Trajectories of Differential Inclusions with State Constraints

Alberto Bressan and Giancarlo Facchi

Department of Mathematics, Penn State University
University Park, Pa. 16802, U.S.A.

e-mails: bressan@math.psu.edu , facchi@math.psu.edu

April 12, 2011

Abstract

The paper deals with solutions of a differential inclusion $\dot{x} \in F(x)$ constrained to a compact convex set Ω. Here F is a compact, possibly non-convex valued, Lipschitz continuous multifunction, whose convex closure coF satisfies a strict inward pointing condition at every boundary point $x \in \partial\Omega$. Given a reference trajectory $x^*(\cdot)$ taking values in an ε-neighborhood of Ω, we prove the existence of a second trajectory $x : [0, T] \mapsto \Omega$ which satisfies $\|x - x^*\|_{W^{1,1}} \leq C\varepsilon(1 + |\ln \varepsilon|)$. As shown by an earlier counterexample, this bound is sharp.

1 Introduction

Let $F : I\!R^n \rightsquigarrow I\!R^n$ be a Lipschitz continuous multifunction with compact values. A Carathéodory solution of the differential inclusion

$$\dot{x} \in F(x) \tag{1.1}$$

will be called an F-trajectory. By definition, this is an absolutely continuous map $x(\cdot)$ from a time interval $[a, b]$ into $I\!R^n$, whose time derivative $\dot{x}(t) = \frac{d}{dt}x(t)$ satisfies the differential inclusion (1.1) at a.e. time t.

Given a closed convex set $\Omega \subset I\!R^n$, we are interested in F-trajectories that remain inside Ω. More precisely, let $t \mapsto x^*(t)$ be an F-trajectory which remains within an ε-neighborhood of Ω, so that

$$d(x^*(t), \Omega) \leq \varepsilon \qquad \text{for all } t \in [0, T]\,. \tag{1.2}$$

Moreover, let an initial data x_0 be given, satisfying

$$x_0 \in \Omega\,, \qquad |x_0 - x^*(0)| \leq \varepsilon\,. \tag{1.3}$$

We seek a second F-trajectory $x : [0, T] \mapsto \Omega$ which satisfies the initial condition

$$x(0) = x_0 \tag{1.4}$$

and remains close to $x^*(\cdot)$ throughout the interval $[0,T]$.

Since we require that $x(t)$ be inside Ω at every time t, a natural assumption is that, at each boundary point $x\in\partial\Omega$, the convex hull of the set of velocities $F(x)$ should contain some vector $\mathbf{a}$ which points strictly in the interior of Ω. If this condition holds, from earlier literature the following facts are known:

- If Ω is a compact set with $\mathcal{C}^1$ boundary, then for every F-trajectory $x^*(\cdot)$ satisfying (1.2) there exists an F-trajectory $x:[0,T]\mapsto\Omega$ with initial data (1.4) such that

$$\|\dot x-\dot x^*\|_{\mathbf{L}^1([0,T])}\ \leq\ C\varepsilon\,, \tag{1.5}$$

  for some constant C independent of x^* and ε. See [15] for details.

- On the other hand, if the boundary of Ω is not smooth, then the estimate (1.5), which is linear in ε, cannot hold in general. This was proved by the counterexample in [4].

- In the special case where $F(x)\equiv F$ is a fixed compact set, independent of x, and Ω is the intersection of two closed half-spaces, the analysis in [5] has proved that (1.5) can be replaced by the weaker estimate

$$\|\dot x-\dot x^*\|_{\mathbf{L}^1([0,T])}\ \leq\ C\varepsilon\,|\ln\varepsilon|\qquad\qquad (0<\varepsilon<1/4)\,. \tag{1.6}$$

Aim of the present paper is to show that the "$\varepsilon\ln\varepsilon$" estimate (1.6) can be achieved for a fully general class of Lipschitz multifunctions F and compact sets Ω. Our main assumption will be the following inward-pointing condition.

(A1) For every $x\in\Omega$ one has $\quad coF(x)\ \cap\ int\Big(T_\Omega(x)\Big)\ \neq\ \emptyset.$

In other words, for every point x on the boundary of Ω, the convex hull of the velocity set $F(x)$ should contain at least one vector $\mathbf{a}$ which lies in the interior of the tangent cone $T_\Omega(x)$. We recall that the tangent cone to the set Ω at the point x is defined as

$$T_\Omega(x)\ \doteq\ \Big\{y\in I\!\!R^n\,;\quad \lim_{h\to 0+}\quad h^{-1}\,d(x+hy\,;\ \Omega)=0\Big\}.$$

Our main result is the following.

Theorem 1. *Fix $T>0$, let $\Omega\subset I\!\!R^n$ be a compact, convex domain, and let $F:I\!\!R^n\mapsto I\!\!R^n$ be a Lipschitz continuous, compact valued multifunction, which satisfies (A1). Then there exists a constant K such that the following holds.*

Given any F-trajectory $x^*:[0,T]\mapsto I\!\!R^n$ and any initial point $x_0\in\Omega$, calling

$$\varepsilon\ \doteq\ |x_0-x^*(0)|\ +\ \max_{t\in[0,T]}\ d(x^*(t),\,\Omega), \tag{1.7}$$

there exists a second F-trajectory $x:[0,T]\mapsto\Omega$ with $x(0)=x_0$ and such that

$$\|x-x^*\|_{\mathcal{C}^0([0,T])}\ \leq\ K\varepsilon\,, \tag{1.8}$$

$$\|\dot x-\dot x^*\|_{\mathbf{L}^1([0,T])}\ \leq\ K\varepsilon(1+|\ln\varepsilon|). \tag{1.9}$$

Remark 1. If $\varepsilon = 0$ in (1.7), one can simply take $x(t) = x^*(t)$. When ε is large, the result is also trivial, by the compactness of Ω. The above estimates have interest when ε is positive but small, say $0 < \varepsilon < 1/4$. In this case, (1.9) is equivalent to (1.6). Estimates such as (1.8)-(1.9) play a key role in determining the regularity of the value function, for optimal control problems with state constraints [4, 5, 13, 16, 15, 17].

The above theorem will be proved in several stages. In Section 2 we show that the same conclusions hold in the case $F(x) = F$ is a compact convex set, independent of x, and Ω is a closed convex set satisfying the following assumption.

