On the Optimal Strategy for an Isotropic Blocking Problem

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Abstract

The paper is concerned with a dynamic blocking problem, originally motivated by the control of wild fires. It is assumed that the region $R(t) \subset \mathbb{R}^2$ burned by the fire is initially a disc, and expands with unit speed in all directions. To block the fire, a barrier Γ can be constructed in real time, so that the portion of the barrier constructed within time t has length $\leq \sigma t$, for some constant $\sigma > 2$. We prove that, among all barriers consisting of a single closed curve, the one which minimizes the total burned area is axisymmetric, and consists of an arc of circumference and two arcs of logarithmic spirals.

1 Introduction

A class of dynamic blocking problems, motivated by the control of wild fires, was introduced in [4]. The region burned by the fire at time $t \geq 0$ is denoted by $R(t) \subset \mathbb{R}^2$. In the case where no blocking strategy is implemented, the set $R(t)$ is described as the reachable set for a differential inclusion

$$\dot{x} \in F(x), \qquad x(0) \in R_0, \tag{1.1}$$

where the upper dot denotes a derivative w.r.t. time. Here $x \mapsto F(x)$ is a Lipschitz continuous multifunction with compact, convex values, containing the origin. The initial set $R(0) = R_0$ is supposed to be open and bounded.

To restrain the growth of the burned region, it is assumed that a barrier can be constructed, in real time. We denote by $\gamma(t) \subset \mathbb{R}^2$ the portion of the barrier constructed within time $t \geq 0$. Given a construction speed $\sigma > 0$, we say that a strategy $t \mapsto \gamma(t)$ is **admissible** if the following conditions hold.

(H1) For any $t_1 < t_2$, one has $\gamma(t_1) \subseteq \gamma(t_2)$.

(H2) For every $t \geq 0$, $\gamma(t)$ is a rectifiable set with length

$$m_1(\gamma(t)) \leq \sigma t.$$

Here m_1 denotes the one-dimensional Hausdorff measure, normalized so that $m_1(\Gamma)$ yields the usual length of a smooth curve Γ.

The **reachable set** (burned by the fire) at time t, determined by the blocking strategy γ, is defined as

$$R^\gamma(t) \doteq \Big\{x(t);\ \ x(\cdot) \text{ is absolutely continuous}, \qquad x(0)\in R_0\,, \\ \dot x(\tau)\in F(x(\tau)) \ \text{ for a.e. } \tau\in[0,t], \qquad x(\tau)\notin \gamma(\tau) \ \text{ for all } \tau\in[0,t]\Big\}. \tag{1.2}$$

In other words, $R^\gamma(t)$ is the set reached by trajectories of the differential inclusion (1.1) which do not cross the constructed portion of the barrier.

As proved in [7], one can equivalently describe a blocking strategy $t\mapsto\gamma(t)$ in terms of one single set

$$\Gamma \doteq \left(\bigcup_{t\geq 0}\gamma(t)\right)\setminus\left(\bigcup_{t\geq 0}R^\gamma(t)\right). \tag{1.3}$$

Given a rectifiable set $\Gamma\subset\mathbb{R}^2$, the corresponding **reachable sets** for the differential inclusion (1.1) restricted to $\mathbb{R}^2\setminus\Gamma$ are defined as

$$R^\Gamma(t) \doteq \Big\{x(t);\ \ x(\cdot) \text{ is absolutely continuous}, \qquad x(0)\in R_0\,, \\ \dot x(\tau)\in F(x(\tau)) \ \text{ for a.e. } \tau\in[0,t], \qquad x(\tau)\notin \Gamma \ \text{ for all } \tau\in[0,t]\Big\}. \tag{1.4}$$

We say that Γ is **admissible** if

$$m_1\left(\Gamma\cap\overline{R^\Gamma(t)}\right) \leq \sigma\, t \qquad \text{for all } t\geq 0. \tag{1.5}$$

Here and in the sequel, an overline denotes the closure of a set. Clearly, the set Γ is admissible if and only if

$$t \mapsto \gamma(t) \doteq \Gamma\cap\overline{R^\Gamma(t)} \tag{1.6}$$

is an admissible strategy. Notice that the set $\gamma(t)$ in (1.6) is the portion of the barrier Γ touched by the fire within time t. We shall denote by

$$R^\Gamma_\infty \doteq \bigcup_{t\geq 0} R^\Gamma(t) \tag{1.7}$$

the whole region burned by the fire. In connection with this model, two issues naturally arise:

Blocking problem: *Find an admissible set Γ such that R^Γ_∞ is bounded.*

Optimization problem: *Given two constants $\kappa_1\geq 0$ and $\kappa_2\geq 0$, find an admissible set Γ which minimizes the cost*

$$J(\Gamma) \doteq \kappa_1\cdot m_1(\Gamma)+\kappa_2\cdot m_2(R^\Gamma_\infty). \tag{1.8}$$

Here $m_2(\cdot)$ denotes the two-dimensional Lebesgue measure of a set. The functional in (1.8) thus takes into account (i) the cost of constructing the barrier, and (ii) the value of the entire region destroyed by the fire.

Various results on the existence or non-existence of an admissible strategy that confines the fire within a bounded set were proved in [4, 5, 8]. A general theorem on the existence of optimal blocking strategies was proved in [6, 10]. Moreover, various necessary conditions for optimality were derived in [4, 9, 17], providing explicit descriptions of the optimal barriers, in some basic cases.

On the other hand, sufficient conditions for optimality are not yet known. In particular, not a single case is known of a blocking strategy which is provably optimal. The difficulty is due to the fact that all necessary conditions given in [4, 9, 17] require some regularity assumptions, which may not be satisfied by a general optimal strategy. The present paper represents a first step in the derivation of sufficient conditions for optimality, developing techniques which allow a direct comparison between different strategies.

