Global Optimality Conditions for a Dynamic Blocking Problem

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Abstract

The paper is concerned with a class of optimal blocking problems in the plane. We consider a time dependent set $R(t) \subset \mathbb{R}^2$, described as the reachable set for a differential inclusion. To restrict its growth, a barrier can be constructed, in real time. This is a one-dimensional rectifiable set which blocks the trajectories of the differential inclusion.

In this paper we introduce a definition of "regular strategy", based on a careful classification of blocking arcs. Moreover, we derive local and global necessary conditions for an optimal strategy, which minimizes the value of the region $\cup_{t\geq 0} R(t)$ plus the cost of constructing the barrier. We show that a Lagrange multiplier, corresponding to the constraint on the construction speed, can be interpreted as the "instantaneous value of time". This value, which we compute by two separate formulas, remains constant when free arcs are constructed and is monotone decreasing otherwise.

1 Introduction

Aim of this paper is to derive global necessary conditions satisfied by an optimal strategy, for the dynamic blocking problem introduced in [5]. As described in [5, 7], these problems were originally motivated by the control of wild fires or the spatial spreading of a contaminating agent.

At each time $t \geq 0$, we denote by $R(t) \subset \mathbb{R}^2$ the region burned by the fire. In absence of control, for each $t \geq 0$ the set $R(t)$ is described as the reachable set for a differential inclusion:

$$\dot{x} \in F(x) \qquad\qquad x(0) \in R_0 \,, \tag{1.1}$$

where the upper dot denotes a derivative w.r.t. time. In other words,

$$R(t) = \Big\{ x(t)\,; \;\; x(\cdot) \;\text{ absolutely continuous}\,, \;\; x(0) \in R_0\,, \;\; \dot{x}(\tau) \in F\big(x(\tau)\big) \;\text{ for a.e. } \tau \in [0,t] \Big\}\,.$$

We assume that the initial set $R_0 \subset I\!R^2$ is open and bounded. Moreover, we assume that $F : I\!R^2 \mapsto I\!R^2$ is a Lipschitz continuous multifunction with compact, convex values, and satisfies

$$0 \in F(x) \qquad\qquad \text{for all } x \in I\!R^2. \tag{1.2}$$

Clearly, this implies

$$R(t_1) \subseteq R(t_2) \qquad\qquad \text{whenever} \quad t_1 < t_2\,. \tag{1.3}$$

In our model, the growth of the reachable set (i.e., the spreading of the fire) can be controlled by constructing barriers, in real time. Let $\psi : I\!R^2 \mapsto R_+$ be a continuous, strictly positive function. Calling $\gamma(t) \subset I\!R^2$ the portion of the wall constructed within time $t \geq 0$, we make the following assumptions:

(H1) For any $t_1 < t_2$ one has $\gamma(t_1) \subseteq \gamma(t_2)$.

(H2) For every $t \geq 0$, the total length of the wall satisfies

$$\int_{\gamma(t)} \psi\, dm_1 \leq t\,, \tag{1.4}$$

where m_1 denotes the one-dimensional Hausdorff measure, normalized so that $m_1(\Gamma)$ yields the usual length of a smooth curve Γ.

In the above formula, $1/\psi(x)$ is the speed at which the wall can be constructed, at the location x. In particular, if $\psi(x) \equiv \sigma^{-1}$ is constant, then (1.4) simply means that the length of the curve $\gamma(t)$ is $\leq \sigma t$. A strategy γ satisfying (H1)-(H2) will be called an **admissible strategy**. In addition, we say that the strategy γ is **complete** if it satisfies

(H3) For every $t \geq 0$ there holds

$$\int_{\gamma(t)} \psi\, dm_1 = t\,, \qquad\qquad \gamma(t) = \bigcap_{s>t} \gamma(s)\,. \tag{1.5}$$

Moreover, if $\gamma(t)$ has positive upper density at a point x, i.e. if

$$\limsup_{r \to 0+} \frac{m_1\Big(B(x,r) \cap \gamma(t)\Big)}{r} > 0\,,$$

then $x \in \gamma(t)$. Here $B(x,r)$ is the open ball centered at x with radius r.

As proved in [7], for every admissible strategy $t \mapsto \gamma(t)$ one can construct a second admissible strategy $t \mapsto \tilde\gamma(t) \supseteq \gamma(t)$, which is complete.

When a barrier is being constructed, the set reached by the fire is reduced. Namely, we define

$$\begin{aligned} R^\gamma(t) \doteq \quad & \Big\{ x(t)\,;\;\; x(\cdot) \;\text{ absolutely continuous}\,, \quad x(0) \in R_0\,, \\ & \dot x(\tau) \in F\big(x(\tau)\big) \;\text{ for a.e. } \tau \in [0,t]\,, \qquad x(\tau) \notin \gamma(\tau) \;\text{ for all } \tau \in [0,t] \Big\}\,. \end{aligned} \tag{1.6}$$

To define an optimization problem, we need to introduce a cost functional. In general, this should take into account:

- The value of the area burned by the fire.

- The cost of building the barrier.

As in [5], we thus consider two continuous, non-negative functions $\alpha, \beta : I\!R^2 \mapsto I\!R_+$ and define the functional

$$J(\gamma) = \int_{R^\gamma_\infty} \alpha \, dm_2 + \int_{\gamma_\infty} \beta \, dm_1 , \tag{1.7}$$

where the sets R^γ_∞, γ_∞ are defined respectively as

$$R^\gamma_\infty \doteq \bigcup_{t\geq 0} R^\gamma(t) , \qquad\qquad \gamma_\infty \doteq \bigcup_{t\geq 0} \gamma(t) . \tag{1.8}$$

In (1.7), m_2 denotes the two-dimensional Lebesgue measure, while m_1 is the one-dimensional Hausdorff measure. In the case of a fire, $\alpha(x)$ is the value of a unit area of land at the point x, while $\beta(x)$ is the cost of building a unit length of wall at the point x. This leads to

(OP1) Optimization Problem 1: find an admissible strategy $t \mapsto \gamma(t)$ for which the corresponding functional $J(\gamma)$ at (1.7) attains its minimum value.

For this problem, the existence of an optimal solution was proved in [7], under the following assumptions:

(A1) The initial set R_0 is open and bounded. Its boundary satisfies $m_2(\partial R_0) = 0$.

(A2) The multifunction F is Lipschitz continuous w.r.t. the Hausdorff distance. For each $x \in I\!R^2$ the set $F(x)$ is nonempty, closed and convex and contains the origin in its interior.

(A3) For every $x \in I\!R^2$ one has $\alpha(x) \geq 0$, $\beta(x) \geq 0$, $\alpha(x) + \beta(x) > 0$, and $\psi(x) \geq \psi_0 > 0$. Moreover, α is locally integrable, while β and ψ are both lower semicontinuous.

In its original formulation, a strategy is a set-valued map $t \mapsto \gamma(t) \subset I\!R^2$ describing the portion of the wall constructed within a given time $t \geq 0$. The subsequent paper [10] showed that the above problem can be reformulated in a simpler way, where a strategy is entirely determined by assigning one single rectifiable set $\Gamma \subset I\!R^2$. We shall briefly review this equivalence result.

Consider a rectifiable set $\Gamma \subset I\!R^2$ which is **complete**, in the sense that it contains all of its points of positive upper density:

$$\limsup_{r\to 0+} \frac{m_1\Big(B(x,r)\cap\Gamma\Big)}{r} > 0 \qquad \Longrightarrow \qquad x \in \Gamma .$$

Define the reachable set for the differential inclusion (1.1) restricted to $I\!R^2 \setminus \Gamma$

$$R^\Gamma(t) \doteq \Big\{ x(t) ; \ \ x(\cdot) \ \text{absolutely continuous} , \ \ x(0) \in R_0 , \\ \dot x(\tau) \in F\big(x(\tau)\big) \ \text{for a.e.}\ \tau \in [0,t] , \quad x(\tau) \notin \Gamma \ \text{for all}\ \tau \in [0,t] \Big\} . \tag{1.9}$$

Throughout the following, $\overline{S}$ will denote the closure of a set S. We say that the rectifiable set Γ is **admissible** in connection with the differential inclusion (1.1) and the bound on the construction speed (1.4) if

$$\int_{\Gamma\cap\overline{R^\Gamma(t)}} \psi\, dm_1 \;\leq\; t \qquad \text{for all } t\geq 0\,. \tag{1.10}$$

Of course, this means that the strategy

$$t \;\mapsto\; \gamma(t) \;\doteq\; \Gamma\cap\overline{R^\Gamma(t)} \tag{1.11}$$

is admissible according to (1.4). One can then consider:

(OP2) Optimization Problem 2: Find an admissible rectifiable set $\Gamma\subset I\!\!R^2$ such that, calling $R^\Gamma_\infty \doteq \bigcup_{t\geq 0} R^\Gamma(t)$, the cost

$$J(\Gamma) = \int_{R^\Gamma_\infty} \alpha\, dm_2 + \int_\Gamma \beta\, dm_1 \tag{1.12}$$

attains the minimum possible value.

As proved in [10], under the assumptions (A1)–(A3) the two formulations are equivalent. Namely, if $t\mapsto\gamma(t)$ is a complete, optimal strategy for (OP1), then the rectifiable set

$$\Gamma \;\doteq\; \left(\bigcup_{t\geq 0}\gamma(t)\right)\setminus\left(\bigcup_{t\geq 0} R^\gamma(t)\right) \tag{1.13}$$

is admissible and provides an optimal solution to the minimization problem (OP2). Viceversa, if the set Γ provides an optimal solution to (OP2), then the strategy $\gamma(\cdot)$ in (1.11) is optimal for (OP1).

Remark 1. For each $t\geq 0$, the set $\gamma(t)$ in (1.11) is the part of the wall Γ touched by the fire at time t. This is the portion that actually needs to be put in place within time t, in order to constrain the fire. The remaining portion $\Gamma\setminus\gamma(t)$ can be constructed at a later time. On the other hand, given a strategy $\gamma(\cdot)$, the set Γ consists of the "useful" part of all walls constructed by γ. Portions of a wall, which are constructed in a region already reached by the fire, are clearly useless.

Remark 2. By the assumption (A2), each velocity set $F(x)$ is a neighborhood of the origin. Hence the set $R^\Gamma_\infty \doteq \bigcup_{t\geq 0} R^\Gamma(t)$ of all points reached by the fire without crossing Γ can be characterized as the union of all connected components of $I\!\!R^2\setminus\Gamma$ which intersect R_0.

Some necessary conditions for optimality were derived in [5], in the special case where $\beta\equiv 0$ and $\psi\equiv 1$, i.e. when there is no construction cost and the construction speed is constant. These conditions were essentially of local nature, obtained by perturbing the optimal strategy in a neighborhood of a given point.

The main goal of the present paper is to derive general optimality conditions, also of global nature. In particular, we study necessary conditions which must be satisfied at points of junction between two different arcs. We also analyze the case where the fire propagates along two or more fronts, and describe the optimal strategy at the time when one of these advancing fronts is extinguished.

As a preliminary, in Section 2 we introduce a concept of "regular strategy", and provide a careful classification of arcs. In particular, we observe that portions of an optimal barrier Γ may be constructed not only to block the fire, but also to slow down its advancement. These will be called "delaying arcs". Their presence increases the time needed for the fire to reach some regions of the plane.

Sections 3 and 4 describe necessary conditions for the optimality of "free arcs", constructed away from the advancing fire front, and "boundary arcs", constructed right along the edge of an advancing front.

Section 6 deals with necessary conditions at junctions. In particular, we show show that optimal arcs must join tangentially and we study the relations between Lagrange multipliers associated to different arcs.

Our analysis shows the existence of a scalar function $W(\cdot)$, which arises naturally as a global Lagrange multiplier, and can be interpreted as the "instantaneous value of time". Roughly speaking, $W(\tau)$ measures by how much the total cost could be reduced if the constraint (1.10) is replaced by

$$\int_{\Gamma\cap\overline{R^\Gamma(t)}} \psi\, dm_1 \ \leq\ \begin{cases} t & \text{if } t < \tau\,, \\ t+\varepsilon & \text{if } t \geq \tau\,. \end{cases}$$

The paper is concluded with two examples, where the optimal strategies and the value of time can be explicitly computed.

2 Regular strategies and classification of arcs

In this section, we introduce the basic framework and the regularity assumptions, in order to derive suitable necessary conditions for optimality.

Let Γ be an admissible barrier for the differential inclusion (1.1), so that (1.10) holds. We observe that the construction of the barrier Γ has two effects, namely: (i) it restricts the fire to the set R^Γ_∞, consisting of all connected components of $I\!R^2 \setminus \Gamma$ which intersect the initial domain R_0, and (ii) within the set R^Γ_∞, it can slow down the advancement of the fire.

This fact, illustrated in fig. 1, can be better described as follows. Given the differential inclusion (1.1) and the barrier Γ, we define the minimum time function as

$$T^\Gamma(x) \ \doteq\ \inf\ \Big\{t\geq 0\,;\quad x\in\overline{R^\Gamma(t)}\Big\}. \tag{2.1}$$

Given a point $y\in I\!R^2$ such that $T^\Gamma(y)<\infty$, there exists a sequence of trajectories $t\mapsto x_k(t)$, $t\in[0,\tau_k]$, such that

$$\dot x_k(t)\in F(x_k(t)) \qquad x_k(0)\in R_0\,, \qquad \tau_k\to T^\Gamma(y)\,, \qquad x_k(\tau_k)\to y.$$

By a compactness argument, we can extract a subsequence converging uniformly to a trajectory $t\mapsto x(t)$ with $x(0)\in\overline{R}_0$ and $x(T^\Gamma(y))=y$. In the following, we call $\mathcal{F}(y)$ the family of all such trajectories, obtained as limit of a convergent sequence. For a given trajectory $x(\cdot) \in \mathcal{F}(y)$, two cases may occur:

(i) All points $x(\tau)$ with $0 < \tau < T^\Gamma(y)$ lie outside Γ.

(ii) Some points $x(\tau)$ with $0 < \tau < T^\Gamma(y)$ lie in Γ.

In the first case, calling

$$T(x) \doteq \inf \left\{ t \geq 0\,; \quad x \in \overline{R(t)} \right\}.$$

the minimum time function for the differential inclusion (1.1) (without any barrier), we clearly have $T^\Gamma(y) = T(y)$.

If the second case arises, at least for some points y, we define the non-empty subset $\Gamma^d \subseteq \Gamma$ of **delaying walls** as

$$\Gamma^d \doteq \left\{ x \in \Gamma\,; \quad x = x(\tau) \ \text{ for some } y \in I\!\!R^2,\ x(\cdot) \in \mathcal{F}(y)\,,\ 0 < \tau < T^\Gamma(y) < \infty \right\}.$$

We think of Γ^d as a portion of the barrier Γ which contributes to slowing down fire propagation. The set of **blocking walls** is defined as

$$\Gamma^b \doteq \Gamma \cap \partial R^\Gamma_\infty\,. \tag{2.2}$$

Remark 3. If Γ is optimal, and the construction cost is strictly positive, then $\Gamma = \Gamma^d \cup \Gamma^b$. Indeed, any arc $\Gamma' \subset \Gamma$ contained in the interior of the reachable set R^Γ_∞ must be part of Γ^d. Otherwise the alternative strategy $\widetilde{\Gamma} \doteq \Gamma \setminus \Gamma'$ would also be admissible, with a smaller cost.

On the other hand, as shown in fig. 1, one can have $\Gamma^d \cap \Gamma^b \neq \emptyset$.

