The Minimum Speed for a Blocking Problem on the Half Plane

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Abstract. We consider a blocking problem: fire propagates on a half plane with unit speed in all directions. To block it, a barrier can be constructed in real time, at speed σ. We prove that the fire can be entirely blocked by the wall, in finite time, if and only if $\sigma > 1$.
The proof relies on a geometric lemma of independent interest. Namely, let $K \subset I\!R^2$ be a compact, simply connected set with smooth boundary. We define $d_K(x, y)$ as the minimum length among all paths connecting x with y and remaining inside K. Then d_K attains its maximum at a pair of points $(\bar{x}, \bar{y})$ both on the boundary of K.

1 Introduction

Aim of this note is to analyze the blocking problem introduced in [4], originally motivated by the control of wild fires or of the spatial spreading of a contaminating agent.

At each time $t \geq 0$, we denote by $R(t) \subset I\!R^2$ the *burned region*. In absence of control, we assume that the set $R(t)$ grows uniformly in all directions, namely

$$R(t) \doteq B(R_0, t) \doteq \Big\{ x \in I\!R^2 \,;\;\; d(x, R_0) \leq t \Big\}.$$

Here $R_0 \subset I\!R^2$ is a fixed (nonempty) bounded open set, describing the region invaded by the fire at the initial time $t = 0$. In our model, the spreading of the fire can be controlled by constructing barriers. In mathematical terms, we thus assume that the controller can construct a one-dimensional rectifiable curve γ which blocks the spreading of the contamination. Calling $\gamma(t) \subset I\!R^2$ the portion of the wall constructed within time $t \geq 0$, we make the following assumptions:

(H1) For every $t_2 > t_1 \geq 0$ one has $\gamma(t_1) \subseteq \gamma(t_2)$.

(H2) For every $t \geq 0$, the total length of the wall satisfies

$$m_1(\gamma(t)) \leq \sigma t \,. \tag{1.1}$$

Here m_1 denotes the one-dimensional Hausdorff measure, normalized so that $m_1(\Gamma)$ yields the usual length of a smooth curve Γ. The constant $\sigma > 0$ is the speed at which the wall can be constructed. A strategy γ satisfying (H1)-(H2) will be called an *admissible strategy.* In addition, we say that the strategy γ is *complete* if it satisfies

(H3) For every $t \geq 0$ there holds

$$m_1(\gamma(t)) = \sigma t \,, \qquad\qquad \gamma(t) = \bigcap_{s>t} \gamma(s) \,. \tag{1.2}$$

Moreover, if $\gamma(t)$ has positive upper density at a point x, i.e. if

$$\limsup_{r\to 0+} \frac{m_1\Big(B(x,r)\cap\gamma(t)\Big)}{r} > 0 \,,$$

then $x \in \gamma(t)$. Here $B(x,r)$ is the ball centered at x with radius r.

As proved in [6], for every admissible strategy $t \mapsto \gamma(t)$ one can construct a second admissible strategy $t \mapsto \tilde{\gamma}(t) \supseteq \gamma(t)$, which is complete.

When a wall is being constructed, the burned set is reduced. Indeed, we define

$$\begin{aligned} R^\gamma(t) \doteq \quad \Big\{ x(t)\,;\;\; & x(\cdot) \;\text{ absolutely continuous}\,, \;\; x(0) \in R_0\,, \\ & |\dot{x}(\tau)| \leq 1 \;\text{ for a.e. } \tau \in [0,t]\,, \qquad x(\tau) \notin \gamma(\tau) \;\text{ for all } \tau \in [0,t] \Big\} \,. \end{aligned} \tag{1.3}$$

In the above setting, we consider the problem

(BP1) Blocking Problem 1. Find an admissible strategy $t \mapsto \gamma(t)$ such that the corresponding reachable sets $R^\gamma(t)$ remain uniformly bounded, for all times $t \geq 0$.

In other words, calling $B_r \doteq \{x \in I\!\!R^2\,;\; |x| < r\}$, we seek a strategy such that

$$R^\gamma(t) \subseteq B_r \qquad \text{for all } t \geq 0$$

for some radius r sufficiently large.

We recall that the Hausdorff distance between two compact sets X, Y is defined as

$$d_H(X,Y) \doteq \max\Big\{ \max_{x\in X}\; d(x,Y) \;,\; \max_{y\in Y} d(y,X) \Big\} \,,$$

where $d(x,Y) \doteq \inf_{y\in Y} d(x,y)$ and $d(x,y) \doteq |x-y|$ is the Euclidean distance on $I\!\!R^2$.

In its original formulation, a strategy is a mapping $t \mapsto \gamma(t)$ describing the walls $\gamma(t) \subset I\!\!R^2$ constructed at any given time $t \geq 0$. This blocking problem can be reformulated in a simpler way, where a strategy is entirely determined by assigning one single rectifiable set $\Gamma \subset I\!R^2$. Indeed, consider a rectifiable set $\Gamma \subset I\!R^2$. We assume that Γ is *complete*, in the sense that it contains all of its points of positive upper density:

$$\limsup_{r\to 0+} \frac{m_1\Big(B(x,r)\cap\Gamma\Big)}{r} > 0 \qquad \Longrightarrow \qquad x\in\Gamma\,.$$

Define the set reached at time $t > 0$ by trajectories which do not cross Γ:

$$R^\Gamma(t) \doteq \Big\{x(t)\,;\quad x(\cdot)\ \text{ absolutely continuous}\,,\quad x(0)\in R_0\,,\\ |\dot x(\tau)|\leq 1\ \text{ for a.e. } \tau\in[0,t]\,,\quad x(\tau)\notin\Gamma\ \text{ for all } \tau\in[0,t]\Big\}\,. \tag{1.4}$$

Throughout the following, $\overline{S}$ will denote the closure of a set S. We say that the rectifiable set Γ is *admissible* for the construction speed σ if, for every $t \geq 0$, the set

$$\gamma(t) \doteq \Gamma\cap\overline{R^\Gamma(t)} \tag{1.5}$$

satisfies (1.1), i.e. it can be constructed within time t. We observe that the set $\gamma(t)$ in (1.5) represents the relevant portion of the barrier Γ which needs to be put in place at time t. The remaining part $\Gamma\setminus\overline{R^\Gamma(t)}$ has not been yet reached by the fire, and can thus be constructed at a later time. We now consider:

(BP2) Blocking Problem 2. Find an admissible rectifiable set $\Gamma \subset I\!R^2$ such that the union of all connected components of $I\!R^2\setminus\Gamma$ which intersect R_0 is bounded.

As proved in [7], the two above formulations of the blocking problem are equivalent. Namely, the problem (BP1) has a solution if and only the same is true for (BP2).

The analysis in [4] shows that in the entire plane a blocking strategy exists if $\sigma > 2$, and cannot exist if $\sigma < 1$. This result is not sharp, leaving a gap between the existence and the non-existence case. In the light of the computations in [5] one is led to

Conjecture: *On the entire plane, a blocking strategy exists if and only if* $\sigma > 2$.

In the present paper we study the case where the fire is constrained to the half plane $I\!R^2_+ = \{(x_1, x_2) \in I\!R^2\,;\ \ x_2 > 0\}$. For this case, the critical speed which discriminates between the existence or non-existence of blocking strategies can be precisely determined.

Theorem 1. *If the fire is restricted to a half plane, then, for any initial open bounded set* $R_0 \subset I\!R^2_+$ *, a blocking strategy exists if and only if* $\sigma > 1$.