We shall consider the isotropic case, where the fire is initially burning on the open unit disc and propagates with unit speed in all directions. This corresponds to (1.1), taking $R_0 = B_1 \doteq \{y \in I\!R^2\,;\ |y| < 1\}$ and letting $F(x) \doteq \overline{B}_1 = \{y \in I\!R^2\,;\ |y| \leq 1\}$ be the closed unit disc, for every $x \in I\!R^2$. Let a construction speed $\sigma > 2$ and constants $\kappa_1 \geq 0$, $\kappa_2 > 0$ be given. As suggested by the necessary conditions derived in [4, 9], the optimal barrier Γ^* which minimizes the cost (1.8) should consist of an arc of circumference and two arcs of logarithmic spirals.

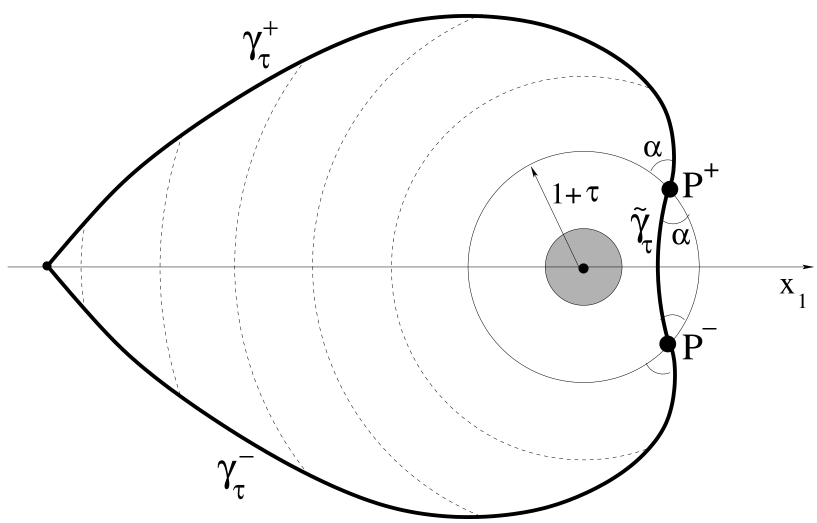

Figure 1: Construction of the barrier $\Gamma_\tau = \tilde{\gamma}_\tau \cup \gamma_\tau^+ \cup \gamma_\tau^-$.

We now describe more precisely this barrier (Fig. 1). For every $\tau > 0$ small enough, there exists a unique arc of circumference $\tilde{\gamma}_\tau$ with the following properties:

(i) $\tilde{\gamma}_\tau$ is symmetric w.r.t. the x_1-axis and has length $m_1(\tilde{\gamma}_\tau) = \sigma\tau$

(ii) the endpoints P^-, P^+ lie on the circumference $\{|x| = 1 + \tau\}$

(iii) the angle α between the two circumferences at P^- and at P^+ satisfies

$$\sin\alpha \ = \ \frac{2}{\sigma}\,. \tag{1.9}$$

In addition, consider the two arcs of logarithmic spirals $\gamma_\tau^+, \gamma_\tau^-$, defined as

$$\gamma_\tau^\pm \ = \ \Big\{(r\cos\theta\,,\ \pm r\sin\theta)\,;\quad r \ = \ r_0\, e^{\lambda\theta}, \qquad r \geq 1+\tau\,, \quad \theta \leq \pi\Big\}. \tag{1.10}$$

Here

$$\lambda = \sqrt{\frac{4}{\sigma^2 - 4}}, \tag{1.11}$$

while the constant r_0 is chosen so that the two arcs start from the points P^+, P^- respectively. Notice that the conditions (1.9)–(1.11) imply that the arcs $\gamma_\tau^\pm$ meet the circular arc $\tilde{\gamma}_\tau$ tangentially at $P^\pm$.

For every fixed τ, the union of these three arcs

$$\Gamma_\tau \doteq \tilde{\gamma}_\tau \cup \gamma_\tau^+ \cup \gamma_\tau^- \tag{1.12}$$

is a simple closed curve. By minimizing the cost $J(\Gamma_\tau)$ over the scalar parameter τ, we single out the curve

$$\Gamma^* \doteq \Gamma_{\tau^*}, \qquad \tau^* \doteq \operatorname*{argmin}_{\tau>0} J(\Gamma_\tau). \tag{1.13}$$

As shown in [9], the optimal value τ^* in (1.13) could also be singled out by the requirement that the Lagrange multiplier $t \mapsto W(t)$, describing the "instantaneous value of time", be continuous at $t = \tau$.

The necessary conditions proved in [4, 9] strongly suggest that Γ^* should be optimal for the optimization problem described at (1.8). In this direction, our main result is

Theorem 1. *The barrier Γ^* is optimal within the family of all admissible Jordan curves with finite length.*

In other words, if Γ is any simple closed curve with $m_1(\Gamma) < \infty$, which is admissible according to (1.5), then

$$J(\Gamma^*) \leq J(\Gamma). \tag{1.14}$$

Indeed, our analysis shows that the inequality in (1.14) is strict, except when Γ is the image of Γ^* by a rotation around the origin. Observe that, if $\kappa_1 > 0$, then any curve providing the minimum to (1.8) must have finite length.

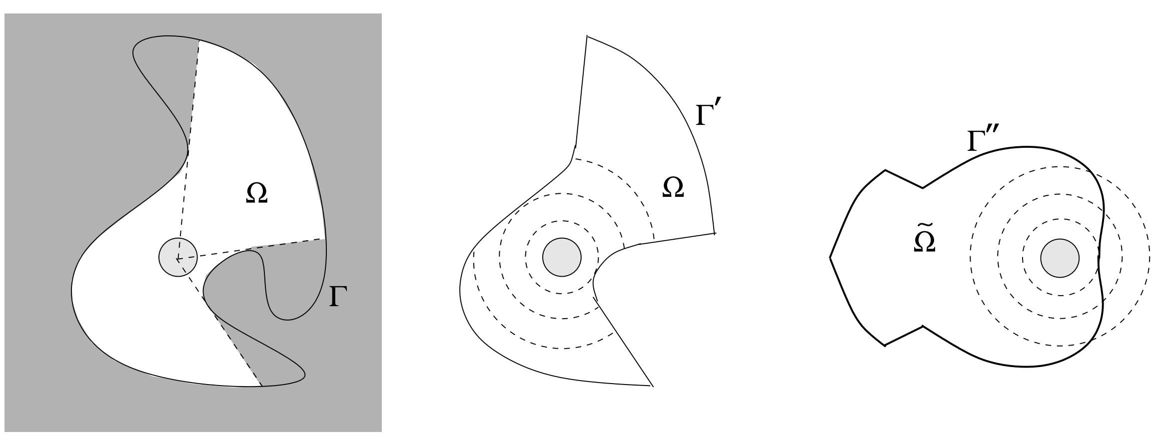

Figure 2: The three mains steps in the proof of Theorem 1.

