Error bounds for a Deterministic Version of the Glimm Scheme

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Abstract. Consider the hyperbolic system of conservation laws $u_t + F(u)_x = 0$. Let u be the unique viscosity solution with initial condition $u(0,x) = \bar u(x)$, and let u^G be an approximate solution constructed by the Glimm scheme, corresponding to the mesh sizes Δx, $\Delta t = O(\Delta x)$. With a suitable choice of the sampling sequence, we prove the estimate

$$\big\| u^G(t,\cdot) - u(t,\cdot)\big\|_{\mathbf{L}^1} = o(1)\cdot \sqrt{\Delta x}\,\big|\ln(\Delta x)\big|.$$

1 - Introduction

Aim of this paper is to investigate the rate of convergence of approximate solutions obtained by the Glimm scheme, in connection with the Cauchy problem

$$u_t + \big[F(u)\big]_x = 0, \tag{1.1}$$

$$u(0,x) = \bar u(x), \tag{1.2}$$

for a nonlinear $N \times N$ system of conservation laws in one space dimension. We assume that the system is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear [13].

Following [4], we shall assume that the system (1.1) generates a *Standard Riemann Semigroup* (SRS). In other words, there exists a continuous semigroup $\{S_t;\ \ t \geq 0\}$, defined on some domain $\mathcal{D} \subset \mathbf{L}^1$ containing all integrable functions with sufficiently small total variation, with the following properties:

(i) For some Lipschitz constant L, one has

$$\left\| S_t \bar{u} - S_t \bar{v} \right\|_{\mathbf{L}^1} \leq L \cdot \|\bar{u} - \bar{v}\|_{\mathbf{L}^1} \qquad \forall \bar{u}, \bar{v} \in \mathcal{D}, \quad t \geq 0. \tag{1.3}$$

(ii) If $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small $S_t \bar{u}$ coincides with the solution of (1.1)-(1.2) which is obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

The existence of a Standard Riemann Semigroup was proved in [1, 3] for certain $N \times N$ systems with coinciding shock and rarefaction curves and in [7] for general 2×2 systems. The construction of a SRS in the general $N \times N$ case is outlined in the survey paper [5]. Details will appear in [6].

If a SRS exists, then it is necessarily unique (up to the domain $\mathcal{D}$) and its trajectories can be characterized as *Viscosity Solutions*, according to the definition introduced in [4]. Moreover, any weak solution of (1.1)-(1.2) obtained in the limit by a wave-front tracking algorithm, or by the Glimm scheme, coincides with the corresponding semigroup trajectory $t \mapsto S_t \bar{u}$.

A brief description of the scheme of Glimm [8] is given below. Consider an open set $\Omega \subseteq I\!\!R^N$ containing the origin, and let $F : \Omega \mapsto I\!\!R^N$ be a smooth map, whose Jacobian matrix $A(u) \doteq DF(u)$ has N real and distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$. By possibly performing a linear change of coordinates in the t-x plane where the solution of (1.1) is defined, it is not restrictive to assume

$$0 < \lambda_i(u) < 1 \qquad \forall i = 1, \ldots, N, \quad u \in \Omega. \tag{1.4}$$

To construct an approximate solution u^ε of the Cauchy problem (1.1)-(1.2), choose mesh lengths $\Delta t = \Delta x = \varepsilon$, and let $(\theta_\ell)_{\ell \geq 0}$ be a sequence of numbers within the interval $[0, 1]$. On the initial strip $0 \leq t < \varepsilon$, the function u^ε is the exact solution of (1.1) with initial condition

$$u^\varepsilon(0, x) = \bar{u}\big((j + \theta_0)\varepsilon\big) \qquad \text{if} \qquad j\varepsilon < x < (j+1)\varepsilon.$$

Now assume that u^ε has been constructed for $0 \leq t < \ell\varepsilon$. Then, on the strip $\ell\varepsilon \leq t < (\ell+1)\varepsilon$, u^ε is the exact solution of (1.1) with starting condition

$$u^\varepsilon(\ell\varepsilon, x) = u^\varepsilon\big(\ell\varepsilon-, (j + \theta_\ell)\varepsilon\big) \qquad \text{if} \qquad j\varepsilon < x < (j+1)\varepsilon.$$

By induction, using suitable a-priori bounds on the total variation, the approximate solution u^ε can be defined for all $t \geq 0$.

Repeating this construction with the same values θ_ℓ but letting the mesh size ε tend to zero, one obtains a sequence of approximate solutions $(u_\nu)_{\nu\geq 1}$. By compactness, there exists a subsequence which converges to some limit function u in $\mathbf{L}^1_{loc}$. If the values θ_ℓ are uniformly distributed, it was proved in [10] that u is a weak solution of (1.1)-(1.2). We recall that the sequence $(\theta_\ell)_{\ell\geq 0}$ is *uniformly distributed* on $[0,1]$ if

$$\lim_{n\to\infty} \left| \lambda - \frac{1}{n} \sum_{\ell=0}^{n-1} \chi_{[0,\lambda]}(\theta_\ell) \right| = 0 \qquad \forall \lambda \in [0,1], \tag{1.5}$$

where $\chi_{[0,\lambda]}$ denotes the characteristic function of the interval $[0,\lambda]$. In order to obtain estimates on the convergence rate of approximate solutions, we now introduce an assumption on the rate at which the limits in (1.5) are attained, uniformly w.r.t. λ. Following [12], for $0 \leq m < n$, the *discrepancy* of the set $\{\theta_m, \ldots, \theta_{n-1}\}$ is defined as

$$D_{m,n} \doteq \sup_{\lambda\in[0,1]} \left| \lambda - \frac{1}{n-m} \sum_{m\leq \ell<n} \chi_{[0,\lambda]}(\theta_\ell) \right| . \tag{1.6}$$

In Section 3 we explicitly construct a sequence $(\theta_\ell)_{\ell\geq 0}$ such that

$$D_{m,n} \leq C \cdot \frac{1+\ln(n-m)}{n-m} \qquad \forall n > m \geq 1 \tag{1.7}$$

for some constant C. When these particular values θ_ℓ are used in the Glimm scheme, estimates can be given on the rate of convergence of approximate solutions, in the $\mathbf{L}^1$ norm.

Theorem 1 *Let $(\theta_\ell)_{\ell\geq 0}$ be a sequence of numbers in $[0,1]$ satisfying (1.7). Given any initial condition $\bar u$ with small total variation, let $u(t,\cdot) = S_t\bar u$ be the unique viscosity solution of (1.1)-(1.2), and let u^ε be the corresponding Glimm approximation with mesh sizes $\Delta x = \Delta t = \varepsilon$, generated by the sampling sequence $(\theta_\ell)_{\ell\geq 0}$. Then, for every $T \geq 0$ one has*

$$\lim_{\varepsilon\to 0} \frac{\big\|u^\varepsilon(T,\cdot) - u(T,\cdot)\big\|_{\mathbf{L}^1}}{\sqrt{\varepsilon}\,|\ln \varepsilon|} \;=\; 0\,. \tag{1.8}$$

The limit (1.8) is uniform w.r.t. $\bar u$, as long as Tot.Var.$(\bar u)$ remains uniformly small.

Remark 1. In the case of scalar conservation laws with random, uniformly distributed sampling, B. Lucier proved in [11] that the expected error in $\mathbf{L}^1$ satisfies

$$E\Big(\big\|u^\varepsilon(t) - u(t)\big\|_{\mathbf{L}^1}\Big) = O(1)\cdot\sqrt{\varepsilon t}\cdot \text{Tot.Var.}(\bar u). \tag{1.9}$$

The estimate (1.8), on the other hand, corresponds to a deterministic choice of the sampling values θ_ℓ.

