Gamma' \setminus B(x,\varepsilon) = \Gamma \setminus B(x,\varepsilon)$ and such that $T^{\Gamma'}(y) \neq T^\Gamma(y)$ at some point $y \notin B(x,\delta)$.

We think of $\Gamma^d$ as a portion of the barrier $\Gamma$ which contributes to slowing down fire propagation. In addition, the barrier $\Gamma$ will contain an outer portion $\Gamma^b$, separating the burned from the unburned region.

**Definition.** The subset $\Gamma^b \subseteq \Gamma$ of **blocking walls** is defined as $\Gamma^b \doteq \Gamma \cap \partial(R_\infty^\Gamma)$.

**Remark 5.** If $\Gamma$ is optimal and the construction cost $\beta$ in (1.13) is strictly positive, then $\Gamma = \Gamma^d \cup \Gamma^b$. Indeed, any arc $\Gamma' \subset \Gamma$ contained in the interior of the reachable set $R_\infty^\Gamma$ must be part of $\Gamma^d$, otherwise the alternative strategy $\widetilde{\Gamma} \doteq \Gamma \setminus \Gamma'$ would also be admissible, with a smaller cost. On the other hand, as shown in Fig. 6, one can have $\Gamma^d \cap \Gamma^b \neq \emptyset$.

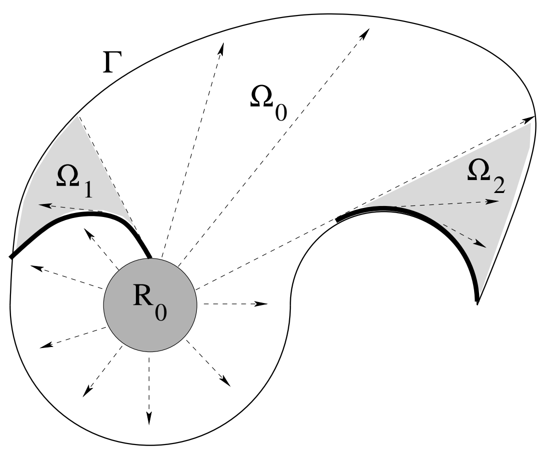


Figure 6: Here we take $R_0 = B_1$, and $F(x) \equiv F = \overline{B}_1$, the unit disc centered at the origin. The two thick arcs denote the portion $\Gamma^d \subset \Gamma$ which contributes to slowing down the propagation of the fire. Notice that $T^\Gamma(x) = T(x)$ for $x \in \Omega_0$, but $T^\Gamma(x) < T(x)$ for $x \in \Omega_1 \cup \Omega_2$. The thick arc next to the shaded region $\Omega_1$ lies in $\Gamma^d \setminus \Gamma^b$, while the thick arc next to $\Omega_2$ lies in $\Gamma^d \cap \Gamma^b$.

Given an admissible barrier $\Gamma$, a further classification of arcs can be achieved as follows. Define the set of times

$$\mathcal{S} \doteq \left\{ t \geq 0\,; \quad \int_{\Gamma \cap \overline{R^\Gamma(t)}} \psi \, dm_1 = t \right\}. \tag{5.1}$$

These are the times where the admissibility constraint is **saturated**, i.e. it is satisfied as an equality. We can further classify points $x \in \Gamma$ by setting

$$\Gamma_{\mathcal{S}} \doteq \{x \in \Gamma\,; \ \ T^\Gamma(x) \in \mathcal{S}\}, \qquad \Gamma_{\mathcal{F}} \doteq \{x \in \Gamma\,; \ \ T^\Gamma(x) \notin \mathcal{S}\}.$$

Following [5], arcs lying in the subset $\Gamma_{\mathcal{F}}$ will be called **free arcs**, while arcs lying in $\Gamma_{\mathcal{S}}$ will be called **boundary arcs**. Notice that boundary arcs are constructed right at the edge of the advancing fire front. On the other hand, free arcs represent a preemptive strategy: they are put in place in advance, at locations which will be reached by the fire only at a later time.

In the following, for simplicity we discuss necessary conditions for free and boundary arcs, assuming that the functions $\beta(x) \equiv \beta$ and $\psi(x) \equiv \sigma^{-1}$ are both constant. Results valid in the general case can be found in [11]. Furthermore, in the case of delaying arcs, necessary conditions for optimality were recently derived in [23].

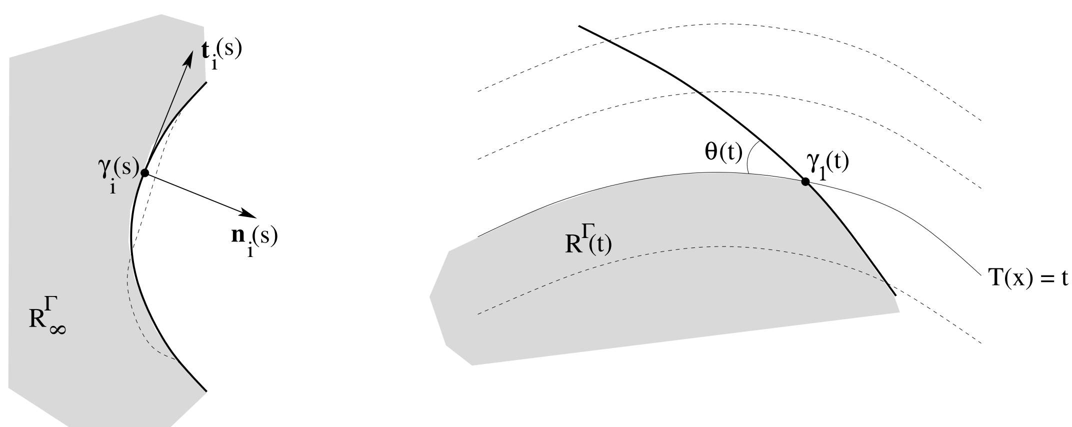


Figure 7: Left: a free arc $\gamma_i$, parameterized by arc-length. Here any small perturbation having the same length (the dotted line) yields another admissible barrier. Hence the optimality conditions are the same as in isoperimetric problems. Right: a single boundary arc $\gamma_1$. In this case the admissibility condition already suffices to determine the arc.