The proof of Theorem 1 is accomplished in three steps, which are outlined below (Fig. 2).

STEP 1. Consider any admissible, simple closed curve Γ. Thinking of Γ as a wall which blocks the light, we let $\Omega \subset I\!R^2$ be the set of points illuminated by a light source located at the origin $0 \in I\!R^2$. As it will be proved in Section 3, the boundary $\Gamma' \doteq \partial\Omega$ is also an admissible, simple closed curve. Moreover, $J(\Gamma') \leq J(\Gamma)$.

STEP 2. By the previous construction, the domain Ω is star-shaped. Indeed, it can be represented in polar coordinates as

$$\Omega = \Big\{(r\cos\theta\,,\ r\sin\theta)\,;\quad r \leq r(\theta) \quad \theta \in [-\pi,\,\pi]\Big\} \tag{1.15}$$

for some (possibly discontinuous) function $\theta \mapsto r(\theta)$ having bounded variation. Letting $\theta \mapsto \tilde{r}(\theta)$ be the symmetric, non-decreasing rearrangement of the map $\theta \mapsto r(\theta)$, we obtain a symmetric domain

$$\widetilde{\Omega} = \Big\{(r\cos\theta\,,\ r\sin\theta)\,;\quad r \leq \tilde{r}(\theta) \quad \theta \in [-\pi,\,\pi]\Big\} \tag{1.16}$$

According to the analysis in Section 4, the boundary $\Gamma'' \doteq \partial\widetilde{\Omega}$ is an admissible, simple closed curve. Moreover, $J(\Gamma'') \leq J(\Gamma')$.

STEP 3. The curve Γ'' has sufficient regularity, so that the necessary conditions for optimality derived in [9] can finally be applied. Relying on these necessary conditions, in Section 4 we show that Γ'' can be optimal only if $\Gamma'' = \Gamma^*$, concluding the proof.

In the remainder of the paper, Section 2 introduces some further definitions and notations. The three main steps in the proof of Theorem 1 are then worked out in Sections 3, 4, and 5.

For an introduction to geometric measure theory and rectifiable sets we refer to [1]. The basic properties of Jordan curves and of simply connected sets in the plane, used in this paper, can be found in most books on algebraic topology, see for example [13, 16]. Regarding set-valued functions, and the Hausdorff distance in the family of compact sets, our basic reference is [3].

2 Preliminaries

Given an admissible barrier Γ, we define the corresponding **minimum time function** as

$$T^\Gamma(x) \doteq \inf\Big\{t \geq 0;\quad x \in \overline{R^\Gamma(t)}\Big\}. \tag{2.1}$$

The set of times where the constraint (1.5) is **saturated**, i.e. it is satisfied as an equality, will be denoted as

$$\mathcal{S} \doteq \Big\{t \geq 0;\quad m_1\Big(\Gamma \cap \overline{R^\Gamma(t)}\Big) = \sigma t\Big\}. \tag{2.2}$$

The **saturated** and the **free** portions of the barrier Γ are defined respectively as

$$\Gamma_{\mathcal{S}} \doteq \big\{x \in \Gamma;\quad T^\Gamma(x) \in \mathcal{S}\big\}, \qquad\qquad \Gamma_{\mathcal{F}} \doteq \big\{x \in \Gamma;\quad T^\Gamma(x) \notin \mathcal{S}\big\}. \tag{2.3}$$

For future use, we now prove

Lemma 2.1. *If Γ is a closed, admissible barrier, then the set $\mathcal{S}$ in (2.2) is closed.*

Proof. Consider a convergent sequence of times $t_n \to t^*$, with $t_n \in \mathcal{S}$ for every $n \geq 1$. We need to show that $t^* \in \mathcal{S}$. Observe that from this sequence we can always extract a monotone subsequence. We thus need to consider two cases.

CASE 1: The sequence is monotone increasing, so that $t_n \leq t_{n+1} \leq t^*$ for all $n \geq 1$.

In this case $R^\Gamma(t_n) \subseteq R^\Gamma(t_{n+1}) \subseteq R^\Gamma(t^*)$. Hence

$$\sigma t^* \;\geq\; m_1\Big(\overline{R^\Gamma(t^*)} \cap \Gamma\Big) \;\geq\; \limsup_{n\to\infty}\, m_1\Big(\overline{R^\Gamma(t_n)} \cap \Gamma\Big) \;=\; \limsup_{n\to\infty}\; \sigma t_n \;=\; \sigma t^*. \tag{2.4}$$

Of course, the first inequality holds because Γ is admissible. Therefore all terms in (2.4) are equal, and $t^* \in \mathcal{S}$.

CASE 2: The sequence is monotone decreasing, so that $t_n \geq t_{n+1} \geq t^*$ for all $n \geq 1$.

In this case $\overline{R^\Gamma(t_n)} \supseteq \overline{R^\Gamma(t_{n+1})} \supseteq \overline{R^\Gamma(t^*)}$ for every n. Hence $\overline{R^\Gamma(t^*)} \subseteq \bigcap_{n\geq 1} \overline{R^\Gamma(t_n)}$. We claim that the opposite inequality also holds:

$$\overline{R^\Gamma(t^*)} \;\supseteq\; \bigcap_{n\geq 1} \overline{R^\Gamma(t_n)}\,. \tag{2.5}$$

Indeed, assume $y \notin \overline{R^\Gamma(t^*)}$. Since this set is closed, the distance must be strictly positive:

$$\delta \;\doteq\; d\Big(y\,;\;\; \overline{R^\Gamma(t^*)}\Big) \;>\; 0$$

Recalling that the fire propagates with unit speed, we conclude

$$y \;\notin\; \overline{R^\Gamma(t)} \qquad\qquad \text{for all} \quad t < t^* + \delta\,.$$

Hence the point y is not contained in the right hand side of (2.5), and our claim is proved.