Remark 2. Let $S : \mathcal{D}\times[0,\infty[\,\mapsto \mathcal{D}$ be a Standard Riemann Semigroup for (1.1). Let $u^\varepsilon : [0,T] \mapsto \mathcal{D}$ be a piecewise continuous approximate solution, with jumps at the times $0 = t_0 < t_1 < \ldots < t_n \leq T$. Recalling (1.3), the difference between $u^\varepsilon(T)$ and the exact solution $u(T) \doteq S_T \bar u$ of (1.1)-(1.2) can be estimated by

$$\big\|u^\varepsilon(T) - u(T)\big\|_{\mathbf{L}^1} \leq L\cdot\sum_{\ell=0}^{n} \big\|u^\varepsilon(t_\ell+) - u^\varepsilon(t_\ell-)\big\|_{\mathbf{L}^1} + L\cdot\int_0^T \left(\limsup_{\eta\to 0+} \frac{\big\|u^\varepsilon(t+\eta) - S_\eta u^\varepsilon(t)\big\|_{\mathbf{L}^1}}{\eta}\right) dt, \tag{1.10}$$

with the convention $u^\varepsilon(0-) = \bar u$. In [4], bounds of the form (1.10) were effectively used to estimate the convergence rate of approximate solutions generated by wave-front tracking. For solutions u^ε generated by the Glimm scheme, however, the bound (1.10) is of little help. Indeed, in this case the integral term vanishes identically, but the quantity

$$\begin{aligned}\sum_{\ell=0}^{n} \big\|u^\varepsilon(t_\ell+) - u^\varepsilon(t_\ell-)\big\|_{\mathbf{L}^1} &= \sum_{\ell=0}^{T/\varepsilon} \big\|u^\varepsilon(\ell\varepsilon+) - u^\varepsilon(\ell\varepsilon-)\big\|_{\mathbf{L}^1} \\ &= \sum_{\ell=0}^{T/\varepsilon}\sum_{j=-\infty}^{\infty} \int_{j\varepsilon}^{(j+1)\varepsilon} \Big|u^\varepsilon(\ell\varepsilon-, x) - u^\varepsilon\big(\ell\varepsilon-, (j+\theta_\ell)\varepsilon\big)\Big|\, dx\end{aligned}$$

does not approach zero as $\varepsilon \to 0$.

The proof of Theorem 1 is based on the analysis of T. P. Liu [10]. We first subdivide the interval $[0,T]$, inserting points $0 = t_0 < t_1 < \cdots < t_\nu = T$. On each subinterval $J_i \doteq [t_i, t_{i+1}]$, a key lemma in [10] shows that the elementary waves in an approximate solution can be partitioned so that their speeds and sizes can be traced. On J_i, the error

$$E_i \doteq \big\|u^\varepsilon(t_{i+1}) - S_{t_{i+1}-t_i} u^\varepsilon(t_i)\big\|_{\mathbf{L}^1}$$

comes from two different sources:

(i) Errors in the speeds assigned to wave-fronts.

(ii) Errors due to the interactions and cancellations of waves.

If $t_i = m\varepsilon$, $t_{i+1} = n\varepsilon$, the difference between the exact speed and the average speed assigned to a wave-front by the Glimm scheme is estimated by (1.6). To reduce the size of errors of type (i), it is thus convenient to choose the intervals J_i suitably large. On the other hand, the new waves generated by interactions and the waves which disappear due to cancellations cannot be traced over the whole time interval $[t_i, t_{i+1}]$. The size of these errors of type (ii) can be reduced only by choosing the intervals J_i suitably small.

As $\varepsilon \to 0$, it is convenient to choose the asymptotic size of the intervals J_i in such a way that the errors in (i) and (ii) have approximately the same order of magnitude. In particular, the estimate (1.8) will be obtained by choosing $|J_i| \approx \sqrt{\varepsilon} \cdot \ln|\ln \varepsilon|$.

2 - Equidistributed Sequences

Aim of this section is to explicitly construct certain equidistributed sequences of points in $[0,1]$ whose discrepancies, defined at (1.6), approach zero sufficiently fast.

Proposition 1. *For every integer $r \geq 2$ there exists a sequence $(\theta_k)_{k\geq 0}$ such that*

$$D_{m,n} \leq \frac{2r-2}{n-m}\left[1 + \frac{\ln(n-m)}{\ln r}\right] \qquad \forall n > m \geq 1. \tag{2.1}$$

The proof is given in several steps.

1. Let $r \geq 2$ be given. Every integer $k \geq 0$ can be uniquely written as a sum of powers of r:

$$k = k_0 + k_1 r + \ldots + k_M r^M, \qquad 0 \leq k_i \leq r-1. \tag{2.2}$$

In connection with (2.2) we then define

$$\theta_k \doteq \frac{k_0}{r} + \frac{k_1}{r^2} + \ldots + \frac{k_M}{r^{M+1}} \ \in [0,1]. \tag{2.3}$$

We claim that the sequence $(\theta_k)_{k\geq 0}$ defined by (2.2)-(2.3) satisfies (2.1).

2. For any integers $\ell_0 = m \leq \ell_1 \leq \cdots \leq \ell_p = n$, there holds

$$D_{m,n} \leq \sum_{j=0}^{p-1} \frac{\ell_{j+1}-\ell_j}{n-m} \cdot D_{\ell_j,\ell_{j+1}}. \tag{2.4}$$

Indeed, for every $\lambda \in [0,1]$ we have

$$\begin{aligned}\left|\lambda - \frac{1}{n-m}\sum_{m\leq \ell<n} \chi_{[0,\lambda]}(\theta_\ell)\right| &\leq \sum_{j=0}^{p-1}\left\{\frac{\ell_{j+1}-\ell_j}{n-m}\cdot\left|\lambda - \frac{1}{\ell_{j+1}-\ell_j}\sum_{\ell_j\leq r<\ell_{j+1}} \chi_{[0,\lambda]}(\theta_r)\right|\right\}\\ &\leq \sum_{j=0}^{p-1}\frac{\ell_{j+1}-\ell_j}{n-m}\cdot D_{\ell_j,\ell_{j+1}}.\end{aligned}$$

3. As the integer k ranges over the half-open interval $[ir^\alpha,\ (i+1)r^\alpha[$, the set of the corresponding values θ_k has the form

$$\left\{\frac{j}{r^\alpha} + \frac{k_\alpha}{r^{\alpha+1}} + \cdots + \frac{k_M}{r^{M+1}}\ ; \qquad j = 0,\ldots,r^\alpha - 1\right\}, \tag{2.5}$$

for suitable integers $k_\alpha, \dots, k_M \in \{0, \dots, r-1\}$.

Let $\lambda \in [0,1]$ be given. Observe that, by (2.5),

$$\sum_{ir^\alpha \le k < (i+1)r^\alpha} \chi_{[0,\lambda]}(\theta_k) = q$$

if and only if

$$\lambda \in \left[\frac{q-1}{r^\alpha} + \beta,\ \frac{q}{r^\alpha} + \beta\right[\qquad \left(\beta = \frac{k_\alpha}{r^{\alpha+1}} + \cdots + \frac{k_M}{r^{M+1}}\right).$$

Hence,

$$\left|\lambda - \frac{q}{r^\alpha}\right| \le \frac{1}{r^\alpha}\,.$$

Since λ was arbitrary, this yields

$$D_{ir^\alpha,(i+1)r^\alpha} \le \frac{1}{r^\alpha}\,. \tag{2.6}$$

4. Now let $n > m \ge 0$ be given. Let $\alpha \ge 0$ be the largest integer such that

$$I_{i,\alpha} \doteq [ir^\alpha,\ (i+1)r^\alpha[\ \subseteq\ [m,n[\tag{2.7}$$

for some i. Clearly,

$$\alpha \le \log_r(n-m) = \frac{\ln(n-m)}{\ln r}\,. \tag{2.8}$$

Denote by $S_\alpha \doteq \{I_{i,\alpha};\ \ i \in \mathcal{J}_\alpha\}$ the family of all intervals for which (2.7) holds. By the maximality of α, there can be at most $2r-2$ such intervals $I_{i,\alpha}$.

Next, call $S_{\alpha-1} \doteq \{I_{i,\alpha-1};\ \ i \in \mathcal{J}_{\alpha-1}\}$ the family of all intervals of the form $I_{i,\alpha-1} \doteq [ir^{\alpha-1},\ (i+1)r^{\alpha-1}[$ which are contained inside the set

$$[m,n[\ \setminus \bigcup_{i\in\mathcal{J}_\alpha} I_{i,\alpha}.$$

Observe that no more than $2r-2$ such intervals exist.

By induction on $\beta \in \{\alpha, \alpha-1, \dots, 1, 0\}$, let

$$S_\beta \doteq \{I_{i,\beta};\ \ i \in \mathcal{J}_\beta\}$$

be the family of all intervals of the form

$$I_{i,\beta} \doteq [ir^\beta,\ (i+1)r^\beta[$$

which are contained inside the set

$$[m,n[\ \setminus \bigcup_{\beta'=\beta+1}^{\alpha} \bigcup_{i\in\mathcal{J}_{\beta'}} I_{i,\beta'}.$$

Once again, observe that no more than $2r-2$ such intervals can exist.