## 5.1 Free arcs

Let $\Gamma$ be an optimal barrier. Assume that, during a time interval $[t_0, t_1]$, this optimal strategy simultaneously constructs $N$ free, blocking arcs: $\gamma_1, \ldots, \gamma_N \subset \Gamma_{\mathcal{F}} \cap \Gamma^b$. Referring to Fig. 7, let $s \mapsto \gamma_i(s)$, $s \in [a_i, b_i]$ be a parameterization of $\gamma_i$ in terms of arc-length, so that $|\dot\gamma_i(s)| \equiv 1$. Consider the unit tangent vector $\mathbf{t}_i(s) \doteq \dot\gamma_i(s)$ and let $\mathbf{n}_i(s)$ be the unit normal vector, oriented toward the exterior of the set $R^\Gamma_\infty$ burned by the fire. Let $\kappa_i(s)$ be the curvature of $\gamma_i$ at the point $\gamma_i(s)$, so that

$$\ddot\gamma_i(s) = \dot{\mathbf{t}}_i(s) = \kappa_i(s)\, \mathbf{n}_i(s). \tag{5.2}$$

Since $\gamma_i$ is a free arc, every curve $\tilde\gamma_i$ sufficiently close to $\gamma_i$, with the same length and the same endpoints, will also yield an admissible barrier. Notice that, if $\gamma_i$ is not a segment, many such perturbations $\tilde{\gamma}$ can be constructed. The necessary conditions for optimality thus take the same form as in the classical isoperimetric problem of the Calculus of Variations.

**Theorem 7.** *Let $\gamma_1, \ldots, \gamma_N \subset \Gamma_{\mathcal{F}} \cap \Gamma^b$ be free arcs, simultaneously constructed by an optimal strategy. Then the curvature of each arc is proportional to the local value $\alpha(\cdot)$ of the land. Indeed, either $\kappa_i(s) \equiv 0$ (hence all arcs are straight segments), or there exists a Lagrange multiplier $\lambda \geq 0$ such that*

$$\left(\beta + \frac{\lambda}{\sigma}\right) \kappa_i(s) \;=\; \alpha(\gamma_i(s)) \qquad \textit{for all } \; i \in \{1, \ldots, N\}, \quad a_i < s < b_i\,. \tag{5.3}$$

In particular, if $\alpha(x) \equiv \alpha$ is a constant, then the curves $\gamma_i$ are arcs of circumferences, all with the same radius $r = \frac{1}{\kappa} = \frac{\beta + (\lambda/\sigma)}{\alpha}$. It is important to notice that the constant $\lambda$ is the same for every arc $\gamma_i$, $i \in \{1, \ldots, N\}$. Calling $r_i(s) = 1/\kappa_i(s)$ the radius of curvature, one has

$$\lambda \;=\; \Big(\alpha(\gamma_i(s)) \cdot r_i(s) - \beta\Big)\sigma\,. \tag{5.4}$$

As shown in [11], the Lagrange multiplier $\lambda$ can be interpreted as the *instantaneous value of time*. The next paragraph provides an intuitive explanation of this concept.

Assume that, in an idealized situation, we could "buy time". In other words, assume that we had at our disposal a short time interval $[t,\, t+\varepsilon]$ to construct an additional portion of barrier, while in the meantime the fire front did not advance. Making the most of this advantage, we could thus reduce the total area eventually burned by the fire, and hence the total cost. Roughly speaking, we say that $V(t)$ is the *instantaneous value of time* (at time $t$) if

$$[\text{reduction of the total cost}] \;=\; \varepsilon \cdot V(t) + o(\varepsilon),$$

where the Landau symbol $o(\varepsilon)$ denotes a higher order infinitesimal as $\varepsilon \to 0$. In general, one can prove that $t \mapsto V(t)$ is a nonincreasing function. If at some time $\tau$ the fire propagation is extinguished, then $V(t) = 0$ for all $t > \tau$. In the situation described by Theorem 7, the value of time is actually constant: $V(t) \equiv \lambda$ during the interval of time when the free arcs $\gamma_1, \ldots, \gamma_N$ are constructed.

## 5.2 A single boundary arc

Next, assume that during a time interval $[t_1, t_2]$ the optimal strategy constructs one single boundary arc $\gamma_1 \subset \Gamma_{\mathcal{S}} \cap \Gamma^b$. We choose a parameterization $t \mapsto \gamma_1(t)$ of this arc so that each point $\gamma(t)$ lies on the level set $\{x\,;\;\; T^\Gamma(x) = t\}$.

In this case, an equation determining the arc $\gamma_1$ can already be derived from the admissibility condition (1.8), without using any optimality condition. Indeed, let $T(\cdot)$ be the minimum time function and let

$$h(x) \;\doteq\; |\nabla T(x)|^{-1} \tag{5.5}$$

be the propagation speed of the fire front in the normal direction, as in (1.4). We then have the identities

$$|\dot{\gamma}_1(t)| \;\equiv\; \sigma\,, \qquad\qquad T(\gamma_1(t)) \;=\; t \qquad\qquad \text{for all } t \in [t_1,\, t_2]\,. \tag{5.6}$$

With reference to Fig. 7, let $\theta_1$ be the angle between the curve $\gamma_1$ and the level curve of the minimum time function $T(\cdot)$, at a point $x$. By (5.6), one has

$$\sigma \cdot \sin\theta_1(x) \;=\; h(x)\,. \tag{5.7}$$

If the initial point $\gamma_1(t_1)$ is known, from (5.7) one can recover the entire curve $\gamma_1$.