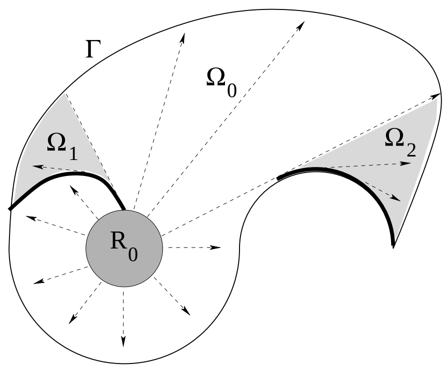

Figure 1: Here we take $R_0 = F(x) = B(0,1)$, the unit disc centered at the origin. The two thick arcs denote the portion $\Gamma^d \subset \Gamma$ which contributes to slowing down the propagation of the fire. Notice that $T^\Gamma(x) = T(x)$ for $x \in \Omega_0$, but $T^\Gamma(x) < T(x)$ for $x \in \Omega_1 \cup \Omega_2$. The thick arc next to the shaded region Ω_1 lies in $\Gamma^d \setminus \Gamma^b$, the thick arc next to Ω_2 lies in $\Gamma^d \cap \Gamma^b$.

Given an admissible barrier Γ, a further classification of arcs can be achieved as follows. Define the set of times

$$\mathcal{S} \doteq \left\{ t \geq 0\,; \quad \int_{\Gamma \cap \overline{R^\Gamma(t)}} \psi\, dm_1 = t \right\}\,. \tag{2.3}$$

These are the times where the constraint is *saturated*, i.e. it is satisfied as an equality. We can further classify points $x \in \Gamma$ by setting

$$\Gamma_{\mathcal{S}} \doteq \{x \in \Gamma\,;\;\; T^\Gamma(x) \in \mathcal{S}\}\,, \qquad\qquad \Gamma_{\mathcal{F}} \doteq \{x \in \Gamma\,;\;\; T^\Gamma(x) \notin \mathcal{S}\}\,,$$

As in [5], arcs lying in the subset $\Gamma_{\mathcal{F}}$ will be called **free arcs**.

A very general result on the existence of optimal blocking strategies was recently proved in [7]. However, this provides little information about the regularity of these optimal strategies. Namely, if Γ is optimal, then Γ must be the union of countably many compact, connected, rectifiable sets, plus a set whose 1-dimensional Hausdorff measure is zero. In order to derive necessary condition for optimality, additional regularity assumptions will be imposed.

Motivated by the definition of regular synthesis for an optimal control problem [3, 4, 8, 13], we consider a decomposition

$$I\!R^2 = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_M \tag{2.4}$$

with the following properties.

(i) Each $\mathcal{M}_j \subset I\!R^2$ is an embedded, connected $\mathcal{C}^2$ submanifold.

(ii) If $j \neq k$, then $\mathcal{M}_j \cap \mathcal{M}_k = \emptyset$.

(iii) If $\mathcal{M}_j \cap \overline{\mathcal{M}}_k \neq \emptyset$, then $\mathcal{M}_j \subset \overline{\mathcal{M}}_k$.

We call $d_k \doteq \dim(\mathcal{M}_k) \in \{0, 1, 2\}$, the dimension of the submanifold $\mathcal{M}_k$. In particular, $d_k = 0$ if $\mathcal{M}_k$ consists of a single point and $d_k = 2$ if $\mathcal{M}_k$ is an open subset of $I\!R^2$. In the case $d_k = 1$, the above assumptions imply that $\mathcal{M}_k$ is a curve admitting a $\mathcal{C}^2$ parameterization in terms of arc-length.

Throughout the following, we assume that there exists a decomposition (2.4) such that the following holds.

(RA1) The barrier Γ admits the decomposition

$$\Gamma = \bigcup_{k\in\mathcal{B}} \mathcal{M}_k\,, \qquad\qquad \text{for some} \quad \mathcal{B} \subset \{1, 2, \ldots, M\}.$$

Moreover, this decomposition is consistent with the previous classifications. Namely, each of the subsets $\Gamma^d, \Gamma^b, \Gamma_S, \Gamma_U$ can be represented as a union of some of the manifolds $\mathcal{M}_k$.

(RA2) Restricted to each submanifold $\mathcal{M}_j$, the minimum time function T^Γ is a $\mathcal{C}^2$ function, or else $T^\Gamma \equiv +\infty$.

Concerning the differential inclusion (1.1) we assume

(RA3) The velocity sets $F(x)$ are uniformly convex, have $\mathcal{C}^2$ boundary, and contain the origin as an interior point. Moreover, denoting by $\langle\cdot,\cdot\rangle$ the Euclidean inner product, the map

$$(p, x) \mapsto \arg\max_{y\in F(x)} \;\langle p\,,\, y\rangle$$

is $\mathcal{C}^2$ on the set where $p \neq 0$.

We observe that, away from the barrier Γ, the minimum time function T^Γ is Lipschitz continuous and provides a viscosity solution to the Hamilton-Jacobi equation

$$H(x, \nabla V) - 1 = 0\,, \qquad\qquad H(x, p) \doteq \max_{y\in F(x)} \;\langle p\,,\, y\rangle. \tag{2.5}$$

We denote by

$$h(x) \doteq \frac{1}{|\nabla T^\Gamma(x)|} = \max_{y\in F(x)} \langle \mathbf{n}(x), y\rangle \qquad \mathbf{n}(x) \doteq \frac{\nabla T^\Gamma(x)}{|\nabla T^\Gamma(x)|}, \tag{2.6}$$

the propagation speed of the fire front, in the normal direction, at the point x.

Remark 4. The assumption **(RA1)** implies that each submanifold $\mathcal{M}_k$ with $k \in \mathcal{B}$ is a portion of the barrier Γ falling in one single class of the above classification. For example, if $\mathcal{M}_k$ contains a point $x \in \Gamma_\mathcal{S}$, then $\mathcal{M}_k \subseteq \Gamma_\mathcal{S}$ and $\mathcal{M}_k \cap \Gamma_\mathcal{F} = \emptyset$.

Remark 5. If $\gamma \subset \Gamma_\mathcal{F} \setminus \Gamma^d$, then, as will be proved in the next section, any local perturbation of the arc γ requiring the same construction time yields another admissible strategy. This is not true if $\gamma \subset \Gamma_\mathcal{F} \cap \Gamma^d$. Indeed, in this case a local perturbation of γ will affect the minimum time function T^Γ in other regions. Therefore, other arcs $\tilde\gamma \subset \Gamma_\mathcal{S}$ may not be admissible any more.

Remark 6. Consider an arc $\gamma \subset \partial R^\Gamma \setminus \Gamma^d$. Then, the minimum time function T^Γ can be extended from R^Γ to a whole neighborhood of each point $x \in \Gamma$. Indeed, this extended value function $\widetilde{T}$ is constructed by solving the Hamilton-Jacobi equation (2.5) with data assigned along γ:

$$\max_{y\in F(x)} \nabla\widetilde{T}(x)\cdot y = 1, \qquad \widetilde{T}(x) = T^\Gamma(x) \quad \text{for} \quad x \in \gamma. \tag{2.7}$$

In particular, the gradient $\nabla T^\Gamma(x)$ and the normal propagation speed $h(x)$ in (2.6) are well defined also for $x \in \gamma$, by a continuous extension.

3 Necessary conditions for free arcs

In this section we consider an arc $\gamma \subset \Gamma_\mathcal{F}$. Intuitively, this means that at the time where this portion of wall is constructed, the fire has not yet reached points in γ. In addition, we assume that $\gamma \subseteq \Gamma^b \setminus \Gamma^d$. This means that γ is a purely blocking arc. All minimum-time trajectories for the fire terminate when they reach a point of γ.

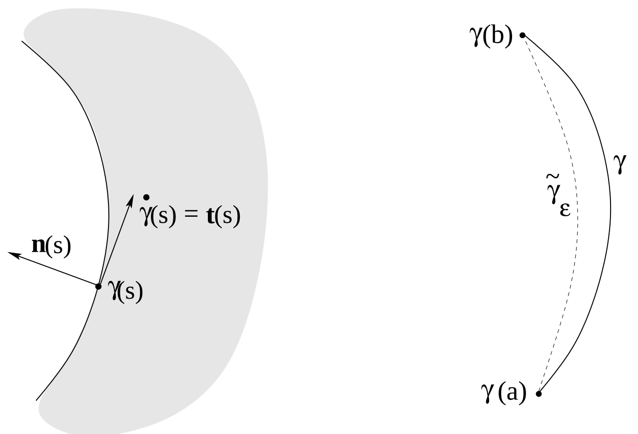

Figure 2: Left: The shaded region denotes the burned set. Since $\dot{\mathbf{n}}(s) = \kappa(s)\mathbf{t}(s)$, here the curvature κ is negative. Right: For a normal extremal, one can find perturbations $\gamma^\sharp_\varepsilon$ that can be constructed in shorter time.

We seek necessary conditions for optimality of the arc γ. To fix the ideas, let $s \mapsto \gamma(s) \in I\!R^2$, $s \in [a,b]$, be a $\mathcal{C}^2$ parametrization of γ in terms of arc-length. Throughout the sequel, an upper dot will denote a derivative w.r.t. s. Given two vectors $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} c \\ d \end{pmatrix}$, their wedge product will be written as $\mathbf{v} \wedge \mathbf{w} \doteq ad - bc$. By $\mathbf{t}(s) \doteq \dot\gamma(s)$ and $\mathbf{n}(s)$ we denote the unit vectors respectively tangent and perpendicular to the curve γ at the point $\gamma(s)$, oriented so that $\mathbf{t} \wedge \mathbf{n} = 1$.

We say that γ is a **normal arc** if there exists a smooth scalar function $\varphi^\sharp : [a,b] \mapsto I\!R$ with $\varphi^\sharp(a) = \varphi^\sharp(b) = 0$ such that, calling $\mathcal{T}(\gamma^\sharp_\varepsilon)$ the time needed to construct the arc $\gamma^\sharp_\varepsilon$ described by

$$s \mapsto \gamma^\sharp_\varepsilon(s) \doteq \gamma(s) + \varepsilon\varphi^\sharp(s)\mathbf{n}(s), \tag{3.1}$$

there holds

$$\left[\frac{d}{d\varepsilon}\mathcal{T}(\gamma^\sharp_\varepsilon)\right]_{\varepsilon=0} \doteq \left[\frac{d}{d\varepsilon}\int_a^b \psi(\gamma^\sharp_\varepsilon(s)) \cdot |\dot\gamma^\sharp_\varepsilon(s)|\, ds\right]_{\varepsilon=0} < 0\,. \tag{3.2}$$

This implies that one can join the endpoints $P = \gamma(a)$ and $Q = \gamma(b)$ with some arc which can be constructed in a slightly shorter time: the curve γ is not a time-minimizer. By possibly taking a small perturbation, it is not restrictive to assume that $\varphi^\sharp(s) = 0$ for s in a neighborhood of a and b.

Given a smooth function $\varphi : [a,b] \mapsto I\!R$ with $\varphi(s) = 0$ for s in a neighborhood of a and b, and given ε, η close to zero, consider the perturbed curve

$$s \mapsto \gamma_{\varepsilon,\eta}(s) \doteq \gamma(s) + (\varepsilon\varphi(s) + \eta\varphi^\sharp(s))\mathbf{n}(s). \tag{3.3}$$

By (3.2) and the implicit function theorem, for every ε in a neighborhood of zero there exists a unique $\eta(\varepsilon)$ such that the time needed to construct the curves γ and $\gamma_{\varepsilon,\eta(\varepsilon)}$ is the same. Calling $\gamma_\varepsilon \doteq \gamma_{\varepsilon,\eta(\varepsilon)}$, we thus have

$$\left[\frac{d}{d\varepsilon}\mathcal{T}(\gamma_\varepsilon)\right]_{\varepsilon=0} = \frac{d}{d\varepsilon}\left[\int_a^b \psi(\gamma_\varepsilon(s)) \cdot |\dot\gamma_\varepsilon(s)|\, ds\right]_{\varepsilon=0} = 0. \tag{3.4}$$

We recall that $\dot{\mathbf{n}}(s) = \kappa(s)\,\mathbf{t}(s)$, where $\kappa(s)$ is the curvature of γ at the point $\gamma(s)$. Since $|\dot\gamma| \equiv 1$, computing a derivative at $\varepsilon = 0$ we find

$$\left[\frac{d}{d\varepsilon}|\dot\gamma_\varepsilon|\right]_{\varepsilon=0} = \left\langle \dot\gamma\,,\ \frac{d}{d\varepsilon}\dot\gamma_\varepsilon\right\rangle_{\varepsilon=0} = \kappa(s)\,\Big(\varphi(s) + \eta'(0)\varphi^\sharp(s)\Big)\,.$$

The equation (3.4) can thus be written as

$$\int_a^b \Big\{\Big\langle \nabla\psi(\gamma(s))\,,\ \mathbf{n}(s)\Big\rangle + \psi(\gamma(s))\,\kappa(\gamma(s))\Big\} \cdot \Big(\varphi(s) + \eta'(0)\varphi^\sharp(s)\Big)\, ds = 0. \tag{3.5}$$

For notational convenience, given a scalar function $g : I\!R^n \to I\!R$ we define

$$\mathcal{G}(g,\gamma,s) \doteq \big\langle \nabla g(\gamma(s)), \mathbf{n}(s)\big\rangle + g(\gamma(s))\,\kappa(s). \tag{3.6}$$

With this notation, from (3.5) it follows

$$\eta'(0) = -\frac{\int_a^b \mathcal{G}(\psi,\gamma,s)\,\varphi(s)\,ds}{\int_a^b \mathcal{G}(\psi,\gamma,s)\,\varphi^\sharp(s)\,ds}\,. \tag{3.7}$$

Notice that in (3.7) the denominator is $\neq 0$, because by (3.2)

$$\int_a^b \mathcal{G}(\psi, \gamma, s)\, \varphi^\sharp(s)\, ds \;=\; \left[\frac{d}{d\varepsilon}\mathcal{T}(\gamma_\varepsilon^\sharp)\right]_{\varepsilon=0} \;<\; 0.$$

Recalling that $\gamma \in \Gamma_{\mathcal{F}} \setminus \Gamma^d$, we now show that, for all ε sufficiently close to zero, the barrier Γ_ε obtained from Γ replacing the arc γ by γ_ε is still admissible. To show this, we first observe that the map

$$t \mapsto \int_{\Gamma\cap\overline{R^\Gamma(t)}} \psi\, dm_1$$

is non-decreasing and right-continuous, hence it is upper semicontinuous. In turn, the *excess map*

$$\mathcal{E}(t) \;\doteq\; t - \int_{\Gamma\cap\overline{R^\Gamma(t)}} \psi\, dm_1 \tag{3.8}$$

is lower semicontinuous.