5. Call $D(I_{i,\beta})$ the discrepancy of the set $\{\theta_\ell;\ \ell \in I_{i,\beta}\}$. By (2.6) we have $D(I_{i,\beta}) \leq r^{-\beta}$. From (2.4) and (2.8) it now follows

$$D_{m,n} \leq \sum_{\beta=0}^{\alpha} \sum_{i\in\mathcal{J}_\beta} \frac{r^\beta}{n-m} \cdot D(I_{i,\beta}) \leq \sum_{\beta=0}^{\alpha} (2r-2)\frac{r^\beta}{n-m} \cdot \frac{1}{r^\beta} \leq \left(1 + \frac{\ln(n-m)}{\ln r}\right) \frac{2r-2}{n-m},$$

proving (2.1).

3 - Some basic notations

In the following, we call $A(u) = DF(u)$ the $N\times N$ Jacobian matrix of F at u, and denote by $\lambda_i(u)$, $l_i(u)$, $r_i(u)$, $i = 1,\ldots,N$, its eigenvalues and left and right eigenvectors, respectively. The parametrized i-shock and i-rarefaction curves through a state $\omega \in \Omega$ are denoted by

$$\sigma \mapsto S_i(\sigma)(\omega), \qquad \sigma \mapsto R_i(\sigma)(\omega).$$

Given two nearby states $u^-, u^+ \in \mathbb{R}^N$, the Riemann problem with initial data

$$u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases} \tag{3.1}$$

is solved by determining the intermediate states $\omega_0,\ldots,\omega_N$ and wave sizes $\sigma_1,\ldots,\sigma_N$ such that

$$\omega_0 = u^-, \quad \ldots \quad \omega_i = \psi_i(\sigma_i)(\omega_{i-1}), \quad \ldots \quad \omega_N = \psi_N(\sigma_N)(\omega_{N-1}) = u^+. \tag{3.2}$$

Here the functions ψ_i are defined as

$$\psi_i(\sigma) = \begin{cases} S_i(\sigma) & \text{if} \quad \sigma < 0, \\ R_i(\sigma) & \text{if} \quad \sigma \geq 0. \end{cases} \tag{3.3}$$

Let now $u : \mathbb{R} \mapsto \mathbb{R}^N$ be a piecewise constant function, with jumps at the points x_α. Call $\sigma_{i,\alpha}$ the size of the i-th wave generated by the Riemann problem at x_α. The total strength of waves in u and the potential for future wave interactions are defined respectively as

$$V(u) \doteq \sum_{i,\alpha} |\sigma_{i,\alpha}|, \qquad Q(u) \doteq \sum_{((i,\alpha),(j,\beta))\in\mathcal{A}} |\sigma_{i,\alpha}\sigma_{j,\beta}|,$$

where the second sum ranges over all couples of approaching waves.

If $u^\varepsilon = u^\varepsilon(t,x)$ is an approximate solution generated by the Glimm scheme with step sizes $\Delta t = \Delta x = \varepsilon$, for every $t \geq 0$ we write

$$V(t) \doteq V\big(u^\varepsilon(t,\cdot)\big), \qquad Q(t) \doteq Q\big(u^\varepsilon(t,\cdot)\big).$$

A fundamental estimate of Glimm [8, 13] shows that there exists a constant C_0 independent of ε such that the function

$$t \mapsto V(t) + C_0 Q(t) \doteq \Upsilon(t) \tag{3.4}$$

is non-increasing, for all approximate solutions with sufficiently small total variation. Moreover, for any given $\tau < \tau'$, the total amount of interaction and cancellation taking place on the interval $[\tau, \tau']$ can be estimated as $O(1) \cdot \big[\Upsilon(\tau) - \Upsilon(\tau')\big]$.

4 - Piecewise constant approximations

Throughout this paper we are concerned with an approximate solution u^ε constructed by the Glimm scheme, with mesh sizes $\Delta t = \Delta x = \varepsilon$, corresponding to the sampling sequence $(\theta_\ell)_{\ell \geq 0}$. It is convenient to redefine u^ε inside the open strips $]i\varepsilon,\ (i+1)\varepsilon[\,\times I\!\!R$ as follows.

$$u(t,x) = \begin{cases} u^\varepsilon(i\varepsilon, x) & \text{if} \quad t \in \big[i\varepsilon,\ (i+1)\varepsilon\big[\,, \quad x \in \big]j\varepsilon + t - i\varepsilon,\ (j+1)\varepsilon\big], \\ u^\varepsilon\big((i+1)\varepsilon, x\big) & \text{if} \quad t \in \big[i\varepsilon,\ (i+1)\varepsilon\big[\,, \quad x \in \big]j\varepsilon,\ j\varepsilon + t - i\varepsilon\big], \end{cases} \tag{4.1}$$

where $i = m, \ldots, n$ and $j \in Z\!\!\!Z$. Observe that the function u is piecewise constant in the t-x plane, and all of its jumps travel with speed 0 or 1. Moreover $u = u^\varepsilon$ at all times $t = i\varepsilon$, $i \in Z\!\!\!Z$.

For fixed integers $0 \leq m < n$, we consider the time interval $[\tau,\ \tau'] \doteq [m\varepsilon,\ n\varepsilon]$ and seek an estimate on the difference

$$\big\|u^\varepsilon(\tau', \cdot) - S_{\tau'-\tau} u^\varepsilon(\tau, \cdot)\big\|_{\mathbf{L}^1}. \tag{4.2}$$

According to Remark 2, this quantity cannot be directly estimated by the formula (1.10). We thus need to introduce an auxiliary piecewise constant function $w = w(t,x)$, with $w(\tau, \cdot) = u(\tau, \cdot)$, and split (4.2) as the sum of two terms:

$$\big\|u(\tau', \cdot) - w(\tau', \cdot)\big\|_{\mathbf{L}^1} + \big\|w(\tau', \cdot) - S_{\tau'-\tau} w(\tau, \cdot)\big\|_{\mathbf{L}^1}. \tag{4.3}$$

The idea behind the construction of w comes from [10]. In the solution u obtained by the Glimm scheme, at every node $(i\Delta t, j\Delta x)$ the outgoing waves can be partitioned into primary waves $\tilde{v}_k^h(i,j)$ and secondary waves $\tilde{\tilde{v}}_k^h(i,j)$. A primary wave originates at time τ and can be traced all the way up to τ'. The changes in its size and speed can be carefully estimated. On the other hand, secondary waves are those produced by interactions occurring after time τ, or waves which disappear before time τ' due to cancellations. Their total strength can be bounded in terms of the total amount of interaction and cancellation occurring within the time interval $[\tau,\ \tau']$.

Relying on this decomposition, we construct a piecewise constant approximate solution $w = w(t,x)$ on the strip $[\tau,\ \tau'] \times I\!\!R$ with the following basic property. For every primary wave in u, there exists a corresponding wave-front of w with the same initial and final position, having constant strength and travelling with constant speed (see fig.1). This construction will imply that the first term in (4.3) is small, because it only accounts for the strengths of secondary waves. The second term will be estimated using (1.10).

After a brief overview, we now turn to details. Call $R_k^+(\omega)$, $S_k^-(\omega)$ respectively the positive k-rarefaction curve and the negative k-shock curve through the state ω. Consider again the approximate solution u in (4.1) determined by the Glimm scheme with mesh sizes $\Delta t = \Delta x = \varepsilon$. Suppose that the couple of states (u_{k-1}, u_k) determines a shock or a rarefaction wave in the k-th characteristic family, at the node $(i\varepsilon, j\varepsilon)$. In case of a shock, we choose any vectors $y_0, y_1, \ldots,\ y_\ell \in S_k^-(u_{k-1})$, with $y_0 = u_{k-1}$, $y_\ell = u_k$, $\lambda_k(y_h) \le \lambda_k(y_{h-1})$ for every $h = 1, 2, \ldots, \ell$, and set

$$v_k^h(i,j) = y_h - y_{h-1}, \qquad \lambda_k^h(i,j) = \lambda_k(u_{k-1}, u_k).$$

If (u_{k-1}, u_k) is a k-rarefaction wave, we choose vectors $y_0,\ y_1,\ \ldots,\ y_\ell \in R_k^+(u_{k-1})$, with $y_0 = u_{k-1}$, $y_\ell = u_k$, $\lambda_k(y_h) > \lambda_k(y_{h-1})$ for every $h = 1, 2, \ldots, \ell$, and set

$$v_k^h(i,j) = y_h - y_{h-1}, \qquad \lambda_k^h(i,j) = \lambda_k(y_{h-1}).$$

In this second case we require that

$$\big|\lambda_k(y_h) - \lambda_k(y_{h-1})\big| \le \varepsilon \tag{4.4}$$

and, to make sure that $\{v_k^h(i,j)\}$ is not partitioned further at $t = (i+1)\varepsilon$, we also require that

$$\theta_{i+1} \notin \ \big]\lambda_k(y_{h-1}), \lambda_k(y_h)\big[\,, \qquad h = 1, 2, \ldots, \ell. \tag{4.5}$$