## 5.3 Several boundary arcs constructed simultaneously

We now consider a more general situation where $\nu$ boundary arcs $\gamma_1, \gamma_2, \ldots, \gamma_\nu \in \Gamma_{\mathcal{S}} \cap \Gamma^b$ are simultaneously constructed, on a time interval $t \in [t_1, t_2]$. Let each arc be parameterized by time, so that

$$T(\gamma_i(t)) \;=\; t \qquad\qquad \text{for all } t \in [t_1, t_2]\,, \quad i \in \{1, \ldots, \nu\}\,. \tag{5.8}$$

The admissibility condition (1.8) yields

$$\sum_{i=1}^{\nu} |\dot\gamma_i(t)| \;=\; \sigma \qquad\qquad \text{for all } t \in [t_1,\, t_2]. \tag{5.9}$$

In the present case where $\nu \geq 2$, the equations (5.8)-(5.9) are not enough to uniquely determine the arcs $\gamma_i$. Additional conditions, derived from the optimality of the barrier $\Gamma$, must be used. Following [5], we will show how the problem of determining these optimal arcs can be reduced to a standard problem of optimal control.

Call $w_i^*(t) \doteq \sigma^{-1}|\dot\gamma_i(t)|$ the portion of overall resources devoted to the construction of the arc $\gamma_i$, at time $t$. We regard the map $t \mapsto w^*(t) = (w_1^*(t), \ldots, w_\nu^*(t)) \in \Delta^\nu$ as a control function, taking values in the unit simplex

$$\Delta^\nu \;\doteq\; \left\{(w_1, \ldots, w_\nu)\,; \qquad w_i \geq 0\,, \qquad \sum_{j=1}^{\nu} w_j = 1\right\}. \tag{5.10}$$

We now choose a set of coordinates $(t, s) \mapsto x_i(t, s)$, on a neighborhood of each arc $\gamma_i$. To fix the ideas (see Fig. 8), for each time $t$ let the map $s \mapsto x_i(t, s)$ provide an arc-length parametrization of the level set $\{x\,;\ T(x) = t\}$, in a neighborhood of $\gamma_i(t)$. Moreover, let $\mathbf{e}_i = \frac{\partial x_i(t,s)}{\partial s}$ be the unit tangent vector to this level set.

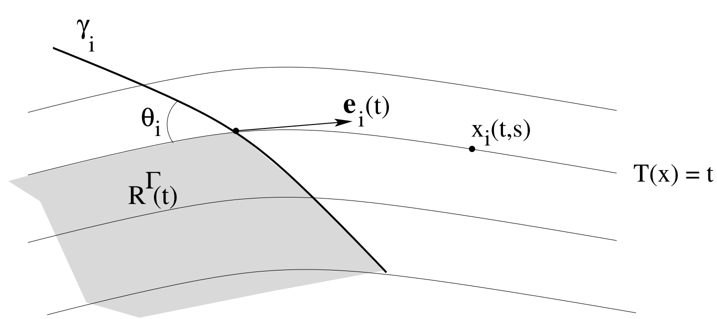


Figure 8: Choice of the coordinates $(t, s)$ in a neighborhood of the arc $\gamma_i$.

Consider a second admissible strategy $t \mapsto w(t) = (w_1(t), \ldots, w_\nu(t))$. This will result in the construction of different arcs $t \mapsto y_i(t)$, determined by the equations

$$|\dot y_i(t)| \;=\; \sigma\, w_i(t), \qquad\qquad T(y_i(t)) \;=\; t\,,$$

with the same initial conditions at time $t = t_1$:

$$y_i(t_1) \;=\; \gamma_i(t_1) \qquad\qquad \text{for all } i \in \{1, \ldots, \nu\}.$$

In our coordinate system, let $y_i(t) = x_i(t, s_i(t))$. For each $i$, the scalar function $s_i(\cdot)$ will satisfy an ODE of the form

$$\dot{s}_i \;=\; f_i(t, s_i(t), w_i(t))\,. \tag{5.11}$$

Here the right hand side $f_i$ is implicitly determined by the scalar constraint

$$\left| \frac{\partial x_i(t, s_i)}{\partial t} + f_i(t, s_i, w_i)\, \frac{\partial x_i(t, s_i)}{\partial s_i} \right| \;=\; \sigma\, w_i. \tag{5.12}$$

This accounts for the fact that $|\dot{y}_i| = \sigma w_i$. We observe that the equation (5.12) admits solutions provided that

$$h(x_i(t, s_i)) \;\leq\; \sigma\, w_i. \tag{5.13}$$

Indeed, the speed $|\dot{y}_i|$ at which the barrier is constructed cannot be smaller than the propagation speed of the fire front, in the normal direction. In the case of a strict inequality, the equation (5.12) has exactly two solutions, say $f_i^- < f_i^+$. Clearly, the correct choice depends on the relative position of the burned region. For example, if the burned region locally has the representation $\{x_i(t, s)\,;\; s < s_i(t)\}$ (as in Fig. 8), then one should choose $f_i = f_i^-$.

Consider the control system consisting of the $\nu$ equations (5.11), supplemented by the initial and terminal constraints

$$x_i(t_1,\, s_i(t_1)) \;=\; \gamma_i(t_1)\,, \qquad\qquad x_i(t_2,\, s_i(t_2)) \;=\; \gamma_i(t_2) \qquad\qquad i = 1, \cdots, \nu\,. \tag{5.14}$$

For this system, consider the optimization problem

$$\text{minimize}: \quad \Lambda(w) \;\doteq\; \sum_{i=1}^{\nu} \int_a^b L_i(t, s_i(t), w_i(t))\, dt\,, \tag{5.15}$$

where the running costs have the form

$$L_i(t, s_i, w_i) \;\doteq\; \beta\, \sigma\, w_i(t) + \int_{\bar{s}_i}^{s_i} h(x_i(t, \xi))\, \alpha(x_i(t, \xi))\, d\xi\,, \tag{5.16}$$

with $h$ as in (5.5). The minimum in (5.15) is sought among all control functions $w : [t_1,\, t_2] \mapsto \Delta^{\nu}$. Notice that the first term in (5.16) accounts for the cost of building the wall, while the second term is related to the value of the burned area. Here the choice of the constants $\bar{s}_i$ is immaterial, because it does not affect the minimizers.