(A2) There exists a non-zero vector $\mathbf{a} \in F$ and a positive number $\rho > 0$ such that, defining the closed convex cone

$$\Gamma_{\mathbf{a},\rho} \doteq \Big\{\lambda y; \quad \lambda \geq 0,\ |y - \mathbf{a}| \leq \rho\Big\} \subset I\!R^n, \tag{1.10}$$

one has

$$\Omega + \Gamma_{\mathbf{a},\rho} = \Omega\,. \tag{1.11}$$

Notice that the above assumption is clearly satisfied if Ω is any convex cone, with $F \cap int(\Omega) \neq \emptyset$. Indeed, in this case it suffices to select $\mathbf{a} \in F \cap int(\Omega)$ and choose $\rho > 0$ small enough so that the closed ball $\overline{B}(\mathbf{a}, \rho)$ centered at $\mathbf{a}$ with radius ρ is contained inside Ω.

If the assumption (A2) holds, we can write an explicit formula for the trajectory $x(\cdot)$, achieving a short, transparent proof.

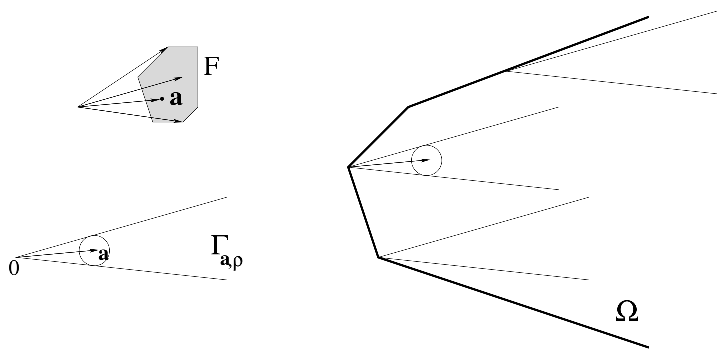

Figure 1: The set of velocities F, the cone $\Gamma_{\mathbf{a},\rho}$, and a set Ω satisfying the condition (A2).

In Sections 3–5 we prove Theorem 1 in full generality. This is achieved in three stages.

1. The case where the velocity sets $F(x)$ depend Lipschitz continuously on x is handled using a standard Gronwall type estimate.

2. The convexity assumption on the velocity sets $F(x)$ is removed using Lyapunov's theorem on the range of a vector measure [10, 14].

3. Finally, a straightforward covering argument allows us to extend the result to the case where Ω is an arbitrary compact convex domain.

The last section of paper provides some straightforward extensions of the main result. Indeed, Theorem 1 remains valid, more generally, on a domain Ω which locally coincides with the diffeomorphic image of a convex set.

We recall that the Hausdorff distance between two compact subsets $X, Y \subset I\!R^n$ is defined as

$$d_H(X,Y) \doteq \max\Big\{ \max_{x\in X} d(x,Y)\,,\ \max_{y\in Y} d(y,X)\Big\}.$$

A compact valued multifunction F is Lipschitz continuous if there exists a Lipschitz constant L such that

$$d_H(F(x),\, F(y)) \leq L\,|x-y| \qquad\qquad \text{for all } x,y\in I\!R^n. \tag{1.12}$$

The multifunction F is bounded if there exists a constant M such that

$$|v|\leq M \qquad \text{for all } v\in F(x)\,,\quad x\in I\!R^n. \tag{1.13}$$

A comprehensive introduction to the theory of set-valued functions and differential inclusions can be found in [1, 2].

2 An explicit formula

In this section, we prove a version of Theorem 1, valid in the case where the set of velocities is convex, and independent of x. In this case, the trajectory $x(\cdot)$ can be described by an explicit formula.

Lemma 1. *The conclusions of Theorem 1 hold when $F(x) = F$ is a fixed compact, convex set, and Ω is closed, convex domain satisfying the assumption (A2).*

Proof. Let $\mathbf{a}\in F$ and $\rho > 0$ be as in assumption (A2). We claim that the trajectory $x(\cdot)$ defined by

$$x(t) \doteq \begin{cases} x_0 + t\,\mathbf{a} & t\in[0,\, C\varepsilon] \\[2mm] x_0 + C\varepsilon\,\mathbf{a} + \left(1-\dfrac{C\varepsilon}{t}\right)(x^*(t)-x^*(0)) & t\in[C\varepsilon,\, T] \end{cases} \tag{2.1}$$

satisfies all requirements, provided that the constant C is chosen large enough. Clearly, C will depend on F and Ω, but not on $x_0,\ x^*(\cdot)$.

1. First, we show that the state constraint is satisfied, namely

$$x(t) \in \Omega \qquad\qquad \text{for all } t\in[0,T]\,. \tag{2.2}$$

For $t\in[0,\, C\varepsilon]$, the inclusion $x(t) = x_0 + t\mathbf{a}\in\Omega$ is obvious.

On the other hand, for $t\in[C\varepsilon,\, T]$ we have $x(t) = \tilde x(t) + C\varepsilon\mathbf{a}$, where

$$\tilde x(t) \doteq x_0 + \left(1-\frac{C\varepsilon}{t}\right)(x^*(t)-x^*(0)) = \frac{C\varepsilon}{t}x^*(0) + \left(1-\frac{C\varepsilon}{t}\right)x^*(t) + \Big(x_0 - x^*(0)\Big). \tag{2.3}$$

For notational convenience, let $B \doteq \{y \in I\!\!R^n\,;\quad |y| \leq 1\}$ be the closed unit ball in $I\!\!R^n$. The set $\Omega + \varepsilon B$ thus describes the closed ε-neighborhood around Ω. Since by assumption $x^*(t) \in \Omega + \varepsilon B$ and this ε-neighborhood is convex, it follows that

$$\frac{C\varepsilon}{t} x^*(0) + \left(1 - \frac{C\varepsilon}{t}\right)(x^*(t)) \ \in\ \Omega + \varepsilon B \qquad\qquad \text{for all } t \geq C\varepsilon\,.$$

By (1.7) one has $x_0 - x^*(0) \in \varepsilon B$. Hence $\tilde{x}(t) \in \Omega + 2\varepsilon B$, i.e.

$$\tilde{x}(t) \ =\ y(t) + 2\varepsilon u(t), \quad \text{for some } y(t) \in \Omega\,,\ u(t) \in B\,. \tag{2.4}$$

Choose $C = 2/\rho$ in (2.1). Without loss of generality, we can assume $\rho < T/2$, so that $C > T$. Then

$$x(t) \ =\ \tilde{x}(t) + C\varepsilon \mathbf{a} \ =\ y(t) + \frac{2\varepsilon}{\rho}\left(\mathbf{a} + \rho u(t)\right). \tag{2.5}$$

Clearly, $(\mathbf{a} + \rho u(t)) \in \Gamma_{\mathbf{a},\rho}$, while $y(t) \in \Omega$. By (A2) we conclude $x(t) \in \Omega$, proving (2.2).