The *strength* σ_k^h of elementary wave v_k^h is defined as follows. If (u_{k-1}, u_k) is a k-shock and $y_h = S_k(s_h)(u_{k-1})$, $h = 1, \ldots, \ell$, we set

$$\sigma_k^h = s_h - s_{h-1}. \tag{4.6}$$

The same definition (4.6) is valid if (u_{k-1}, u_k) is a k-rarefaction and $y_h = R_k(s_h)(u_{k-1})$. In the following, for $0 \le m < n$ we write

$$\Delta\Upsilon_{m,n} \doteq V\big(u(m\varepsilon,\cdot)\big) + C_0 Q\big(u(m\varepsilon,\cdot)\big) - V\big(u(n\varepsilon,\cdot)\big) - C_0 Q\big(u(n\varepsilon,\cdot)\big). \tag{4.7}$$

We recall that, with a suitable choice of the constant C_0, the total amount of wave interaction and cancellation on the time interval $[m\varepsilon,\ n\varepsilon]$ can be estimated as $O(1) \cdot \Delta\Upsilon_{m,n}$.

Proposition 2. *There exists a partition of elementary waves $\big\{v_k^h(i,j), \lambda_k^h(i,j)\big\}$ which satisfies (4.4)–(4.6) and, moreover, $\big\{v_k^h(i,j), \lambda_k^h(i,j)\big\}$ is a disjoint union of $\big\{\tilde v_k^h(i,j), \tilde\lambda_k^h(i,j)\big\}$ and $\big\{\tilde{\tilde v}_k^h(i,j), \tilde{\tilde\lambda}_k^h(i,j)\big\}$, so that, for every $i \in \{m, m+1, \ldots, n\}$, the following holds.*

$$\sum_{h,j,k} \big\|\tilde{\tilde v}_k^h(i,j)\big\| = O(1) \cdot \Delta\Upsilon_{m,n}, \tag{4.8}$$

and there is a one-to-one correspondence between $\{\tilde v_k^h(m,j),\,\tilde\lambda_k^h(m,j)\}$ and $\{\tilde v_k^h(i,j),\,\tilde\lambda_k^h(i,j)\}$:

$$\{\tilde v_k^h(m,j),\ \tilde\lambda_k^h(m,j)\}\ \longleftrightarrow\ \{\tilde v_k^h(i,\ell_{(i,j,h,k)}),\ \tilde\lambda_k^h(i,\ell_{(i,j,h,k)})\} \tag{4.9}$$

such that the strengths $\tilde\sigma_k^k$ *and the speeds* $\tilde\lambda_k^h$ *of the corresponding waves satisfy*

$$\sum_{h,j,k}\left(\max_{m\le i\le n}\left|\tilde\sigma_k^h(m,j)-\tilde\sigma_k^h(i,\ell_{(i,j,h,k)})\right|\right)=O(1)\cdot\Delta\Upsilon_{m,n}, \tag{4.10}$$

$$\sum_{h,j,k}\left(\left|\tilde\sigma_k^h(m,j)\right|\cdot\max_{m\le i\le n}\left|\tilde\lambda_k^h(i,\ell_{(i,j,h,k)})-\tilde\lambda_k^h(m,j)\right|\right)=O(1)\cdot\Delta\Upsilon_{m,n}, \tag{4.11}$$

Roughly speaking, the correspondence (4.9) means that the portion $\tilde v_k^h(m,j)$ of the k-wave issuing from the node $(m\varepsilon,\ j\varepsilon)$ travels along an approximate characteristic (see [9]) and reaches the node $(i\varepsilon,\ \ell_{(i,h,j,k)}\varepsilon)$ at time $t=i\varepsilon$.

The construction of the elementary waves and of the bijection (4.9) was carried out in the proof of Lemma 3.2 in [10]. Retracing the argument in [10], we see that the elementary waves $\tilde v_k^h$ have the additional properties:

(P1) *If at the node* $(m\varepsilon,j\varepsilon)$ *the wave* (u_k,u_{k-1}) *is a shock, then there exists at most one primary wave* $\tilde v_k^h(m,j)$ *issuing from this node.*

(P2) The map (4.9) is order-preserving. More precisely, among the k*-waves present at a fixed time* $t=i\varepsilon$*, define the ordering*

$$v_k^h(i,j)\prec v_k^{h'}(i,j')\qquad \text{iff}\qquad j<j'\ \text{ or }\ j=j'\ \text{and}\ h<h'. \tag{4.12}$$

Then the correspondence (4.9), mapping the primary k*-waves at time* $t=m\varepsilon$ *onto the primary* k*-waves at time* $t=i\varepsilon$*, preserves the ordering (4.12).*

On the strip $[\tau,\tau']\times I\!R$ we now construct a piecewise constant function $w=w(t,x)$ with the following properties. At the initial time τ one has $w(\tau,\cdot)=u(\tau,\cdot)$. For each primary wave $\tilde v_k^h(m,j)$ originating from the node $(m\varepsilon,j\varepsilon)$ and eventually reaching the node $(n\varepsilon,\ell_{(n,h,j,k)}\varepsilon)$, the function w has a jump along the segment joining these two nodes. The left and right states across this jump determine a k-wave of constant strength $\tilde\sigma_k^h(m,j)$. Let $\{\Gamma_\alpha\}$ be the collection of all segments constructed above, and let $\{(\bar t_\beta,\bar x_\beta)\}$ be the set containing all points where two of the segments Γ_α intersect, together with all nodes $(m\varepsilon,j\varepsilon)$, with j integer. The set of jumps of w will consist of the segments Γ_α together with the lines

$$\Gamma_\beta\doteq\{(t,x);\qquad t\in[\bar t_\beta,\ \tau],\quad x=\bar x_\beta+2(t-\bar t_\beta)\}.$$

In analogy with the wave-front tracking algorithm [2], we shall refer to the segments Γ_α and Γ_β as wave-fronts of order 1 and of order 2, respectively. The construction of w goes as follows. At the initial time $t = m\varepsilon$ we set $w(m\varepsilon, x) = u(m\varepsilon, x)$. To define w in a neighborhood of a given node $(m\varepsilon, j\varepsilon)$, for each h, k we consider the primary wave $\tilde{v}_k^h(m, j)$ issuing from $(m\varepsilon, j\varepsilon)$ and look at the corresponding node $(n\varepsilon, \ell_{(n,j,h,k)}\varepsilon)$ reached by this wave at time $t = n\varepsilon$. The slope of the segment joining these two nodes, given by

$$\bar{\lambda}_k^h(j) \doteq \frac{\ell_{(n,j,h,k)} - j}{n - m}\,, \tag{4.13}$$

can be regarded as the average speed of the wave-front. Call $u^-(m, j)$ and $u^+(m, j)$ respectively the values of $u(\tau, \cdot)$ on the left and on the right of $(m\varepsilon, j\varepsilon)$. Define the auxiliary state

$$u^*(m, j) \doteq \psi_N\left(\sum_h \tilde{\sigma}_N^h(m, j)\right) \circ \ldots \circ \psi_1\left(\sum_h \tilde{\sigma}_1^h(m, j)\right)\big(u^-(m, j)\big), \tag{4.14}$$

where $\tilde{\sigma}_k^h$ are the strengths of the primary waves, defined as in (4.6). In a neighborhood of the node $(m\varepsilon, j\varepsilon)$, the function w has wave fronts with strengths $\tilde{\sigma}_k^h(m, j)$, travelling with the speeds $\bar{\lambda}_k^h(j)$ in (4.13). These fronts connect the state u^- with u^*. In turn, the states u^* and u^+ are connected by a non-physical wave-front travelling with speed 2, located on the line $x = j\varepsilon + 2(t - m\varepsilon)$. Observe that the strength of this jump can be estimated by

$$\big|u^*(m, j) - u^-(m, j)\big| \le C_1 \sum_{h,k} \big|\tilde{\tilde{\sigma}}_k^h(m, j)\big|, \tag{4.15}$$

for some constant C_1. The piecewise constant function w can now be prolonged up to the first time where two wave-fronts interact. At a time $\tau > m\varepsilon$ where an interaction occurs, the new Riemann problem is solved without changing the size and the speed of any wave-front of order 1. This can be accomplished by introducing an artificial wave-front of order 2, travelling with speed $\dot{x} = 2$. More precisely, let $(\bar{t}, \bar{x})$ be a point in the t-x plane where two incoming fronts interact. Call $u^\flat$, $u^\natural$ and $u^\sharp$ respectively the left, middle and right states before the interaction time. Assume that the jumps $(u^\flat, u^\natural)$ and $(u^\natural, u^\sharp)$ have strengths σ', σ and travel with speeds λ', λ, respectively.