The system of ODEs (5.11) and boundary conditions (5.14), together with the integral functional at (5.15)-(5.16) yields an optimal control problem in standard form. The optimal control $t \mapsto w(t) = (w_1^*(t), \ldots, w_\nu^*(t))$ thus satisfies the Pontryagin maximum principle [8, 16, 22]. In turn, this yields a set of necessary conditions for the optimal arcs $\gamma_i$.

**Theorem 8.** *Let $\Gamma$ be an optimal barrier. Let $\gamma_1, \ldots, \gamma_\nu \subset \Gamma_{\mathcal{S}} \cap \Gamma^b$ be boundary arcs simultaneously constructed during the time interval $[t_1,\, t_2]$, parameterized as in (5.8). Call $\mathbf{e}_i(t)$ the unit vector tangent to the boundary of the reachable set $R(t)$ at the point $\gamma_i(t)$, oriented toward the exterior of $R_\infty^\Gamma$ (see Fig. 8).*

*Then there exists a constant* $\lambda_0 \geq 0$ *and nontrivial solutions to the adjoint equations*

$$\dot{p}_i(t) \;=\; \frac{\langle \dot\gamma_i(t)\,,\;\dot{\mathbf{e}}_i(t)\rangle}{\langle \gamma_i(t)\,,\;\mathbf{e}_i(t)\rangle}\, p_i(t) - \lambda_0\, h(\gamma_i(t))\,\alpha(\gamma_i(t)) \tag{5.17}$$

*for* $i = 1, \ldots, \nu$*, such that the functions*

$$V_i(t) \;\doteq\; \left\langle \frac{\dot\gamma_i(t)}{|\dot\gamma_i(t)|}\;,\;\mathbf{e}_i(t)\right\rangle^{-1}\cdot p_i(t) \tag{5.18}$$

*all coincide, at each time* $t \in [t_1,\, t_2]$*.*

A proof of this theorem was first obtained in [5]. For a more general result, valid for general functions $\beta(\cdot)$ and $\psi(\cdot)$, we refer to [11].

With reference to Fig. 8, the inner product in (5.17) is related to the angle $\theta_i$ between the barrier and the fire front. Namely

$$\left\langle \frac{\dot\gamma_i(t)}{|\dot\gamma_i(t)|}\;,\;\mathbf{e}_i(t)\right\rangle \;=\; \cos\theta_i(t)\,.$$

According to Theorem 8, the quantity

$$V(t) \;\doteq\; \left(\frac{p_i(t)}{\cos\theta_i(t)} - \beta\right)\sigma \tag{5.19}$$

is independent of $i = 1, \ldots, \nu$. For a suitable choice of the boundary conditions for the $p_i$ in (5.17), the function $V(\cdot)$ can be interpreted as the *instantaneous value of time* [11].

## 5.4 Necessary conditions at junctions

The optimality conditions derived in the previous subsections provide a set of ODEs satisfied by the optimal arcs. In order to uniquely determine these arcs, one needs suitable boundary conditions. These can be obtained by studying what happens at points where arcs originate, or at junctions between arcs of different types. Referring to Fig. 9, we recall here two results proved in [5].

- *A barrier containing two boundary arcs* $\gamma_1, \gamma_2$ *originating from the same point cannot be optimal.*

- *If an optimal barrier contains a free arc* $\gamma_1$ *and a boundary arc* $\gamma_2$ *with a point in common, then they must be tangent at the point of junction.*

Further necessary conditions, valid at junction points, can be found in [11]. In particular, at the time $t$ when a free arc joins a boundary arc, the two expressions for the instantaneous value of time (5.4) and (5.19) coincide.

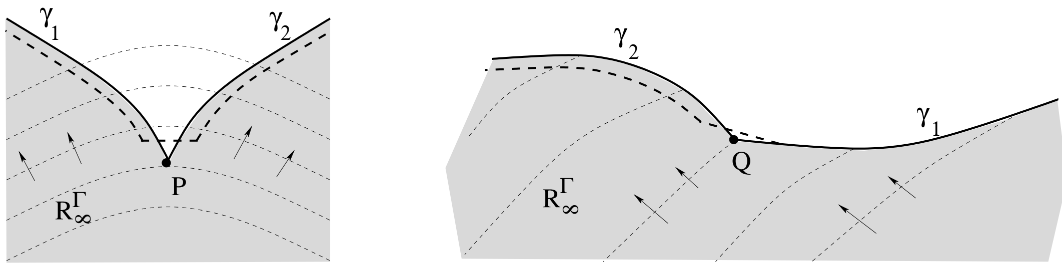


Figure 9: Left: The boundary arcs $\gamma_1, \gamma_2$ originate at the same point $P$. They can be replaced by a new admissible barrier (the thick dotted line), reducing the total cost. Right: the free arc $\gamma_1$ and the boundary arc $\gamma_2$ join non-tangentially at the point $Q$. They can again be replaced by a new admissible barrier (the thick dotted line), reducing the total cost. Here the thin lines are the level set of the minimum time function, while the arrows give the direction of fire propagation.

# 6 Sufficient conditions for optimality

We shall consider the isotropic case, where the fire is initially burning on the open unit disc $B_1 \subset I\!R^2$ and propagates with unit speed in all directions. Let a construction speed $\sigma > 2$ and constant costs $\alpha > 0$, $\beta \geq 0$ be given. As suggested by the necessary conditions derived in [5, 11], the optimal barrier $\Gamma^*$ which minimizes the cost $J$ in (2.5) should consist of an arc of circumference and two arcs of logarithmic spirals.