2. Next, we check that $x(\cdot)$ is indeed an F-trajectory. On the initial time interval one has

$$\dot{x}(t) \ =\ \mathbf{a} \in F \qquad\qquad \text{for all } t \in [0, C\varepsilon[\,.$$

Moreover, for $t > C\varepsilon$ there holds

$$\dot{x}(t) \ =\ \left(\frac{C\varepsilon}{t}\right)\frac{x^*(t) - x^*(0)}{t} + \left(1 - \frac{C\varepsilon}{t}\right)\dot{x}^*(t) \ \in\ F \qquad \text{for a.e. } \ t \in [C\varepsilon,\, T]. \tag{2.6}$$

Indeed, $\dot{x}(t)$ is a convex combination of two vectors in the convex set F.

3. Toward the estimate (1.9), consider the upper bound on the velocities:

$$M \ \doteq\ \max_{v \in F} |v|\,.$$

Observe that

$$\begin{aligned}
\int_{C\varepsilon}^{T} |\dot{x}(t) - \dot{x}^*(t)|\,dt \ &=\ \int_{C\varepsilon}^{T} \left| -\frac{C\varepsilon}{t}\dot{x}^*(t) + \frac{C\varepsilon}{t^2}(x^*(t) - x^*(0))\right| dt \\
&\leq\ \int_{C\varepsilon}^{T} \frac{C\varepsilon}{t}\ M\,dt + \int_{C\varepsilon}^{T} \frac{C\varepsilon}{t^2}\ Mt\,dt \\
&=\ 2CM\varepsilon \ln\left(\frac{T}{C\varepsilon}\right) \ \leq\ 2CM\varepsilon|\ln\varepsilon|,
\end{aligned}$$

where in the last inequality we used the fact that $\varepsilon < 1$ and $C > T$. This yields

$$\begin{aligned}
\|\dot{x}(\cdot) - \dot{x}^*(\cdot)\|_{\mathbf{L}^1([0,T])} \ &\leq\ \int_0^{C\varepsilon} |\dot{x}(t) - \dot{x}^*(t)|\,dt + \int_{C\varepsilon}^{T} |\dot{x}(t) - \dot{x}^*(t)|\,dt \\
&\leq\ C\varepsilon\, 2M + 2CM\varepsilon|\ln\varepsilon| \ =\ 2CM\varepsilon(1 + |\ln\varepsilon|)\,,
\end{aligned}$$

proving (1.9) with $K = 2CM$.

4. It remains to prove the estimate (1.8). For $t \in [0,\, C\varepsilon]$ one has

$$|x(t) - x^*(t)| \;=\; |x_0 + t\mathbf{a} - x^*(t)| \;\leq\; |x_0 - x^*(0)| + |x^*(0) - x^*(t)| + t|\mathbf{a}|$$

$$\leq\; \varepsilon + MC\varepsilon + MC\varepsilon \;=\; (1 + 2MC)\varepsilon.$$

On the other hand, for $t \in [C\varepsilon,\, T]$ there holds

$$|x(t) - x^*(t)| \;=\; \left| C\varepsilon\mathbf{a} + x_0 - x^*(0) - \frac{C\varepsilon}{t}(x^*(t) - x^*(0)) \right|$$

$$\leq\; C\varepsilon|\mathbf{a}| + \varepsilon + MC\varepsilon \;=\; (1 + 2MC)\varepsilon.$$

Hence (1.8) holds with $K = 1 + 2MC$. □

Remark 2. Choosing the larger constant $C = 4/\rho$ in (2.1), the estimate (2.2) can be improved to

$$B(x(t), 2\varepsilon) \;\subseteq\; \Omega \qquad \text{for all } t \in [C\varepsilon,\, T]\,. \tag{2.7}$$

Indeed, if $|\xi| < \varepsilon$ and $C = 4/\rho$, the estimate (2.5) can be replaced by

$$x(t) + \xi \;=\; (\tilde{x}(t) + \xi) + C\varepsilon\mathbf{a} \;\in\; \Omega + 4\varepsilon B + \frac{4\varepsilon}{\rho}\mathbf{a} \;\subseteq\; \Omega + \Gamma_{\mathbf{a},\rho} \;=\; \Omega\,. \tag{2.8}$$

3 Some auxiliary results

In order to extend the result in Lemma 1 to the case of a Lipschitz continuous multifunction F, possibly with non-convex values, some tools from measure theory and set-valued analysis will be needed. For convenience of the reader, these results are collected in the present section.

We first recall a version of Lyapunov's theorem on the range of a non-atomic vector measure. For a proof, see [10, 14].

Lemma 2. *Let $g_1, \ldots, g_N : [a,b] \mapsto I\!R^n$ be Lebesgue integrable functions. Let $\theta_1, \ldots, \theta_N : [a,b] \mapsto [0,1]$ be measurable functions, with $\sum_{i=1}^N \theta_i(t) \equiv 1$. Then there exists a partition of the interval $[a,b]$ into disjoint measurable subsets $J_1, \ldots, J_N$ such that*

$$meas(J_k) \;=\; \int_a^b \theta_k(t)\,dt \qquad \textit{for all } k = 1, \ldots, N\,, \tag{3.1}$$

$$\int_a^b \sum_{i=1}^N \theta_i(t) g_i(t)\,dt \;=\; \sum_{i=1}^N \int_{J_i} g_i(t)\,dt\,. \tag{3.2}$$

For convenience we shall use the notation

$$B_0(v,r) \;\doteq\; \begin{cases} B(v,r) \;\doteq\; \{w \in I\!R^n\,;\;\; |w - v| < r\} & \text{if} \quad r > 0\,, \\ \qquad\qquad\qquad\quad \{v\} & \text{if} \quad r = 0\,. \end{cases} \tag{3.3}$$

Moreover, an upper bar indicates the closure of a set. A version of the next result can be found in [7].

Lemma 3. *Let $F : I\!R^n \rightsquigarrow I\!R^n$ be a multifunction with compact, possibly non-convex values, Lipschitz continuous with constant L. Let $J \subset I\!R$ be a compact set of times and let $t \mapsto \xi(t)$, $t \mapsto v(t)$ be continuous functions on J, such that*

$$v(t) \in F(\xi(t)) \qquad \text{for all } t \in J\,. \tag{3.4}$$

Then the multifunction

$$G(t,x) \doteq \overline{B_0\Big(v(t),\, 2L|x - \xi(t)|\Big) \cap F(x)} \tag{3.5}$$

is lower semicontinuous on $J \times I\!R^n$, with compact, nonempty values.