From(2.5) it follows

$$\sigma t^* \;\geq\; m_1\big(\overline{R^\Gamma(t_*)} \cap \Gamma\big) \;=\; \liminf_{n\to\infty}\, m_1\big(\overline{R^\Gamma(t_n)} \cap \Gamma\big) \;=\; \liminf_{n\to\infty}\; \sigma t_n \;=\; \sigma t^*,$$

hence $t^* \in \mathcal{S}$. □

3 A barrier bounding a star-shaped domain

Let Γ be an admissible barrier, consisting of a simple closed curve of length $\ell \doteq m_1(\Gamma)$. Notice that the open unit disc B_1 must be contained in the bounded component of $\mathbb{R}^2 \setminus \Gamma$. Otherwise the set R^Γ_∞ is unbounded and the cost $J(\Gamma)$ is infinite.

Let $s \mapsto \gamma(s)$ be a Lipschitz parameterization of Γ, with $s \in [0,1]$, $\gamma(0) = \gamma(1)$, and $|\dot\gamma(s)| = \ell$ for a.e. $s \in [0,1]$. It is convenient to extend γ by periodicity to the entire real axis, setting $\gamma(s) = \gamma(s+N)$ for every integer N. We shall denote by $(\rho(s), \theta(s))$ the polar coordinates of the point $\gamma(s) \in \mathbb{R}^2$. Thinking of a light source located at the origin, we define the illuminated region (see Fig. 2) as

$$\Omega \;\doteq\; \Big\{x \in I\!R^2\,;\;\; \lambda x \notin \Gamma \qquad \text{for all}\;\; 0 \leq \lambda < 1\Big\}\,. \tag{3.6}$$

The boundary of this region will be denoted by $\Gamma' \doteq \partial\Omega$. In this section we work toward a proof of the following properties:

(P1) Γ' is an admissible, simple closed curve.

(P2) $J(\Gamma') \leq J(\Gamma)$, with equality holding only if $\Gamma' = \Gamma$.

3.1 The shaded set

We define the **shaded set** as

$$Z \doteq \Big\{ s \in \mathbb{R}\,;\ \ \gamma(s) \notin \Omega \Big\} \ = \ \Big\{ s \in \mathbb{R}\,;\ \ \lambda\gamma(s) = \gamma(s') \ \ \text{for some} \ \ s' \in I\!R\,,\ 0 < \lambda < 1 \Big\}. \tag{3.7}$$

This is the set of parameter values s such that the point $\gamma(s)$ is in the shade.

Lemma 3.1. *The set Z is a countable union of disjoint nontrivial intervals.*

Proof. Since $Z \subset \mathbb{R}$, every connected component of Z is an interval, possibly reduced to a single point.

Suppose that Z contains a connected component consisting of a single point, say $\{s^*\}$. In this case, we can find two monotone sequences $s_k \to s^*$, $s'_k \to s^*$, with

$$s_k < s^* < s'_k\,, \qquad s_k \notin Z, \quad s'_k \notin Z \qquad \text{for all} \ \ k \geq 1\,. \tag{3.8}$$

The following steps show that this assumption leads to a contradiction.

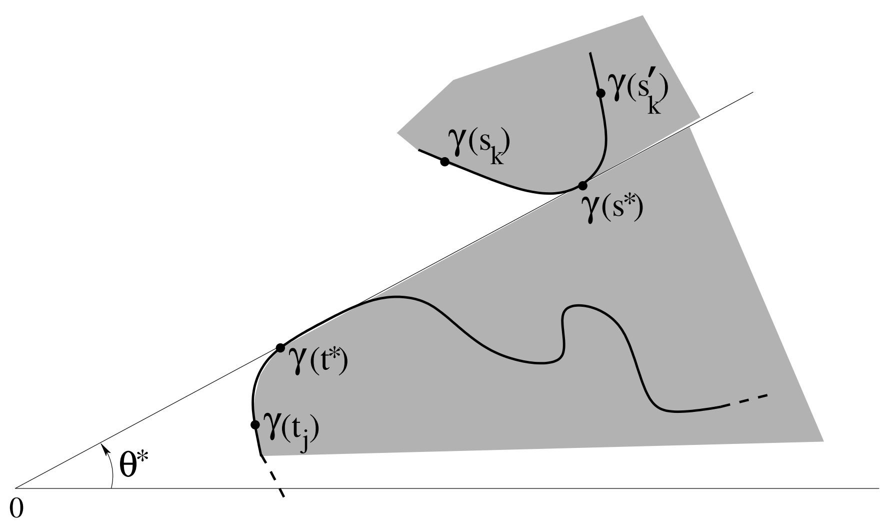

Figure 3: If $\gamma(s^*)$ is in the shade, then it must be part of a nontrivial arc of points $\gamma(s)$ all in the shade.

1. Call $(\rho(s), \theta(s))$ the polar coordinates of the point $\gamma(s)$ along the curve. Since $\gamma(s^*)$ is in the shade, as shown in Fig. 3 we can find $t^* \in [0,1] \setminus Z$ such that $\theta^* \doteq \theta(t^*) = \theta(s^*)$ and $\rho(t^*) < \rho(s^*)$. Observe that t^* cannot be in the interior of the set $\{s\,;\ \ \theta(s) = \theta^*\}$. Otherwise, since the map $s \mapsto \gamma(s)$ is one-to-one, the point $\gamma(t^*)$ would also be in the shade, contrary to the choice of t^*. We can thus assume that there exists a sequence $t_j \to t^*$ such that $\theta(t_j) < \theta^*$

for all $j \geq 1$ (the case $\theta(t_j) > \theta^*$ is entirely similar). Set $\delta \doteq \frac{\rho(s^*)-\rho(t^*)}{2} > 0$. Choosing t_j sufficiently close to t^*, by continuity we have $\rho(s) \leq \rho(t^*) + \delta$ for all $s \in [t^*, t_j]$. Hence all the points with polar coordinates (r, θ) such that

$$r > \rho(s^*) - \delta\,, \qquad\qquad \theta \in [\theta(t_j),\, \theta(t^*)]$$

lie in the shade. Hence there exists a neighborhood $\mathcal{N}$ of $P^* = \gamma(s^*)$ such that every point $\gamma(s) \in \mathcal{N}$ with $\theta(s) \leq \theta^*$ lies in the shade.