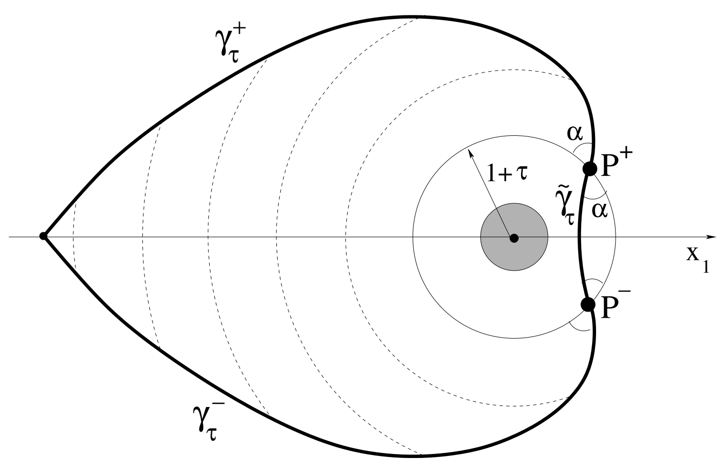


Figure 10: Construction of the barrier $\Gamma_\tau = \tilde{\gamma}_\tau \cup \gamma_\tau^+ \cup \gamma_\tau^-$.

We now describe more precisely this barrier (Fig. 10). For every $\tau > 0$ small enough, there exists a unique arc of circumference $\tilde{\gamma}_\tau$ with the following properties:

(i) $\tilde{\gamma}_\tau$ is symmetric w.r.t. the $x_1$-axis and has length $m_1(\tilde{\gamma}_\tau) = \sigma\tau$

(ii) the endpoints $P^-, P^+$ lie on the circumference $\{|x| = 1 + \tau\}$

(iii) the angle $\alpha$ between the two circumferences at $P^-$ and at $P^+$ satisfies $\sin\alpha = 2/\sigma$.

In addition, consider the two arcs of logarithmic spirals $\gamma_\tau^+, \gamma_\tau^-$, defined as

$$\gamma_\tau^\pm = \Big\{(r\cos\theta,\ \pm r\sin\theta);\quad r = r_0\, e^{\lambda\theta}, \qquad r \geq 1+\tau, \quad \theta \leq \pi\Big\}. \tag{6.1}$$

Here $\lambda = \sqrt{\frac{4}{\sigma^2-4}}$, while the constant $r_0$ is chosen so that the two arcs start from the points $P^+, P^-$ respectively. The above choice of the constants $\alpha, \lambda$ implies that the arcs $\gamma_\tau^\pm$ meet the circular arc $\tilde{\gamma}_\tau$ tangentially at $P^\pm$.

For every fixed $\tau$, the union $\Gamma_\tau \doteq \tilde{\gamma}_\tau \cup \gamma_\tau^+ \cup \gamma_\tau^-$ of these three arcs is a simple closed curve. By minimizing the cost $J(\Gamma_\tau)$ over the scalar parameter $\tau$, we single out the curve

$$\Gamma^* \doteq \Gamma_{\tau^*}, \qquad \tau^* \doteq \underset{\tau>0}{\operatorname{argmin}} \ J(\Gamma_\tau). \tag{6.2}$$

At present, it is not known whether the barrier $\Gamma^*$ is a global minimizer. A partial result in this direction was recently proved in [12].

**Theorem 9.** *The barrier $\Gamma^*$ is optimal within the family of all admissible Jordan curves with finite length.*

In other words, if $\Gamma$ is any simple closed curve with length $m_1(\Gamma) < \infty$, which is admissible according to (1.8), then

$$J(\Gamma^*) \leq J(\Gamma). \tag{6.3}$$

In fact, one can show that the inequality in (6.3) is strict, except when $\Gamma$ is the image of $\Gamma^*$ by a rotation around the origin. Observe that, if the construction cost $\beta$ is strictly positive, then any optimal curve must have finite length.

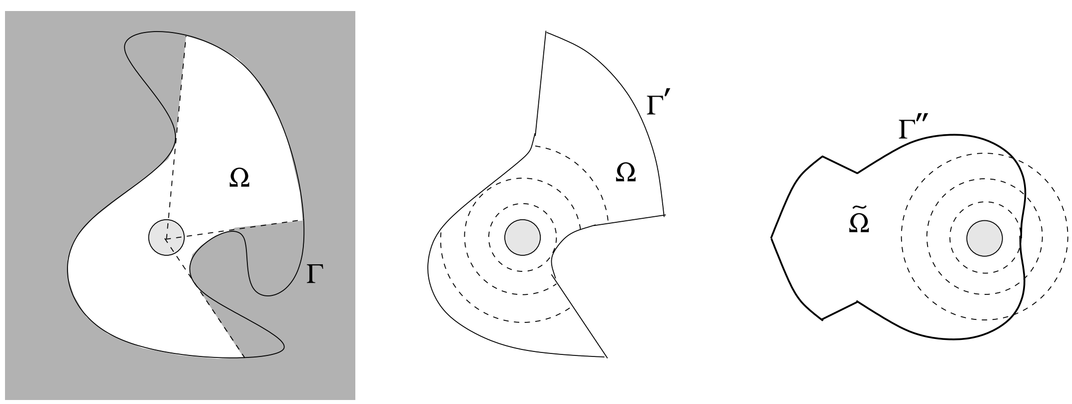


Figure 11: The mains steps in the proof of Theorem 9.

Referring to Fig. 11, we sketch the three main steps in the proof of Theorem 9.