To fix the ideas, assume $T^\Gamma(\gamma(s)) \in]\tau_0, \tau[$ for $s \in]a, b[$, with $]\tau_0, \tau[\cap \mathcal{S} = \emptyset$. Moreover, assume that the perturbations $\varphi, \varphi^\sharp$ in (3.3) are supported in the compact subset $[a+\sigma,\, b-\sigma]$. By lower semicontinuity and compactness, there exists $\delta > 0$ such that

$$\mathcal{E}(T(\gamma(s)) > \delta \qquad\qquad s \in [a+\sigma,\, b-\sigma]\,.$$

Choose times $\tau_0 < \tau_1 < \cdots < \tau_N = \tau$ such that $\tau_i - \tau_{i-1} < \delta/2$ for every $i = 1, \ldots, N$. For each i, we can now choose $\varepsilon_i > 0$ small enough such that, for $|\varepsilon| \leq \varepsilon_i$, there holds

$$\int_{\Gamma_\varepsilon\cap\overline{R^{\Gamma_\varepsilon}(\tau_i)}} \psi\, dm_1 \;<\; \frac{\delta}{2} + \int_{\Gamma\cap\overline{R^\Gamma(\tau_i)}} \psi\, dm_1\,,$$

Setting $\bar\varepsilon \doteq \min\{\varepsilon_1, \ldots, \varepsilon_N\}$, we now prove that the barrier Γ_ε is admissible whenever $|\varepsilon| \leq \bar\varepsilon$. Indeed, when $\tau_{i-1} < t \leq \tau_i$ we have

$$\int_{\Gamma_\varepsilon\cap\overline{R^{\Gamma_\varepsilon}(t)}} \psi\, dm_1 \;\leq\; \int_{\Gamma_\varepsilon\cap\overline{R^{\Gamma_\varepsilon}(\tau_i)}} \psi\, dm_1 \;<\; \frac{\delta}{2} + \int_{\Gamma\cap\overline{R^\Gamma(\tau_i)}} \psi\, dm_1 \;=\; \frac{\delta}{2} + \tau_i - \mathcal{E}(\tau_i) \;\leq\; \tau_i - \frac{\delta}{2} \;\leq\; t\,. \tag{3.9}$$

Next, call $J(\Gamma_\varepsilon)$ the total cost associated with this perturbed strategy. If Γ is optimal, then

$$\left[\frac{d}{d\varepsilon}\, J(\Gamma_\varepsilon)\right]_{\varepsilon=0} = 0. \tag{3.10}$$

Assuming that the normal vector $\mathbf{n}$ points toward the outside of the burned region, by the previous analysis (3.10) can be written as

$$\int_a^b \Big\{\alpha(\gamma(s)) + \mathcal{G}(\beta, \gamma, s)\Big\} \cdot \Big(\varphi(s) + \eta'(0)\varphi^\sharp(s)\Big)\, ds \;=\; 0. \tag{3.11}$$

Inserting the value of $\eta'(0)$ given at (3.7), and adopting the shorter notation $\alpha(s) \doteq \alpha(\gamma(s))$, from the above equation we obtain

$$\int_a^b \{\alpha(s) + \mathcal{G}(\beta, \gamma, s)\}\, \varphi(s)\, ds + \lambda \cdot \int_a^b \mathcal{G}(\psi, \gamma, s)\varphi(s)\, ds \;=\; 0. \tag{3.12}$$

Here the constant λ has the role of a Lagrange multiplier:

$$\lambda \;=\; \frac{-\int_a^b \{\alpha(s) + \mathcal{G}(\beta,\gamma,s)\}\,\varphi^\sharp(s)\,ds}{\int_a^b \mathcal{G}(\psi,\gamma,s)\,\varphi^\sharp(s)\,ds}\,. \tag{3.13}$$

Since (3.12) holds for all smooth functions φ with $\varphi(a) = \varphi(b) = 0$, recalling the definition of $\mathcal{G}(\cdot)$ we conclude

$$\alpha(s) + \Big\langle \nabla(\beta(s) + \lambda\psi(s))\,,\, \mathbf{n}(s)\Big\rangle + \big(\beta(s) + \lambda\psi(s)\big)\kappa(s) \;=\; 0 \tag{3.14}$$

for all $a < s < b$. Written as

$$-\,(\beta(s) + \lambda\psi(s))\,\kappa(s) \;=\; \alpha(\gamma(s)) + \Big\langle \nabla(\beta(s) + \lambda\psi(s))\,,\, \mathbf{n}(s)\Big\rangle, \tag{3.15}$$

this necessary condition takes the form of a second order nonlinear O.D.E., determining the curvature of γ. In the special case where ψ and β are constant, the above equation reduces to

$$\kappa(s) \;=\; -\,\frac{\alpha(s)}{\beta + \lambda\psi}\,, \tag{3.16}$$

showing that the curvature of γ must be proportional to the local value of land α. In particular, if α is also constant, then the free arc γ is an arc of circumference.

Remark 7. The Lagrange multiplier λ can be interpreted as the (constant) value of time, during the construction of the arc γ. Indeed, given any smooth scalar function $\varphi : [a,b] \mapsto I\!R$ with $\varphi(a) = \varphi(b) = 0$, for ϵ in a neighborhood of zero consider the perturbed arc $\tilde\gamma_\epsilon(s) \doteq \gamma(s) + \epsilon\varphi(s)\mathbf{n}(s)$. The time $\mathcal{T}(\tilde\gamma_\epsilon)$ needed to construct this arc satisfies

$$\left[\frac{d}{d\epsilon}\mathcal{T}(\tilde\gamma_\epsilon)\right]_{\epsilon=0} \;=\; \int_a^b \mathcal{G}(\psi,\gamma,s)\,\varphi(s)\,ds. \tag{3.17}$$

Calling Γ_ϵ the barrier obtained from Γ by replacing the arc γ with $\tilde\gamma_\epsilon$, the cost $J(\Gamma_\epsilon)$ satisfies

$$\left[\frac{d}{d\epsilon}\,J(\Gamma_\epsilon)\right]_{\epsilon=0} \;=\; \int_a^b \{\alpha(s) + \mathcal{G}(\beta,\gamma,s)\}\,\varphi(s)\,ds. \tag{3.18}$$

The ratio $\dfrac{\text{[decrease of the total cost]}}{\text{[increase in the construction time]}}$ now yields the value of time. Assuming that the quantity in (3.17) does not vanish, this ratio can be computed as

$$\left[\frac{-\frac{d}{d\varepsilon}J(\Gamma_\varepsilon)}{\frac{d}{d\varepsilon}\mathcal{T}(\gamma_\varepsilon)}\right]_{\varepsilon=0} \;=\; \frac{-\int_a^b \{\alpha(s) + \mathcal{G}(\beta,\gamma,s)\}\,\varphi(s)\,ds}{\int_a^b \mathcal{G}(\psi,\gamma,s)\,\varphi(s)\,ds} \;=\; \lambda. \tag{3.19}$$

Indeed, by (3.13) and (3.12), the ratio does not depend on the choice of the function φ.

In the special case where the construction speed $\sigma = 1/\psi$ and the construction cost β are constant, calling $r = -1/\kappa$ the radius of curvature, from (3.16) it follows

$$\lambda \;=\; (\alpha r - \beta)\sigma\,. \tag{3.20}$$

Notice that, in an optimal strategy, one must have $\alpha r - \beta \geq 0$. Otherwise, the cost of building the barrier would be larger than the value of the region shielded from the fire.

Next, we consider the case where $]\tau_0, \tau_1[\ \cap\ \mathcal{S}\ =\ \emptyset$, and the portion of wall constructed during this time interval consists of not just one but several free arcs, say

$$\{x \in \Gamma;\ \ T^\Gamma(x) \in]\tau_0, \tau_1[\,\} \ = \ \gamma_1 \cup \cdots \cup \gamma_\nu \ \subseteq \ \Gamma_{\mathcal{F}} \setminus \Gamma^d .$$

Let the i-th arc be parameterized by arc-length, say $s \mapsto \gamma_i(s)$, $s \in]a_i, b_i[$. Assume that at least one of these arcs, say γ_1, is normal. Then we can find a compactly supported perturbation $\varphi_1^\sharp$ so that (3.2) holds.

Given a set of smooth perturbations with compact support $\varphi_i :]a_i, b_i[\mapsto I\!R$, $i = 1, \ldots, \nu$, for any ε sufficiently close to zero we can find $\eta(\varepsilon)$ such that the total time needed to construct the ν perturbed curves

$$\gamma_{1,\varepsilon}(s) = \gamma_1(s) + [\varepsilon\varphi_1(s) + \eta(\varepsilon)\varphi_1^\sharp(s)]\mathbf{n}_1(s), \qquad \gamma_{i,\varepsilon}(s) = \gamma_i(s) + \varepsilon\varphi_i(s)\mathbf{n}_i(s) \quad i = 2, 3, \ldots, \nu,$$

is the same for all ε. Hence

$$\frac{d}{d\varepsilon} \sum_{i=1}^{\nu} \mathcal{T}(\gamma_{i,\varepsilon}) \ = \ \frac{d}{d\varepsilon} \left[\sum_{i=1}^{\nu} \int_{a_i}^{b_i} \psi(\gamma_{i,\varepsilon}(s)) \cdot |\dot\gamma_{i,\varepsilon}(s)|\, ds \right] \ \equiv \ 0. \tag{3.21}$$

A similar argument as in (3.7) yields

$$\eta'(0) \ = \ - \sum_{i=1}^{\nu} \frac{\int_{a_i}^{b_i} \mathcal{G}(\psi, \gamma_i, s)\, \varphi_i(s)\, ds}{\int_{a_1}^{b_1} \mathcal{G}(\psi, \gamma_1, s)\, \varphi_1^\sharp(s)\, ds} . \tag{3.22}$$

As before, one can show that the strategy Γ_ε obtained by replacing each arc γ_i with $\gamma_{i,\varepsilon}$ is still admissible, as long as ε remains sufficiently small. Since Γ is optimal, the identity (3.10) must hold. In the present case, this yields

$$\int_{a_1}^{b_1} \big(\alpha(\gamma_1(s)) + \mathcal{G}(\beta, \gamma_1, s)\big) \cdot \eta_1'(0)\varphi_1^\sharp(s)\, ds + \sum_{i=1}^{\nu} \int_{a_i}^{b_i} \big(\alpha(\gamma_i(s)) + \mathcal{G}(\beta, \gamma_i, s)\big)\varphi_i(s)\, ds \ = \ 0. \tag{3.23}$$

Hence there exists a Lagrange multiplier

$$\lambda \ = \ - \frac{\int_{a_1}^{b_1} \{\alpha(\gamma_1(s)) + \mathcal{G}(\beta, \gamma_1, s)\}\, \varphi_1^\sharp(s)\, ds}{\int_{a_1}^{b_1} \mathcal{G}(\psi, \gamma_1, s)\, \varphi_1^\sharp(s)\, ds} \tag{3.24}$$

such that

$$\int_a^b \{\alpha(\gamma_i(s)) + \mathcal{G}(\beta, \gamma_i, s)\}\, \varphi_i(s)\, ds + \lambda \cdot \int_a^b \mathcal{G}(\psi, \gamma_i, s)\varphi_i(s)\, ds \ = \ 0. \tag{3.25}$$

for all $i = 1, \ldots, \nu$ and all perturbations φ_i with compact support in $]a_i, b_i[$. As in (3.14)-(3.15), setting $\alpha_i(s) \doteq \alpha(\gamma_i(s))$, $\beta_i(s) \doteq \beta(\gamma_i(s)) \ldots$, and recalling the definition of $\mathcal{G}$, we conclude that

$$\alpha_i(s) + \Big(\langle \nabla(\beta_i(s) + \lambda\psi_i(s))\, ,\, \mathbf{n}_i(s)\rangle + (\beta_i(s) + \lambda\psi_i(s))\, \kappa_i(s)\Big) \ = \ 0 \tag{3.26}$$

for all $a_i < s < b_i$.

Summarizing the previous analysis, we now state a necessary condition for optimality, valid when several free arcs are simultaneously constructed.

Theorem 1 (Necessary conditions for free arcs). *Let $\gamma_1, \ldots, \gamma_\nu \subset \Gamma_{\mathcal{F}} \setminus \Gamma^d$ be free arcs, simultaneously constructed by an optimal strategy Γ during the time interval $t \in]\tau_0, \tau_1[$. Assume that at least one of these arcs is normal, and let $s \mapsto \gamma_i(s)$ be a parametrization of γ_i by arc-length, with $s \in]a_i, b_i[$. Then there exists a Lagrange multiplier $\lambda \geq 0$ such that*

$$-\left(\beta_i(s) + \lambda\psi_i(s)\right)\kappa_i(s) \;=\; \alpha_i(s) + \Big\langle \nabla\left(\beta_i(s) + \lambda\psi_i(s)\right), \, \mathbf{n}_i(s)\Big\rangle \tag{3.27}$$

for all $i = 1, \ldots, \nu$, $a_i < s < b_i$.

We recall that all curvatures κ_i are negative, as explained in fig. 2.

As in Remark 7, the Lagrange multiplier λ can be interpreted as the *value of time*. Note that this value is the same for all arcs $\gamma_1, \ldots, \gamma_\nu$, and remains constant throughout the time interval $]\tau_0, \tau_1[$ where the constraint (1.4) is unsaturated.

4 Necessary conditions for boundary arcs

Let Γ be an optimal barrier, and assume that for $t \in [a, b]$ the constraint (1.10) is saturated i.e. it is satisfied as an equality. To fix the ideas, assume that the subset

$$\Gamma_{[a,b]} \;\doteq\; \{x \in \Gamma\,; \;\; a \leq T^\Gamma(x) \leq b\}$$

consists of ν boundary arcs $\gamma_1, \ldots, \gamma_\nu \subset \Gamma_{\mathcal{S}} \setminus \Gamma^d$, simultaneously constructed. Let each of these arcs be parameterized by time: $t \mapsto \gamma_i(t)$, $t \in [a, b]$, so that $T^\Gamma(\gamma_i(t)) = t$ for each $i = 1, \ldots, \nu$. We seek here necessary conditions for the optimality of these arcs. These will extend the conditions derived in [5] to the case where the cost functions α, β and the construction speed $1/\psi$ are allowed to depend on the space variable x.

We say that γ_i is a **normal arc** if, at every $t \in [a, b]$, the tangent vector $\dot\gamma_i(t)$ is not parallel to the gradient of the value function $\nabla T^\Gamma(\gamma_i(t))$. Since $T^\Gamma(\gamma_i(t)) = t$ for all t, this is equivalent to the strict inequality

$$|\dot\gamma_i(t)| \;>\; h(x) \tag{4.1}$$

In other words, the speed at which the arc γ_i is constructed is strictly greater than the local propagation speed $h(x)$ of the fire front, defined at (2.6). Throughout the following, we assume that all the arcs $\gamma_1, \ldots, \gamma_\nu$ are normal.

We begin by choosing a suitable set of coordinates, around each arc γ_i . Let $\widetilde{T}$ be the minimum time function, extended to a neighborhood of each arc γ_i, as in Remark 6. The assumption that the arcs γ_i are normal guarantees that these extensions are well defined. For $t \in [a, b]$ and s close to zero, define a coordinate system $(t, s) \mapsto x_i(t, s)$ so that $x_i(t, 0) = \gamma_i(t)$, while, for each fixed time t, the map $s \mapsto x_i(t, s)$ provides an arc-length parametrization of the curve $\{x\,;\; \widetilde{T}(x) = t\}$. To fix the ideas, we choose the orientation so that the points $x_i(t, s)$ with $s > 0$ fall outside the set R^Γ reached by the fire, as in fig.3. This implies

$$\big\langle \mathbf{e}_i(t), \, \dot\gamma_i(t)\big\rangle \;<\; 0, \tag{4.2}$$

where $\mathbf{e}_i(t) \doteq \frac{\partial x_i(t,s)}{\partial s}\Big|_{s=0}$ denotes the unit vector tangent to the curve $\{x\,;\ \widetilde{T}(x) = t\}$ at the point $\gamma_i(t)$. In addition, we let $\mathbf{n}_i(t)$ be the unit vector parallel to $\nabla\widetilde{T}$ (hence perpendicular to $\mathbf{e}_i(t)$) at the point $\gamma_i(t)$, as in Figure 3.

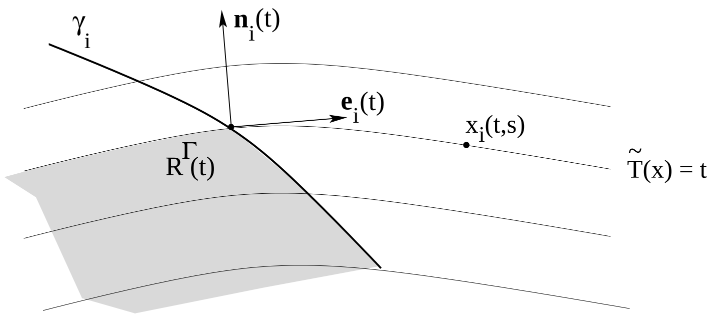

Figure 3: Choice of the coordinates (t, s) in a neighborhood of the arc γ_i.