Proof. For every (t,x), by (3.4) and the Lipschitz continuity of F, one can find an element $y \in F(x)$ such that $|y - v(t)| \leq L|x - \xi(t)|$. Therefore the right hand side of (3.5) is non-empty.

To show that G is lower semicontinuous, for any open set $V \subset I\!R^n$ we need to prove that the set

$$V_G \doteq \{(t,x) \in J \times I\!R^n\,;\quad G(t,x) \cap V \neq \emptyset\}$$

is open in $J \times I\!R^n$. Assume $(\tilde t, \tilde x) \in V_G$. We consider two cases.

Case 1: $\tilde x = \xi(\tilde t)$. By definition, this means $v(\tilde t) \in V$. Since V is open, there exists $\rho > 0$ and

$$B(v(\tilde t),\, \rho) \subseteq V.$$

By continuity, there exists $\delta > 0$ such that

$$B\Big(v(t),\; 2L|x - \xi(t)|\Big) \subseteq B(v(\tilde t),\, \rho)\,, \qquad F(x) \cap B(v(\tilde t), \rho) \neq \emptyset\,,$$

whenever

$$(t,x) \in J \times I\!R^n, \qquad |t - \tilde t| < \delta\,, \qquad |x - \tilde x| < \delta\,. \tag{3.6}$$

Hence all points (t,x) satisfying (3.6) lie in V_G, and $(\tilde t, \tilde x)$ is in the relative interior of V_G.

Case 2: $\tilde x \neq \xi(\tilde t)$. Since V is open, there exists $\rho > 0$ and $\tilde y \in F(\tilde x)$ such that

$$B(\tilde y, \rho) \subseteq V \cap B\Big(v(t),\; 2L|\tilde x - \xi(t)|\Big).$$

By continuity, there exists $\delta > 0$ such that (3.6) implies

$$B(\tilde y,\, \rho/2) \subseteq V \cap B\Big(v(t),\; 2L|x - \xi(t)|\Big)\,, \qquad F(x) \cap B(\tilde y, \rho/2) \neq \emptyset\,.$$

We again conclude that $(\tilde t, \tilde x)$ is in the relative interior of V_G. Hence V_G is relatively open. □

Next, we recall an existence theorem for lower semicontinuous differential inclusions [6, 8]. A multifunction $G = G(t,x)$ with (nonempty) compact values is called *Scorza-Dragoni lower semicontinuous* on $[0,T] \times I\!R^n$ if the following holds. For every $\varepsilon > 0$ there exists a compact set $J \subseteq [0,T]$ with $\text{meas}(J) > T - \varepsilon$ such that F is lower semicontinuous restricted to $J \times I\!R^n$. A result proved in [8] is

Lemma 4. *Let $G = G(t,x)$ be a bounded, Scorza-Dragoni lower semicontinuous multifunction on $[0,T] \times I\!R^n$. Then for every $x_0 \in I\!R^n$ the multivalued Cauchy problem*

$$\dot x(t) \in G(t, x(t)), \qquad x(0) = x_0$$

has at least one Caratheodory solution.

4 The Lipschitz continuous case

In this section we extend the arguments of Lemma 1 to the case where the multifunction F has non-convex values, possibly depending on x. The key step is the following local version of Theorem 1, valid on a sufficiently short time interval.

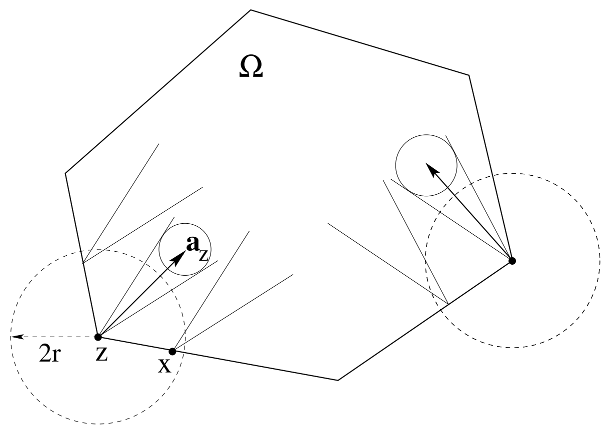

Figure 2: A set Ω which locally satisfies condition (A2).

Lemma 5. *Let $\Omega \subset I\!R^n$ be a compact, convex domain, and let $F : I\!R^n \mapsto I\!R^n$ be a Lipschitz continuous, compact valued multifunction. Let $z \in \Omega$ be a point such that*

$$coF(z) \cap int(T_\Omega(z)) \neq \emptyset \,. \tag{4.1}$$

Then there exists constants $r > 0$, $T > 0$ small enough, and K suitably large, so that the conclusions of Theorem 1 hold whenever $|x_0 - z| < r$ and $|x^*(0) - z| < r$.

Proof. 1. In the case where $z \in int\,\Omega$, the result is straightforward. Indeed, let M be an upper bound on the velocities, as in (1.13). Choose $r > 0$ so that $B(z, 2r) \subset \Omega$, and let $T = r/M$. Then every F-trajectory that starts at a point $x_0 \in B(z, r)$ will remain inside Ω during the time interval $[0, T]$. In this case, the existence of a trajectory $x(\cdot)$ satisfying

$$x(0) = x_0 \,, \qquad \|x(\cdot) - x^*(\cdot)\|_{W^{1,1}([0,T])} \leq K\varepsilon \tag{4.2}$$

follows directly from Filippov's theorem [11], valid when constraints are not present.