**1.** Consider any admissible, simple closed curve $\Gamma$. Thinking of $\Gamma$ as a wall which blocks the light, let $\Omega \subset I\!R^2$ be the set of points illuminated by a light source located at the origin $0 \in I\!R^2$. Then the boundary $\Gamma' \doteq \partial\Omega$ is also an admissible, simple closed curve. Moreover, $J(\Gamma') \leq J(\Gamma)$.

**2.** By the previous construction, the domain $\Omega$ is star-shaped. Indeed, it can be represented in polar coordinates as

$$\Omega = \Big\{(r\cos\theta, \ r\sin\theta); \quad r \leq r(\theta) \quad \theta \in [-\pi, \pi]\Big\} \tag{6.4}$$

for some (possibly discontinuous) function $\theta \mapsto r(\theta)$ having bounded variation. Let $\theta \mapsto \tilde{r}(\theta)$ be the symmetric, non-decreasing rearrangement of the map $\theta \mapsto r(\theta)$. Consider the symmetric domain

$$\widetilde{\Omega} = \Big\{(r\cos\theta\,,\ r\sin\theta)\,;\quad r \leq \tilde{r}(\theta) \quad \theta \in [-\pi,\,\pi]\Big\}. \tag{6.5}$$

The boundary $\Gamma'' \doteq \partial\widetilde{\Omega}$ of this new domain is an admissible, simple closed curve. Moreover, $J(\Gamma'') \leq J(\Gamma')$.

**3.** Since the barrier $\Gamma''$ is optimal within the class of simple closed curves, it must satisfy the necessary conditions stated in Section 5. As a result, we conclude that the symmetric curve $\Gamma''$ is a concatenation of

- free arcs, which must be arcs of circumferences

- boundary arcs, which must be arcs of logarithmic spirals of the form

$$\Big\{(r\cos\theta\,,\ \pm r\sin\theta)\,;\quad r = c\,e^{\lambda\theta}, \qquad r \in [r_1,\,r_2]\Big\}, \qquad \text{with} \quad \lambda = \sqrt{\frac{4}{\sigma^2-4}}\,.$$

Moreover, each arc must join tangentially with the previous one.

A simple geometric argument now shows that the curves $\Gamma_\tau$ considered at (6.2), and their images under rigid rotations around the origin, are the only symmetric curves with $\theta \mapsto r(\theta)$ nondecreasing, which satisfy all these necessary conditions. This concludes the proof. For all details we refer to [12].

# 7 Numerical computation of optimal barriers

A first algorithm for the computation of optimal barriers was developed in [9]. This construction relies on two basic assumptions about the optimal barrier, namely:

(i) The optimal barrier $\Gamma$ is a simple closed curve, which can be represented as

$$\Gamma = \{(r\cos\theta, r\sin\theta)\,;\quad r = \rho(\theta)\}$$

for some positive continuous function $\rho(\cdot)$. In particular, the corresponding set $R^\Gamma_\infty$ burned by the fire is star-shaped.

(ii) The optimal barrier does not contain delaying arcs. Hence, restricted to the set $R^\Gamma_\infty$ enclosed by the curve $\Gamma$, the minimal time function satisfies $T^\Gamma(x) = T(x)$. In particular, the function $T(\cdot)$ can be determined in advance, before the computation of $\Gamma$.

Under the assumptions (i)-(ii), barriers can be approximated by polygonal curves, whose vertices lie on a family of rays through the origin (see Fig. 12). To define one of these approximations, we fix an integer $n \geq 3$ and consider closed polygonal curves $\mathcal{P}$ having vertices at points $Q_k = (\rho_k \cos k\theta,\ \rho_k \sin k\theta)$, $0 \leq k \leq n$. Here $\rho_0, \ldots, \rho_n$ are positive numbers with $\rho_0 = \rho_n$, while $\theta \doteq 2\pi/n$. We call $S_k$ the edge of the polygonal joining $Q_{k-1}$ with $Q_k$. Its length is computed by

$$\|S_k\| = \sqrt{\rho_k^2 + \rho_{k-1}^2 - 2\rho_k\rho_{k-1}\cos\theta}\,.$$

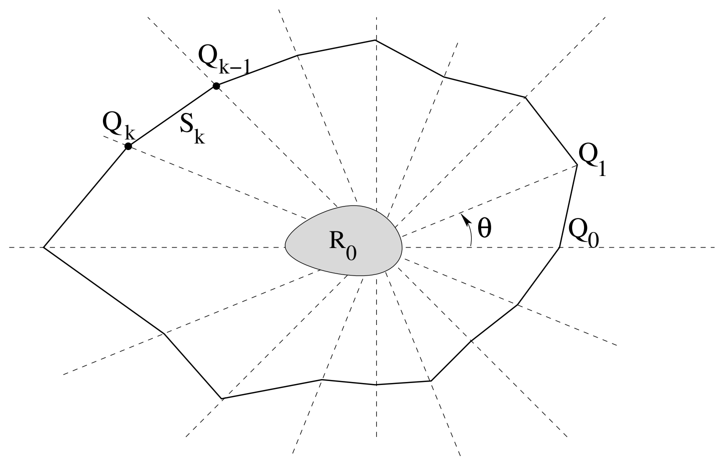


Figure 12: A polygonal approximation to the optimal barrier.