CASE 1: *both incoming waves have order 1.*

The Riemann problem is then solved in terms of three outgoing wave-fronts. If $u^\natural = \psi_{k'}(\sigma')u^\flat$ and $u^\sharp = \psi_k(\sigma)u^\natural$, then for $t > \bar{t}$ the solution w will contain the four states

$$u^\flat, \quad u^* \doteq \psi_k(\sigma)u^\flat, \quad u^{**} \doteq \psi_{k'}(\sigma')u^*, \quad u^\sharp.$$

The three jumps separating these states travel with speeds $\lambda', \lambda, 2$, respectively. The strength of the jump $(u^{**}, u^\sharp)$ is estimated by

$$|u^{**} - u^\sharp| \le C_2|\sigma\sigma'|. \tag{4.16}$$

CASE 2: *one of the incoming waves has order 2.*

The Riemann problem is then solved in terms of two outgoing wave-fronts. If $u^\sharp = \psi_k(\sigma)u^\natural$, then for $t > \bar t$ the solution w will contain the three states $u^\flat$, $u^* \doteq \psi_k(\sigma)u^\flat$, $u^\sharp$. The two jumps separating these states travel with speeds $\lambda, 2$, respectively. The size of the jump $(u^*, u^\sharp)$ is estimated by

$$|u^* - u^\sharp| \le C_3|\sigma|\big|u^\flat - u^\natural\big|. \tag{4.17}$$

Remark 3. In the above construction, it may happen that two primary (rarefaction) waves v_k^h, $v_k^{h'}$ start from the same node (m, j) at time $t = m\varepsilon$ and reach the same node (n, j_n) at time $t = n\varepsilon$, with $j_n = \ell_{(n,j,h,k)} = \ell_{(n,j,h',k)}$. One should thus consider also the case where two or more elementary waves in u correspond to the same front of w. To avoid this additional technicality, we change the speed of some of the wave-fronts in w by an arbitrarily small amount, so that this situation does not happen. In the same way, we can assume that, in the construction of w, every interaction involves exactly two incoming wave-fronts. All these interactions then fall within the two cases described above. Indeed, all waves of order 2 travel with the same speed $\dot x = 2$ and never interact with each other.

We conclude this section with some estimates, for later use. Referring to the decomposition in elementary waves described in Proposition 2, we say that the two primary waves $\tilde v_k^h(m, j)$, $\tilde v_{k'}^{h'}(m, j')$ *cross each other* during the time interval $]m\varepsilon, n\varepsilon]$ if $j < j'$, $k > k'$ and $\ell_{(n,j,h,k)} \ge \ell_{(n,j',h',k')}$. By CW we denote the set of all couples of Crossing Waves. Moreover, we say that two negative waves of the same family $\tilde\sigma_k^h(m, j)$, $\tilde\sigma_k^{h'}(m, j')$ *join together* during the time interval $]m\varepsilon, n\varepsilon]$ (thus forming a single shock) if $j < j'$ and $\ell_{(n,j,h,k)} = \ell_{(n,j',h',k)}$. By JS we denote the set of all couples of Joining Shocks. Observing that the total amount of interaction during the interval $]m\varepsilon, n\varepsilon]$ is $O(1) \cdot \Delta\Upsilon_{m,n}$, from (4.10) we deduce

$$\sum_{CW} \big|\tilde\sigma_k^h(m, j)\tilde\sigma_{k'}^{h'}(m, j')\big| = O(1) \cdot \Delta\Upsilon_{m,n}\,, \tag{4.18}$$

$$\sum_{JS} \big|\tilde\sigma_k^h(m, j)\tilde\sigma_k^{h'}(m, j')\big| = O(1) \cdot \Delta\Upsilon_{m,n}\,. \tag{4.19}$$

For book-keeping purposes, it is convenient to relabel the various jumps in w. We denote by $\big\{x_\alpha(\cdot);\ \ \alpha \in \mathcal{R} \cup \mathcal{S}\big\}$ the set of first order wave-fronts. Each front is classified as a Rarefaction or a Shock depending on its size σ_α. Its speed is $\bar\lambda_\alpha \doteq \dot x_\alpha$. The set of second order (Non-physical) wave-fronts is written $\big\{x_\beta(\cdot);\ \ \beta \in \mathcal{N}\big\}$. By construction, all these fronts travel with speed $\dot x_\beta = 2$. Their strength is defined as

$$\sigma_\beta(t) \doteq \Big|\Delta w\big(t, x_\beta(t)\big)\Big| \doteq \Big|w\big(t, x_\beta(t)+\big) - w\big(t, x_\beta(t)-\big)\Big|\,.$$

Lemma 1. *At every time $t \in [\tau, \tau']$, the total strength of waves in w of order 2 is*

$$\tilde{\tilde{V}}(t) \doteq \sum_{\beta \in \mathcal{N}} \Big|\Delta w\big(t, x_\beta(t)\big)\Big| = \sum_{\beta \in \mathcal{N}} \sigma_\beta(t) = O(1) \cdot \Delta\Upsilon_{m,n}. \tag{4.20}$$

Proof. For each $\beta \in \mathcal{N}$, let $(\bar{t}_\beta, \bar{x}_\beta)$ be the initial location and let $\sigma_\beta(\bar{t}_\beta)$ be the initial strength of the corresponding wave-front. For $t > \bar{t}_\beta$, from (4.17) it follows

$$\sigma_\beta(t) \le \sigma_\beta(\bar{t}_\beta) \cdot \exp\Big\{C_3 \sum |\sigma_\alpha|\Big\},$$

where the summation extends to all wave-fronts x_α (of order 1) which cross the front x_β during the interval $]\bar{t}_\beta, t]$. Since the total strength of all such waves is uniformly bounded, for some constant C_4 one has

$$\sum_{\beta \in \mathcal{N}} \sigma_\beta(t) \le C_4 \cdot \sum_{\beta \in \mathcal{N}} \sigma_\beta(\bar{t}_\beta) \qquad \forall t \in [\tau, \tau']. \tag{4.21}$$

We now split the sum on the right hand side of (4.21), considering separately those waves which originate at time τ and those which are generated by the interaction of two (first order) wave-fronts at some time $\bar{t}_\beta > \tau$. Recalling (4.15) and (4.16), then (4.8) and (4.18) we conclude

$$\begin{aligned} \sum_{\bar{t}_\beta = \tau} \sigma_\beta(\bar{t}_\beta) + \sum_{\bar{t}_\beta > \tau} \sigma_\beta(\bar{t}_\beta) &\le C_1 \sum_{h,j,k} \big|\tilde{\tilde{\sigma}}_k^h(m,j)\big| + C_2 \sum_{CW} \big|\tilde{\sigma}_k^h(m,j)\tilde{\sigma}_{k'}^{h'}(m,j')\big| \\ &= O(1) \cdot \Delta\Upsilon_{m,n}. \end{aligned} \tag{4.22}$$

Together, (4.21) and (4.22) yield (4.20). //

From (4.18) and (4.20) we also obtain

Lemma 2. *For each $\alpha \in \mathcal{R} \cup \mathcal{S}$, call Q_α the total amount of waves in w that cross the line $x_\alpha(\cdot)$ over the interval $]\tau, \tau']$. Then*

$$\sum_{\alpha \in \mathcal{R} \cup \mathcal{S}} |\sigma_\alpha| Q_\alpha = O(1) \cdot \Delta\Upsilon_{m,n}. \tag{4.23}$$

5 - The key estimates

We begin this section by estimating the second term in (4.3), for $\tau = m\varepsilon$, $\tau' = n\varepsilon$.