2. In the remainder of the proof, we thus concentrate on the case where z lies on the boundary of Ω. By assumption, we can find $\mathbf{a}_z \in coF(z)$ and $\rho > 0$ such that

$$B(\mathbf{a}_z, 3\rho) \subset T_\Omega(z) \,.$$

By the lower semicontinuity of the tangent cone to a convex set, there exists $r > 0$ such that

$$B(\mathbf{a}_z, 2\rho) \subset T_\Omega(x) \qquad \text{whenever} \quad x \in \Omega \,, \ |x - z| < 2r \,.$$

Therefore, a local version of (1.11) holds, namely

$$(\Omega + \Gamma_{\mathbf{a}_z, 2\rho}) \cap B(z, 2r) \subseteq \Omega. \tag{4.3}$$

Since the multifunction F is continuous by possibly reducing the size of $r > 0$, we can also assume the following: if $|x_0 - z| < r$ then there exists a vector $\mathbf{a} \in coF(x_0)$ with $|\mathbf{a} - \mathbf{a}_z| < \rho$. By (4.3), this implies

$$(\Omega + \Gamma_{\mathbf{a},\rho}) \cap B(z, 2r) \ \subseteq \ \Omega. \tag{4.4}$$

Since $\mathbf{a} \in coF(x_0) \subset I\!R^n$ is a linear combination of vectors in $F(x_0)$, by a theorem of Carathéodory there exist $n+1$ points $\mathbf{a}_1, \ldots, \mathbf{a}_{n+1} \in F(x_0)$ and coefficients $\theta_i \in [0,1]$ such that

$$\sum_{i=1}^{n+1} \theta_i \ = 1 \ , \qquad\qquad \sum_{i=1}^{n+1} \theta_i \mathbf{a}_i \ = \ \mathbf{a}\,. \tag{4.5}$$

By choosing $T < r/M$ we guarantee that every F-trajectory starting inside $B(z,r)$ will remain inside $B(z, 2r)$ for all $t \in [0,T]$.

3. Given $x_0 \in \Omega \cap B(z,r)$ and an F-trajectory $t \mapsto x^*(t)$ with $x^*(0) \in B(z,r)$ and $|x^*(0) - x_0| \leq \varepsilon$, choose $\mathbf{a} \in coF(x_0) \cap intT_\Omega(x_0)$ as in the previous step. Consider the auxiliary trajectory

$$y^*(t) \ \doteq \ \begin{cases} x_0 + t\,\mathbf{a} & t \in [0,\, C\varepsilon] \\ \\ x_0 + C\varepsilon\,\mathbf{a} + \left(1 - \dfrac{C\varepsilon}{t}\right)(x^*(t) - x^*(0)) & t \in [C\varepsilon,\, T] \end{cases} \tag{4.6}$$

Notice that this coincides with the definition (2.1). Since now we are not assuming that the multifunction F is constant or convex valued, in general $y^*(\cdot)$ will not be an F-trajectory. However, the steps **1.**, **3.**, and **4.** in the proof of Lemma 1 do not rely on these properties of F. Hence they remain valid in the present situation. According to Remark 2, by choosing $C = 4/\rho$, we thus achieve

$$B(y^*(t), 2\varepsilon) \ \subseteq \ \Omega \qquad\qquad \text{for all } t \in [C\varepsilon,\, T]\,, \tag{4.7}$$

together with

$$|y^*(t) - x^*(t)| \ \leq \ K_0 \varepsilon \qquad\qquad \text{for all } t \in [0,T]\,, \tag{4.8}$$

$$\|\dot y^* - \dot x^*\|_{\mathbf{L}^1([0,T])} \ \leq \ K_0 \varepsilon (1 + |\ln \varepsilon|)\,, \tag{4.9}$$

for a suitable constant K_0.

4. In the remainder of the proof, we construct an F-trajectory $x(\cdot)$ which remains close to $y^*(\cdot)$. This step yields the construction on the initial time interval $[0, C\varepsilon]$, while the next two steps deal with the remaining interval $[C\varepsilon, T]$.

Let $(\tau_k)_{k \geq 1}$ be a decreasing sequence of times, satisfying

$$\tau_1 = C\varepsilon\,, \qquad\qquad \lim_{k\to\infty} \tau_k = 0\,, \qquad\qquad 0 \ < \ \tau_k - \tau_{k+1} \ < \ \frac{\rho}{4M}\,\tau_k\,. \tag{4.10}$$

Let the vectors $\mathbf{a}_i$ and the coefficients θ_i be as in (4.5). We divide each interval $J_k \doteq [\tau_{k+1},\, \tau_k]$ into $n+1$ subintervals $J_{k,i}$ with lengths proportional to the coefficients θ_i in (4.5). On the time interval $[0, C\varepsilon]$, let the functions $y(\cdot)$ and $x(\cdot)$ provide solutions to the following Cauchy problems:

$$\begin{cases} y(0) \ = \ x_0 \\ \dot y(t) \ = \ \mathbf{a}_i \qquad\qquad \text{if } \ t \in \bigcup_{k=1}^\infty J_{k,i} \end{cases} \tag{4.11}$$

$$\begin{cases} x(0) &= x_0 \\ \dot x(t) &\in \overline{F(x(t)) \cap B_0(\mathbf{a}_i\,,\ 2L|x(t)-x_0|)} \qquad\qquad \text{if } \ t \in \bigcup_{k=1}^\infty J_{k,i}\,. \end{cases} \tag{4.12}$$

By construction, the trajectory $y(\cdot)$ satisfies

$$y(\tau_k) \ = \ y^*(\tau_k) \ = \ x_0 + \tau_k\,\mathbf{a} \qquad\qquad \text{for all } k \geq 1\,. \tag{4.13}$$

On the other hand, by Lemmas 3 and 4, the multivalued Cauchy problem (4.12) has at least one solution. If $x(\cdot)$ is any such solution, recalling the definitions of the constants L, M at (1.12)-(1.13) we obtain

$$\frac{d}{dt}\,|x(t)-y(t)| \ \leq \ 2L\,d(x(t),x_0) \ \leq \ 2LMt\,,$$

$$|x(t)-y(t)| \ \leq \ LMt^2 \ \leq \ LMC\varepsilon\, t \ \leq \ \frac{\rho}{4}\,t \qquad\qquad t\in[0,\,C\varepsilon]\,, \tag{4.14}$$

provided that $\varepsilon > 0$ is sufficiently small. The inclusion (4.4) implies

$$B(y^*(t),\ \rho t) \ = \ B(x_0 + t\mathbf{a},\, \rho t) \ \subset \ \Omega\,.$$

Therefore, if $t \in [\tau_{k+1}\,,\,\tau_k]$, from (4.14) and (4.13) it follows

$$\begin{aligned} x(t) \ &\in \ B(x(\tau_k),\ M|\tau_k-\tau_{k+1}|) \ \subseteq \ B\Big(y^*(\tau_k),\ M|\tau_k-\tau_{k+1}| + \frac{\rho t}{4}\Big) \\ &\subseteq \ B\Big(x_0+\tau_k\mathbf{a},\ \frac{\rho\tau_k}{4}+\frac{\rho\tau_k}{4}\Big) \ \subseteq \ \Omega\,. \end{aligned} \tag{4.15}$$

In particular, for $t = \tau_1 = C\varepsilon$ we have

$$|x(\tau_1) - y^*(\tau_1)| \ \leq \ LMC^2\varepsilon^2 \ \leq \ \frac{\varepsilon}{4}\,, \tag{4.16}$$

provided that $\varepsilon > 0$ is sufficiently small.