Let $w_i^*(t) > 0$ be the amount of resources allocated at time t to the construction of the arc γ_i, so that

$$|\dot\gamma_i(t)| = \frac{w_i^*(t)}{\psi(\gamma_i(t))}, \qquad \sum_{i=1}^{\nu} w_i^*(t) \equiv 1 \qquad t \in [a, b].$$

Consider an alternative strategy $w = (w_1, \ldots, w_\nu)$. This will result in the construction of different arcs $t \mapsto y_i(t)$, determined by the equations

$$|\dot y_i(t)| = \frac{w_i(t)}{\psi(y_i(t))}, \qquad \widetilde{T}(y_i(t)) = t\,.$$

Using our previous coordinate system, let $y_i(t) = x_i(t, s_i(t))$. For each $i = 1, \ldots, \nu$, the scalar function $s_i(t)$ will then satisfy an O.D.E. of the form

$$\dot s_i = f_i(t, s_i(t), w_i(t))\,. \tag{4.3}$$

Here the right hand side f_i is implicitly determined by the scalar constraint

$$\left| \frac{\partial x_i(t, s_i)}{\partial t} + f_i(t, s_i, w_i)\, \frac{\partial x_i(t, s_i)}{\partial s_i} \right| \;=\; \frac{w_i}{\psi(x_i(t, s_i))}. \tag{4.4}$$

We observe that the equation (4.4) admits solutions provided that

$$h(x_i(t, s_i)) \;\leq\; \frac{w_i}{\psi(x_i(t, s_i))}. \tag{4.5}$$

Indeed, the speed at which the barrier is constructed cannot be smaller than the propagation speed of the fire front, in the normal direction. In the case of a strict inequality, the equation (4.5) has exactly two solutions. The choice of the solution clearly depends on the side occupied by the burned region (see fig. 4).

Assuming that the strategy $t \mapsto w^*(t) = (w_1^*, \ldots, w_\nu^*)(t)$ is optimal for the fire blocking problem, we now construct an auxiliary control problem for which w^* is optimal as well. Consider the control system consisting of the ν equations (4.3), supplemented by the initial and terminal constraints,

$$s_i(a) = s_i(b) = 0 \qquad i = 1, \cdots, \nu\,. \tag{4.6}$$

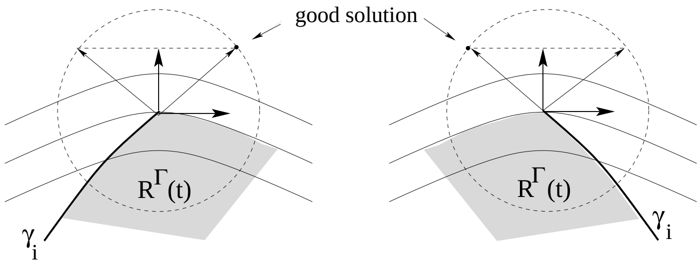

Figure 4: In equation (4.4), one should take f_i as the larger or the smaller solution, respectively if the burned region lies to the right or to the left of the barrier γ_i.

Calling

$$I\!R^\nu_+ \doteq \Big\{ w = (w_1, \ldots, w_\nu)\,; \quad w_i \geq 0 \quad \text{for all } i = 1, \ldots, \nu \Big\},$$

the family of admissible controls functions is defined as

$$\mathcal{W} \doteq \left\{ w : [a,b] \mapsto I\!R^\nu_+\,; \quad w \text{ measurable}, \qquad \sum_{i=1}^{\nu} \int_a^t w_i(\tau)\,d\tau \;\leq\; t - a \qquad \text{for all } \; t \in [a,b] \right\}. \tag{4.7}$$

Now consider the optimization problem

$$\text{minimize :} \quad \Lambda(w) \;\doteq\; \sum_{i=1}^{\nu} \int_a^b L_i(t, s_i(t), w_i(t))\,dt\,, \tag{4.8}$$

where the running costs are

$$L_i(t, s_i, w_i) \;\doteq\; \beta(x_i(t,s_i))\,\frac{w_i(t)}{\psi(x_i(t,s_i))} + \int_0^{s_i} h(x_i(t,\xi))\,\alpha(x_i(t,\xi))\,d\xi\,, \tag{4.9}$$

with $h(\cdot)$ as in definition (2.6). The minimum in (4.8) is sought among all control functions $w \in \mathcal{W}$. Notice that the first term in (4.9) accounts for the cost of building the wall, while the second term is related to the value of the burned area. We are assuming here that the burned region has the representation $\{x_i(t,s)\,;\; s < s_i(t)\}$.

It is convenient to introduce two additional state variables, to account for the cost functional (4.8) and for the integral constraint in (4.7), which will be reformulated as a pointwise state constraint. We thus consider the variables $s_0, s_{\nu+1}$, governed by the equations

$$s_0(a) = 0\,, \qquad \dot{s}_0(t) \;=\; f_0(w(t)) \;\doteq\; -1 + \sum_{i=1}^{\nu} w_i(t), \tag{4.10}$$

$$s_{\nu+1}(a) = 0\,, \qquad \dot{s}_{\nu+1}(t) \;=\; f_{\nu+1}(t, s(t), w(t)) \;\doteq\; \sum_{i=1}^{\nu} L_i(t, s_i(t), w_i(t))\,. \tag{4.11}$$

For a system with state variables $s = (s_0, s_1, \ldots, s_{\nu+1})$ and dynamics (4.3), (4.6), (4.10), (4.11), we now consider the optimization problem

$$\text{minimize:} \;\; s_{\nu+1}(b) \tag{4.12}$$

with state constraint

$$s_0(t) \leq 0 \qquad\qquad \text{for all} \;\; t \in [a,b]. \tag{4.13}$$

The minimum is sought among all measurable controls $w = (w_1, \ldots, w_\nu) : [a, b] \mapsto I\!R^\nu_+$.

By construction, the control $w^*(t) = (w_1^*(t), \ldots, w_\nu^*(t))$ corresponding to the trajectory $t \mapsto (s_0(t),\, s_1(t), \ldots, s_\nu(t)) \equiv (0, 0, \ldots, 0)$ is optimal for this auxiliary optimal control problem. We recall that

$$w_i^*(t) > 0, \qquad \sum_{i=1}^{\nu} w_i^*(t) = 1 \qquad t \in [a, b].$$

Using a version of the Pontryagin maximum principle in the presence of the state constraints (see [15, 12]), we conclude that there exists $\lambda_0 \geq 0$, and a map $t \mapsto q(t) = (q_0(t), \cdots, q_\nu(t))$, not both equal to zero, such that the following holds. The map q_0 satisfies

$$q_0(b) = 0, \qquad q_0(t) = q_0(a) - \int_a^t d\mu$$

where μ is any positive measure supported on the set where $s_0 = 0$. Since by assumption this set is the entire interval $[a, b]$, this is equivalent to

$$t \mapsto q_0(t) \quad \text{is bounded, non-increasing,} \qquad q_0(b) = 0. \tag{4.14}$$

Moreover, the other components $q_1, \ldots q_\nu$ are absolutely continuous functions such that

$$\dot q_i(t) \;=\; -\, q_i \frac{\partial f_i}{\partial s_i}(t, 0, w_i^*(t)) - \lambda_0 \frac{\partial L_i}{\partial s_i}(t, 0, w_i^*(t)) \qquad i = 1, \ldots, \nu \tag{4.15}$$

for a.e. $t \in [a, b]$. Finally, for a.e. $t \in [a, b]$ there holds

$$\begin{aligned} q_0(t) + \lambda_0 \sum_{i=1}^{\nu} L_i(t, 0, w_i^*(t)) + \sum_{i=1}^{\nu} q_i(t)\, f_i(t, 0, w_i^*(t)) \\ = \min_{w \in I\!R^\nu_+} \left(q_0(t) \sum_{i=1}^{\nu} w_i + \lambda_0 \sum_{i=1}^{\nu} L_i(t, 0, w_i) + \sum_{i=1}^{\nu} q_i(t)\, f_i(t, 0, w_i) \right). \end{aligned} \tag{4.16}$$

Differentiating w.r.t. $w_1, \ldots, w_\nu$, from (4.16) we deduce

$$-q_i(t) \frac{\partial f_i}{\partial w_i}(t, 0, w_i^*(t)) - \lambda_0 \frac{\partial L_i}{\partial w_i}(t, 0, w_i^*(t)) \;=\; q_0(t) \qquad i = 1, \ldots, \nu\,. \tag{4.17}$$

We now work out a more explicit form of the equations (4.15) and of the conditions (4.17). From the definition of L_i at (4.9) it follows

$$\frac{\partial L_i}{\partial s_i}(t, 0, w_i^*(t)) \;=\; \left\langle \nabla \left(\frac{\beta}{\psi} \right) (\gamma_i(t))\,,\; \mathbf{e}_i(t) \right\rangle w_i^*(t) + h_i(\gamma_i(t))\, \alpha(\gamma_i(t)). \tag{4.18}$$

Toward the computation of $\partial f_i / \partial s_i$, consider a family of perturbed trajectories of the form

$$t \to \gamma_i^\varepsilon(t) \;=\; \gamma_i(t,\, \varepsilon\zeta(t) + \mathcal{O}(\varepsilon^2)) \;=\; \gamma_i(t) + \varepsilon\zeta(t)\mathbf{e}_i(t) + \mathcal{O}(\varepsilon^2), \tag{4.19}$$

such that for all ε, t

$$|\dot\gamma_i^\varepsilon(t)| \cdot \psi(\gamma_i^\varepsilon(t)) \;=\; |\dot\gamma_i(t)| \cdot \psi(\gamma_i(t)) \;=\; w_i^*(t).$$

In particular, the above identities imply

$$\frac{d}{d\varepsilon}\Big[|\dot\gamma_i^\varepsilon(t)|^2\,\psi^2(\gamma_i^\varepsilon(t))\Big]_{\varepsilon=0} \;=\; 0. \tag{4.20}$$

Using (4.19) in the above equation, we obtain

$$\Big\langle\dot\gamma_i(t)\,,\;\dot\zeta(t)\mathbf{e}_i(t)+\zeta(t)\dot{\mathbf{e}}_i(t)\Big\rangle\,\psi(\gamma_i(t))+\Big\langle\nabla\psi(\gamma_i(t))\,,\;\zeta(t)\mathbf{e}_i(t)\Big\rangle\,|\dot\gamma_i(t)|^2=0. \tag{4.21}$$

Solving (4.21) for $\dot\zeta$ and recalling (4.20), we derive the first order linear O.D.E.:

$$\dot\zeta \;=\; -\,\frac{\big\langle\dot\gamma_i(t),\dot{\mathbf{e}}_i(t)\big\rangle\,\psi(\gamma_i(t))+\big\langle\nabla\psi(\gamma_i(t))\,,\;\mathbf{e}_i(t)\big\rangle\,\big|\dot\gamma_i(t)\big|^2}{\big\langle\dot\gamma_i(t),\mathbf{e}_i(t)\big\rangle\,\psi(\gamma_i(t))}\,\zeta\,. \tag{4.22}$$

On the other hand, we observe that the scalar functions $s_i^\varepsilon = \varepsilon\zeta(t)+\mathcal{O}(\varepsilon^2)$ in (4.19) are solutions to the same O.D.E.

$$\dot s_i^\varepsilon(t) = f_i(t, s_i^\varepsilon(t), w_i^*(t)),$$

with possibly different initial data. Hence the first order term $\zeta(\cdot)$ in the expansion provides a solution to the linear equation

$$\dot\zeta = \frac{\partial f_i}{\partial s_i}(t,0,w_i^*(t))\,\zeta. \tag{4.23}$$

Comparing (4.22) with (4.23) we conclude

$$\frac{\partial f_i}{\partial s_i}(t,0,w_i^*(t)) \;=\; -\,\frac{\big\langle\dot\gamma_i(t),\dot{\mathbf{e}}_i(t)\big\rangle\,\psi(\gamma_i(t))+\big\langle\nabla\psi(\gamma_i(t))\,,\;\mathbf{e}_i(t)\big\rangle\,\big|\dot\gamma_i(t)\big|^2}{\big\langle\dot\gamma_i(t),\mathbf{e}_i(t)\big\rangle\,\psi(\gamma_i(t))}\,. \tag{4.24}$$

The equations (4.18) and (4.24) provide a more explicit expression of the right hand side of (4.15).

To compute the partial derivative $\partial f_i/\partial w_i$ at points $(t,0,w_i^*(t))$, we start by writing (4.4) in the equivalent form

$$\left|\frac{\partial x_i(t,s_i)}{\partial t}+f_i(t,s_i,w_i)\,\frac{\partial x_i(t,s_i)}{\partial s_i}\right|^2 \;=\; \left(\frac{w_i}{\psi(x_i(t,s_i))}\right)^2. \tag{4.25}$$

Since our choice of coordinates (t,s_i) implies $\frac{\partial x_i}{\partial s_i}(t,0)=\mathbf{e}_i(t)$ and $f_i(t,0,w_i^*(t))\equiv 0$, differentiating (4.25) w.r.t. w_i we obtain

$$\big\langle\dot\gamma_i(t)\,,\;\mathbf{e}_i(t)\big\rangle\,\frac{\partial f_i}{\partial w_i}(t,0,w_i^*(t)) \;=\; \frac{w_i^*(t)}{\psi^2(\gamma_i(t))}. \tag{4.26}$$

For convenience, we denote by $\theta_i(t)$ the angle between the barrier γ and the level set $\{\tilde T(x)= t\}$, as in fig. 3. Notice that this implies

$$\left\langle\frac{\dot\gamma_i(t)}{|\dot\gamma_i(t)|}\,,\;\mathbf{e}_i(t)\right\rangle \;=\; -\cos\theta_i(t).$$

Recalling that the optimal control is $w_i^*(t) = |\dot\gamma_i(t)|\cdot\psi(\gamma_i(t))$, from the identity (4.26) we deduce

$$\frac{\partial f_i}{\partial w_i}(t,0,w_i^*(t)) \;=\; -\,\frac{1}{\psi(\gamma_i(t))}\cdot\frac{1}{\cos\theta_i(t)} \;<\; 0. \tag{4.27}$$

We observe that the map $w_i \mapsto f_i(t, 0, w_i)$ is well defined and monotone decreasing, for w_i in a neighborhood of $w_i^*(t)$. Intuitively, if we increase the amount w_i of resources allocated to the construction of the barrier γ_i, then the construction speed $|\dot\gamma|$ increases. As a result, the angle θ_i decreases and the barrier will be shifted toward the left, further reducing the region burned by the fire (see fig. 3).

We also notice that (4.14) implies that $q_0(t) \geq 0$ for all $t \in [a, b]$. Summarizing the the above arguments, we now state a set of necessary conditions for the optimality of multiple arcs which are constructed simultaneously. Recall that f_i, L_i are the functions in (4.3) and (4.9), while θ_i is the angle between the barrier and the fire front.

Theorem 2 (Necessary Conditions for Boundary Arcs). *Let $\gamma_1, \cdots, \gamma_\nu \subset \Gamma_{\mathcal{S}} \backslash \Gamma^d$ be the boundary arcs simultaneously constructed by an optimal strategy Γ, during the time interval $t \in [\tau_1, \tau_2]$. Assume that each arc $t \mapsto \gamma_i(t)$ is normal, parameterized by time $t = T^\Gamma(\gamma_i(t))$, and call $\mathbf{e}_i(t)$ the unit vector tangent to the boundary of the reachable set $R^\Gamma(t)$ at the point $\gamma_i(t)$, oriented toward the outside of the reachable set. Then there exist a constant $\lambda_0 \geq 0$ and an absolutely continuous adjoint vector $q = (q_1, \ldots, q_\nu)$, not both equal to zero, satisfying*

$$\begin{aligned} \dot q_i(t) \;&=\; q_i(t) \cdot \frac{\big\langle \dot\gamma_i(t), \dot{\mathbf{e}}_i(t)\big\rangle\, \psi(\gamma_i(t)) + \big\langle \nabla\psi(\gamma_i(t))\,,\; \mathbf{e}_i(t)\big\rangle\, \big|\dot\gamma_i(t)\big|^2}{\big\langle \dot\gamma_i(t), \mathbf{e}_i(t)\big\rangle\, \psi(\gamma_i(t))} \\ &\qquad - \lambda_0 \left[\left\langle \nabla\left(\frac{\beta}{\psi}\right)\big(\gamma_i(t)\big)\,,\; \mathbf{e}_i(t) \right\rangle w_i^*(t) + h_i(\gamma_i(t))\, \alpha(\gamma_i(t)) \right], \end{aligned} \tag{4.28}$$

and such that the optimality conditions

$$\begin{aligned} & q_0(t) w_i^*(t) + \lambda_0\, L_i(t, 0, w_i^*(t)) + q_i(t)\, f_i(t, 0, w_i^*(t)) \\ &\qquad = \min_{\omega \geq 0} \Big\{ q_0(t)\omega + \lambda_0\, L_i(t, 0, \omega) + q_i(t)\, f_i(t, 0, \omega) \Big\} \end{aligned} \tag{4.29}$$

hold for every $i = 1, \ldots, \nu$ and $\tau_1 < t < \tau_2$. Moreover, the functions

$$W_i(t) \doteq \frac{1}{\cos\theta_i(t)} \frac{q_i(t)}{\psi(\gamma_i(t))} - \lambda_0 \cdot \frac{\beta(\gamma_i(t))}{\psi(\gamma_i(t))} \qquad i = 1, \ldots, \nu, \tag{4.30}$$

are non-negative, non-increasing, and all equal to each other.