Setting $\vec{\rho} \doteq (\rho_1, \ldots, \rho_n)$, the area enclosed by the polygonal is computed as

$$A(\vec{\rho}) \;=\; \frac{1}{2}\sin\theta \cdot \sum_{k=1}^{n} \rho_k\rho_{k-1}\,,$$

while the total length is

$$L(\vec{\rho}) \;=\; \sum_{k=1}^{n} \sqrt{\rho_k^2 + \rho_{k-1}^2 - 2\rho_k\rho_{k-1}\,\cos\theta}.$$

Moreover, the minimum time needed by the fire to reach some point on the segment $S_k$ is defined as

$$T(S_k) \;\doteq\; \inf\;\{T(x)\,;\; x \in S_k\}.$$

A discrete approximation to the constrained optimization problem (2.5), (1.8) with $\alpha(x) \equiv \alpha$ and $\beta(x) \equiv \beta$ constant, is given by

$$\min_{\vec{\rho}}\;\Big\{\alpha \cdot A(\vec{\rho}) + \beta \cdot L(\vec{\rho})\Big\}\,, \tag{7.1}$$

subject to the family of constraints

$$\sum_{k=1}^{m} \|S_{i_k}\| - \sigma \cdot T(S_{i_m}) \;\le\; 0 \qquad \text{for all} \quad m = 1, 2, \cdots, n. \tag{7.2}$$

Here $(i_1, i_2, \ldots, i_n)$ is some permutation of the indices $(1, 2, \ldots, n)$ such that

$$T(S_{i_1}) \;\le\; T(S_{i_2}) \;\le\; \quad\cdots\quad \;\le\; T(S_{i_n}).$$

We denote by $\mathcal{F}_{\vec{\rho}}$ the collection of all such possible permutations, for a given $\vec{\rho}$. Observe that, if the constraints (7.2) are satisfied for one permutation $\alpha \in \mathcal{F}_{\vec{\rho}}$, then they are necessarily satisfied for every permutation $\beta \in \mathcal{F}_{\vec{\rho}}$.

In a first step, the algorithm constructs a local minimum for the finite-dimensional problem (7.1)-(7.2). In the next step, the number of vertices is doubled, replacing $\theta$ with $\theta/2$. A further minimization process is carried out, starting with the local minimizer constructed at the previous step, etc...
The results of some numerical experiments using this algorithm are reported in [9].

## 8 Open problems

**1 - Isotropic blocking problem.** On the entire plane $\mathbb{R}^2$, assume that $F(x) \equiv \overline{B}_1$, so that the fire propagates with unit speed in all directions. Let the barrier be constructed at speed $\sigma > 0$. For any (nonempty, open, bounded) initial set $R_0$, it is then known that a blocking strategy exists if $\sigma > 2$, and does not exist if $\sigma \leq 1$. The case $1 < \sigma \leq 2$ remains open. A reasonable conjecture is that, if $\sigma \leq 2$, then the fire cannot be blocked. In this direction, the following observations may be useful.

- As shown in Section 3, it is not restrictive to assume that the initial set $R_0 \subset \mathbb{R}^2$ is the unit disc centered at the origin.

- If the barrier $\Gamma$ is a simple closed curve, then the estimate (3.5) can be replaced by

$$m_1(\gamma_2) + m_1(\gamma_3) \ \leq \ m_1(\Gamma).$$

The same is true if $\Gamma$ is the union of finitely many simple closed curves. In this case, the argument given at (3.4)–(3.6) already implies that the construction speed for a blocking strategy must be $\sigma > 2$. We conclude that, if a blocking strategy exists for $\sigma \leq 2$, the barrier $\Gamma$ must also contain purely delaying arcs.

- Since the fire front propagates with unit speed in the normal direction, to be effective any boundary arc $\gamma_i$ must be constructed at speed $\sigma_i(t) > 1$. Indeed, let $t \mapsto \gamma_i(t)$ be a parameterization of this arc according to the construction time, so that $T^\Gamma(\gamma_i(t)) = t$. Then (see Fig. 8), the angle $\theta_i$ between the barrier and the level curve $\{x\,;\ T^\Gamma(x) = t\}$ is determined by

$$|\dot\gamma_i(t)| \ = \ \sigma_i(t) \ = \ \frac{1}{\sin\theta_i(t)}\,.$$

If $\sigma \leq 2$, then only one boundary arc can be constructed, at any given time.

For example, assume that the initial set $R_0$ is the unit disc. Then the construction of one single arc $\gamma_1$ along the edge of the advancing fire front produces a spiraling curve (see Fig. 13). As shown in [6], this curve eventually closes on itself, blocking the fire, only if the construction speed is $|\dot\gamma_1(t)| = \sigma > \sigma^\dagger = 2.614430844\ldots$ More precisely, the constant $\sigma^\dagger$ is here defined as

$$\sigma^\dagger \ \doteq \ \max_{\lambda>0} \ \frac{1}{\cos\theta(\lambda)}\,,$$

where $\lambda \mapsto \theta(\lambda) \in [2\pi,\ 5\pi/2]$ is the function implicitly defined by

$$e^{\lambda\theta}\cos\theta - 1 \ = \ \lambda e^{\lambda\theta}\sin\theta\,.$$

In particular, this analysis shows that a strategy constructing a single spiraling wall cannot block the fire if $\sigma \leq 2$.

For additional remarks on this problem, and a cash prize for its solution, one may look at the author's web page: http://www.math.psu.edu/bressan/

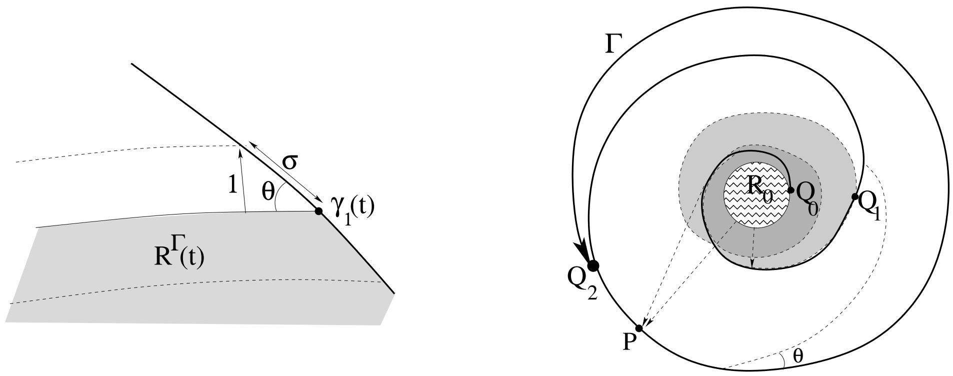


Figure 13: Left: if the fire front advances with unit speed, and the barrier $\gamma_1$ is constructed at speed $\sigma$, then the angle $\theta$ is determined by $\sin\theta = 1/\sigma$. Right: a fire starting on the unit disc is encircled by a spiraling barrier. Shaded areas denote regions reached by the fire at various times. The first portion of the wall, between $Q_0$ and $Q_1$, is exactly a logarithmic spiral. This is a delaying arc. For example, the minimum time needed to reach the point $P$ without crossing $\Gamma$ is $T^\Gamma(P) > T(P) = |P| - 1$. As proved in [6], if $\sigma > \sigma^\dagger \approx 2.614430844$, then the spiraling barrier eventually closes on itself.