The remainder of the paper is organized as follows. In Section 2 we give further equivalent forms of the blocking properties (BP1), (BP2). In particular we show that, if any blocking strategy exists, then a solution of (BP2) can be found, consisting of a rectifiable set Γ with finite length and finitely many compact connected components. In Section 3 we prove a geometric lemma on the distance function on a simply connected domain. Namely, call $d_K(x, y)$ the minimum length among all continuous paths $\gamma : [0,1] \mapsto K$ which connect the points $x, y \in K$. If $K \subset \mathbb{R}^2$ is a simply connected, compact domain whose boundary ∂K is a smooth Jordan curve, we show that the maximum of the path-distance function d_K is attained at a couple of points $\bar{x}, \bar{y}$ both lying on the boundary of K. We remark that this conclusion is generally false if the domain K is not simply connected. Using this geometric lemma, in Section 4 we give a proof of Theorem 1.

For an introduction to geometric measure theory and rectifiable sets we refer to [1]. The basic properties of Jordan curves and of simply connected sets in the plane, used in this paper, can be found in any textbook on algebraic topology, for example [11, 12, 13].

2 Equivalent properties

Throughout this section, we assume that the initial set R_0 is contained in the half plane $\mathbb{R}^2_+ \doteq \{(x_1, x_2)\,;\ \ x_2 > 0\}$, and that all trajectories of the fire are restricted to $\mathbb{R}^2_+$ as well. The following lemma collects various equivalent formulations of the blocking property. In the following, we call $B^+_\rho \doteq B(0,\rho) \cap \mathbb{R}^2_+$ the upper half of the open disc centered at the origin, with radius ρ. The reachable set at time t, starting from R_0 and avoiding the wall Γ will be denoted as $R^\Gamma(t, R_0)$.

Lemma 1. *Given a construction speed $\sigma > 0$, the following statements are equivalent:*

(i) A blocking strategy for (BP1) exists when $R_0 = B^+_\rho$, for some $\rho > 0$.

(ii) For every initial bounded open set R_0, a blocking strategy for (BP1) exists.

(iii) For every initial bounded open set R_0 there exists a set $\Gamma \subset \mathbb{R}^2$ with $m_1(\Gamma) < \infty$, which solves the blocking problem (BP2) and consists of finitely many compact connected components.

Proof. The implication (ii)$\Longrightarrow$ (i) is trivial, while (iii) $\Longrightarrow$ (ii) is an immediate consequence of Theorem 1 in [7].

To prove that (i) $\Longrightarrow$ (ii), let $R_0 \subset \mathbb{R}^2_+$ be any bounded set. Choose a radius $r > 0$ such that $R_0 \subseteq B^+_r$. Let $t \to \gamma(t)$ be a complete, admissible strategy which blocks a fire initially starting from the half disc B^+_ρ. As observed in [4], the rescaling

$$\tilde{\gamma}(t) \doteq \frac{r}{\rho} \cdot \gamma\left(\frac{\rho}{r}\,t\right)$$

defines another complete admissible strategy. Indeed, for every $t \geq 0$ one has

$$m_1(\tilde{\gamma}(t)) \;=\; \frac{r}{\rho} \cdot m_1\left(\gamma\Big(\frac{\rho}{r}\,t\Big)\right) \;\leq\; \sigma\, t.$$

Moreover, the corresponding sets reached by the fire starting from $R_0 \subseteq B^+_r$ satisfy

$$R^{\tilde{\gamma}}(t,\, R_0) \;\subseteq\; R^{\tilde{\gamma}}(t,\, B^+_r) \;=\; \frac{r}{\rho} \cdot R^\gamma\left(\frac{\rho}{r}\,t,\, B^+_\rho\right) \tag{2.1}$$

for every $t \geq 0$. Since by assumption the sets on the right hand side of (2.1) remain uniformly bounded, we conclude that $\tilde{\gamma}$ is a blocking strategy, when the fire starts from R_0.

Finally, we prove the implication (ii) $\Longrightarrow$ (iii). Let (ii) hold. In particular, there must exist some admissible strategy $t \mapsto \gamma(t)$ which solves the blocking problem (BP2) in connection with the initial set $R_0 = B_3^+$. Say,

$$R^\gamma(t, B_3^+) \subseteq B_r^+ ,$$

for some radius $r > 3$ and all $t \geq 0$. It is clearly not restrictive to assume that all walls $\gamma(t)$ are contained in the closed half disc $\overline{B_r^+}$. Choose a time T large enough such that

$$T \;>\; \frac{2}{\sigma}\, m_2(B_r^+) = \frac{\pi r^2}{\sigma}\,. \tag{2.2}$$

Consider the complete rectifiable set $\Gamma \doteq \gamma(T+1)$. By the analysis in [7], we can assume that the totally disconnected component of Γ has 1-dimensional measure zero. The set Γ can thus be decomposed as a countable union of connected components with positive length, plus a totally disconnected set Γ_0 of zero length, say

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_N \cup \Gamma_{N+1} \cup \cdots$$

Here we choose the integer N large enough so that, setting $\Gamma'' \doteq \bigcup_{j>N} \gamma_j$, there holds

$$m_1(\Gamma'') \;<\; \frac{1}{2}\,. \tag{2.3}$$

Let S be the union of all bounded connected components of the complement $\mathbb{R}^2_+ \setminus \Gamma''$. We observe here that the boundary of S satisfies $\partial S \subseteq \Gamma''$. Hence, every component Γ_i with $1 \leq i \leq N$ is either entirely contained in S, or else it does not intersect S at all. We now define

$$\Gamma' \doteq \bigcup_{1\leq i\leq N} \Gamma_i \setminus S$$

as the union of those among the first N components which are outside S.

For $1 < t < T+1$, the analysis in [7] yields

$$R^{\Gamma'}(t, B_1^+) \;\subseteq\; R^{\Gamma'}(t-1, B_2^+) \;\subseteq\; R^{\Gamma}\Big(t-1+m_1(\Gamma'')\,,\; B_2^+\Big)\cup S \;\subseteq\; R^{\Gamma}(t\,,\; B_2^+)\cup S \;\subseteq\; B_r^+\,. \tag{2.4}$$

By (2.2) and (2.4) there exists a time $\tau \in [1, T]$ such that

$$m_2\Big(R^{\Gamma'}(\tau+1, B_1^+)\Big) - m_2\Big(R^{\Gamma'}(\tau, B_1^+)\Big) \;<\; \sigma. \tag{2.5}$$

Since Γ' is the union of finitely many compact, connected, rectifiable sets, the minimum time function

$$V(x) \;\doteq\; \inf\; \Big\{t \geq 0\,;\;\; x \in R^{\Gamma'}(t, B_1^+)\Big\}$$

is locally Lipschitz continuous outside Γ'. Indeed (see for example [2]), it provides a viscosity solution to the eikonal equation $|\nabla V(x)| - 1 = 0$. An application of the coarea formula yields

$$m_2\Big(R^{\Gamma'}(\tau+1, B_1^+) \;\setminus\; R^{\Gamma'}(\tau, B_1^+)\Big) \;=\; \int_\tau^{\tau+1} m_1\Big(\{x \notin \Gamma'\,;\;\; V(x) = \tau'\}\Big)\, dt\,.$$

Therefore, by (2.5) there exists a time $\tau' \in [\tau, \tau+1]$ such that

$$m_1\Big(\{x \notin \Gamma'\,;\quad V(x) = \tau'\}\Big) \;<\; \sigma. \tag{2.6}$$

We claim that the set

$$\Gamma^* \doteq \Gamma' \cup \{x \notin \Gamma'\,;\quad V(x) = \tau'\}$$

is a closed, admissible set which solves the blocking problem (BP2) when $R_0 = B_1^+$.