Proposition 3. *The map $t \mapsto w(t, \cdot)$ from $[m\varepsilon, n\varepsilon]$ into $\mathbf{L}^1$ is Lipschitz continuous. Moreover*

$$\big\|S_{(n-m)\varepsilon} w(m\varepsilon, \cdot) - w(n\varepsilon, \cdot)\big\|_{\mathbf{L}^1} = O(1) \cdot \Big[\Delta\Upsilon_{m,n} + \frac{1 + \ln(n-m)}{n-m} + \varepsilon\Big](n-m)\varepsilon. \tag{5.1}$$

Proof. The first assertion clearly holds because w has bounded variation and all of its jumps travel with speed ≤ 2. Using (1.10) we deduce

$$\big\|S_{(n-m)\varepsilon}w(m\varepsilon,\cdot)-w(n\varepsilon,\cdot)\big\|_{\mathbf{L}^1}\leq L\cdot\int_{m\varepsilon}^{n\varepsilon}\limsup_{\eta\to 0}\frac{\big\|S_\eta w(t,\cdot)-w(t+\eta,\cdot)\big\|_{\mathbf{L}^1}}{\eta}\,dt. \tag{5.2}$$

Denote by $x_\alpha(\cdot)$ the lines of discontinuity of w and let $\mathcal{S}$, $\mathcal{R}$, $\mathcal{N}$ (Shock, Rarefaction, Non-physical) be respectively the set of indices α corresponding to waves of negative strength, positive strength and to second order waves. The constant speeds of these fronts are written $\bar\lambda_\alpha\doteq\dot x_\alpha(t)$.

Assume that at time t no interaction takes place. Then, as in [4, p.214], we can find $\rho>0$ such that

$$\limsup_{\eta\to 0+}\frac{\big\|S_\eta w(t,\cdot)-w(t+\eta,\cdot)\big\|_{\mathbf{L}^1}}{\eta}=\limsup_{\eta\to 0+}\sum_{\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{N}}\frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho}\big|S_\eta w(t)(x)-w(t+\eta,x)\big|\,dx\,. \tag{5.3}$$

Call $w(x_\alpha-)$ and $w(x_\alpha+)$ respectively the left and right limits of $w(t,\cdot)$ at $x=x_\alpha(t)$. Concerning the non-physical wave-fronts of w, by (4.20) for $\eta>0$ small we have

$$\sum_{\alpha\in\mathcal{N}}\frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho}\big|S_\eta w(t)(x)-w(t+\eta,x)\big|\,dx=\sum_{\alpha\in\mathcal{N}}O(1)\cdot\big|w(x_\alpha+)-w(x_\alpha-)\big|=O(1)\cdot\Delta\Upsilon_{m,n}. \tag{5.4}$$

Next, consider the case $\alpha\in\mathcal{S}\cup\mathcal{R}$. For some k_α, σ_α we thus have $w(x_\alpha+)=\psi_{k_\alpha}(\sigma_\alpha)w(x_\alpha-)$. Assume that the jump $x_\alpha(\cdot)$ of w corresponds to the primary wave $\tilde v_k^h(m,j)$ in u, having strength $\tilde\sigma_k^h(m,j)=\sigma_\alpha$. Of course, we must have $k=k_\alpha$, $j\varepsilon=x_\alpha(m\varepsilon)$. By construction, the speed $\bar\lambda_\alpha=\dot x_\alpha$ satisfies

$$\bar\lambda_\alpha=\frac{\#\Big\{i\in I\!N;\quad m<i\leq n,\quad \theta_i\leq\tilde\lambda_k^h(i,\ell_{(i,j,h,k)})\Big\}}{n-m}\,, \tag{5.5}$$

where # denotes the cardinality of a set. By the assumption (1.7),

$$\begin{aligned}\bar\lambda_\alpha&\geq\min_{m<i\leq n}\tilde\lambda_{k_\alpha}^{h_\alpha}(i,\ell_{(i,j,h,k)})-C\cdot\frac{1+\ln(n-m)}{n-m}\,,\\ \bar\lambda_\alpha&\leq\max_{m<i\leq n}\tilde\lambda_{k_\alpha}^{h_\alpha}(i,\ell_{(i,j,h,k)})+C\cdot\frac{1+\ln(n-m)}{n-m}\,.\end{aligned} \tag{5.6}$$

From (4.11) and (5.6) we deduce

$$\sum_{\alpha\in\mathcal{S}\cup\mathcal{R}}|\sigma_\alpha|\big|\bar\lambda_\alpha-\tilde\lambda_{k_\alpha}^h(m,j)\big|=O(1)\cdot\left[\Delta\Upsilon_{m,n}+\frac{1+\ln(n-m)}{n-m}\right]. \tag{5.7}$$

In the following, if the states w^-,w^+ are joined by a k-shock, we denote by $\lambda_k(w^-,w^+)$ the speed of this shock, determined by the Rankine-Hugoniot equations. As usual, $\lambda_k(w)$ denotes the k-th characteristic speed at the point w. Recalling the definition of Q_α in Lemma 2, by construction we have the estimates

$$\Big|\tilde\lambda_{k_\alpha}^h(m,j)-\lambda_{k_\alpha}\big(w(x_\alpha-),w(x_\alpha+)\big)\Big|=O(1)\cdot\left(\sum_h\big|\tilde{\tilde\sigma}_{k_\alpha}^h(m,j)\big|+Q_\alpha\right), \tag{5.8}$$

$$\left|\tilde\lambda^h_{k_\alpha}(m,j)-\lambda_{k_\alpha}\big(w(x_\alpha-)\big)\right| = O(1)\cdot Q_\alpha, \tag{5.9}$$

valid for $\alpha\in\mathcal{S}$ and for $\alpha\in\mathcal{R}$, respectively. In the case $\alpha\in\mathcal{S}$ we now have

$$\begin{aligned}\frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho}\big|S_\eta w(t)(x)-w(t+\eta,x)\big|\,dx &= O(1)\cdot|\sigma_\alpha|\Big|\lambda_{k_\alpha}\big(w(x_\alpha-),w(x_\alpha+)\big)-\bar\lambda_\alpha\Big|\\ &= O(1)\cdot|\sigma_\alpha|\Big\{\Big|\lambda_{k_\alpha}\big(w(x_\alpha-),w(x_\alpha+)\big)-\tilde\lambda^h_{k_\alpha}(m,j)\Big|+\big|\tilde\lambda^h_{k_\alpha}(m,j)-\bar\lambda_\alpha\big|\Big\}.\end{aligned}$$

Therefore, from (5.8) and (5.7), using (4.8) and (4.23) we deduce

$$\sum_{\alpha\in\mathcal{S}}\frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho}\big|S_\eta w(t)(x)-w(t+\eta,x)\big|\,dx = O(1)\cdot\left[\Delta\Upsilon_{m,n}+\frac{1+\ln(n-m)}{n-m}\right]. \tag{5.10}$$

Finally, in the case $\alpha\in\mathcal{R}$, we have

$$\begin{aligned}\frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho}\big|S_\eta w(t)(x)-w(t+\eta,x)\big|\,dx &= O(1)\cdot|\sigma_\alpha|\Big\{\big|\lambda_{k_\alpha}(w(x_\alpha-))-\bar\lambda_\alpha\big|+|\sigma_\alpha|\Big\}\\ &= O(1)\cdot|\sigma_\alpha|\Big\{\big|\lambda_{k_\alpha}(w(x_\alpha-))-\tilde\lambda^h_{k_\alpha}(m,j)\big|+\big|\tilde\lambda^h_{k_\alpha}(m,j)-\bar\lambda_\alpha\big|+|\sigma_\alpha|\Big\}.\end{aligned}$$

Recalling that $|\sigma_\alpha|\le\varepsilon$, from (5.9) and (5.7), using (4.23) we deduce

$$\sum_{\alpha\in\mathcal{R}}\frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho}\big|S_\eta w(t)(x)-w(t+\eta,x)\big|\,dx = O(1)\cdot\left[\Delta\Upsilon_{m,n}+\frac{1+\ln(n-m)}{n-m}+\varepsilon\right]. \tag{5.11}$$

Using (5.2) and estimating the right hand side of (5.3) by means of (5.4), (5.10) and (5.11), we finally obtain (5.1). //

In the remainder of this section we seek an estimate on the first term in (4.3). The basic strategy is as follows. For every y, to estimate the difference $\big|u(\tau',y)-w(\tau',y)\big|$ we look at the behavior of the functions u and w along the segment

$$\Gamma_y \doteq \big\{(t,x);\quad x=y+2(t-\tau'),\quad t\in[\tau,\tau']\big\}. \tag{5.12}$$

By construction, $u=w$ at the initial point of Γ_y, i.e. when $t=\tau$. Since both u and w are piecewise constant in the t-x plane, we can evaluate the quantities $u(\tau',y)$, $w(\tau',y)$ by keeping track of the wave-fronts which cross the segment Γ_y during the interval $[\tau,\tau']$. Observing that no wave-front of w of order 2 ever crosses the line Γ_y, setting

$$\bar\omega \doteq u\big(\tau,y-2(\tau'-\tau)\big)=w\big(\tau,y-2(\tau'-\tau)\big)$$

and recalling (3.3) we can write

$$\begin{aligned}u(\tau',y)&=\psi_{p(\mu)}(\sigma_\mu)\circ\cdots\circ\psi_{p(1)}(\sigma_1)(\bar\omega),\\ w(\tau',y)&=\psi_{q(\nu)}(\sigma'_\nu)\circ\cdots\circ\psi_{q(1)}(\sigma'_1)(\omega),\end{aligned} \tag{5.13}$$

for suitable wave strengths $\sigma_\alpha, \sigma'_\alpha$, and indices $p(\alpha), q(\alpha) \in \{1, \dots, N\}$.