**2 - Existence of optimal strategies.** The existence theorem proved in [7, 15] already covers very general situations. However, it relies on one key assumption: every velocity set $F(x)$ should contain a neighborhood of the origin. In other words, the fire should propagate with strictly positive speed in every direction. This assumption guarantees the Lipschitz continuity of the minimum time function $T^\Gamma$, away from the barrier.

It is not known whether Theorem 6, on the existence of optimal strategies, remains valid if we only assume that $0 \in F(x)$ for every $x \in \mathbb{R}^2$. This extension would cover situations where the wind pushes the fire only in one direction. For example, consider the "ice-cream cone" case:

$$F(x) \;=\; \Big\{(\lambda x_1\,,\,\lambda x_2)\,;\quad (x_1-2)^2 + x_2^2 \;\leq\; 1\,,\quad \lambda \in [0,1]\Big\} \;\subset\; \mathbb{R}^2.$$

**3 - Sufficient conditions for optimality.** At the present date, not one single example is known of a blocking strategy which is provably optimal.

In the isotropic case, according to Theorem 9, the barrier $\Gamma^*$ consisting of an arc of circumference and two arcs of logarithmic spirals is the one which encloses the minimum area, among all simple closed curves satisfying the admissibility condition (2.2). One conjectures that $\Gamma^*$ is the global minimizer among all admissible curves, regardless of their topological structure.

In the classical setting of Calculus of Variations and optimal control, sufficient conditions for optimality are obtained by studying the value function [4, 8, 13, 16]. If the state space is finite-dimensional, this value function can be often characterized as the unique (viscosity) solution to a Hamilton-Jacobi PDE.

For a dynamic blocking problem, the "state" of the system at time $t \geq 0$ can be described by the couple $(R^\gamma(t),\ \gamma(t))$. Here $R^\gamma(t)$ is the set burned by the fire at time $t$, as in (1.11), while $\gamma(t)$ is the portion of the barrier constructed up to time $t$. One can now introduce a value function: $V = V(R_0, \gamma_0)$, defined as the infimum among the costs of all admissible strategies, assuming that at the initial time $t = 0$ the fire is burning on the region $R_0$ and a barrier is already in place along the rectifiable set $\gamma_0$.

We remark that the space of all couples $(R_0, \gamma_0)$, where $R_0 \subset I\!R^2$ is Lebesgue measurable and $\gamma_0 \subset I\!R^2$ is a rectifiable set, does not have the structure of a vector space. A characterization of the value function $V$ in terms of a PDE is out of the question. Yet, it may be of interest to study some properties of the function $V$, possibly related to a dynamic programming principle.

An alternative approach relies on the observation that the couple $(R^\gamma(t), \gamma(t))$ can be recovered from the truncated minimum time function $x \mapsto \min\{T^\gamma(x),\ t\}$, which is a Special function of Bounded Variation. One may thus consider a corresponding value function $V(\cdot)$ defined on elements of the functional space SBV.

**4 - Regularity.** According to the existence results proved in [7, 15], under the general hypotheses of Theorem 6 the optimal barrier $\Gamma^*$ is a complete rectifiable set, which can be decomposed as

$$\Gamma^* = \left(\bigcup_{i\geq 1} \Gamma_i\right) \cup \Gamma_0 \,.$$

Here the countably many sets $\Gamma_i$ are disjoint, compact, and connected, while $\Gamma_0$ is a set with 1-dimensional Hausdorff measure $m_1(\Gamma_0) = 0$.

Unfortunately, the above result does not allow us to derive any of the necessary conditions for optimality in [5, 11, 23], which require stronger regularity assumptions. Indeed, at the present date these optimality conditions are known to hold only for an optimal barrier which is the union of finitely many regular arcs.

It would be of interest to close this gap, proving that any optimal barrier $\Gamma^*$ must satisfy additional regularity conditions. In particular, assuming that the initial set $R_0$ has a smooth boundary and the cost functions $\alpha(\cdot)$, $\beta(\cdot)$ are smooth, the following questions naturally arise:

- What is the regularity of an optimal barrier? Is $\Gamma^*$ the union of finitely many $\mathcal{C}^1$ arcs?
- If $R_0$ is connected, does this imply that the optimal barrier $\Gamma^*$ is connected?
- Does the optimal barrier contain purely delaying arcs $\gamma_i \subset \Gamma^d \setminus \Gamma^b$ ?

All the above problems are open even in the case where $\alpha, \beta$ are constant, and $R_0$ is a convex set. In particular, no example is known of an optimal barrier containing a purely delaying arc.

Results in this direction would be relevant also toward an understanding of the isotropic blocking problem. Indeed, for a given speed $\sigma \leq 2$, assume that a blocking strategy exists. Then there exists also an optimal blocking strategy, minimizing the total area of the burned region. Hence the search for blocking strategies can be restricted to barriers which satisfy necessary optimality conditions. Of course, this approach is useful only if some a priori regularity of optimal barriers is known, in order to apply the necessary conditions in [5, 11, 23].

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