To check that Γ^* is closed, let $x_n \to x$ be a convergent sequence with $x_n \in \Gamma^*$ for every n. Then, either $x \in \Gamma' \subset \Gamma^*$, or else $x \notin \Gamma'$. In this second case, $V(x_n) = \tau'$ for all n sufficiently large, hence $V(x) = \tau'$ by continuity. In both cases $x \in \Gamma^*$.

By construction, we have

$$R^{\Gamma^*}(t, B_1^+) = R^{\Gamma'}(\tau', B_1^+) \qquad \text{for all } \; t \geq \tau'. \tag{2.7}$$

To prove that Γ^* is admissible, for $t < \tau'$ we estimate

$$\begin{aligned} m_1\left(\Gamma^* \cap \overline{R^{\Gamma^*}(t, B_1^+)}\right) \;&\leq\; m_1\left(\Gamma^* \cap \overline{R^{\Gamma'}(t, B_1^+)}\right) \;=\; m_1\left(\Gamma' \cap \overline{R^{\Gamma'}(t, B_1^+)}\right) \\ \leq\; m_1\left(\Gamma' \cap \overline{R^{\Gamma}(t, B_2^+)}\right) \;&\leq\; m_1\left(\Gamma \cap \overline{R^{\Gamma}(t-1, B_3^+)}\right) \;\leq\; \sigma(t-1)\,. \end{aligned} \tag{2.8}$$

Here we used (2.4) and the fact that Γ' does not intersect S. Moreover, for $t \geq \tau'$ a similar argument yields

$$\begin{aligned} m_1\left(\Gamma^* \cap \overline{R^{\Gamma^*}(t, B_1^+)}\right) \;&=\; m_1\left(\Gamma^* \cap \overline{R^{\Gamma^*}(\tau', B_1^+)}\right) \\ &\leq\; m_1\left(\Gamma \cap \overline{R^{\Gamma}(\tau'-1, B_3^+)}\right) + m_1\Big(\{x \notin \Gamma'\,;\quad V(x) = \tau'\}\Big) \\ &<\; \sigma(\tau'-1) + \sigma \;\leq\; \sigma t. \end{aligned} \tag{2.9}$$

Therefore, Γ^* is admissible and provides a solution to the blocking problem. In general, however, the level set in (2.6) may have infinitely many connected components. In order to satisfy the last property in (iii), we consider the compact connected set $\overline{R^{\Gamma^*}(\tau', B_1^+)}$ and call Ω_∞ the unbounded component of its complement. Then the boundary of Ω_∞ is connected and satisfies $\partial\Omega_\infty \subseteq \Gamma^*$. Replacing Γ^* with its closed subset

$$\Gamma^\sharp \;\doteq\; \Gamma' \cup \partial\Omega_\infty\,,$$

it is clear that $\Gamma^\sharp$ satisfies all the properties in (iii).

We have thus established the implication (ii) $\Longrightarrow$ (iii) in the case where the initial set is $R_0 = B_1^+$. The general case follows from a simple rescaling argument. □

3 The distance function on a simply connected domain

Let $K \subset \mathbb{R}^2$ be a compact, path-connected set. For $x, y \in K$ we define the distance $d_K(x,y)$ as the minimum length among all absolutely continuous paths $\gamma : [0,1] \mapsto K$ with $\gamma(0) = x$, $\gamma(1) = y$. As shown in fig. 1, if the set K is not simply connected the function d_K may attain its global maximum at a couple of interior points.

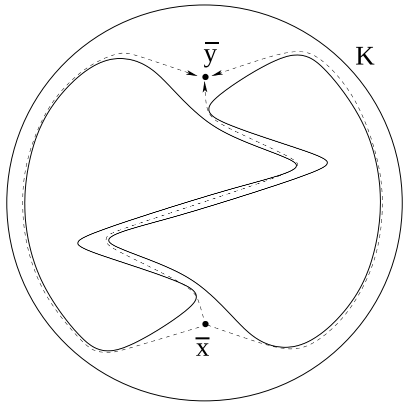

Figure 1: For the multiply connected domain K, $\max_{x,y\in K} d_K(x,y)$ is attained at the couple of interior points $(\bar{x},\bar{y})$.

In this section we will prove that, if K is simply connected, then the maximum of the distance d_K is always attained at boundary points. We recall that a *Jordan curve* is a homeomorphic image of the circumference $\{(x_1,x_2)\,;\;\; x_1^2+x_2^2=1\}$, see [11, 13].

Lemma 2. *Let $K\subset \mathbb{R}^2$ be the compact region enclosed by a smooth Jordan curve $\Gamma \doteq \partial K$. Then*

$$\max_{x,y\in K} d_K(x,y) \;=\; \max_{x,y\in\Gamma} d_K(x,y). \tag{3.1}$$

As a consequence,

$$\max_{x,y\in K} d_K(x,y) \;\leq\; \frac{1}{2}\, m_1(\partial K). \tag{3.2}$$

Of course, the assumption implies that K is simply connected, with smooth boundary. The lemma will be proved in various steps. We begin by proving an intermediate result.

Lemma 3. *Let K be a (compact, simply connected) polygon. Then for any two points $x,y\in K$, the shortest path $\gamma:[0,1]\mapsto K$ joining x with y is unique, up to reparametrizations.*

Proof. 1. Let $p_1,\ldots,p_n$ be the vertices of the polygon, with $n\geq 3$. Then K can be covered with triangles $\Delta_1,\ldots,\Delta_{n-2}$, whose vertices lie in the set $\{p_1,\ldots,p_n\}$. This is a well known result, used in numerical analysis [3]. For reader's convenience, we sketch her a proof.

The result is trivially true when $n=3$ By induction, assume that the result holds for all polygons whose number n of vertices satisfies $3\leq n<m$. Let K be a polygon with m vertices $p_1,\ldots,p_m$. For notational convenience, set $p_0\doteq p_m$. Choose three consecutive vertices p_{j-1},p_j,p_{j+1} such that the angle at p_j formed by the consecutive edges $\overline{p_{j-1},p_j}$ and $\overline{p_j,p_{j+1}}$ has amplitude $<\pi$. Such an index j certainly exists. Two cases can occur (see figure 2).

CASE 1: The triangle Δ_j with vertices p_{j-1},p_j,p_{j+1} does not contain any other vertex. Then we can add the edge $\overline{p_{j-1},p_{j+1}}$. The original polygon K is thus decomposed as the union of a triangle and a sub-polygon K' having $m-1$ vertices. We can thus apply the inductive assumption to K'.

CASE 2: The triangle Δ_j with vertices p_{j-1}, p_j, p_{j+1} contains some other vertices, say p_α for $\alpha \in \mathcal{I} \subset \{1, \ldots, m\}$. Call $\mathbf{n}$ the outer normal to the triangle Δ at the point p_j, perpendicular to the opposite edge $\overline{p_{j-1}, p_{j+1}}$. Choose an index $\beta \in \mathcal{I}$ such that

$$\langle \mathbf{n}, p_\beta \rangle \;=\; \max_{\alpha \in \mathcal{I}} \langle \mathbf{n},\, p_\alpha \rangle. \tag{3.3}$$

Here $\langle \cdot, \cdot \rangle$ denotes an inner product. We claim that the segment $\overline{p_j p_\beta}$ does not intersect any other edge of the polygon. Indeed, assume on the contrary that this segment intersected the edge $\overline{p_{\ell-1}, p_\ell}$. Since $\overline{p_{\ell-1}, p_\ell}$ does not intersect the two edges $\overline{p_{j-1}, p_j}$ and $\overline{p_j, p_{j+1}}$, at least one of the vertices $p_{\ell-1}$, p_ℓ must be contained in the interior of the smaller triangle

$$\Delta'_j \;\doteq\; \Delta_j \cap \Big\{ p \in \mathbb{R}^2\,; \quad \langle \mathbf{n}, p \rangle \geq \langle \mathbf{n}, p_\beta \rangle \Big\}.$$

But this contradicts the maximality condition (3.3).