In order to compare the two quantities in (5.13), two technical lemmas are needed.

Lemma 3. *Let ω, ω' be connected to a given state $\bar\omega$ by a sequence of waves:*

$$\begin{aligned} \omega &= \psi_{p(\mu)}(\sigma_\mu) \circ \dots \circ \psi_{p(1)}(\sigma_1)(\bar\omega) \doteq \bigcirc_{i=1}^{\mu} \psi_{p(i)}(\sigma_i)(\bar\omega), \\ \omega' &= \psi_{q(\nu)}(\sigma'_\nu) \circ \dots \circ \psi_{q(1)}(\sigma'_1)(\bar\omega) \doteq \bigcirc_{j=1}^{\nu} \psi_{q(j)}(\sigma'_j)(\bar\omega). \end{aligned} \tag{5.14}$$

Assume that there exists a nondecreasing, surjective map $\phi : \{1, \dots, \mu\} \mapsto \{1, \dots, \nu\}$ such that $p(i) = q(\phi(i))$ for all i. Then the following estimate holds

$$|\omega - \omega'| = O(1) \cdot \sum_{j=1}^{\nu} \left(\Big| \sigma'_j - \sum_{\phi(i)=j} \sigma_i \Big| + \sum_{\substack{i \neq \ell \\ \phi(i)=\phi(\ell)=j}} |\sigma_i \sigma_\ell| \right), \tag{5.15}$$

provided that the total strength of waves, measured by $\sum |\sigma_i| + \sum |\sigma'_j|$, remains uniformly bounded.

Proof. For $j = 1, \dots, \nu$, consider the intermediate states

$$\omega_j \doteq \Big(\bigcirc_{\phi(i)>j} \psi_{p(i)}(\sigma_i) \Big) \circ \Big(\bigcirc_{\ell=1}^{j} \psi_{q(\ell)}(\sigma'_\ell) \Big)(\bar\omega).$$

Clearly, $\omega_0 = \omega$, $\omega_\nu = \omega'$. Hence

$$|\omega - \omega'| \leq \sum_{j=1}^{\nu} |\omega_j - \omega_{j-1}|. \tag{5.16}$$

To estimate each term on the right hand side of (5.16), define

$$\omega_j^* \doteq \Big(\bigcirc_{\ell=1}^{j-1} \psi_{q(\ell)}(\sigma'_\ell) \Big)(\bar\omega).$$

Assume that $\phi(i) = j$ for those indices i such that $\alpha(j) \leq i \leq \beta(j)$. By assumption, $p(i) = q(j)$ for all such indices. Therefore, standard interaction estimates yield

$$\begin{aligned} |\omega_j - \omega_{j-1}| &\leq C \cdot \left| \psi_{q(j)}(\sigma'_j)(\omega_j^*) - \Big(\bigcirc_{i=\alpha(j)}^{\beta(j)} \psi_{q(j)}(\sigma_i) \Big)(\omega_j^*) \right| \\ &\leq C' \cdot \left\{ \Big| \sigma'_j - \sum_{i=\alpha(j)}^{\beta(j)} \sigma_i \Big| + \sum_{\alpha \leq i < i' \leq \beta} |\sigma_i \sigma_{i'}| \right\}, \end{aligned} \tag{5.17}$$

for some constants C, C', as long as the total strength of waves remains uniformly bounded. Using (5.17) in (5.16) we obtain (5.15). //

Lemma 4. *Let ϕ be a permutation of the set of indices $\{1, \dots, \nu\}$. Assume that*

$$\begin{aligned} \omega &= \psi_{p(\nu)}(\sigma_\nu) \circ \dots \circ \psi_{p(1)}(\sigma_1)(\bar\omega), \\ \omega' &= \psi_{p(\phi(\nu))}(\sigma_{\phi(\nu)}) \circ \dots \circ \psi_{p(\phi(1))}(\sigma_{\phi(1)})(\bar\omega). \end{aligned} \tag{5.18}$$

Then, as long as the total amount of waves remains uniformly bounded, one has the estimate

$$|\omega - \omega'| = O(1) \cdot \sum_{(i,j)\in\mathcal{E}} |\sigma_i \sigma_j|, \tag{5.19}$$

where

$$\mathcal{E} \doteq \big\{(i,j); \quad i<j, \quad \phi(i) > \phi(j)\big\}.$$

Proof. We construct a chain of permutations $\phi_0, \ldots, \phi_h$ with $h = \#\mathcal{E}$, $\phi_0 =$Id, $\phi_h = \phi$, such that each ϕ_ℓ is obtained from $\phi_{\ell-1}$ by switching the position of two adiacent elements. More precisely, we choose the intermediate permutations ϕ_ℓ so that, setting

$$\mathcal{E}_\ell \doteq \big\{(i,j); \quad i<j, \quad \phi_\ell(i) > \phi_\ell(j)\big\},$$

one has

$$\mathcal{E}_\ell = \mathcal{E}_{\ell-1} \cup \big\{(i_\ell, j_\ell)\big\} \qquad \ell = 1, \ldots, h,$$

for some couple of indices (i_ℓ, j_ℓ). Calling

$$\omega_\ell \doteq \psi_{p(\phi_\ell(\nu))}(\sigma_{\phi_\ell(\nu)}) \circ \ldots \circ \psi_{p(\phi_\ell(1))}(\sigma_{\phi_\ell(1)})(\bar\omega),$$

we now have

$$|\omega' - \omega| = |\omega_h - \omega_0| \le \sum_{\ell=1}^{h} |\omega_\ell - \omega_{\ell-1}| \le \sum_{\ell=1}^{h} C \cdot |\sigma_{i_\ell} \sigma_{j_\ell}|,$$

for some constant C. This yields (5.19).

Proposition 4. *The first term in (4.3) satisfies the estimate*

$$\big\| u(n\varepsilon, \cdot) - w(n\varepsilon, \cdot)\big\|_{\mathbf{L}^1} = O(1) \cdot \Delta\Upsilon_{m,n} (n-m)\varepsilon. \tag{5.20}$$

Proof. Recall that the function u, defined at (4.1), is piecewise constant in the t-x plane, and all its jumps travel with speed 0 or 1. More precisely, the elementary wave $v_k^h(i,j)$ issuing from the node $(i\varepsilon, j\varepsilon)$ reaches either $\big((i+1)\varepsilon, j\varepsilon\big)$ or $\big((i+1)\varepsilon, (j+1)\varepsilon\big)$, depending on whether its speed $\lambda_k^h(i,j)$ is $< \theta_{i+1}$ or $\ge \theta_{i+1}$, respectively.

Consider any point $y \in I\!R$, not coinciding with one of the nodes $j\varepsilon$ or with a point reached at time τ' by a second order wave-front in w. By our construction, there is a one-to-one correspondence between the primary wave-fronts in u that cross Γ_y and the fronts (of order 1) in w that cross Γ_y on the interval $[\tau, \tau']$. Denote by $C(y)$ the set of all wave-fronts of w which cross Γ_y. The strength σ_α of any such front is constant in time. By construction, it coincides with the strength $\tilde\sigma_k^h(m,j)$

of the corresponding wave-front of u at the initial time $t = \tau = m\varepsilon$. Call $\tilde{\sigma}^y_\alpha = \tilde{\sigma}^h_k(i, \ell_{(i,j,h,k)})$ the strength of the corresponding wave-front of u at the time $t \in [i\varepsilon, (i+1)\varepsilon[$ when it crosses Γ_y. Set

$$\omega \doteq w(\tau', y), \qquad \omega' \doteq u(\tau', y), \qquad \bar{\omega} \doteq u\big(\tau,\ y - 2(\tau' - \tau)\big) = w\big(\tau,\ y - 2(\tau' - \tau)\big).$$

We then have a representation of the form (5.13). Observe that, in this case, the two quantities in (5.13) may differ because:

(i) The strengths of the waves σ_α, $\tilde{\sigma}^y_\alpha$ may be different.