Indeed, the first part of the theorem is a reformulation of the Pontryagin maximum principle. Concerning the last statement, recalling (4.27), (4.9), and the above definition of W_i, by (4.17) we conclude

$$W_i(t) \;=\; -\, q_i(t) \frac{\partial f_i}{\partial w_i}(t, 0, w_i^*(t)) - \lambda_0 \frac{\partial L_i}{\partial w_i}(t, 0, w_i^*(t)) \;=\; q_0(t) \qquad i = 1, \ldots, \nu\,. \tag{4.31}$$

5 Necessary conditions at junctions

The necessary conditions for optimality derived in the previous two sections were of local nature. Indeed, we always used perturbations of free arcs or of boundary arcs which kept the endpoints fixed. In this section, we shall obtain stronger optimality conditions, of global nature, by allowing changes also at the endpoints of the various arcs.

We recall that, for a free arc γ, the Lagrange multiplier λ introduced at (3.19) could be interpreted as the **value of time**, which is constant during the interval when the free arc is constructed. If the construction cost β and the construction speed $\sigma = 1/\psi$ are constant, then by (3.20) this value of time is computed as

$$W(t) \;=\; (\alpha r - \beta)\sigma. \tag{5.1}$$

Here α is the unit value of the land, while r is the radius of curvature of the barrier γ. According to Theorem 1, if several free arcs $\gamma_1, \ldots, \gamma_\nu$ are simultaneously constructed, the values $(\alpha(\gamma_i(x))r_i(x) - \beta)\sigma$ are all equal to each other.

On the other hand, when boundary arcs are constructed, the functions $W_i(t)$ in (4.30) are defined only up to a positive constant. Indeed, they depend on the choice of the the adjoint variables q_i, λ_0. In the present section we consider some particular configurations of optimal barriers, where one can take $\lambda_0 = 1$ and let $(q_1, \ldots, q_\nu) = \nabla V$ be the gradient of a value function. In this case, the instantaneous value of time is well defined as

$$W(t) \;=\; W_i(t) \;=\; \frac{1}{\cos\theta_i(t)}\,\frac{q_i(t)}{\psi(\gamma_i(t))} - \frac{\beta(\gamma_i(t))}{\psi(\gamma_i(t))} \qquad i = 1, \ldots, \nu. \tag{5.2}$$

5.1 Two boundary arcs joining together

We start by examining the case where two boundary arcs γ_1, γ_2 join together at the terminal point P at time T, thus completing the wall construction, as in fig. 5.

We show that this situation can be modelled by an optimal control system in standard form, with free terminal time. Indeed, let $(t, s) \mapsto x(t, s)$ be a system of coordinates, chosen so that

- *for each fixed t, the map $s \mapsto x(t, s)$ is an arc-length parameterization of the boundary $\partial R^\Gamma(t)$, so that*

$$T^\Gamma(x(t, s)) = t, \qquad\qquad \left|\frac{\partial}{\partial s} x(t, s)\right| \equiv 1\,.$$

We introduce an auxiliary optimal control problem, with state $s = (s_0, s_1, s_2)$ and control variable $w = (w_1, w_2)$.

$$\text{Minimize:} \qquad s_0(T) \tag{5.3}$$

for the system with dynamics

$$\begin{cases} \dot s_0 &= f_0(t, s_1, s_2, w_1, w_2) \\ \dot s_1 &= f_1(t, s_1, w_1) \\ \dot s_2 &= f_2(t, s_2, w_2) \end{cases} \tag{5.4}$$

and terminal constraints

$$s_1(T) - s_2(T) \;=\; 0. \tag{5.5}$$

Here the controls w_1, w_2 satisfy the constraints

$$w_i(t) \in [0, 1]\,, \qquad\qquad w_1(t) + w_2(t) \;\le\; 1\,. \tag{5.6}$$

Moreover,

$$
\begin{aligned}
f_0(t, s_1, s_2, w_1, w_2) \ &\doteq\ \beta(x(t, s_1(t)))\,\frac{w_1(t)}{\psi(x(t, s_1(t)))} + \beta(x(t, s_2(t)))\,\frac{w_2(t)}{\psi(x(t, s_2(t)))} \\
&\qquad + \int_{s_1(t)}^{s_2(t)} h(x(t,\xi))\,\alpha(x(t,\xi))\,d\xi\,,
\end{aligned}
\tag{5.7}
$$

The first two terms in the definition of f_0 account for the cost of building the two walls, while the integral term keeps track of the increase in the burned area. As in (2.6), $h(x)$ is the normal velocity of the advancing fire front, at the point x. The functions f_1, f_2 are implicitly determined by the identities

$$
\left|\frac{\partial x(t, s_i)}{\partial t} + f_i(t, s_i, w_i)\cdot\frac{\partial x(t, s_i)}{\partial s_i}\right| \ =\ \frac{w_i}{\psi(x(t, s_i))} \qquad\qquad i = 1, 2. \tag{5.8}
$$

As remarked in the previous section, each equation in (5.8) has two solutions. If the orientation of the vector $\mathbf{e} = \frac{\partial x(t,s)}{\partial s}$ is as shown in fig. 5, one should choose the larger solution for $i = 1$ and the smaller solution for $i = 2$.

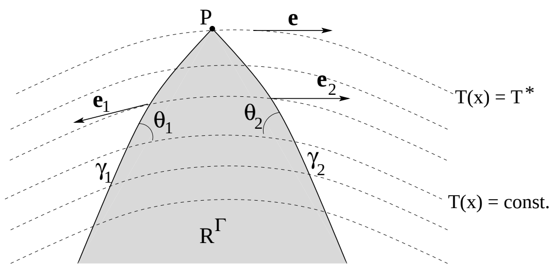

Figure 5: Left: two boundary arcs joining at the terminal point. Right: if the construction is completed as the two walls meet at P, one can consider the perturbed problem, where at time t the position of the wall γ_2 is shifted by ε along the fire front $\{x\,;\ \ T(x) = t\}$. This yields a slightly different cost J_ε. One can then uniquely determine the adjoint variable by setting $q_2(t) = \left[\frac{d}{d\varepsilon} J_\varepsilon\right]_{\varepsilon=0}$.

Let $t \mapsto w^*(t) = (w_1^*, w_2^*)(t)$ be an optimal control, and let $t \mapsto s^*(t) = (s_1^*, s_2^*)(t)$ be the corresponding optimal trajectory, Then, by the Pontryagin maximum principle [9], there exists an absolutely continuous adjoint vector $p(t) = (p_0, p_1, p_2)(t)$ such that the following holds.

$$
\dot p_0 \ \equiv\ 0\,, \qquad\qquad \dot p_i \ =\ -\sum_{k=0}^{2}\frac{\partial f_k}{\partial s_i}\,p_k \qquad\qquad i = 1, 2\,. \tag{5.9}
$$

$$
\sum_{i=0}^{2} p_i(t)\cdot f_i(t, s^*(t), w^*(t)) \ =\ \min_{w_1+w_2\le 1}\ \sum_{i=0}^{2} p_i(t)\cdot f_i(t, s^*(t), w_i) \tag{5.10}
$$

at almost every time t. Assuming that both arcs are normal, as in (4.1), we can here normalize the adjoint vector by taking $p_0 \equiv 1$. By (5.5), at the terminal time $t = T^*$ one has

$$
(p_0, p_1, p_2)(T^*) \ =\ (1, -\eta,\ \eta) \tag{5.11}
$$

for some real number η. This Lagrange multiplier can be determined using the further relation

$$
\min_{w_1+w_2\le 1}\ \Big\{ f_0(T^*, s^*(T^*), w_1, w_2) - \eta f_1(T^*, s_1^*(T^*), w_1) + \eta f_2(T^*, s_2^*(T^*), w_2)\Big\} \ =\ 0\,. \tag{5.12}
$$

By the first equation in (5.9) and the terminal conditions (5.11) it follows $p_0(t) \equiv 1$. Setting

$$\xi_i(t) \doteq \sum_{k=0}^{2} p_k(t) \frac{\partial}{\partial w_i} f_k(t, s^*(t), w^*(t)) = \frac{\beta(\gamma_i^*(t))}{\psi(\gamma_i^*(t))} - \frac{p_i(t)}{\psi(\gamma_i^*(t)) \cdot \cos\theta_i(t)} \qquad i = 1, 2,$$

from the optimality condition (5.10) it follows

$$\xi_1(t) = \xi_2(t) = \frac{d}{d\varepsilon} \left[\min_{w_1 + w_2 \leq 1+\varepsilon} \sum_{i=0}^{2} p_i(t) \cdot f_i(t, s^*(t), w) \right]_{\varepsilon=0}. \tag{5.13}$$

The positive quantity $W(t) \doteq -\xi_i(t)$ is the instantaneous value of time.

At the terminal time $t = T^*$ one has $s_1 = s_2$ and $\gamma_1(T^*) = \gamma_2(T^*) = P$. From the necessary condition (4.30), taking into account the orientations of $\mathbf{e}_1 = -\mathbf{e}$ and $\mathbf{e}_2 = \mathbf{e}$, it thus follows

$$\theta_1(T^*) = \theta_2(T^*).$$

In particular, the control that achieves the minimum in (5.10) is $w_1 = w_2 = 1/2$, hence $f_0 = \beta/\psi$. To compute the difference $f_1 - f_2$, observe that at the terminal point P one has

$$f_1\Big(T^*, s_1(T^*), \frac{1}{2}\Big) - f_2\Big(T^*, s_2(T^*), \frac{1}{2}\Big) = \frac{\cos\theta_i(T^*)}{\psi(P)},$$

where θ_1, θ_2 are the angles between the barriers γ_1, γ_2 and the fire front, as in fig. 5. At the terminal time $t = T^*$ these two angles are equal, and can be determined by the identity

$$h(P) = \frac{\sin\theta_i(T^*)}{2\psi(P)}. \tag{5.14}$$

Using these relations in (5.12) we obtain

$$\frac{\beta(P)}{\psi(P)} - \eta \frac{\cos\theta_i(T^*)}{\psi(P)} = 0, \qquad \eta = \frac{\beta(P)}{\cos\theta_i(T^*)}.$$

According to (4.30), the terminal value of time is computed by

$$W(T^*) = \frac{1}{\cos\theta_i} \frac{\eta}{\psi} - \frac{\beta}{\psi} = \left(\frac{1}{\cos^2\theta_i(T^*)} - 1 \right) \frac{\beta(P)}{\psi(P)} = \frac{4\psi^2(P)h^2(P)}{1 - 4\psi^2(P)h^2(P)} \cdot \frac{\beta(P)}{\psi(P)} > 0.$$

Indeed, by (5.14) we have $\cos^2\theta_i(T^*) = 1 - 4\psi^2(P)h^2(P)$.

For future use, we shall denote by $V(\tau, \bar{s}_1, \bar{s}_2)$ the value function, i.e. the minimum cost corresponding to initial data

$$s(\tau) = \bar{s}_1, \qquad s_2(\tau) = \bar{s}_2. \tag{5.15}$$

5.2 Four boundary arcs joining at different times

Next, we study the case of four boundary arcs, shown in fig. 6. The two arcs γ_3, γ_4 join together at a time τ_1, while γ_1 and γ_2 join at a later time $\tau_2 > \tau_1$.

This can be modelled by a control system with state $s = (s_0, s_1, s_2, s_3, s_4)$ and control functions $w = (w_1, w_2, w_3, w_4)$.

$$\text{Minimize:} \quad s_0(T) + V(T, s_1(T), s_2(T)) \tag{5.16}$$

for the system with dynamics

$$\begin{cases} \dot{s}_0 &= f_0(t, s, w)\,, \\ \dot{s}_i &= f_i(t, s_i, w_i) \qquad\qquad i = 1, 2, 3, 4. \end{cases} \tag{5.17}$$

Here $V(\tau, \bar{s}_1, \bar{s}_2)$ is the value function corresponding to the previous problem with two walls, considered at (5.15). Moreover,

$$\begin{aligned} f_0(t, s, w) \;&=\; \sum_{i=1}^{4} \beta(x(t, s_i(t)))\, \frac{w_i(t)}{\psi(x(t, s_i(t)))} \\ &\quad + \left(\int_{s_1(t)}^{s_2(t)} + \int_{s_3(t)}^{s_4(t)} \right) h(x(t, \xi))\, \alpha(x(t, \xi))\, d\xi\,, \end{aligned} \tag{5.18}$$

while the functions $f_1, \ldots, f_4$ are implicitly determined by the identities

$$\left| \frac{\partial x(t, s_i)}{\partial t} + f_i(t, s_i, w_i) \cdot \frac{\partial x(t, s_i)}{\partial s_i} \right| \;=\; \frac{w_i}{\psi(x(t, s_i))} \qquad\qquad i = 1, 2, 3, 4. \tag{5.19}$$

The controls satisfy the constraints

$$w_i(t) \in [0, 1]\,, \qquad\qquad \sum_{i=1}^{4} w_i(t) \;\leq\; 1\,, \tag{5.20}$$

while the terminal set is described by the identity.

$$s_3(T) - s_4(T) \;=\; 0. \tag{5.21}$$

Let $t \mapsto w^*(t)$ be an optimal control, with optimal trajectory $t \mapsto s^*(t)$. Then by the Pontryagin necessary conditions [9], there exists a nontrivial absolutely continuous adjoint vector $p(t) \doteq (p_0, p_1, p_2, p_3, p_4)(t)$ such that

$$\dot{p}_0(t) = 0 \qquad\qquad \dot{q}_i(t) \;=\; -\sum_{k=0}^{4} p_k \frac{\partial f_k}{\partial s_i}, \qquad i = 1, \ldots, 4\,, \tag{5.22}$$

$$\sum_{i=0}^{4} p_i(t) \cdot f_i(t, s^*(t), w^*(t)) \;=\; \min_{w_1 + w_2 + w_3 + w_4 \leq 1} \; \sum_{i=0}^{4} p_i(t) \cdot f_i(t, s^*(t), w) \tag{5.23}$$

at almost every time t. In addition, at the terminal time $t = \tau_1$ one has

$$(p_0, p_1, p_2, p_3, p_4)(\tau_1) \;=\; (1, V_{s_1}, V_{s_2}, -\eta, \eta) \tag{5.24}$$

for some real number η. This Lagrange multiplier can be determined using the further relation

$$\min_{w_1 + w_2 + w_3 + w_4 \leq 1} \; \sum_{i=0}^{4} p_i(\tau_1) \cdot f_i(\tau_1, s^*(\tau_1), w) \;=\; 0. \tag{5.25}$$

From the minimality condition (5.23) it follows that, by setting

$$\xi_j(t) \doteq \sum_{i=0}^{4} p_i(t) \cdot \frac{\partial}{\partial w_j} f_i(t, s^*(t), w^*(t)) = \frac{-p_j(t)}{\cos\theta_j(t)\,\psi(\gamma_j(t))} - \frac{\beta(\gamma_j(t)}{\psi(\gamma_j(t))}, \tag{5.26}$$

one has

$$\xi_1(t) = \xi_2(t) = \xi_3(t) = \xi_4(t) = \frac{d}{d\varepsilon}\left[\min_{w_1+w_2+w_3+w_4 \leq 1+\varepsilon} \sum_{i=0}^{4} p_i(t) \cdot f_i(t, s^*(t), w(t))\right]_{\varepsilon=0}. \tag{5.27}$$

As in the previous case, the positive quantity $W(t) \doteq -\xi_i(t)$ yields the instantaneous value of time.