Adding the edge $\overline{p_j, p_\beta}$, the original polygon K is decomposed in two sub-polygons, each with a number of edges $< m$. By the inductive hypothesis, each of these can be triangulated.

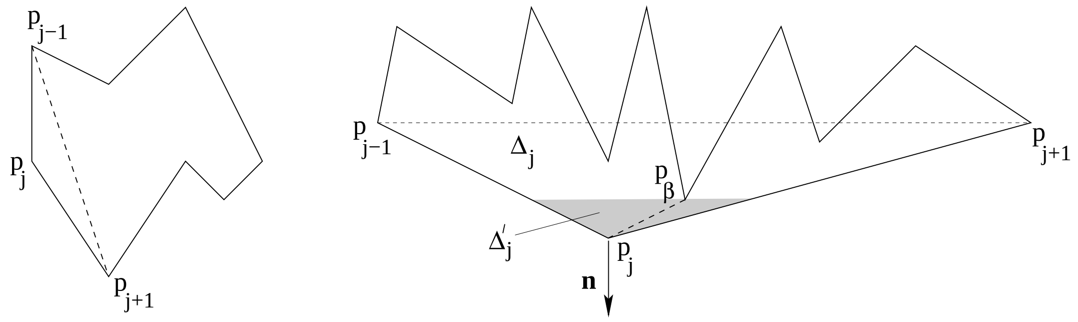

Figure 2: The two cases of the inductive step.

2. Let now $\Delta_1, \ldots, \Delta_{n-2}$ be a triangulation of the n-polygon K. We say that two triangles are connected if they have an edge in common. With this relation, we claim that the set of triangles is a simply connected graph. To establish the simple connectedness, we argue by contradiction. Assume that the graph contains a nontrivial cycle, say

$$\Delta_{i(0)}\,,\ \Delta_{i(1)}\,,\quad \ldots\quad,\ \Delta_{i(\nu)}\,,$$

with $\Delta_{i(\nu)} = \Delta_{i(0)}$, and such that each two consecutive triangles $\Delta_{i(k-1)}$, $\Delta_{i(k)}$ have an edge in common, say $\Gamma_{i(k)}$. For every $k = 1, \ldots, \nu$, fix a point $q_k \in \Gamma_{i(k)}$, different from the endpoints (see fig. 3)

Consider the closed polygonal γ having vertices $q_1, \ldots, q_\nu$. This is a simple, closed curve, contained in the interior of K. The set $\mathbb{R}^2 \setminus \gamma$ has two connected components, one bounded and the other unbounded. Since K is simply connected, all points in the bounded component of $\mathbb{R}^2 \setminus \gamma$ belong to the interior of K. Hence this bounded component does not contain any of the vertices $p_1, \ldots, p_n$. This yields a contradiction, because every edge $\Gamma_{i(k)}$ has two endpoints, say $p^-_{i(k)}$ and $p^+_{i(k)}$, located in distinct components of $\mathbb{R}^2 \setminus \gamma$.

3. Let any two points $x, y \in K$ be given. Choose indices $\alpha, \beta \in \{1, \ldots, n-2\}$ such that $x \in \Delta_\alpha$, $y \in \Delta_\beta$. We observe that these indices may not be unique. However, any choice will suffice for our purposes.

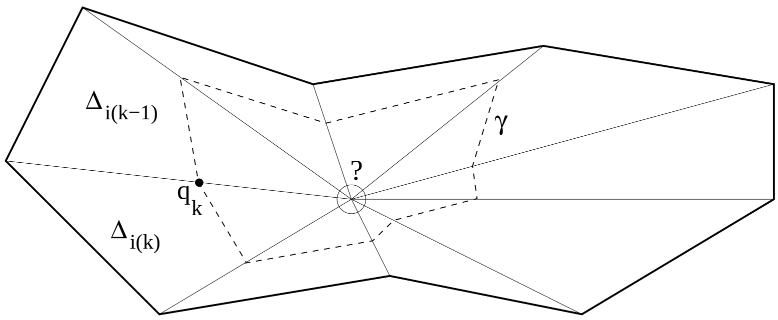

Figure 3: Proving the simple connectedness of the set of triangles.

By the previous step, the set of triangles $\{\Delta_1, \ldots, \Delta_{n-2}\}$ forms a simply connected graph. Therefore, there exists a unique chain of triangles $\Delta_\alpha = \Delta_{i(0)}, \Delta_{i(1)}, \ \ldots \ , \Delta_{i(\nu)} = \Delta_\beta$ connecting Δ_α with Δ_β.

Given a path $\gamma : [0,1] \mapsto K$ connecting x with y, for $\ell = 0, 1, \ldots, \nu$ we define the times

$$t_\ell^- \doteq \min \left\{ t \in [0,1] \,; \ \ \gamma(t) \in \Delta_{i(\ell)} \right\}, \qquad\qquad t_\ell^+ \doteq \max \left\{ t \in [0,1] \,; \ \ \gamma(t) \in \Delta_{i(\ell)} \right\}.$$

Since $\gamma(0) = x \in \Delta_{i(0)}$ and $\gamma(1) \in \Delta_{i(\nu)}$, we clearly have $t_0^- = 0$, $t_\nu^+ = 1$.

Assume that, among all paths joining x with y and remaining inside K, the path γ has minimum length. Since the points $\gamma(t_\ell^-)$ and $\gamma(t_\ell^+)$ are both contained in the convex triangle $\Delta_{i(\ell)}$, the restriction of γ to each sub-interval $[t_\ell^-, t_\ell^+]$ must be a segment.

Up to a reparameterization, the curve γ must therefore be a polygonal line, say with vertices $x = z_0, z_1, \ldots, z_\nu = y$. Moreover, for $\ell = 1, \ldots, \nu - 1$, the vertex z_ℓ lies on the edge $\Gamma_\ell = \Delta_{i(\ell-1)} \cap \Delta_{i(\ell)}$.

4. Now let $\gamma', \gamma'' : [0,1] \mapsto K$ be two distinct paths joining x with y, both with minimum length. Call $z'_0, \ldots, z'_\nu$ and $z''_0, \ldots, z''_\nu$ the corresponding vertices, as in the previous step.