Since all points $\gamma(s_k)$, $\gamma(s'_k)$ are illuminated, we must have

$$\theta(s_k) \;>\; \theta^*\,, \qquad\qquad \theta(s'_k) \;>\; \theta^* \tag{3.9}$$

for all k sufficiently large.

2. For notational convenience we write $P^* \doteq \gamma(s^*)$. For $\varepsilon > 0$ sufficiently small, consider the two arcs $\gamma_1, \gamma_2 \subset \Gamma$, defined as

$$\gamma_1 \;\doteq\; \Big\{\gamma(s);\;\; s \in]s^* - \varepsilon,\, s^*[\,\Big\}, \qquad\qquad \gamma_2 \doteq \Big\{\gamma(s);\;\; s \in]s^*,\, s^* + \varepsilon[\,\Big\}. \tag{3.10}$$

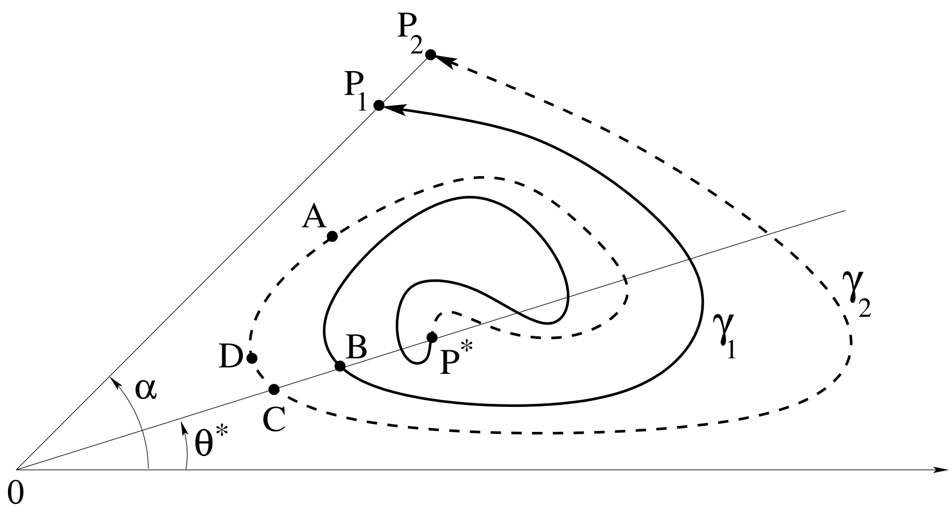

Figure 4: The construction used in the proof of Lemma 3.1.

Choose an angle $\alpha > \theta^*$ close to θ^*. In view of (3.9), the arcs γ_1 and γ_2 must both contain points along the ray

$$\vec{\zeta}_\alpha \;\doteq\; \Big\{(r\cos\alpha,\, r\sin\alpha)\,; \quad r \geq 0\Big\}\,,$$

provided that $\alpha - \theta^*$ is chosen sufficiently small.

For i=1,2, let $P_i \;\doteq\; \gamma(s_{P_i})$ be the first point where γ_i intersects the ray $\vec{\zeta}_\alpha$, as shown in Fig. 4. In other words, choose

$$s^* - \varepsilon \;<\; s_{P_1} \;<\; s^* \;< s_{P_2} \;<\; s^* + \varepsilon$$

such that

$$\theta(s_{P_1}) \;=\; \theta(s_{P_2}) \;=\; \alpha\,, \qquad \text{and} \qquad \theta(s) < \alpha \quad \text{for all } s \in]s_{P_1},\, s_{P_2}[\,.$$

To fix the ideas, assume that $\rho(s_{P_1}) < \rho(s_{P_2})$, the other case being entirely similar.

By (3.8) and (3.9), we can choose a point A on γ_2 with $A = \gamma(s_A) = \gamma(s'_j)$ for some $j \geq 1$, such that $\theta^* < \theta(s_A) < \alpha$. We then consider the point $B = \gamma(s_B)$ on γ_1 which satisfies

$$\rho(s_B) \;=\; \min\big\{\rho(s);\ \theta(s) = \theta^*,\ s \in [s_{P_1}, s^*]\big\}.$$

This choice of B implies that the union

$$\Sigma \;\doteq\; \overline{OP_1} \cup \overline{OB} \cup \big\{\gamma(s);\quad s \in [s_{P_1}\,,\ s_B]\big\}. \tag{3.11}$$

is a Jordan curve. Let U be the open, bounded connected component of $I\!R^2 \setminus \Sigma$, so that $\Sigma = \partial U$. We observe that $A \in U$ while $P_2 \notin \overline{U}$. Therefore, the connected arc

$$\omega_{AP_2} \;\doteq\; \{\gamma_2(s)\,;\quad s \in [s_A\,,\, s_{P_2}]\} \;\subseteq\; \gamma_2$$

must intersect the boundary $\partial U = \Sigma$ at some point C. Since $\gamma_2 \cap \overline{OP_1} \;=\; \gamma_2 \cap \gamma_1 = \emptyset$, this intersection must lie on the segment $\overline{OB}$. Choose the intersection point $C \;\doteq\; \gamma(s_C)$ such that

$$s_C \;\doteq\; \min\Big\{s \in [s_A,\, s_{P_2}];\ \ \theta(s) = \theta^* \ \text{ and }\ \rho(s) \;<\; \rho(s_B)\Big\}. \tag{3.12}$$

By (3.12) we can now choose a new point $D = \gamma(s_D)$ sufficiently close to C, such that

$$s_D < s_C\,, \qquad \theta(s_D) > \theta^*, \qquad \rho(s) < \rho(s^*) \quad \text{for all } s \in [s_D, s_C].$$

Hence there exists a neighborhood $\mathcal{N}$ of $P^* = \gamma(s^*)$ such that every point $\gamma(s) \in \mathcal{N}$ with $\theta(s) \geq \theta^*$ lies in the shade produced by the arc $\gamma_{CD} = \{\gamma(s)\,;\quad s \in [s_D, s_C]\}$.