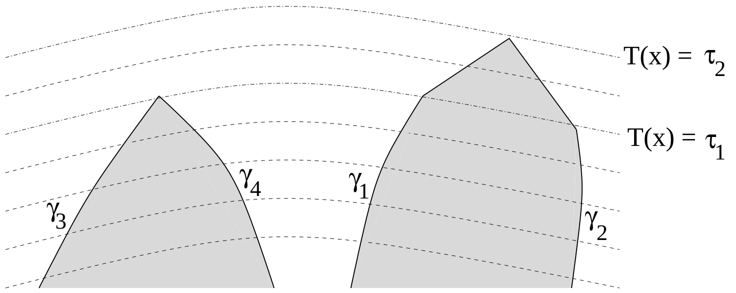

Figure 6: Four boundary arcs, joining at different times.

5.3 Junctions between a free arc and a boundary arc

We now consider a free arc γ and a boundary arc $\gamma^\sharp$, joining at a point P, as shown in fig. 7. In [5] it was proved that, if these arcs are part of an optimal strategy minimizing the total burned area, then they must be tangent at P. Here we study a more general case and derive further necessary conditions for optimality.

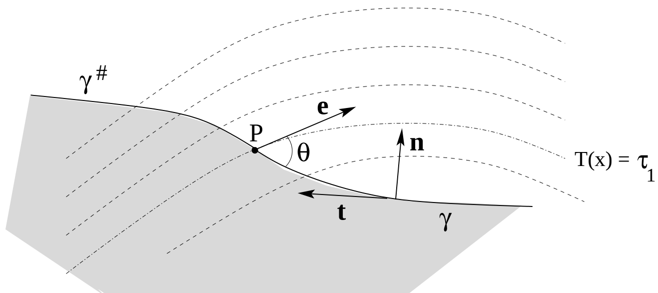

Figure 7: A junction between a free arc γ and a boundary arc $\gamma^\sharp$.

To fix the ideas, we assume that $T^\Gamma(x) \leq \tau_1$ for all $x \in \gamma$ and $T^\Gamma(x) \geq \tau_1$ for all $x \in \gamma^\sharp$. We assume that the free arc γ is normal and parameterized by arc-length: $s \mapsto \gamma(s)$ with $s \in [a, b]$. It joins the boundary arc $\gamma^\sharp$ at the endpoint $P = \gamma(b)$. As in fig. 2, we denote by $\mathbf{t}(s)$ and by $\mathbf{n}(s)$ respectively the unit tangent vector an the unit normal vector to γ and the at the point $\gamma(s)$. As in (3.1), let $\varphi^\sharp : [a, b] \mapsto I\!R$ be a smooth function which vanishes in a neighborhood of a and b, and such that (3.2) holds.

Given any vector $\mathbf{v} \in I\!R^2$ and scalar $\rho \in I\!R$, we shall construct a family of perturbed curves $\gamma_\varepsilon : [a,b] \mapsto I\!R^2$ having endpoints

$$\gamma_\varepsilon(a)\,, \qquad\qquad \gamma_\varepsilon(b) + \varepsilon\mathbf{v}\,, \tag{5.28}$$

and such that the total time needed to construct each γ_ε is

$$\mathcal{T}(\gamma_\varepsilon) \;=\; \mathcal{T}(\gamma) + \varepsilon\rho + o(\varepsilon). \tag{5.29}$$

Here and in the sequel, the Landau symbol $o(\varepsilon)$ denotes an infinitesimal of higher order w.r.t. ε. Toward this goal, choose a smooth function $\varphi : [a,b] \mapsto I\!R^2$ such that

$$\varphi(b) = \mathbf{v}\,, \qquad\qquad \varphi(s) = 0 \quad \text{for } s \text{ in a neighborhood of } a.$$

and let φ_1, φ_2 be the normal and the tangential components of φ, so that

$$\varphi(s) = \varphi_1(s)\mathbf{n}(s) + \varphi_2(s)\mathbf{t}(s).$$

Moreover, define

$$\eta_0' \;\doteq\; \left(\int_a^b \mathcal{G}(\psi,\gamma,s)\,\varphi^\sharp(s)\,ds\right)^{-1}\left(\rho - \psi(\gamma(b))\,\varphi_2(b) - \int_a^b \mathcal{G}(\psi,\gamma,s)\,\varphi_1(s)\,ds\right), \tag{5.30}$$

$$\gamma_\varepsilon(s) \;\doteq\; \gamma(s) + \varepsilon\varphi(s) + \varepsilon\eta_0'\varphi^\sharp(s)\mathbf{n}(s) \qquad\qquad s \in [a,b]. \tag{5.31}$$

Recalling that $\varphi^\sharp$ vanishes at $s = a$ and at $s = b$, it is clear that the end-point conditions (5.28) are satisfied. We claim that (5.29) also holds.

Since now we are perturbing γ also in the tangential direction, computing the derivative of the time needed to construct γ_ε as in (3.5) we obtain an additional term. Namely, calling $\kappa(s)$ the curvature of γ, in place of (3.5) one has

$$\begin{aligned}\left[\frac{d}{dt}\mathcal{T}(\gamma_\varepsilon)\right]_{\varepsilon=0} \;&=\; \int_a^b \Big\{\Big\langle \nabla\psi(\gamma(s))\,,\;\mathbf{n}(s)\Big\rangle + \psi(\gamma(s))\,\kappa(\gamma(s))\Big\}\cdot\Big(\varphi_1(s) + \eta_0'\varphi^\sharp(s)\Big)\;ds \\ &\quad + \int_a^b \Big\{\Big\langle \nabla\psi(\gamma(s))\,,\;\mathbf{t}(s)\Big\rangle\,\varphi_2(s) + \psi(\gamma(s))\,\Big\langle \mathbf{t}(s)\;,\;\frac{d}{ds}[\varphi_2(s)\mathbf{t}(s)]\Big\rangle\Big\}\,ds \\ &=\; I_1 + I_2\,.\end{aligned} \tag{5.32}$$

We observe that the identity (3.17) was true for every scalar function ϕ vanishing at the endpoints a,b. By a density argument, it still holds for a smooth function φ_1 which does not vanish at $s = b$. Using this identity, together with $\langle \mathbf{t}(s),\,\dot{\mathbf{t}}(s)\rangle \equiv 0$, we obtain

$$I_1 \;=\; \int_a^b \mathcal{G}(\psi,\gamma,s)\,(\varphi_1(s) + \eta_o'\varphi^\sharp(s))\,ds \;=\; \rho - \psi(\gamma(b))\varphi_2(b)\,, \tag{5.33}$$

$$I_2 \;=\; \int_a^b \left\{\frac{d}{ds}\Big(\psi(\gamma(s))\varphi_1(s)\Big)\right\}ds \;=\; \psi(\gamma(b))\,\varphi_2(b)\,. \tag{5.34}$$

In connection with (5.32), this yields (5.29).

We now consider the cost $J(\gamma_\varepsilon)$ associated with the perturbed arc γ_ε. The change in the cost of building the wall is described by

$$\begin{aligned}
&\frac{d}{d\varepsilon}\left[\int_a^b \beta(\gamma_\varepsilon(s))\,|\dot\gamma_\varepsilon(s)|\,ds\right]_{\varepsilon=0} \\
&= \int_a^b \Big\{\Big\langle \nabla\beta(\gamma(s))\,,\ \mathbf{n}(s)\Big\rangle + \beta(\gamma(s))\,\kappa(\gamma(s))\Big\}\cdot\Big(\varphi_1(s)+\eta_0'\varphi^\sharp(s)\Big)\,ds \\
&\quad + \int_a^b \Big\{\frac{d}{ds}[\beta(\gamma(s)\varphi_2(s)]\Big\}\,ds\,.
\end{aligned} \tag{5.35}$$

On the other hand, the change in the cost related to the burned area is estimated by

$$\varepsilon\int_a^b \alpha(\gamma(s))\,\Big(\varphi_1(s)+\eta_0'\varphi^\sharp(s)\Big)\,ds + o(\varepsilon)\,. \tag{5.36}$$

We now use the fact that the free arc γ is optimal, hence (3.12) holds (with φ replaced by φ_1). Comparing (5.35)-(5.36) with (5.30), and using (3.19), we obtain

$$\frac{d}{d\varepsilon}J(\gamma_\varepsilon)\bigg|_{\varepsilon=0} \;=\; \varphi_2(b)(\psi\cdot\eta+\beta)-\rho\eta\,, \tag{5.37}$$

where η is the Lagrange multiplier defined at (3.13). It is understood that the functions ψ,β are here computed at the terminal point $\gamma(b)$.

Remark 8. The value λ in (3.13) and (3.19) describes for the value of time, which remains constant as long as the free arc is being built. The formula (5.37) ths has a simple interpretation. If the free arc γ is part of an optimal strategy, the only change in the cost functional associated to the perturbation γ_ε are due to (i) the cost of building the additional portion of wall near the endpoint $\gamma(b)$, and (ii) the change in the construction time.

We can now state the main result of this section.

Theorem 3 (necessary conditions at junctions). *Let $\gamma_1,\ldots,\gamma_\nu\subset\Gamma_{\mathcal F}\setminus\Gamma^d$ be free arcs, simultaneously constructed by an optimal strategy Γ during the time interval $t\in\,]\tau_0,\tau_1[\,$. Assume that at least one of these arcs is normal, and let $s\mapsto\gamma_i(s)$ be a parameterization of γ_i by arc-length, with $s\in\,]a_i,b_i[\,$.*

Let $\gamma_1^*,\cdots,\gamma_\nu^*\subset\Gamma_S\setminus\Gamma^d$ are boundary arcs, all normal, simultaneously constructed by the optimal strategy during the time interval $t\in[\tau_1,\,\tau_2]$.

Assume that each pair of arcs γ_i,γ_i^* have a common endpoint

$$P_i \;=\; \gamma_i(b_i) \;=\; \gamma_i^*(\tau_1), \tag{5.38}$$

and that the angle θ_i between the barrier γ_i and the fire front $\{T^\Gamma(x)=\tau_1\}$ at the junction point P_i (see fig. 7) satisfies

$$0<\theta_i<\frac{\pi}{2}\qquad\qquad i=1,\ldots,\nu. \tag{5.39}$$

Then there exists constants $\lambda_0 = 1$ and λ, and adjoint variables $q_i(t)$, such that all necessary conditions stated in Theorems 1 and 2 hold, together with the following matching conditions, valid for $i = 1, \ldots, \nu$.

$$q_i(\tau_1) \;=\; \Big(\beta(P_i) + \lambda\,\psi(P_i)\Big)\,\cos\theta_i\,, \tag{5.40}$$

$$W_i(\tau_1) \;=\; \lambda\,. \tag{5.41}$$

Moreover, for each i the curves γ_i and γ_i^* meet tangentially at the point P_i.

Remark 9. We recall that $W_i(t)$ is the instantaneous value of time for the boundary arc γ_i^*, defined at (4.30). This function is the same for all $i = 1, \ldots, \nu$, but can decrease in time. On the other hand, λ is the constant value of time corresponding to the free arcs. The identity (5.41) says that the value of time is continuous at $t = \tau_1$, when the junction occurs.

To motivate the identities (5.40), we observe that the Lagrange multiplier $q_i(t)$ describes the increase in the total cost produced by shifting the initial point of the boundary arc $\gamma_i^*(t)$ in the direction of the unit vector $\mathbf{e}_i(\tau_1)$. On the other hand (5.37) implies that, if we shift the terminal point of the free arc γ_i in the direction $\mathbf{e}_i(\tau_1) = v_1\mathbf{n} - v_2\mathbf{t}$ without changing the total construction time (i.e. with $\rho = 0$), the cost related to the free arc decreases at the rate

$$v_2\beta(\gamma_i(b_i)) + \lambda v_2\psi(\gamma_i(b_i)) \;=\; \Big(\beta(P_i) + \eta\,\psi(P_i)\Big)\,\cos\theta_i\,.$$

Proof. 1. We begin by showing that, for every $i = 1, \ldots, \nu$, the arcs γ_i and $\gamma_i^\sharp$ are tangent at the junction point P_i.

If not, we claim that there exists at least one index i such that at P_j the arcs γ_i, γ_i^* produce an inward corner (see fig. 8).

Indeed, consider the portion of the arc γ_i which is reached by the fire during the interval $[\tau - \varepsilon, \tau]$. This has length

$$m_1\Big(\big\{x \in \gamma_i\,;\quad T(x) \in [\tau_1 - \varepsilon\,,\,\tau_1]\big\}\Big) \;=\; \frac{\varepsilon\,h(P_i)}{\cos\theta_i} + o(\varepsilon),$$

where $h(P_i)$ is the speed at which the fire front is advancing, at the point P_i, as in (2.6). Similarly,

$$m_1\Big(\big\{x \in \gamma_i^*\,;\quad T(x) \in [\tau_1\,,\,\tau_1 + \varepsilon]\big\}\Big) \;=\; \frac{\varepsilon\,h(P_i)}{\cos\theta_i^*} + o(\varepsilon).$$

By assumption, the constraint (1.10) is satisfied as an equality for $t \geq \tau_1$ and as a strict inequality for $t < \tau_1$. For $\varepsilon > 0$ small, this yields

$$\varepsilon \;=\; \int_{\Gamma\cap\{T(x)\in[\tau_1,\,\tau_1+\varepsilon]\}} \psi\,dm_1 \;=\; \sum_{i=1}^{\nu} \psi(P_i)\,\frac{\varepsilon\,h(P_i)}{\cos\theta_i^*} + o(\varepsilon)\,,$$

$$\varepsilon \;<\; \int_{\Gamma\cap\{T(x)\in[\tau_1-\varepsilon,\,\tau_1]\}} \psi\,dm_1 \;=\; \sum_{i=1}^{\nu} \psi(P_i)\,\frac{\varepsilon\,h(P_i)}{\cos\theta_i} + o(\varepsilon)\,,$$

If now $0 \leq \theta_i \leq \theta_i^* \leq \pi/2$ for every $i = 1, \ldots, \nu$, and $\theta_j < \theta_j^*$ for at least one j, then the two above conditions yield a contradiction.

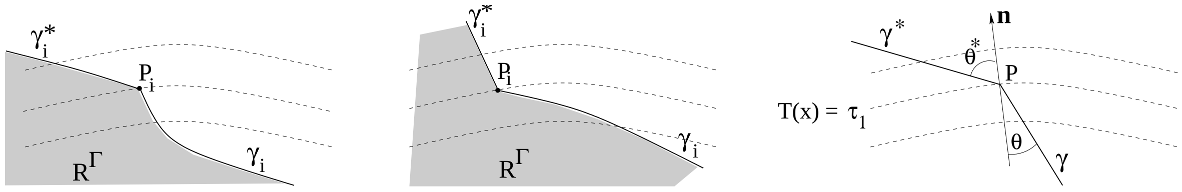

Figure 8: Left: the two arcs $\gamma_i, \gamma_i^\sharp$ produce an outward corner. Center: two arcs producing an inward corner. Right: the angles θ, θ^* formed by the arcs $\gamma, \gamma^\sharp$ with the normal vector $\mathbf{n}$ to the level set $T(x) = \tau_1$ at P.

2. By the previous step, we can now assume that at least one pair of arcs, say γ_i and γ_i^*, form an inward corner. We claim that this blocking strategy Γ is not optimal.