A contradiction will be obtained by showing that the polygonal γ with vertices

$$z_\ell \doteq \frac{z'_\ell + z''_\ell}{2}, \qquad\qquad \ell = 0, \ldots, \nu$$

has strictly smaller length. Clearly $z_0 = x$ and $z_\nu = y$. Moreover, each segment $[z_{\ell-1}, z_\ell]$ is entirely contained in the triangle $\Delta_{j(\ell)}$. We observe that, for each $\ell = 0, \ldots, \nu$, the convexity of the euclidean norm yields

$$|z_\ell - z_{\ell-1}| \ = \ \left| \frac{z'_\ell + z''_\ell}{2} - \frac{z'_{\ell-1} + z''_{\ell-1}}{2} \right| \ \leq \ \frac{|z'_\ell - z'_{\ell-1}|}{2} + \frac{|z''_\ell - z''_{\ell-1}|}{2}.$$

Moreover, equality holds if and only if the two vectors $z'_\ell - z'_{\ell-1}$ and $z''_\ell - z''_{\ell-1}$ are parallel. If we assume that the polygonals γ' and γ'' are distinct, the above vectors cannot be parallel for every $\ell = 0, \ldots, \nu$. Hence the length of γ is strictly less than the length of γ' and of γ''. This contradiction completes the proof of the lemma. □

Lemma 4. *Let $K \subset I\!\!R^2$ be the compact region enclosed by a smooth Jordan curve Γ. Fix any interior point $q \in int K$. Then the function $x \mapsto V_K^q(x) \doteq d_K(x,q)$ is $\mathcal{C}^1$ on $(int K) \setminus \{q\}$.*

Proof. We first establish the result assuming that K is a polygon, then in the general setting considered in the lemma. We observe that the distance function V_K^q provides a viscosity solution of the eikonal equation

$$|\nabla V| - 1 = 0\,, \tag{3.4}$$

on $(\text{int}\, K) \setminus \{q\}$. In the interior of K, the function V_K^q is locally Lipschitz continuous with constant 1. By Rademacher's theorem, the gradient $\nabla V_K^q(x)$ is well defined for a.e. x.

1. If K is a polygon and $q \in \text{int}\, K$, we argue by contradiction. If the conclusion of the lemma fails, there exists a point $\bar{x} \in \text{int}\, K \setminus \{q\}$ where the gradient ∇V_K^q is not continuous. By a translation of coordinates we can assume that $\bar{x} = 0$. In this setting, there exist sequences of points $x_n' \to 0$, $x_n' \to 0$ and unit vectors

$$\mathbf{v}_n \;\doteq\; \nabla V_K^q(x_n) \;\to\; \mathbf{v}, \qquad\qquad \mathbf{v}_n' \;\doteq\; \nabla V_K^q(x_n') \;\to\; \mathbf{v}', \tag{3.5}$$

with $\mathbf{v} \neq \mathbf{v}'$. For each n, let γ_n be the unique shortest path connecting x_n to q, parameterized by arc-length. Similarly, let γ_n' be the shortest path connecting x_n' to q. Since 0 lies in the interior of K, we can choose $r > 0$ such that the open disc centered at the origin with radius r satisfies $B(0,r) \subset K$. The necessary conditions for the optimality of the paths γ_n, γ_n' imply

$$\frac{d}{ds}\gamma_n(s) \;=\; -\,\mathbf{v}_n\,, \qquad\qquad \frac{d}{ds}\gamma_n'(s) \;=\; -\,\mathbf{v}_n' \tag{3.6}$$

as long as $\gamma_n(s), \gamma_n'(s) \in B(0,r)$. By a compactness argument, after extracting a subsequence we can assume $\gamma_n \to \gamma$, $\gamma_n' \to \gamma'$, for some paths γ, γ'. By (3.6) and (3.5) it follows

$$\gamma(s) \;=\; -\,s\,\mathbf{v} \;\neq\; \gamma'(s) \;=\; -\,s\,\mathbf{v}' \qquad\qquad 0 < s < r.$$

Hence the two paths are distinct. Moreover, both γ and γ' connect 0 with q, and their lengths satisfy

$$\|\gamma\| \;\leq\; \liminf_{n\to\infty} \|\gamma_n\| \;=\; \liminf_{n\to\infty} V_K^q(x_n) \;=\; V_K^q(0),$$

$$\|\gamma'\| \;\leq\; \liminf_{n\to\infty} \|\gamma_n'\| \;=\; \liminf_{n\to\infty} V_K^q(x_n') \;=\; V_K^q(0).$$

Therefore both paths are optimal. Since this would be in contradiction with Lemma 3, we conclude that the distance function V_K^q must be continuously differentiable in the interior of the polygon K.

2. Next, assume that K is a compact set whose boundary is a smooth Jordan curve, say $\partial K = \Gamma$. We can then construct a sequence of (simply connected) polygons K_n which invade K. More precisely, given any compact subset K' contained in the interior of K, there exists an integer N such that $K' \subseteq K_n$ for all $n \geq N$.

Let any point $q \in \text{int}\, K$ be given. Then, for all n large enough, the functions $V_{K_n}^q : K_n \mapsto I\!\!R_+$ are well defined. We claim that, for each $x \in \text{int}\, K$, the sequence $V_{K_n}^q(x)$ decreases monotonically to $V_K^q(x)$. Indeed, the fact that the sequence is non-negative and monotone decreasing follows immediately from the definition. To compute its limit, let $\gamma : [0,1] \mapsto K$

be a path of minimum length connecting x with q. For each $\varepsilon > 0$ we can approximate γ with a second path $\tilde{\gamma}$, with the same initial and terminal point, taking values strictly in the interior of K. Choose an integer N such that $\tilde{\gamma}$ is entirely contained in the polygon K_N. For all $n \geq N$ we thus have

$$V^q_{K_n}(x) \;\leq\; \|\tilde{\gamma}\| \;\leq\; \|\gamma\| + \varepsilon \;=\; V^q_K(x) + \varepsilon\,.$$

This proves the pointwise convergence

$$\lim_{n\to\infty} V^q_{K_n}(x) \;=\; V^q_K(x) \qquad\qquad x \in \text{int}\, K\,.$$

3. Still assuming that $\partial K = \Gamma$ is a smooth Jordan curve, we now argue by contradiction. If V^q_K is not $\mathcal{C}^1$ in the interior of K, the same construction as in step **1** shows that, after a translation of coordinates, there exists a radius $r > 0$ and two unit vectors $\mathbf{v}' \neq \mathbf{v}''$ such that

$$V^q_K(-r\mathbf{v}') \;=\; V^q_K(-r\mathbf{v}'') \;=\; V^q_K(0) - r\,. \tag{3.7}$$

On the other hand, all functions $V^q_{K_n}$ are $\mathcal{C}^1$ on a neighborhood of the closed disc $\overline{B}(0,r)$. By possibly taking a subsequence we can assume the convergence $\mathbf{v}_n \doteq \nabla V^q_{K_n}(0) \to \mathbf{v}$, for some unit vector $\mathbf{v}$. From the identities

$$V^q_{K_n}(s\mathbf{v}_n) = V^q_{K_n}(0) + s \qquad\qquad s \in [-r, r]\,,$$

taking the limit as $n \to \infty$ we deduce

$$V^q_K(r\mathbf{v}) = V^q_K(0) + r\,. \tag{3.8}$$

However, from (3.7) it follows

$$\begin{aligned} V^q_K(r\mathbf{v}) \;&\leq\; \min\Big\{V^q_K(-r\mathbf{v}') + |r\mathbf{v} - r\mathbf{v}'|\,,\; V^q_K(-r\mathbf{v}'') + |r\mathbf{v} - r\mathbf{v}''|\Big\} \\ &=\; V^q_K(0) - r + r\cdot\min\Big\{|\mathbf{v} - \mathbf{v}'|\,,\; |\mathbf{v} - \mathbf{v}''|\Big\} \;<\; V^q_K(0) + r\,. \end{aligned}$$

This yields a contradiction with (3.8), completing the proof of Lemma 4.

Proof of Lemma 2. Using the regularity result stated in Lemma 4, we now provide a proof of Lemma 2.