Together with the previous step, this implies that an entire neighborhood of P^* lies in the shade. Hence s^* is an interior point of Z, contrary to our initial assumption. This contradiction proves the lemma. □

The next lemma shows that each connected component of Z is precisely a half-open interval.

Lemma 3.2. *Let I be a maximal interval contained in Z, with closure $\overline{I} \;=\; [s_a,\, s_b]$. Then*

$$\theta(s_a) \;=\; \theta(s_b). \tag{3.13}$$

Furthermore, I is half-open. Indeed,

$$\begin{array}{lcl} \rho(s_a) \;<\; \rho(s_b) & \Longrightarrow & I \;=\;]s_a\,,\, s_b]\,, \\ \rho(s_a) \;>\; \rho(s_b) & \Longrightarrow & I \;=\; [s_a\,,\, s_b[\,. \end{array} \tag{3.14}$$

Proof. 1. Assume that, on the contrary, $\theta(s_a) \neq \theta(s_b)$. Choose $s'_a \leq s_a < s_b \leq s'_b$ such that $s'_a, s'_b \notin Z$, with $|s'_a - s_a|$ and $|s'_b - s_b|$ sufficiently small. Consider the points

$$P_a = \gamma(s_a), \quad P_b = \gamma(s_b), \quad P'_a = \gamma(s'_a), \quad P'_b = \gamma(s'_b).$$

As in Fig. 5, call Q_a, Q_b respectively the intersections of the unit circumference with the segments $\overline{OP'_a}, \overline{OP'_b}$. These points divide the unit circumference into two sub-arcs, which we call ω^+, ω^-. We shall also split the original curve Γ as

$$\Gamma \;\doteq\; \Gamma' \cup \Gamma'', \qquad \Gamma' \;\doteq\; \Big\{\gamma(s)\,;\quad s'_a \leq s \leq s'_b\Big\}\,, \qquad \Gamma'' = \Gamma \setminus \Gamma'.$$

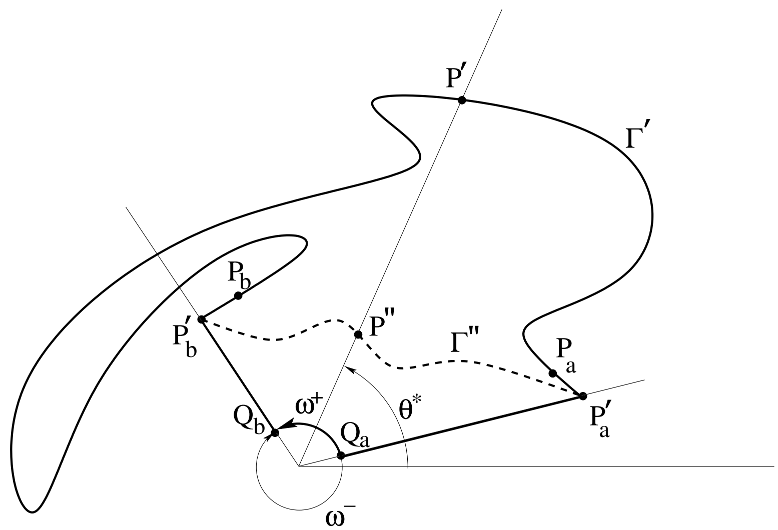

Figure 5: The construction used in the proof of (3.13) in Lemma3.2. In this case, one would choose $\tilde{\omega} = \omega^+$.

Consider the curve

$$\Gamma^\sharp \doteq \overline{P'_a Q_a} \cup \tilde{\omega} \cup \overline{P'_b Q_b} \cup \Gamma'$$

where we choose either $\tilde{\omega} = \omega^+$ or $\tilde{\omega} = \omega^-$. Observe that, regardless of the choice of the arc $\tilde{\omega}$, the curve $\Gamma^\sharp$ is a simple, closed curve. We now choose the arc $\tilde{\omega} \in \{\omega^+, \omega^-\}$ so that the origin remains in the unbounded component of $I\!R^2 \setminus \Gamma^\sharp$.

Let θ^* be the angle corresponding to the mid-point of the arc $\tilde{\omega}$. We claim that the arc Γ' must contain a point $P' = \gamma(s')$ with $\theta(s') = \theta^*$. Indeed, in the opposite case, by our choice of $\tilde{\omega}$ all points along the ray

$$\Big\{(r\cos\theta^*,\, r\sin\theta^*)\,;\quad r \geq 0\Big\} \tag{3.15}$$

(except for $r = 1$) would belong to the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma^\sharp$. But this is impossible, because $\Gamma^\sharp$ is a Jordan curve, and intersects the ray (3.15) exactly once. This proves the claim.

If $|s'_a - s_a|$ and $|s'_b - s_b|$ were chosen sufficiently small, then this intersection point $P' = \gamma(s')$ satisfies $s_a < s' < s_b$ and hence $s' \in Z$. Since P' is in the shade, there exists a point in the light, say $P'' = \gamma(s'') \in \Gamma''$, with

$$s'' \notin Z, \qquad \theta(s'') = \theta(s') = \theta^*, \qquad \rho(s'') < \rho(s').$$

Next, observe that Γ'' contains the point P'' and does not intersect $\Gamma^\sharp$. Therefore, it is entirely contained in the bounded component of $\mathbb{R}^2 \setminus \Gamma^\sharp$. We conclude that the bounded component of $\mathbb{R}^2 \setminus \Gamma = \mathbb{R}^2 \setminus (\Gamma' \cup \Gamma'')$ is contained in the bounded component of $\mathbb{R}^2 \setminus \Gamma^\sharp$, hence it does not contain the origin. This is a contradiction, proving (3.13).