5. We now focus on the remaining subinterval $[C\varepsilon,\,T]$.

Choose $0 < \delta < \varepsilon/2M$ and construct a finite partition of the interval $[C\varepsilon,\,T]$, say

$$C\varepsilon \ = \ t_0 \ < \ t_1 \ < \ \cdots \ < \ t_N \ = \ T, \tag{4.17}$$

such that

$$t_{k+1} - t_k \ \leq \ \delta \ < \ \frac{\varepsilon}{2M} \ < \ \frac{1}{M}\,d(y^*(t_k),\partial\Omega). \tag{4.18}$$

Notice that here the last inequality follows from (4.7).

For every $0 < t \leq T$ one has

$$\frac{x^*(t)-x^*(0)}{t} \ = \ \frac{1}{t}\int_0^t \dot x^*(s)\,ds \ \in \ co\overline{B(F(x^*(0)),\,LMt)}\,.$$

By Carathéodory's theorem, for each $k = 0,\ldots,N-1$ we can thus choose vectors $\mathbf{a}_{k,i} \in \overline{B(F(x^*(0),\,LMt_k)}$ and coefficients $\theta_i \in [0,1]$, $i = 1,\ldots,n+1$ so that

$$\frac{x^*(t_k)-x^*(0)}{t_k} \ = \ \sum_{i=1}^{n+1}\theta_{k,i}\mathbf{a}_{k,i} \qquad\qquad \sum_{i=1}^{n+1}\theta_{k,i} \ = \ 1\,. \tag{4.19}$$

We now set

$$\lambda_k \doteq \frac{C\varepsilon}{t_k} \in [0,1] \tag{4.20}$$

and apply Lemma 2 (i.e., Lyapunov's theorem) on the interval $[t_k\,,\,t_{k+1}]$, in connection with the $n+2$ coefficients $(1-\lambda_k), \lambda_k\theta_{k,1}, \ldots, \lambda_k\theta_{k,n+1}$. This yields a measurable partition

$$[t_k\,,\,t_{k+1}] = J_{k,0} \cup J_{k,1} \cup \cdots \cup J_{k,n+1}$$

with the following properties:

$$\text{meas}(J_{k,0}) = (1-\lambda_k)(t_{k+1}-t_k)\,, \qquad \text{meas}(J_{k,i}) = \lambda_k\theta_{k,i}(t_{k+1}-t_k) \qquad i = 1,\ldots,n+1\,, \tag{4.21}$$

$$\int_{J_{k,0}} \dot{x}^*(s)\,ds = \left(1 - \frac{C\varepsilon}{t_k}\right)[x^*(t_{k+1}) - x^*(t_k)]\,. \tag{4.22}$$

Finally, we let $x : [C\varepsilon,\, T] \mapsto I\!R^n$ be a solution to the following differential inclusion

$$\dot{x}(t) \in \begin{cases} \overline{F(x(t)) \cap B_0\Big(\dot{x}^*(t),\ 2L|x(t) - x^*(t)|\Big)} & t \in \bigcup_k J_{k,0}\,, \\ \\ \overline{F(x(t)) \cap B_0\Big(\mathbf{a}_{k,i},\ 2L(2Mt + \varepsilon)\Big)} & t \in \bigcup_k J_{k,i}\,, \qquad i = 1,\ldots,n+1. \end{cases} \tag{4.23}$$

Here the initial value $x(C\varepsilon)$ is set to be equal to the terminal value of the F-trajectory $x : [0, C\varepsilon] \mapsto \Omega$ constructed in step **4.**

We claim that, if $T > 0$ and $\delta > 0$ are suitably small (with T depending on Ω, F but not on $x^*(\cdot)$ or ε), then the F-trajectory $x(\cdot)$ satisfies all requirements.

Indeed, consider the auxiliary function $w : [C\varepsilon, T] \mapsto I\!R^n$ defined by

$$w(C\varepsilon) = y^*(C\varepsilon) = x_0 + C\varepsilon\mathbf{a}\,, \tag{4.24}$$

$$\dot{w}(t) \in \begin{cases} \dot{x}^*(t) & t \in \bigcup_k J_{k,0} \\ \\ \mathbf{a}_{k,i} & t \in J_{k,i}\,. \end{cases} \tag{4.25}$$

As the mesh δ of our partition approaches zero, comparing (4.6) with (4.19)–(4.22), it is clear that $w(t)$ converges to $y^*(t)$, uniformly for $t \in [C\varepsilon, T]$. We can thus assume that $\delta > 0$ was chosen so that

$$|w(t) - y^*(t)| < \frac{\varepsilon}{4} \qquad \text{for all } t \in [C\varepsilon,\, T]\,. \tag{4.26}$$

Next, we work toward an estimate of $|x - w|$. For $t \in J_{k,0}$, recalling (4.8)) we obtain

$$\begin{aligned} \frac{d}{dt}|x(t) - w(t)| &\leq 2L|x(t) - x^*(t)| \leq 2L\Big(|x(t) - w(t)| + |w(t) - y^*(t)| + |y^*(t) - x^*(t)|\Big) \\ &\leq 2L|x(t) - w(t)| + (1 + K_0)\varepsilon\,. \end{aligned} \tag{4.27}$$

On the other hand, for $t \in J_{k,i}$ we have

$$\frac{d}{dt}|x(t) - w(t)| \leq 2L(2Mt + \varepsilon). \tag{4.28}$$

We now combine the two previous estimates and use Gronwall's lemma. More precisely, set $Z(t) \doteq |x(t) - w(t)|$. Then we can write

$$Z(C\varepsilon) \;\leq\; \frac{\varepsilon}{4}\,, \qquad \frac{d}{dt}Z(t) \;\leq\; 2LZ(t) + (1+K_0)\varepsilon + \psi(t),$$

where

$$\psi(t) \;=\; \begin{cases} 0 & \text{if } \; t \in \bigcup_k J_{k,0}\,, \\ 2L(2Mt+\varepsilon) & \text{if } \; t \notin \bigcup_k J_{k,0}\,. \end{cases}$$