Let K be a compact, simply connected domain, whose boundary ∂K is a smooth Jordan curve. Choose points $\bar{x}, \bar{y} \in K$ such that

$$d_K(\bar{x}, \bar{y}) \;=\; \max_{x,y\in K}\, d_K(x, y).$$

If $\bar{x}, \bar{y} \in \partial K$, we are done. Otherwise, assume $\bar{x} \in \text{int}\, K$. Observe that this implies $\bar{y} \in \partial K$. Indeed, by Lemma 4 the function $x \mapsto V^{\bar{x}}_K(x) \doteq d_K(\bar{x}, y)$ provides a $\mathcal{C}^1$ solution to the eikonal equation (3.4) on the open set $\text{int}\, K \setminus \{\bar{x}\}$. Therefore, it cannot have any local maximum in the interior of K.

Granted that $\bar{y} \in \partial K$, we now consider a sequence of points $y_n \to \bar{y}$ with $y_n \in \text{int}\, K$ for every $n \geq 1$. For each n, by Lemma 4 the function

$$x \mapsto V^{y_n}_K(x) \doteq d_K(x, y_n)$$

provides a $\mathcal{C}^1$ solution to the eikonal equation (3.4) on the open set $\mathrm{int}\, K \setminus \{y_n\}$. Hence, it cannot have local maximum on this set. Choose $x_n \in K$ such that

$$d_K(x_n, y_n) = \max_{x \in K} d_K(x, y_n)\,.$$

By the previous argument, $x_n \in \partial K$.

By taking a subsequence, we can assume $x_n \to x^*$ for some $x^* \in K$. Clearly $x^* \in \partial K$, because $x_n \in \partial K$ for every n. By the uniform continuity of the distance function d_K on $K \times K$, we now have

$$\begin{aligned}\max_{x,y \in \partial K} d_K(x,y) \;\geq\; d_K(x^*, \bar y) \;&=\; \lim_{n\to\infty} d_K(x_n, y_n)\\ &\geq\; \lim_{n\to\infty} d_K(\bar x, y_n) = d_K(\bar x, \bar y) = \max_{x,y \in K} d_K(x,y),\end{aligned}$$

proving (3.1). The statement (3.2) is now clear. □

4 Proof of the main theorem

Relying on the lemmas proved in the previous sections, we give here a proof of Theorem 1, in several steps.

1. Assume $\sigma > 1$. By Lemma 1, it suffices to prove that a blocking strategy exists in the case where $R_0 = B_1^+$ is the upper half of the unit disc centered at the origin.

Set $\lambda \doteq (\sigma^2 - 1)^{-1/2}$, so that $\sigma = \sqrt{1+\lambda^2}/\lambda$, and consider the strategy

$$\gamma(t) = \Big\{ (\, e^{\lambda\theta}\, \cos\theta\,,\;\; e^{\lambda\theta}\, \sin\theta)\,;\;\; 0 \leq \theta \leq \lambda^{-1}\, \ln(1+t) \Big\}\,.$$

This strategy is admissible because

$$m_1\big(\gamma(t)\big) \;=\; \int_0^{\lambda^{-1}\cdot \ln(1+t)} e^{\lambda\theta}\, \sqrt{1+\lambda^2}\, d\theta \;=\; \sigma t\,.$$

Next, consider the time T such that $\lambda^{-1} \ln(1+T) = \pi$. We claim that, for $t \in [0, T]$, the reachable set is

$$R_t^\gamma = \Big\{ (r\cos\theta,\, r\sin\theta)\,;\;\; 0 < \theta < \pi\,,\;\; 0 < r < \min\{1+t,\, e^{\lambda\theta}\} \Big\}. \tag{4.1}$$

Indeed, call $S(t)$ the set on right hand side of (4.1). Since for each $t \geq 0$ the wall $\gamma(t)$ is contained inside the arc of spiral

$$\Sigma \doteq \Big\{ (e^{\lambda\theta} \cos\theta,\, e^{\lambda\theta} \sin\theta)\,;\;\; 0 \leq \theta \leq \pi \Big\},$$

the inclusion $S(t) \subseteq R_t^\gamma$ is clear. To prove the converse inclusion, consider any point $p = (r\cos\theta,\, r\sin\theta)$ with $r \geq e^{\lambda\theta}$. If $p \in R_t^\gamma$, then there exists an absolutely continuous path $s \mapsto x(t)$ with

$$|x(0)| < 1\,, \qquad x(t) = p\,, \qquad |\dot x(s)| \leq 1 \quad \text{for a.e. } s \in [0,t],$$

$$x(s) \notin \gamma(s) \qquad \text{for all } s \in [0,t]\,.$$

Since Σ splits the upper half plane in two connected components, there must be some $\tau \in]0,t]$ such that $x(\tau) \in \Sigma$. We now observe that $|x(\tau)| < 1+\tau$, and hence

$$x(\tau) \in \gamma(\tau) \;=\; \{x \in \Sigma\,;\;\; |x| \leq 1+\tau\}.$$

This contradiction shows that $R_t^\gamma \subseteq S(t)$, completing the proof of (4.1).

In particular, when $t = T$ we have

$$R_T^\gamma = \Big\{(r\cos\theta,\, r\sin\theta)\,;\;\; 0<\theta<\pi\,,\;\;\; 0<r<e^{\lambda\theta}\Big\}, \qquad\qquad \gamma(T) = \Sigma\,.$$

We observe that the boundary of R_T^γ is entirely contained in the union of the arc $\gamma(T) = \Sigma$ and the x_1-axis. Hence the reachable set cannot become any larger: $R_\infty^\gamma = R_t^\gamma = R_T^\gamma$ for every $t \geq T$. This shows that the strategy γ blocks the fire within a bounded set.

2. Next, assume $\sigma \leq 1$. We argue by contradiction. If some blocking strategy exists, then by Lemma 1 we can again assume $R_0 = B_1^+$ and we can find a blocking strategy consisting of finitely many compact connected components $\Gamma \doteq \Gamma_1 \cup \cdots \cup \Gamma_N$, with $m_1(\Gamma) < \infty$. By assumption, the reachable set $R_\infty^\Gamma \doteq \bigcup_{t\geq 0} R_t^\Gamma$ is bounded. It is not restrictive to assume that

$$\Gamma \;\subset\; \overline{R_\infty^\Gamma}\,, \tag{4.2}$$

otherwise we can simply replace Γ by the intersection $\Gamma \cap \overline{R_\infty^\Gamma}$. We can also assume that $\Gamma \cap B_1^+ = \emptyset$. Notice that the reachable set R_∞^Γ is precisely the union of all connected components of the open set $I\!R_+^2 \setminus \Gamma$ which intersect $R_0 = B_1^+$. For each $x \in R_\infty^\Gamma$, let

$$V(x) \;\doteq\; \inf\,\Big\{t \geq 0\,;\;\; x \in \overline{R^\Gamma(t)}\Big\}$$

be the minimum time needed for the fire to reach x. We observe that admissibility condition (1.1), (1.5) implies

$$t^* \;\doteq\; \sup_{x \in R_\infty^\Gamma} \; V(x) \;\geq\; m_1(\Gamma)\,. \tag{4.3}$$

Otherwise, by (4.2) we would have

$$m_1\Big(\Gamma \cap \overline{R^\Gamma(t^*, B_1^+)}\Big) \;=\; m_1\Big(\Gamma \cap \overline{R_\infty^\Gamma}\Big) \;=\; m_1(\Gamma) \;>\; t^*,$$

against the assumption that Γ is admissible.

To prove Theorem 1, a contradiction will be achieved by showing that every point $x \in R_\infty^\Gamma$ can be reached from B_1^+ in time

$$V(x) \leq m_1(\Gamma) - \frac{1}{4}\,. \tag{4.4}$$

More precisely, consider the point $y = (0, 1/2) \in B_1^+$. If we show that every point $x \in R_\infty^\Gamma$ can be connected to y by a path γ of length $\leq m_1(\Gamma) + 1/4$, without crossing the wall Γ, we are done. Indeed, if $x \notin B_1^+$, the path γ must contain a portion of length $\geq 1/2$ inside B_1^+. Hence (4.4) must hold.