(ii) The order in which two primary wave-fronts of u and w cross Γ_y may be inverted.

(iii) Two primary shocks in u may first collapse into a single shock, then cross Γ_y.

(iv) The secondary wave-fronts in u which cross Γ_y have no counterpart in w.

Using Lemma 4 to estimate the contributions due to (ii), and Lemma 3 to estimate the contributions due to (i), (iii) and (iv), we obtain

$$\big|u(\tau', y) - w(\tau', y)\big| \leq C \cdot \left\{ \sum_{C(y)} \big|\sigma_\alpha - \tilde{\sigma}^y_\alpha\big| + \sum_{CW(y)} |\sigma_\alpha \sigma_{\alpha'}| + \sum_{JS(y)} |\sigma_\alpha \sigma_{\alpha'}| + \sum_{C'(y)} \big|\tilde{\tilde{\sigma}}^h_k(i,j)\big| \right\}. \tag{5.21}$$

In (5.21), the first sum is over all waves in w which cross Γ_y, the second is over all couples of waves in w that cross each other and also cross Γ_y. The third sum ranges over all couples of negative primary waves in u that join together and also cross Γ_y, while the fourth sum ranges over all secondary waves in u that cross Γ_y during the interval $[\tau, \tau']$. We now observe that

- For any wave $\alpha \in \mathcal{S} \cup \mathcal{R}$, the set of points y for which the line $x_\alpha(\cdot)$ crosses Γ_y during the interval $[m\varepsilon, n\varepsilon]$ is an interval of length $\leq 2(n-m)\varepsilon$.
- If $\tilde{\tilde{v}}^h_k(i,j)$ is any secondary wave-front in u, issuing from the node $(i\varepsilon, j\varepsilon)$, than it can reach either $\big((i+1)\varepsilon, j\varepsilon\big)$ or $\big((i+1)\varepsilon, (j+1)\varepsilon\big)$. In both cases, the set of points $y \in I\!\!R$ such that Γ_y crosses such wave-front is an interval of length $\leq 2\varepsilon$.

Integrating (5.21), we thus obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \big|u(\tau', y) - w(\tau', y)\big|\, dy \leq C \cdot \Bigg\{ & 2(n-m)\varepsilon \sum_{h,j,k} \left(\max_{m \leq i \leq n} \big|\tilde{\sigma}^h_k(m,j) - \tilde{\sigma}^h_k(i, \ell_{(i,j,h,k)})\big| \right) \\ & + 2(n-m)\varepsilon \sum_{CW} |\sigma_\alpha \sigma_{\alpha'}| + 2(n-m)\varepsilon \sum_{JS} |\sigma_\alpha \sigma_{\alpha'}| + 2\varepsilon \sum_{i,j,h,k} \big|\tilde{\tilde{\sigma}}^h_k(i,j)\big| \Bigg\}.\end{aligned} \tag{5.22}$$

Because of (4.10), (4.18), (4.19) and (4.8), each of the above terms is estimated by $O(1) \cdot \Delta\Upsilon_{m,n}(n-m)\varepsilon$. This yields (5.20).

6 - Proof of Theorem 1

Let $T, \varepsilon > 0$ be given, say with $T = \bar{m}\varepsilon + \varepsilon'$ for some integer $\bar{m}$ and some $\varepsilon' \in [0, \varepsilon[$. In connection with a constant $\delta > 2\varepsilon$ (whose precise value will be specified later), we construct a partition of the interval $[0, \bar{m}\varepsilon]$ into finitely many subintervals $J_i \doteq [t_i, t_{i+1}]$, inserting the points $t_i = m_i\varepsilon$ inductively as follows. Set $m_0 = 0$. If the integers $m_0 < m_1 < \cdots < m_i < \bar{m}$ have already been defined, then

(i) If $\Upsilon(m_i\varepsilon) - \Upsilon\big((m_i+1)\varepsilon\big) \leq \delta$, let m_{i+1} be the largest integer $\leq \bar{m}$ such that $(m_{i+1} - m_i)\varepsilon \leq \delta$ and $\Upsilon(m_i\varepsilon) - \Upsilon\big(m_{i+1}\varepsilon\big) \leq \delta$.

(ii) If $\Upsilon(m_i\varepsilon) - \Upsilon\big((m_i+1)\varepsilon\big) > \delta$, define $m_{i+1} \doteq m_i + 1$.

Here Υ is the function in (3.4). Clearly, $m_\nu = \bar{m}$ for some integer $\nu \leq \bar{m}$. Call $\mathcal{I}, \mathcal{I}'$ respectively the set of indices i for which the alternative (i), (ii) holds. Observe that, for some constant C_5, the cardinalities of these sets can be bounded by

$$\#\mathcal{I} \leq \frac{C_5}{\delta}, \qquad\qquad \#\mathcal{I}' \leq \frac{C_5}{\delta}. \tag{6.1}$$

On each subinterval J_i, $i \in \mathcal{I}$, we construct the auxiliary function w as in the previous sections. Using Propositions 3 and 4 with $[\tau, \tau'] = [m_i\varepsilon, m_{i+1}\varepsilon]$, we obtain an estimate of the form

$$\big\|u(m_{i+1}\varepsilon) - S_{(m_{i+1}-m_i)\varepsilon}u(m_i\varepsilon)\big\|_{\mathbf{L}^1} \leq C_6\left\{\Delta\Upsilon_{m_i,m_{i+1}} + \frac{1+\ln(m_{i+1}-m_i)}{m_{i+1}-m_i} + \varepsilon\right\}(m_{i+1}-m_i)\varepsilon. \tag{6.2}$$

On the other hand, on each interval J_i with $i \in \mathcal{I}'$, the Lipschitz continuity of $u : [0,T] \mapsto \mathbf{L}^1$ implies

$$\big\|u(m_{i+1}\varepsilon) - S_{(m_{i+1}-m_i)\varepsilon}u(m_i\varepsilon)\big\|_{\mathbf{L}^1} \leq C_7(t_{i+1} - t_i) = C_7\varepsilon. \tag{6.3}$$

Recalling the Lipschitz property (1.3) of the semigroup, the bounds (6.2) and (6.3) yield

$$\begin{aligned}
\big\|u(\bar{m}\varepsilon) - S_{\bar{m}\varepsilon}u(0)\big\|_{\mathbf{L}^1} &\leq \sum_{i=0}^{\nu-1}\Big|S_{(\bar{m}-m_{i+1})\varepsilon}u(m_{i+1}\varepsilon) - S_{(\bar{m}-m_i)\varepsilon}u(m_i\varepsilon)\Big| \\
&\leq L\cdot\sum_{i=0}^{\nu-1}\Big|u(m_{i+1}\varepsilon) - S_{(m_{i+1}-m_i)\varepsilon}u(m_i\varepsilon)\Big| \\
&\leq LC_6\cdot\sum_{i\in\mathcal{I}}\left\{\Delta\Upsilon_{m_i,m_{i+1}} + \frac{1+\ln(m_{i+1}-m_i)}{m_{i+1}-m_i} + \varepsilon\right\}\varepsilon(m_{i+1}-m_i) + LC_7\sum_{i\in\mathcal{I}'}\varepsilon
\end{aligned} \tag{6.4}$$

By (6.1) and the choice of m_{i+1} when $i \in \mathcal{I}$, from (6.4) we deduce

$$\begin{aligned}
\big\|u^\varepsilon(T) - S_T\bar{u}\big\|_{\mathbf{L}^1} &\leq LC_6\cdot\frac{C_5}{\delta}\left\{\delta^2 + \varepsilon\left(1+\ln\frac{\delta}{\varepsilon}\right) + \varepsilon\delta\right\} + LC_7\cdot\frac{C_5}{\delta}\varepsilon \\
&\leq C_8\left\{\delta + \frac{\varepsilon}{\delta}\ln\left(\frac{\delta}{\varepsilon}\right) + \varepsilon\left(1+\frac{1}{\delta}\right)\right\},
\end{aligned} \tag{6.5}$$

for a suitable constant C_8. Since (6.5) is valid for every $\delta > 2\varepsilon$, choosing $\delta = \delta(\varepsilon) \doteq \sqrt{\varepsilon} \cdot \ln |\ln \varepsilon|$ we finally obtain (1.8). //

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