By (4.21) and the definition of λ_k in (4.20), it follows

$$\int_{C\varepsilon}^T \psi(t)\,dt \;\leq\; \int_{C\varepsilon}^T 2L(2Mt+\varepsilon)\frac{C\varepsilon}{t}\,dt \;\leq\; \frac{\varepsilon}{5}\,,$$

provided that the time $T > 0$ is chosen small enough. Gronwall's lemma now yields

$$|x(t) - w(t)| \;=\; Z(t) \;\leq\; \left(\frac{\varepsilon}{4} + (1+K_0)\varepsilon T + \frac{\varepsilon}{5}\right)e^{2LT} \;\leq\; \frac{3\varepsilon}{4} \qquad\qquad t \in [C\varepsilon, T]\,, \tag{4.29}$$

provided that $T > 0$ is small enough. Together, the estimates (4.26), (4.29), and (4.8) yield

$$|x(t) - y^*(t)| \;\leq\; |x^*(t) - w(t)| + |w(t) - y^*(t)| \;\leq\; \varepsilon\,, \tag{4.30}$$

$$|x(t) - x^*(t)| \;\leq\; (K_0+1)\varepsilon \qquad\qquad t \in [C\varepsilon,\, T]\,. \tag{4.31}$$

Recalling (4.7), from (4.30) we also deduce

$$x(t) \in \Omega \qquad\qquad t \in [C\varepsilon, T]\,.$$

6. It now only remains to estimate the difference in the velocities $\|\dot{x} - \dot{x}^*\|_{\mathbf{L}^1}$. On the initial interval $[0, C\varepsilon]$ we trivially have

$$\int_0^{C\varepsilon} |\dot{x}(t) - \dot{x}^*(t)|\,dt \;\leq\; 2MC\varepsilon\,. \tag{4.32}$$

On the remaining interval $[C\varepsilon, T]$, recalling (4.23), (4.20), (4.21), and (4.31), we obtain

$$\begin{aligned} \int_{C\varepsilon}^T |\dot{x}(t) - \dot{x}^*(t)|\,dt \;&=\; \left(\int_{t\in\bigcup_k J_{0,k}} + \int_{t\notin\bigcup_k J_{0,k}}\right)|\dot{x}(t) - \dot{x}^*(t)|\,dt \\ &\leq\; \int_{C\varepsilon}^T 2L|x(t) - x^*(t)|\,dt + \int_{C\varepsilon}^T 2M\frac{C\varepsilon}{t}\,dt \\ &\leq\; 2MT(K_0+1)\varepsilon + 2MC\varepsilon(\ln T - \ln(C\varepsilon))\,. \end{aligned} \tag{4.33}$$

Together, (4.32) and (4.33) yield an estimate of the form (1.9), for a suitably large constant K. □

5 Proof of the Theorem.

Using Lemma 5 together with a covering argument, we can now conclude the proof of Theorem 1. Consider a covering of Ω consisting of open balls $B(z, r/2)$, where $z \in \Omega$ and r satisfies (4.3), for some $\mathbf{a}_z \in coF(z)$ and $\rho > 0$. Since Ω is compact we can extract a finite subcovering, say $\{B(x_j, r_j/2)\,;\ \ j = 1, \ldots, N\}$. For each j, by Lemma 5 there exist constants

$$0 \;<\; T_j \;\leq\; \frac{r_j}{2M}\,, \qquad\qquad K_j \;\geq\; 1\,,$$

such that the following holds. Whenever $x_0 \in B(z_j, r_j)$ and $x^* : [0, T_j] \mapsto I\!R^n$ is an F-trajectory with $x^*(0) \in B(z_j, r_j)$, one can construct a second F-trajectory $x : [0, T_j] \mapsto \Omega$ with $x(0) = x_0$ and such that

$$\|x - x^*\|_{\mathcal{C}^0([0,T_j])} \;\leq\; K_j\varepsilon\,, \tag{5.1}$$

$$\|\dot{x} - \dot{x}^*\|_{\mathbf{L}^1([0,T_j])} \;\leq\; K_j\varepsilon(1 + |\ln\varepsilon|). \tag{5.2}$$

As in (1.7), ε is here defined as

$$\varepsilon \;\doteq\; |x_0 - x^*(0)| + \max_{t\in[0,T_j]} d(x^*(t),\, \Omega),$$

We now choose constants $K_* \geq 1$ and $T_0 > 0$ such that

$$K_* \;\doteq\; \max_j\; K_j\,, \qquad\qquad T_0 \;\doteq\; \frac{T}{m} \;\leq\; \min_j\; T_j\,,$$

for some integer m large enough.

Let $x_0 \in \Omega$ and an F-trajectory $x^* : [0, T] \mapsto I\!R^n$ be given. Assume $x_0 \in B(x_{i(1)},\, r_{i(1)}/2)$, for some index $i(1) \in \{1, \ldots, N\}$. An application of Lemma 5 yields an F-trajectory $x : [0, T_0] \mapsto \Omega$ such that

$$x(0) = x_0\,, \qquad \|x - x^*\|_{\mathcal{C}^0([0,T_0])} \;\leq\; K_*\varepsilon\,, \qquad \|\dot{x} - \dot{x}^*\|_{\mathbf{L}^1([0,T_0])} \;\leq\; K_*\varepsilon(1 + |\ln\varepsilon|)\,.$$

Next, choose an index $i(2) \in \{1, \ldots, N\}$ such that $x(T_0) \in B(z_{i(2)}, r_{i(2)}/2)$. We can now apply Lemma 5 on the interval $[T_0,\, 2T_0]$, with the initial point x_0 replaced by $x(T_0)$ and with ε replaced by

$$|x(T_0) - x^*(T_0)| + \max_{t\in[T_0,\,2T_0]} d(x^*(t),\, \Omega) \;\leq\; 2K_*\varepsilon\,.$$

This yields a new trajectory $x : [T_0, 2T_0] \mapsto \Omega$ such that

$$\|x - x^*\|_{\mathcal{C}^0([T_0,\,2T_0])} \leq K_*(2K_*\varepsilon)\,, \qquad \|\dot{x} - \dot{x}^*\|_{\mathbf{L}^1([T_0,\,2T_0])} \leq K_*(2K_*\varepsilon)(1 + |\ln(2K_*\varepsilon)|)\,.$$

Iterating the above construction, after m steps we obtain an F-trajectory $x : [(m-1)T_0,\, T] \mapsto \Omega$ such that

$$\|x - x^*\|_{\mathcal{C}^0([(m-1)T_0,\,T])} \;\leq\; K_*(2K_*)^{m-1}\varepsilon\,,$$

$$\|\dot{x} - \dot{x}^*\|_{\mathbf{L}^1([(m-1)T_0,\,T])} \;\leq\; K_*(2K_*)^{m-1}\varepsilon\,(1 + |\ln(2K_*)^{m-1}\varepsilon|).$$