3. Consider the interval $[a,b]$ where

$$a \doteq \inf\Big\{x_1\,;\quad (x_1,0)\in \overline{R^\Gamma_\infty}\Big\}\,,\qquad\qquad b \doteq \sup\Big\{x_1\,;\quad (x_1,0)\in \overline{R^\Gamma_\infty}\Big\}\,.$$

For later purposes, we define the segment $S_0 = \{(x_1,0)\,;\quad x_1\in[a,b]\}$.

Next, let Ω_∞ be the (unique) unbounded connected component of the open set $I\!R^2\setminus\overline{R^\Gamma_\infty}$. Since $\overline{R^\Gamma_\infty}$ is a compact connected set, the boundary $\partial\Omega_\infty$ is a connected set, contained in $\Gamma\cup S_0$. Let Γ_0 denote the union of $\partial\Omega_\infty$ and of all connected components of Γ which intersect $\partial\Omega_\infty$. Moreover, let $\Gamma_1,\ldots,\Gamma_m$ be a list of all the connected components of Γ which do not intersect Γ_0. We observe that

$$m_1(\Gamma_0)+\sum_{j=1}^m m_1(\Gamma_j)\ \leq\ m_1(\Gamma)+(b-a). \tag{4.5}$$

Let any point $x\in R^\Gamma_\infty$ be given. In the remainder of the proof we will show that there exists a path γ joining the point $y=(0,1/2)$ to the point x, without crossing $\Gamma_0\cup\Gamma_1\cup\cdots\cup\Gamma_m$, with length $\|\gamma\|\leq m_1(\Gamma)+1/4$.

4. Since $x\notin\Gamma\cup S_0$ and $\Gamma_0,\ldots,\Gamma_m$ are disjoint compact sets, we can choose $0<\varepsilon<1/4$ such that

$$d(x,\Gamma\cup S_0)\ >\ \varepsilon,\qquad\qquad d(\Gamma_i,\Gamma_j)>2\varepsilon \tag{4.6}$$

for all $i,j=0,\ldots,m$ with $i\neq j$.

For each $i=2,\ldots,m$, we construct a smooth Jordan curve γ_i surrounding Γ_i such that

$$\gamma_i\subset B(\Gamma_i,\varepsilon),\qquad\qquad m_1(\gamma_i)\ <\ 2m_1(\Gamma_i)+\frac{1}{5m}\,. \tag{4.7}$$

Since each Γ_i is a compact, rectifiable set, this can be done as in [7]. Namely, we first choose a radius $0<r_2<\varepsilon/2$ such that the neighborhood of radius r_2 around the set Γ_i has measure

$$m_2(B(\Gamma_i,r_2))\ <\ 2r_2\left(m_1(\Gamma_i)+\frac{1}{20m}\right).$$

Choosing r_1 sufficiently small, with $0<r_1<<r_2$, we achieve

$$m_2(B(\Gamma_i,r_2)\setminus B(\Gamma_i,r_1))\ <\ 2(r_2-r_1)\left(m_1(\Gamma_i)+\frac{1}{10m}\right)\,. \tag{4.8}$$

Define the distance function with cutoff

$$V_i(x)\ \doteq\ \left\{\begin{array}{ccl} d(x,\Gamma_i) & \text{if} & r_1\ <\ d(x,\Gamma_i)\ <\ r_2\,,\\ r_1 & \text{if} & d(x,\Gamma_i)\leq r_1\,,\\ r_2 & \text{if} & d(x,\Gamma_i)\geq r_2\,.\end{array}\right. \tag{4.9}$$

Let $V_{i,\varepsilon'}\doteq\varphi_{\varepsilon'}*V_i$ be a mollification of V_i, where the smooth kernel $\varphi_{\varepsilon'}$ is supported inside the disc $B(0,\varepsilon')$, for some ε' with $0<\varepsilon'<r_1$. The functions $V_i, V_{i,\varepsilon'}$ are both Lipschitz continuous with constant one. Using the co-area formula and then (4.8), we obtain

$$\begin{aligned}\int_{r_1}^{r_2} m_1\Big(\{x\,;V_{i,\varepsilon'}(x)=s\}\Big)\,ds\ &=\ \int_{I\!R^2}|\nabla V_{i,\varepsilon'}|\,dx\ \leq\ \int_{I\!R^2}|\nabla V_i|\,dx\\ &=\ m_2\Big(B(\Gamma_i,r_2)\setminus B(\Gamma_i,r_1)\Big)\ <\ 2(r_2-r_1)\left(m_1(K)+\frac{1}{10m}\right)\,.\end{aligned} \tag{4.10}$$

Since $V_{i,\varepsilon'}$ is smooth, by Sard's theorem almost every level set $\Sigma_r \doteq \{x\,;\; V_{i,\varepsilon'}(x) = r\}$ is the union of finitely many smooth curves. By (4.10), there exists some ρ, with $r_1 < \rho < r_2$ such that the level set Σ_ρ is the finite union of smooth curves an moreover

$$m_1(\Sigma_\rho) \;<\; 2m_1(\Gamma_i) + \frac{1}{5m}\,. \tag{4.11}$$

The construction of $V_{i,\varepsilon'}$ clearly implies

$$\Sigma_\rho \;\subset\; B(\Gamma_i,\, r_2+\varepsilon') \setminus B(\Gamma_i,\, r_1-\varepsilon')\,. \tag{4.12}$$

Consider the sub-level set

$$\Sigma_\rho^- \doteq \{x\,;\; V_{i,\varepsilon'}(x) < \rho\}\,.$$

Observe that this open set need not be connected. However, since Σ_ρ^- contains the connected neighborhood $B(\Gamma_i,\, r_1-\varepsilon')$, we can uniquely define the set $\widetilde{\Sigma}_\rho^-$ as the connected component of Σ_ρ^- which contains $B(\Gamma_i,\, r_1-\varepsilon')$. Clearly, its boundary satisfies

$$\partial\widetilde{\Sigma}_\rho^- \;\subseteq\; \Sigma_\rho\,. \tag{4.13}$$

The curve $\gamma_i \doteq \partial\widetilde{\Sigma}_\rho^-$ is then a Jordan curve, satisfying all our requirements.

5. We now perform a similar construction for the set Γ_0, but in a more careful way. Let a continuous path be given, say $\gamma_{xy} : [0,1] \mapsto I\!R^2 \setminus \Gamma_0$, joining x with y without crossing Γ_0. By possibly shrinking the value of ε in (4.6), we can assume that

$$\min_{s\in[0,1]} d(\gamma(s),\, \Gamma_0) > 2\varepsilon\,. \tag{4.14}$$

The same procedure used in the previous step now yields radii $0 < r_1 < r_2 < \varepsilon$ and a finite family of smooth closed curves $\gamma_{0,0}, \ldots, \gamma_{0,k} \subset B(\Gamma_0, \varepsilon)$ such that the following holds:

$$\sum_{\ell=1}^{k} m_1(\gamma_{0,\ell}) \;\le\; 2m_1(\Gamma_0) + \frac{1}{4} \tag{4.15}$$

and moreover, if $p_1, p_2 \in I\!R^2$ and $d(p_1, \Gamma_0) < r_1$ while $d(p_2, \Gamma_0) > r_2$, then p_1 and p_2 lie inside distinct connected components of the complement $I\!R^2 \setminus \bigcup_{\ell=1}^{k} \gamma_{0,\ell}$.