2. To prove (3.14), assume that $\rho(s_a) < \rho(s_b)$, the other case being similar. Then by (3.13) the point P_b is in the shade of the point P_a. Hence $s_b \in Z$.

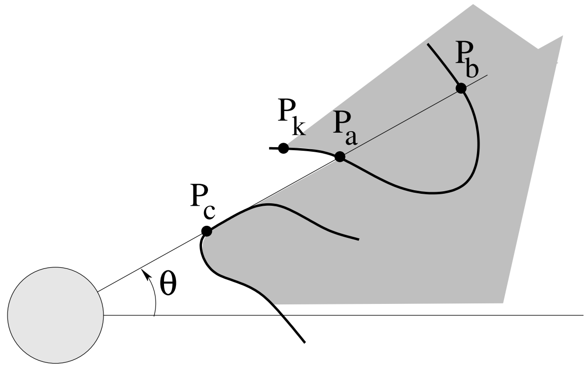

Figure 6: The construction used in the proof of (3.14) in Lemma 3.4.

Finally, assume that $s_a \in Z$. A contradiction is then obtained as follows.

Choose $s_c \notin Z$, such that $\theta(s_c) = \theta(s_a) \doteq \theta_a$ and

$$\rho(s_c) \;=\; \min\Big\{\rho(s)\,;\;\; \theta(s) = \theta(s_a)\Big\}.$$

As in Fig. 6, we set $P_c \doteq \gamma(s_c)$. Then, as s ranges in a small neighborhood of s_c, we can assume that the angle $\theta(s)$ covers a right neighborhood of the angle θ_a (the case of a left neighborhood is entirely similar). As shown in (Fig. 6), all points $\gamma(s)$ in a neighborhood of $\gamma(s_a)$ with $\theta(s) \leq \theta_a$ thus lie in the shade.

On the other hand, by assumption, there exists a sequence $s_k \to s_a$ with $s_k \notin Z$. This is possible only if $\theta(s_k) > \theta_a$. By continuity, there exists k large enough and $\delta > 0$ such that

$$\rho(s) \;<\rho(s_b) - \delta \qquad \text{for all } s \in [s_k, s_a]\,.$$

Therefore, all points $\gamma(s)$ in a neighborhood $\mathcal{N}$ of P_b with $\theta(s) > \theta_a$ lie in the shade. But we already know that points with $\theta(s) \leq \theta_a$ also lie in the shade. Hence $\gamma(s) \in Z$ for all s in a neighborhood of s_b, against the assumptions. This completes the proof of the lemma. □

3.2 Cutting out a shaded portion

Let Γ be an admissible, simple closed curve, parameterized as $s \mapsto \gamma(s)$. Recalling (3.6)-(3.7), assume that

$$\gamma^\sharp \doteq \Big\{\gamma(s);\;\; s \in \,]s_0,\, s_1]\Big\} \;\subset\; \Gamma \tag{3.16}$$

is a maximal arc contained in the shaded region. One can then replace $\gamma^\sharp$ with the segment $\gamma^\flat \doteq \overline{P_0P_1}$ connecting the two endpoints $P_0 = \gamma(s_0)$, $P_1 = \gamma(s_1)$. Aim of this section is to prove

Lemma 3.3. *In the above setting, the curve*

$$\Gamma^\flat \;\doteq\; \gamma^\flat \cup (\Gamma \setminus \gamma^\sharp) \tag{3.17}$$

is an admissible, simple closed curve. Moreover

$$J(\Gamma^\flat) \;\leq\; J(\Gamma) \tag{3.18}$$

with equality holding only if $\Gamma^\flat = \Gamma$.

Proof. 1. If $\gamma^\sharp$ is itself a segment, then $\Gamma^\flat = \Gamma$ and the conclusion is trivial. Otherwise, it s clear that

$$m_1(\gamma^\flat) \ < \ m_1(\gamma^\sharp).$$

Moreover, the global burned areas satisfy $R^\Gamma_\infty \ \subset \ R^{\Gamma^\flat}_\infty$. The difference in the areas is computed as

$$m_2(R^\Gamma_\infty) \ = \ m_2(R^{\Gamma^\flat}_\infty) + m_2(W)\,,$$

where W is the union of all bounded components of $\mathbb{R}^2 \setminus (\gamma^\sharp \cup \gamma^\flat)$. This proves (3.18).

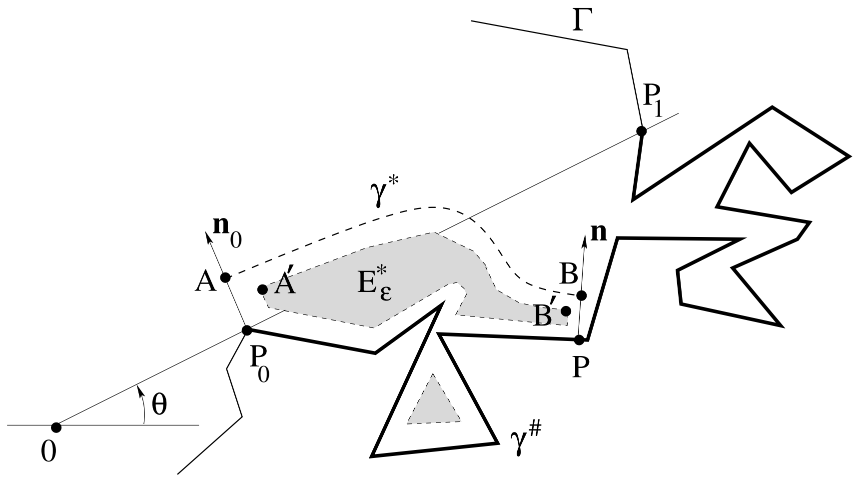

Figure 7: The construction used to show that $\Gamma^\flat$ is admissible.