Indeed, if the construction cost is strictly positive, i.e. $\beta(P_i) > 0$, then a strictly better strategy Γ' is as follows (fig. 9, left). Fix $\varepsilon > 0$ small. Let A be the point along γ_i at distance ε from P_i. Let B be the point along γ_i^* such that the segment AB is perpendicular to the bisectrix $\mathbf{v}$ of the angle at P_i. Replacing the portion of walls in $\gamma_i \cup \gamma_i^*$ between A and B by this segment. The new barrier Γ' thus obtained is admissible. The length of Γ' satisfies $m_1(\Gamma') \leq m_1(\Gamma) - c\varepsilon$, for some $c > 0$ an all $\varepsilon > 0$ small enough. On the other hand, the additional area burned by the fire is of order $\mathcal{O}(\varepsilon^2)$. Hence, if $\beta(P_i) > 0$, for $\varepsilon > 0$ small the total cost associated with Γ' is $J(\Gamma') < J(\Gamma)$.

On the other hand, if $\beta(P_i) = 0$ but $\alpha(P_i) > 0$, then we can reduce both the total length of the wall and the total area burned by the fire, as shown in fig. 9, right. Fix $\varepsilon > 0$ small. Let A be the point along γ_i at distance ε from P_i. Construct a segment AB' perpendicular to the bisectrix $\mathbf{v}$. Prolong this segment to a point B such that (with obvious meaning of notation) the length of the various arcs satisfy

$$m_1(AB) + 2m_1(B'B) = m_1(AP_i) + m_1(P_iB')\,.$$

Then construct an arc $B'C$ of length $\kappa\varepsilon$ (with $\kappa \gg 1$), having constant distance to γ_i^*. Finally, connect the point C with a point D on γ_i^*.

The new barrier Γ' obtained by replacing the arcs AP_i and P_iD with $AB \cup BC \cup CD$ is still admissible. Its total length is smaller, and the total burned area has also decreased. Indeed, by choosing κ large enough, the area of the region $B'BCD$ is strictly smaller than the area of the triangle AP_iB'. Hence, for $\varepsilon > 0$, we again conclude that $J(\Gamma') < J(\Gamma)$, against the optimality of the strategy Γ. The above arguments prove the last statement of the theorem: for every $i = 1, \ldots, \nu$, the arcs γ_i and γ_i^* are tangent at the point of junction.

3. Toward a proof of the matching conditions (5.40), we remark that the optimality conditions for boundary arcs in Theorem 2 were obtained by considering an auxiliary optimal control problem with fixed endpoints. However, the analysis in (5.28)–(5.37) shows that, for any choice of the numbers $r_1, \ldots, r_\nu$, we can replace the free arcs γ_i by perturbed arcs $\gamma_{i,\varepsilon}$, terminating at the endpoints $x_i(\tau_1, \varepsilon r_i)$. Here we use the coordinates $(t,s) \mapsto x_i(t,s)$ as in figure 3. More precisely, the following holds.

(i) The arc $\gamma_{i,\varepsilon}$ starts at $\gamma_i(a_i)$ and terminates at the point

$$\gamma_{i,\varepsilon}(b_i) = x_i(\tau_1, \varepsilon r_i) = P_i + \varepsilon r_i \mathbf{e}_i(\tau_1) + o(\varepsilon).$$

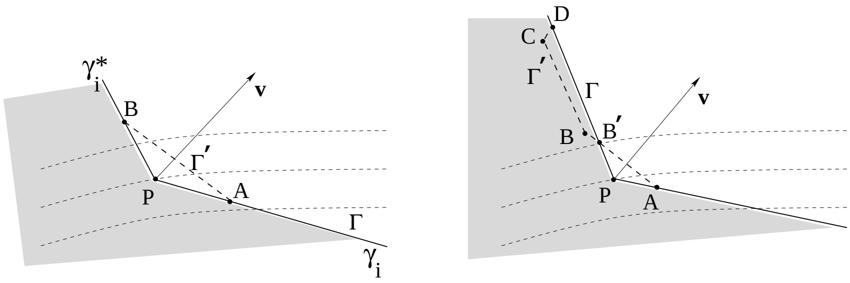

Figure 9: If a barrier Γ contains an inward corner, it can be replaced by a barrier Γ' yielding a smaller cost.

(ii) The total amount of time needed to construct these arcs is the same:

$$\sum_{i=1}^{\nu} \mathcal{T}(\gamma_{i,\varepsilon}) \doteq \sum_{i=1}^{\nu} \int_{a_i}^{b_i} \psi_i(\gamma_{i,\varepsilon}(s))\, ds = \sum_{i=1}^{\nu} \int_{a_i}^{b_i} \psi_i(\gamma_i(s))\, ds \doteq \sum_{i=1}^{\nu} \mathcal{T}(\gamma_i)\,. \tag{5.42}$$

(iii) As in (5.37), the change in cost associated with these perturbed arcs is

$$\frac{d}{d\varepsilon} J(\gamma_{i,\varepsilon})\bigg|_{\varepsilon=0} = (\beta(P_i) + \lambda\psi(P_i))r_i \cos\theta_i\,. \tag{5.43}$$

At this point, the conditions (5.40) become clear. For an optimal control problem with free initial point, the initial values of the adjoint variables $q_1, \ldots, q_\nu$ should equal the gradients of the cost associated with these initial values.

To make this argument completely rigorous, however, we must clarify a technical point. Indeed, in principle the perturbed strategies consisting of free arcs $\gamma_{i,\varepsilon}$, then of boundary arcs $\gamma^\sharp_{i,\varepsilon}$ starting at the points $\gamma_{i,\varepsilon}(b_i) = x_i(\tau_1, \varepsilon\alpha_i)$, may not be admissible. Indeed, since now we are perturbing the free arcs γ_i also at points where the constraint (1.10) is saturated, the arguments used at (3.8) and (3.9) now break down.

To take care of this difficulty, for any fixed $\delta > 0$ we replace the arcs $\gamma_{i,\varepsilon}$ with free arcs $\tilde{\gamma}_{i,\varepsilon}$ having the same endpoints

$$\tilde{\gamma}_{i,\varepsilon}(a_i) = \gamma_{i,\varepsilon}(a_i), \qquad \tilde{\gamma}_{i,\varepsilon}(b_i) = \gamma_{i,\varepsilon}(b_i),$$

but requiring slightly shorter time to construct. Namely

$$\sum_{i=1}^{\nu} \mathcal{T}(\tilde{\gamma}_{i,\varepsilon}) = \sum_{i=1}^{\nu} \mathcal{T}(\gamma_i) - \delta|\varepsilon|\,. \tag{5.44}$$

According to (5.37), the cost associated with these new free arcs is

$$\sum_{i=1}^{\nu} J(\tilde{\gamma}_{i,\varepsilon}) = \sum_{i=1}^{\nu} J(\gamma_i) + \varepsilon \sum_{i=1}^{\nu} r_i(\beta(P_i) + \lambda\psi(P_i)) \cos\theta_i + \lambda\delta|\varepsilon|\,.$$

Because of (5.44), for any fixed $\delta > 0$ these alternative strategies will be admissible, for ε in a neighborhood of the origin (possibly shrinking to zero as $\delta \to 0$).

For $(s_1, \ldots, s_\nu) \approx (0, \ldots, 0)$, we now define

$$V(s_1, \ldots, s_\nu) \doteq \inf \sum_{i=1}^{\nu} J(\tilde{\gamma}_i)$$

where the infimum is taken over all admissible ν-tuples of free arcs $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_\nu$, with

$$\tilde{\gamma}_i(a_i) = \gamma_i(a_i), \qquad \tilde{\gamma}_i(b_i) = x_i(\tau_1, s_i).$$

Since $\delta > 0$ was arbitrary, the previous argument shows that

$$\frac{\partial V}{\partial s_i}\bigg|_{(s_1,\ldots,s_\nu)=(0,\ldots,0)} = \big(\beta(P_i) + \lambda\psi(P_i)\big) \cos\theta_i$$

Since this partial derivative must coincide with the initial value of the adjoint variable q_i, the identities in (5.40) hold.

4. Finally, by (4.30) and (5.43) one finds

$$W_i(\tau_1) = \frac{1}{\cos\theta_i} \frac{q_i}{\psi} - \frac{\beta}{\psi} = \frac{1}{\cos\theta_i} \frac{(\beta + \lambda\psi)\cos\theta_i}{\psi} - \frac{\beta}{\psi} = \lambda\,,$$

proving the matching condition (5.41). □

6 Examples

This final section provides three examples, where the value of time and the matching conditions can be directly computed.

Example 1. Assume that $F(x) \equiv B(0, 1)$, so that the fire propagates with unit speed in all directions.

Assume that, without barriers, the minimal time function is $T(x) = x_2$. We take here $\alpha(x) \equiv 1$ and $\beta(x) \equiv 0$, so that we simply seek to minimize the total burned area. Moreover, we assume that the construction speed $\sigma = 1/\psi$ is a constant. We consider two cases.

CASE 1: At time t, the boundary of the burned region where fire is advancing is a single segment:

$$\{(x_1, x_2)\,; \quad x_1 \in [s_1(t),\ s_2(t)], \quad x_2 = t\}.$$

In this case, assuming that $\sigma > 2$, the optimal strategy is to construct the two walls at same speed $\sigma/2$. If at time $t = 0$ we have $[s_1(0), s_2(0)] = [\bar{s}_1, \bar{s}_2]$, then the time needed to block the fire is

$$T = \frac{\bar{s}_2 - \bar{s}_1}{2\sqrt{(\sigma/2)^2 - 1}} = \frac{\bar{s}_2 - \bar{s}_1}{\sqrt{\sigma^2 - 4}}\,.$$

The total burned area is

$$V(\bar{s}_2 - \bar{s}_1) = \frac{(\bar{s}_2 - \bar{s}_1)^2}{2\sqrt{\sigma^2 - 4}}\,. \tag{6.1}$$

The adjoint variables are

$$q_2(t) \;=\; \frac{\partial V}{\partial \bar s_2} \;=\; \frac{s_2(t)-s_1(t)}{\sqrt{\sigma^2-4}} \;=\; -\,\frac{\partial V}{\partial \bar s_1} \;=\; -\,q_1(t)\,.$$

The angle θ between each wall and the fire front is determined by

$$\sin\theta = \frac{2}{\sigma}\,, \qquad\qquad \cos\theta = \frac{\sqrt{\sigma^2-4}}{\sigma}\,.$$

The instantaneous value of time is computed as

$$W(t) \;=\; \frac{q_2(t)}{\psi\,\cos\theta} \;=\; \frac{\sigma^2}{\sigma^2-4}\cdot(s_2(t)-s_1(t)). \tag{6.2}$$

CASE 2: Assume that at a given time t, the boundary of the burned set consists of two segments:

$$\partial R(t) \;=\; \Big\{(x_1,x_2)\,; \quad x_1\in[s_1(t),\;s_2(t)]\cup[s_3(t),\;s_4(t)]\,,\quad x_2=t\Big\}.$$

We are thus constructing four walls, at the points $P_i(t)=(s_i(t),t)$. Assume that $\sigma>4$ and, to fix the ideas, let $s_4-s_3\le s_2-s_1$ (see fig. 10). We shall reformulate the above problem as an optimal control problem with free terminal time (the time where the walls at P_3 and P_4 join together).

We set $y_1=s_2-s_1$, $y_2=s_4-s_3$, while y_0 will keep track of the burned area up to time t. It is clear that an optimal strategy will satisfy $\dot s_2=-\dot s_1$, $\dot s_4=-\dot s_3$. Therefore, the above variables evolve in time according to

$$\left\{\begin{array}{rcl}\dot y_0 &=& y_1+y_2\\ \dot y_1 &=& -\sqrt{u_1^2-4}\\ \dot y_2 &=& -\sqrt{u_2^2-4}\end{array}\right. \qquad\qquad u_i\;\in\;[2,\sigma]\,,\quad u_1+u_2\le\sigma\,. \tag{6.3}$$

with initial data

$$y_1(0)=\bar s_2-\bar s_1\,,\qquad y_2(0)=\bar s_4-\bar s_3\,,\qquad y_0(0)=0\,. \tag{6.4}$$

The terminal set is

$$S \;=\; \Big\{(y_0,y_1,y_2)\,;\;\; \phi_1(y_0,y_1,y_2)\;\doteq\; y_2\;=\;0\Big\} \tag{6.5}$$

and the terminal payoff is

$$\phi_0(y_0,y_1,y_2) \;=\; y_0+\frac{y_1^2}{2\sqrt{\sigma^2-4}}\,. \tag{6.6}$$

Indeed, after the time where the first couple of walls join together, the problem is reduced to optimizing the construction of the two remaining barriers. For this problem, discussed in CASE 1, the corresponding optimal value function was computed at (6.1).

Applying the Pontryagin maximum principle to the optimal control problem (6.3)–(6.6), we obtain an adjoint vector $p = (p_0, p_1, p_2)$ such that

$$\begin{cases} \dot p_0 &= 0 \\ \dot p_1 &= -\, p_0 \\ \dot p_2 &= \;-\, p_0 \end{cases} \tag{6.7}$$

with terminal conditions

$$(p_0, p_1, p_2)(T) \;=\; \nabla\phi_0 + \lambda_1 \nabla\phi_1 \;=\; \left(1\,,\; \frac{y_1}{\sqrt{\sigma^2-4}}\,,\; \lambda_1\right) \tag{6.8}$$

for some constant λ_1. Moreover, at every time t one has

$$\min_{u_1,u_2\in[2,\sigma]\,,\; u_1+u_2\le\sigma} \left(p_0(y_1+y_2) - p_1\sqrt{u_1^2-4} - p_2\sqrt{u_2^2-4}\right) \;=\; 0\,. \tag{6.9}$$

Since $p_0(t) \equiv 1$, at the terminal time T this yields

$$\min_{\omega_1,\omega_2\in[2,\sigma]\,,\; \omega_1+\omega_2\le\sigma} \left(y_1 - \frac{\sqrt{\omega_1^2-4}}{\sqrt{\sigma^2-4}} y_1 - \lambda_1\sqrt{\omega_2^2-4}\right) \;=\; 0\,. \tag{6.10}$$

The three variables $\omega_1 = u_1(T)$, $\omega_2 = u_2(T)$, and λ_1 can be determined from the equations

$$\begin{cases} \dfrac{\omega_1}{\sqrt{\omega_1^2-4}\,\sqrt{\sigma^2-4}}\, y_1 &= \dfrac{\lambda_1\,\omega_2}{\sqrt{\omega_2^2-4}}\,, \\[2ex] \omega_1+\omega_2 &= \sigma\,, \\[2ex] y_1 - \dfrac{\sqrt{\omega_1^2-4}}{\sqrt{\sigma^2-4}}\, y_1 - \lambda_1\sqrt{\omega_2^2-4} &= 0\,. \end{cases} \tag{6.11}$$

Observe that, if $y_1(\tau) = y_2(\tau)$ at some time τ, then by symmetry the optimal control is $u_1(t) = u_2(t) = \sigma/2$ for all times t. In this case $y_1(t) = y_2(t)$ for all t. In particular, both couples of walls terminate at the same time: $y_1(T) = y_2(T)$. Throughout the following, we shall consider the case where $y_1(t) > y_2(t)$ for all t.

We now study the existence and uniqueness of solutions to the system (6.11). Given p_1, p_2, if u_1, u_2 achieve the minimum in (6.9) then

$$p_1 \cdot \frac{u_1}{\sqrt{u_1^2-4}} \;=\; p_2 \cdot \frac{u_2}{\sqrt{u_2^2-4}}\,. \tag{6.12}$$

At the terminal time $t = T$, since $\omega_2 = \sigma - \omega_1$, from (6.12) it follows

$$\frac{\omega_1}{\sqrt{\omega_1^2-4}} \cdot \frac{\sqrt{(\sigma-\omega_1)^2-4}}{\sigma-\omega_1} \;=\; \frac{p_2}{p_1}\,. \tag{6.13}$$

We observe that the left hand side of (6.13) is a monotonically decreasing function, for $\omega_1 \in\,]2, \sigma-2[$. It approaches $+\infty$ as $\omega_1 \to 2+$ and it approaches $-\infty$ as $\omega_1 \to (\sigma-2)-$. Therefore for a given ratio $p_2/p_1 > 0$, the equation (6.13) has exactly one solution $\omega_1 \in (2, \sigma-2)$.