We claim that, among the closed curves $\gamma_{0,\ell}$, there exists at least one curve, say $\gamma_{0,\mu}$ such that Γ_0 is contained in the bounded connected component of $I\!R^2 \setminus \gamma_{0,\mu}$. Indeed, let Ω_0 be the connected component of the complement $I\!R^2 \setminus \bigcup_{\ell=1}^{k} \gamma_{0,\ell}$ which contains the compact connected set Γ_0. The boundary of the unbounded connected component of $I\!R^2 \setminus \Omega_0$ is the desired curve.

Next, we claim that there exists a second curve, say $\gamma_{0,\nu}$, such that the path γ_{xy} is contained in the bounded component of $I\!R^2 \setminus \gamma_{0,\nu}$ while Γ_1 is contained in the unbounded connected component of $I\!R^2 \setminus \gamma_{0,\nu}$. Indeed, let Ω_1 be the connected component of the complement $I\!R^2 \setminus \bigcup_{\ell=1}^{k} \gamma_{0,\ell}$ which contains the path γ_{xy}. Notice that the open set Ω_1 must be disjoint from the previous component Ω_0. The boundary of the unbounded connected component of $I\!R^2 \setminus \Omega_1$ now provides the desired curve.

By a simple relabeling, we can assume that $\mu = 1$ and $\nu = 2$. We thus have

$$m_1(\gamma_{0,1}) + m_1(\gamma_{0,2}) \;\le\; \sum_{\ell=1}^{k} m_1(\gamma_{0,\ell}) \;\le\; 2m_1(\Gamma_0) + \frac{1}{4} \tag{4.16}$$

while

$$m_1(\gamma_{0,1}) \ > \ 2\,\mathrm{diam}\,(\Gamma_0) \ \geq \ 2(b-a)\,. \tag{4.17}$$

6. Since $\gamma_{0,2}$ is a Jordan curve, the bounded connected component of $\mathbb{R}^2 \setminus \gamma_{0,2}$ is simply connected. Moreover it contains the path γ_{xy}. By Lemma 2, we conclude that there exists a path $\gamma^\dagger : [0,1] \mapsto \mathbb{R}^2 \setminus \Gamma_0$, connecting x with y, whose length satisfies

$$\|\gamma^\dagger\| \ \leq \ \frac{1}{2}\, m_1(\gamma_{0,2})\,. \tag{4.18}$$

Combining (4.18) with (4.16) and (4.17), and recalling (4.5), we conclude

$$\|\gamma^\dagger\| \ \leq \ \frac{1}{2}\left(2m_1(\Gamma_0) - m_1(\gamma_{0,1}) + \frac{1}{4}\right) \ \leq \ m_1(\Gamma_0) - (b-a) + \frac{1}{8} \ \leq \ m_1(\Gamma) - \sum_{j=1}^{m} m_1(\Gamma_j) + \frac{1}{8}\,. \tag{4.19}$$

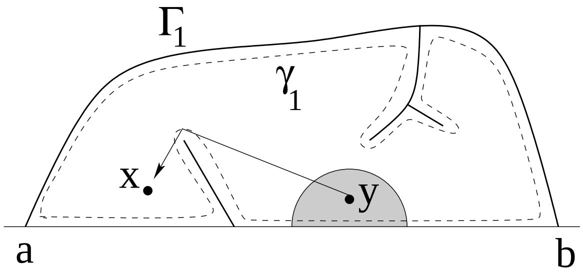

Figure 4: The shortest path $\gamma^\dagger$ joining y with x is not longer than the maximum path-distance between any two points in $\gamma_{0,2}$.

7. The path $\gamma^\dagger$ constructed in the previous step does not intersect Γ_0, but it may cross the other components $\Gamma_1, \ldots, \Gamma_m$. To avoid these crossings, we need to go around the walls Γ_j, using the paths $\gamma_1, \ldots, \gamma_m$ constructed at (4.7).

For each index $i \in \{1, \ldots, m\}$ such that the closed curve γ_i intersects the path $\gamma^\dagger$, we define

$$t_i^- = \inf\Big\{t \in [0,1]\,; \quad \gamma^\dagger(t) \in \gamma_i\Big\}, \qquad\qquad t_i^+ = \sup\Big\{t \in [0,1]\,; \quad \gamma^\dagger(t) \in \gamma_i\Big\},$$

An increasing sequence of times $t^-_{i(1)} < t^-_{i(2)} < \cdots < t^-_{i(\nu)}$ is now defined by the following inductive procedure:

• Begin by setting $t^-_{i(1)} \doteq \min_{1\leq j\leq m} \ t_j^-$.

• Let $t^-_{i(\ell)}$ be given. If $t_j^- < t^+_{i(\ell)}$ for every $j = 1, \ldots, m$, then we set $\nu = \ell$ and the induction terminates. Otherwise we define $t^-_{i(\ell+1)} \doteq \min\{t_j^-\,;\ \ t_j^- > t^+_{i(\ell)}\}$ and continue.

It is clear that the indices $i(1), i(2), \ldots, i(\nu)$ must be all distinct, and that the intervals $[t^-_{i(\ell)},\, t^+_{i(\ell)}]$, $\ell = 1, \ldots, \nu$ are mutually disjoint. In particular, $\nu \leq m$. Since each γ_i is a smooth, closed curve, for each ℓ there is an arc of $\gamma_{i(\ell)}$ joining $\gamma^\dagger(t^-_{i(\ell)})$ with $\gamma^\dagger(t^+_{i(\ell)})$ having length $\leq \frac{1}{2}\, m_1(\gamma_{i(\ell)})$.

A new path $\gamma^\sharp$ joining x with y is now obtained as follows. Given the path $\gamma^\dagger$, we replace each arc

$$\Big\{\gamma^\dagger(t)\,;\quad t\in [t^-_{i(\ell)},\, t^+_{i(\ell)}]\Big\}$$

with an arc of the Jordan curve $\gamma_{i(\ell)}$ having the same initial and terminal points.

It is clear that the path $\gamma^\sharp$ does not cross any of the sets $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$. By the previous estimates (4.19) and (4.7), its length satisfies

$$\begin{aligned} m_1(\gamma^\sharp) \;&\leq\; m_1(\gamma^\dagger)+\frac{1}{2}\sum_{\ell=1}^{\nu} m_1(\gamma_{i(\ell)}) \;\leq\; m_1(\Gamma)-\sum_{j=1}^m m_1(\Gamma_j)+\frac{1}{8}+\sum_{j=1}^m \frac{1}{2}\left(2m_1(\Gamma_j)+\frac{1}{5m}\right)\\ &<\; m_1(\Gamma)+\frac{1}{4}\,. \end{aligned}$$

This yields the desired contradiction, proving the theorem. □

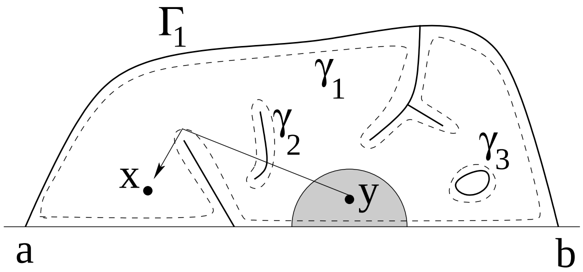

Figure 5: If the path $\gamma^\dagger$ as in fig. 4 crosses some of the walls $\Gamma_1, \ldots, \Gamma_m$, one has to go around these walls following portions of the smooth paths $\gamma_1, \ldots, \gamma_m$.

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