The concatenation of these m trajectories yields an F-trajectory $x : [0, T] \mapsto \Omega$ which satisfies $x(0) = x_0$ together with

$$\|x - x^*\|_{\mathcal{C}^0([0,T])} \;\leq\; K_*(2K_*)^{m-1}\varepsilon\,,$$

$$\|\dot{x} - \dot{x}^*\|_{\mathbf{L}^1([0,T])} \;\leq\; \sum_{i=1}^{m} K_*(2K_*)^{i-1}\varepsilon\,\left(1 + \Big|\ln((2K_*)^{i-1}\varepsilon)\Big|\right).$$

The above two estimates are equivalent to (1.8)-(1.9), for a suitable constant K. □

6 Non-convex domains

It is clear that the result stated in Theorem 1 remains valid under smooth changes of coordinates. In particular, if Ω is the image of a compact convex set $\widetilde{\Omega}$ under a smooth diffeomorphism, then the estimates (1.8)-(1.9) still hold. More generally, consider the following assumption:

(A3) For every $z \in \Omega$ there exists a $\mathcal{C}^{1,1}$ diffeomorphism defined on a neighborhood V_z of z which maps $\Omega \cap V_z$ into a convex set.

More precisely, we assume that there exists a neighborhood V_z of z and one-to-one map $\phi : V_z \mapsto I\!R^n$, such that both ϕ and ϕ^{-1} are continuously differentiable with Lipschitz continuous first derivatives, and such that the image $\phi(\Omega \cap V_z)$ is convex. Relying on this assumption, Theorem 1 can be extended to domains Ω which need not be convex.

Corollary 1. *Theorem 1 remains valid if the assumption that Ω is convex is replaced by the assumption (A3).*

Proof. For each $z \in \Omega$, let the neighborhood V_z and the $\mathcal{C}^{1,1}$ transformation of coordinates $\phi : V_z \mapsto I\!R^n$ be as in (A3). We can then rewrite the original system (1.1) locally in terms of the coordinates $y = \phi(x)$. This yields an equivalent differential inclusion

$$\dot{y} \in \widetilde{F}(y), \qquad\qquad y(t) \in \widetilde{\Omega} \doteq \phi(\Omega).$$

By the assumption (A3), the multifunction $\widetilde{F}$ is Lipschitz continuous, and the intersection of the image $\widetilde{\Omega}$ with a small neighborhood of $\phi(z)$ is convex. By the assumption (A1) it follows

$$co\widetilde{F}(\phi(z)) \cap int\Big(T_{\widetilde{\Omega}}(\phi(z))\Big) \neq \emptyset.$$

Therefore, Lemma 5 can be applied. In turn, taking a finite covering of the compact set Ω, the same argument used in Section 5 yields the result. □

Remark 3. Let $\Omega \subset I\!R^n$ be a compact domain satisfying the following property.

(A4) For every point $z \in \Omega$, there exists a neighborhood V_z of z and $\mathcal{C}^{1,1}$ scalar functions $\phi_1, \ldots, \phi_m$ (with $m \geq 0$) such that

$$\Omega \cap V_z = \{x \in V_z\,;\ \ \phi_i(x) \geq 0\,, \quad i = 1, \ldots m\} \tag{6.1}$$

and such that the gradients $\nabla\phi_1(z), \ldots, \nabla\phi_m(z)$ are linearly independent.

Then Ω satisfies the assumption (A3). Indeed, given $z \in \Omega$, we can construct $n-m$ additional linear maps $\phi_{m+1}, \ldots, \phi_n$, so that the gradients $\nabla\phi_1(z), \ldots, \nabla\phi_n(z)$ are linearly independent, and hence form a basis of $I\!R^n$. In view of (6.1), the map $x \mapsto \phi(x) = (\phi_1(x), \ldots, \phi_n(x))$ satisfies the assumptions (A3), for a suitable choice of the neighborhood V_z.

Acknowledgment. The first author was partially supported by NSF through grant DMS-0807420, "New problems in nonlinear control".

References

[1] J. P. Aubin and A. Cellina, *Differential inclusions.* Springer-Verlag, Berlin, 1984.

[2] J. P. Aubin and H. Frankowska, *Set-valued Analysis.* Birkhäuser, Boston, 1990.

[3] P. Bettiol and H. Frankowska, Regularity of solution maps of differential inclusions for systems under state constraints. *Set-Valued Analysis*, **15** (2007), 21–45.

[4] P. Bettiol, A. Bressan and R. B. Vinter, On trajectories satisfying a state constraint: $W^{1,1}$ estimates and counter-examples. *SIAM J. Control Optim.*, to appear.

[5] P. Bettiol, A. Bressan and R. B. Vinter, Estimates for trajectories confined to a cone in $I\!R^n$. *SIAM J. Control Optim.*, to appear.

[6] A. Bressan, On differential relations with lower continuous right-hand side. An existence theorem, *J. Differential Equations* **37** (1980), 89-97.

[7] A. Bressan, On a bang-bang principle for nonlinear systems, *Suppl. Boll. Un. Mat. Ital.* **1** (1980), 53-59.

[8] A. Bressan, On the qualitative theory of lower semicontinuous differential inclusions. *J. Differential Equations* **77** (1989), 379–391.

[9] A. Bressan, Upper and lower semicontinuous differential inclusions: a unified approach, in *Nonlinear Controllability and Optimal Control*, H. Sussmann ed., M. Dekker (1990), 21-31.

[10] L. Cesari, *Optimization. Theory and Applications. Problems with Ordinary Differential Equations.* Springer-Verlag, New York, 1983.

[11] A. F. Filippov, Classical solutions of differential equations with multi-valued right-hand side. *SIAM J. Control Optim.* **5** (1967), 609-621.

[12] H. Frankowska and F. Rampazzo, Filippov's and Filippov-Wazewski's theorems on closed domains. *J. Differential Equations*, **161** (2000), 449-478.

[13] H. Frankowska and R. B. Vinter, Existence of neighbouring feasible trajectories: applications to dynamic programming for state constrained optimal control Problems. *J. Optim. Theory Appl.*, **104** (2000), 21-40.

[14] P. R. Halmos, The range of a vector measure. *Bull. Amer. Math. Soc.* **54** (1948), 416–421.

[15] F. Rampazzo and R. B. Vinter, A theorem on existence of neighboring trajectories satisfying a state constraint, with applications to optimal control. *IMA J. Math. Control Inform.*, **16** (1999), 335-351.

[16] H. M. Soner, Optimal control problems with state-space constraints. *SIAM J. Control Optim.*, **24** (1986), 552-562 and 1110-1122.

[17] R. B. Vinter, *Optimal Control*, Birkhäuser, Boston, 2000.