We now observe that $p_1(T)$ is determined by $y_1(T)$, but $p_2(T) = \lambda_1$ still needs to be determined. At the terminal time T, the minimum value

$$\min_{u_1+u_2\leq\sigma} \left(y_1 - \frac{y_1}{\sqrt{\sigma^2-4}} \sqrt{u_1^2-4} - \lambda_1 \sqrt{u_2^2-4} \right)$$

is a decreasing function of λ_1. Hence Then there exists unique λ_1 satisfying (6.9) at $t = T$. Together with the previous analysis, this shows that the system (6.11) has a unique solution.

We now claim that $p_2(T) > p_1(T)$, i.e. $\lambda_1 > \frac{y_1}{\sqrt{\sigma^2-4}}$.

On the contrary, assume that $\lambda_1 \leq \frac{y_1}{\sqrt{\sigma^2-4}}$. In view of (6.11),

$$\frac{y_1}{\sqrt{\sigma^2-4}} \cdot \left\{ \sqrt{\sigma^2-4} - \sqrt{\omega_1^2-4} - \sqrt{(\sigma-\omega_1)^2-4} \right\} \leq 0. \tag{6.14}$$

Since the right hand side of (6.14) is convex w.r.t. ω_1 and attains the minimum

$$\frac{y_1}{\sqrt{\sigma^2-4}} \cdot \left\{ \sqrt{\sigma^2-4} - 2\sqrt{\left(\frac{\sigma}{2}\right)^2 - 4} \right\} \geq 0 \tag{6.15}$$

at $\omega_1 = \sigma/2$, the above leads to a contradiction with (6.14). Hence $p_2(T) > p_1(T)$.

In turn, this implies $\omega_2 > \sigma/2 > \omega_1$. Moreover, $p_1(t) = p_1(T)+T-t$ and $p_2(t) = p_2(T)+T-t$. Therefore the ratio satisfies $p_2(t)/p_1(t) > 1$ and is increasing in time, achieving a maximum at the terminal time $t = T$. By (6.12), the optimal controls satisfy $u_2(t) > u_1(t)$, with $t \mapsto u_2(t)$ increasing and $t \mapsto u_1(t)$ decreasing, up to time T.

Finally, consider the value of time

$$W(t) = \frac{\sigma\, q_i(t)}{\cos\theta_i(t)},$$

where $\theta_i(t)$ is the angle formed by the wall γ_i and the fire front. Notice that, for $i = 1, 2$, this can be well defined also for $t > T$. The adjoint variables $q_1(t), q_2(t)$ remain continuous at the time $t = T$, while the angles θ_1, θ_2 suddenly decrease at the time T where the walls γ_3 and γ_4 meet. Therefore, one expects that $W(\cdot)$ should have a downward jump at $t = T$. This is confirmed by the following computations.

$$q_1(T) = q_2(T) = \frac{y_1(T)}{\sqrt{\sigma^2-4}} = \frac{s_2(T)-s_1(T)}{\sqrt{\sigma^2-4}}.$$

For $t > T$ we have

$$\theta_1(t) = \theta_2(t) = \arcsin\frac{2}{\sigma},$$

hence the value of time is provided by (6.2). In particular

$$\lim_{t\to T+} W(t) = \frac{\sigma^2}{\sigma^2-4}\left(s_2(T)-s_1(T)\right). \tag{6.16}$$

On the other hand, for $t < T$ we have $p_2(t) \geq p_1(t)$, hence $u_2(t) \geq u_1(t)$ and

$$w_1(t) = w_2(t) = \frac{u_1(t)}{2} \leq \frac{\sigma}{4}.$$

Hence the angles $\theta_1(t) = \theta_2(t)$ between the walls γ_1, γ_2 and the fire front satisfy

$$\sin\theta_i(t) \;\geq\; \frac{4}{\sigma}\,, \qquad\qquad \frac{1}{\cos\theta_i(t)} \;\geq\; \frac{\sigma}{\sqrt{\sigma^2 - 16}}\,.$$

Therefore

$$\lim_{t\to T-} W(t) \;=\; \frac{y_1(T)}{\sqrt{\sigma^2-4}} \cdot \frac{\sigma}{\cos\theta_1(T-)} \;\geq\; \frac{\sigma^2}{\sqrt{(\sigma^2-4)(\sigma^2-16)}}\,(s_2(T) - s_1(T))\,. \tag{6.17}$$

Comparing (6.17) with (6.16), it is clear that the value of time has a downward jump at the time $t = T$ when the arcs γ_3 and γ_4 meet.

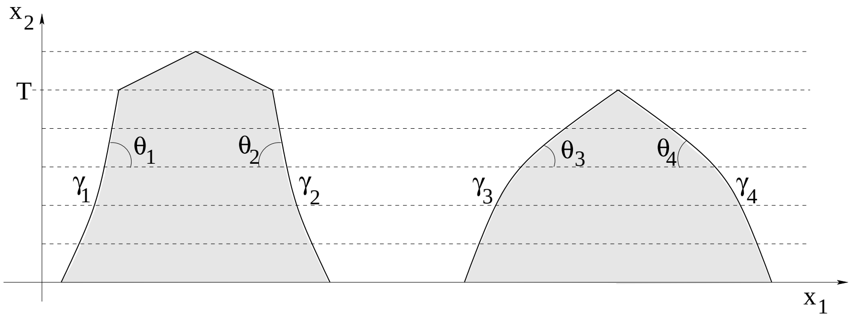

Figure 10: At time $t = T$ the two walls γ_3, γ_4 join together. The angles $\theta_1 = \theta_2$ increase as $t \to T-$, while $\theta_3 = \theta_4$ decrease. For $t > T$, all resources are allocated to the construction of the remaining barriers γ_1, γ_2. In this case, $\theta_1(t) = \theta_2(t) = \arcsin(2/\sigma)$.

Example 2. We again consider the problem of minimizing the total area burned by the fire, assuming that walls constructed at a constant speed $\sigma > 2$. Here we assume that at the initial time $t = 0$ the fire occupies the unit disc $R_0 = B_1$. Moreover, we assume that $F(x) = B_1$ for all $x \in I\!R^2$, so that the fire propagates at unit speed in all directions.

As described in [5], an optimal strategy is as follows. First construct an arc of circumference Γ_1. Then construct two arcs of logarithmic spirals Γ_2, Γ_3 along the boundary of the burned region (see fig. 11). In this case, Γ_1 is a free arc, while Γ_2, Γ_3 are boundary arcs. Here the length of the arc Γ_1 should satisfy

$$m_1(\Gamma_1) = \sigma \cdot d(Q_2,\, R_0) = \sigma \cdot d(Q_3,\, R_0),$$

so that the two end-points Q_2, Q_3 are reached by the boundary of the burned region $R^\gamma(\tau)$ exactly at the time τ when the construction of the arc Γ_1 is completed. According to Theorem 3, the junctions at Q_2 and at Q_3 must be $\mathcal{C}^1$, i.e. the arcs must join tangentially. For each time τ, the above conditions determine a unique strategy $\Gamma^{(\tau)}$. These conditions reduce the problem to an optimization problem over the scalar parameter τ.

We parameterize the boundary arcs Γ_2, Γ_3 by the time t, using polar coordinates. The radius is $\rho(t) = 1 + t$, while the angle angle $\theta(t)$ is the angle between $O\Gamma_i$ and the axis of symmetry. As shown in figure 11, r denotes the radius of the arc of circumstance Γ_1, while θ_1 denotes half of the corresponding angle. Moreover, $\theta_0 = \theta(\tau)$ is the angular coordinate of the point of junction Q_2, while

$$\theta_\sigma = \arcsin\left(\frac{2}{\sigma}\right).$$

denotes the constant angle between the arcs of spirals Γ_2, Γ_3 and the circumferences centered at the origin. At the junction point Q_2 we have the identities

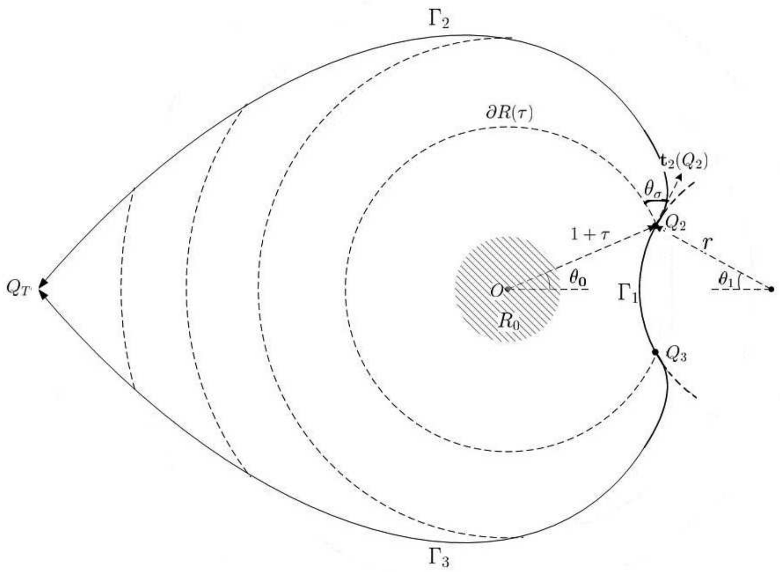

Figure 11: The burned region is enclosed by the arc of circumference Γ_1 and by two arcs of logarithmic spirals Γ_2, Γ_3.

$$\begin{cases} \theta_\sigma &= \theta_0 + \theta_1 \\ \sigma \cdot \tau &= 2r\theta_1 \\ r \cdot \sin\theta_1 &= (1+\tau)\cdot \sin\theta_0 \end{cases} \tag{6.18}$$

In (6.18), θ_0 and r can be written in terms of θ_1 from the first two equations. Inserting these values in the last equation, the system reduces to the single equation

$$\Phi(\theta_1) \doteq \frac{\sigma\tau}{2\theta_1}\cdot \sin\theta_1 - (1+\tau)\cdot \sin(\theta_\sigma - \theta_1) = 0. \tag{6.19}$$

We claim that under the assumption $\tau < \tau_\sigma \doteq \frac{4}{\sigma^2-4}$, the function Φ has unique root $\theta_1 \in [0, \theta_\sigma]$. Indeed, the assumption implies $\frac{\sigma\tau}{2} < (1+\tau)\sin\theta_\sigma$, hence $\Phi(0) < 0$. A differentiation yields

$$\begin{aligned} \frac{d}{d\theta_1}\Phi(\theta_1) &= \frac{\sigma\tau\cos\theta_1}{2\theta_1^2}\cdot(\theta_1 - \tan\theta_1) + (1+\tau)\cdot\cos(\theta_\sigma - \theta_1) \\ &= \frac{\sigma\tau\cos\theta_1}{2\theta_1^2}\cdot(\theta_1 - \tan\theta_1) + (1+\tau)\cdot(\cos\theta_\sigma\cos\theta_1 + \sin\theta_\sigma\sin\theta_1) \\ &\geq \frac{\sigma\tau\cos\theta_1}{2\theta_1^2}\cdot(\theta_1 - \tan\theta_1) + (1+\tau)\sin\theta_\sigma\sin\theta_1 \\ &\geq \frac{\sigma\tau\cos\theta_1}{2\theta_1^2}\cdot(\theta_1 - \tan\theta_1) + \frac{\sigma\tau}{2}\sin\theta_1 \\ &\geq \frac{\sigma\tau\cos\theta_1}{2\theta_1^2}\cdot(\sin\theta_1 - \tan\theta_1) + \frac{\sigma\tau}{2}\sin\theta_1 \\ &= \frac{\sigma\tau\sin\theta_1}{2\theta_1^2}\cdot\left(\cos\theta_1 - 1 + \theta_1^2\right) > 0, \quad \text{for all } \theta_1 \in (0, \pi/2), \end{aligned}$$

showing that Φ strictly increases.

Moreover, $\Phi(\theta_\sigma) = \frac{\sigma\tau}{2\theta_\sigma} \cdot \sin\theta_\sigma > 0$, and $\Phi(0) < 0$. Hence Φ has unique zero inside the interval $]0,\, \theta_\sigma[$.

If τ is given and sufficiently small, the parameters θ_0, θ_1 and r are thus uniquely determined by the system (6.18). From now on, we consider them as functions of τ. The terminal time T can be determined implicitly from the identity

$$\pi - \theta_0 \;=\; \int_\tau^T \frac{\sqrt{(\sigma/2)^2 - 1}}{1+t}\, dt \;=\; \sqrt{(\sigma/2)^2 - 1} \cdot \ln\left(\frac{1+T}{1+\tau}\right). \tag{6.20}$$

The total area burned by the fire is computed by

$$A(\tau) \doteq \int_\tau^T (1+t)^2 \cdot \frac{\sqrt{\sigma^2/4 - 1}}{1+t}\, dt + (1+\tau)^2 \cdot \sin\theta_0 \cos\theta_0 - r^2 \cdot (\theta_1 - \sin\theta_1 \cos\theta_1). \tag{6.21}$$

Notice that this expression can be regarded as a function of the scalar variable τ. In order to find the optimal strategy, one could simply minimize (6.21) w.r.t. the scalar variable τ.

Alternatively, one can determine the optimal value of τ from a matching condition. Indeed, as proved in Theorem 3, the value of time must be a continuous function, constant for $t \in [0, \tau]$, then decreasing to zero for $t \in [\tau, T]$. Along the free arc Γ_1, one has

$$W(t) \;=\; \sigma r \qquad\qquad t \in [0, \tau]\,. \tag{6.22}$$

To compute the value of time along the boundary arcs Γ_2, Γ_3, we need to determine by how much a small perturbation of the data can increase the total cost. Namely, assume that at time t we shift the edge of the wall $\gamma_2(t)$ along the boundary of the burned region. Working in polar coordinates, this means that at time t, instead of being at $(\rho, \vartheta) = (1+t, \theta(t))$, we move it to the point

$$(\rho, \theta) \;=\; (1+t,\; \theta_\varepsilon(t)) \;\doteq\; \left(1+t,\; \theta(t) - \frac{\varepsilon}{1+t}\right).$$

The rate of increase of the corresponding burned area is measured by

$$\begin{aligned} q_2(t) \;&=\; \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_t^T \rho(s) \cdot (\theta(s) - \theta_\varepsilon(s))\, ds \\ &=\; \int_t^T \frac{1+s}{1+t}\, ds \;=\; \frac{(1+T)^2 - (1+t)^2}{2(1+t)}. \end{aligned}$$

By (4.30), the value of time along the boundary arc Γ_2 is

$$W(t) \;=\; \frac{\sigma q_2(t)}{\cos\theta_\sigma} \;=\; \frac{\sigma^2}{\sqrt{\sigma^2 - 4}} \cdot \left\{\frac{(1+T)^2}{2(1+t)} - \frac{1+t}{2}\right\} \qquad\qquad t \in [\tau, T]\,. \tag{6.23}$$

Of course, the same value is valid along the boundary arc Γ_3. It is easily checked that the above function is monotonically decreasing and vanishes for $t = T$.

Imposing that the value of time along the free arc and the boundary arcs coincide at the junction time, i.e. that the right hand sides in (6.23)-(6.22) coincide at time $t = \tau$, we obtain an additional equation to determine τ. The following table displays the various parameters of an optimal strategy, which have been numerically computed for different values of the

σ	τ_σ	τ	θ_0	r	θ_1	T	$A(\tau)$
2.2	4.76190	4.76037	1.14097	42772.4	0.00012	452.370	47126.4
2.4	2.27273	2.26594	0.98373	1973.44	0.00138	83.4908	2374.59
3.0	0.80000	0.77928	0.71676	90.1463	0.01297	14.5653	135.926
4.0	0.33333	0.30351	0.48427	15.4335	0.03933	5.04522	30.9903
6.0	0.12500	0.09588	0.26533	3.86080	0.07450	2.02972	11.5975
8.0	0.06667	0.04295	0.16528	1.96597	0.08740	1.24912	7.86743
10.0	0.04167	0.02286	0.11180	1.27599	0.08956	0.89847	6.38192

construction speed σ. Clearly, as the construction speed increases, the time T needed to block the fire and the total burned area A decrease.

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