2. It remains to prove that $\Gamma^\flat$ is admissible. Relying on the fact that the original curve Γ is admissible, it suffices to show that

$$m_1\Big(\Gamma^\flat \cap \overline{R^{\Gamma^\flat}(t)}\Big) \ \leq \ m_1\Big(\Gamma \cap \overline{R^{\Gamma}(t)}\Big) \ \leq \ \sigma\, t \qquad \text{for all } t \geq 0\,. \tag{3.19}$$

As described in Lemma 3.2, in the present situation we have $\rho(s_0) = |P_0| \ < \ |P_1| \ = \ \rho(s_1)$. Define the times

$$t_0 \ \doteq \ \rho(s_0) - 1 \qquad\qquad t_1 \ \doteq \ \rho(s_1) - 1\,.$$

For $t < t_0$, the reachable sets $R^\Gamma(t)$ and $R^{\Gamma^\flat}(t)$ coincide. Indeed, their closure does not intersect $\gamma^\sharp$, nor $\gamma^\flat$.

For $t \geq t_0$, the portion of the segment $\gamma^\flat$ touched by the fire has length

$$m_1\Big(\gamma^\flat \cap \overline{R^{\Gamma^\flat}(t)}\Big) \ = \ \min\{t - t_0\,,\ t_1 - t_0\}\,. \tag{3.20}$$

To prove (3.19) it thus suffices to show that

$$m_1\Big(\gamma^\sharp \cap \overline{R^{\Gamma}(t)}\Big) \ \geq \ t - t_0 \qquad\qquad \text{for all } t \in [t_0, t_1]\,. \tag{3.21}$$

Let $\zeta \mapsto \gamma^\sharp(\zeta)$ be an arc-length parameterization of the curve $\gamma^\sharp$, starting at the point $\gamma^\sharp(0) = P_0$. We claim that

$$\gamma^\sharp_\tau \ \doteq \ \Big\{\gamma^\sharp(\zeta)\,;\ \ \zeta \in [0,\tau]\Big\} \ \subset \ \overline{R^\Gamma(t_0+\tau)} \qquad \text{whenever} \quad 0 < \tau < t_1 - t_0\,. \tag{3.22}$$

In other words, given any point $P = \gamma^\sharp(\tau)$, there exists a curve Σ of length $\leq t_0 + \tau$ (a trajectory for the fire) which starts inside the unit disc R_0, does not cross Γ, and reaches a point arbitrarily close to P. If the curve $\gamma^\sharp$ is smooth, this fact would be obvious. Indeed, one could define Σ as the union of a segment from R_0 to a point close to P_0, together with a curve of points having constant distance $\varepsilon > 0$ from $\gamma^\sharp$ (Fig. 7). In general, however, we only know that $\gamma^\sharp$ is a Lipschitz continuous curve. In this case, the set of points at a small distance $\varepsilon > 0$ from $\gamma^\sharp$ may not be a connected curve. A more careful analysis is thus needed.

Since the map $\zeta \mapsto \gamma^\sharp(\zeta)$ is Lipschitz continuous, it is differentiable for a.e. ζ. By slightly moving the point P along the curve, we can thus assume that the tangent and the normal vectors to the curve $\gamma^\sharp$ are well defined at the point P. As in Fig. 7, we denote by $\mathbf{n}$ the unit normal to $\gamma^\sharp$ at P, oriented toward the interior of the set Ω enclosed by Γ. We recall that Γ is a Jordan curve, and Ω is the unique bounded connected component of $\mathbb{R}^2 \setminus \Gamma$. Moreover, we denote by $\mathbf{n}_0$ the unit vector perpendicular to the segment $\overline{P_0P_1}$, oriented so that, at P_0, it points toward the interior of Ω. By choosing $\varepsilon_0 > 0$ sufficiently small, the two points

$$A \doteq P_0 + \varepsilon_0 \mathbf{n}_0\,, \qquad\qquad B \doteq P + \varepsilon_0 \mathbf{n}$$

are both contained in the interior of Ω. Since the open set Ω is connected, there exists a smooth path γ^*, entirely contained inside Ω, with endpoints A, B. Consider the curve

$$\Sigma \doteq \overline{P_0A} \cup \gamma^* \cup \overline{PB} \cup \{\gamma^\sharp(\zeta)\,;\quad 0 \leq \zeta \leq \tau\}\,. \tag{3.23}$$

Notice that the last portion of Σ is the portion of the arc $\gamma^\sharp$ from P_0 to P. By construction, Σ is a Jordan curve. Call E the bounded connected component of $\mathbb{R}^2 \setminus \Sigma$. By Theorem 5 in [2] on the one-sided Minkowski content, one has

$$\lim_{\varepsilon\to 0+} m_1\Big(\{x \in E\,;\quad d(x, \Sigma) = \varepsilon\}\Big) = m_1(\partial E) = m_1(\Sigma) = \varepsilon_0 + m_1(\gamma^*) + \varepsilon_0 + \tau\,. \tag{3.24}$$

In general, the open set

$$E_\varepsilon \doteq \{x \in E\,;\quad d(x, \Sigma) > \varepsilon\}$$

has many connected components. Since γ^* is smooth, there will be one connected component, which we call E^*_ε, whose boundary contains a smooth curve γ^*_ε of points remaining at a distance ε from γ^*. Let A', B' be the endpoints of γ^*_ε. Since both E^*_ε and its complement are connected, the boundary ∂E^*_ε is a Jordan curve. Therefore, one can start from A' and reach B', moving along the curve ∂E^*_ε in two different ways: either along the curve γ^*_ε, or along the complement $\partial E^*_\varepsilon \setminus \gamma^*_\varepsilon$. In the second case the total length traveled is

$$m_1(\partial E^*_\varepsilon) - m_1(\gamma^*_\varepsilon) \leq m_1(\partial E_\varepsilon) - m_1(\gamma^*_\varepsilon) = m_1(\Sigma) - m_1(\gamma^*) + o(1) = 2\varepsilon_0 + \tau + o(1)\,, \tag{3.25}$$

where the Landau symbol $o(1)$ denotes a quantity that tends to zero as $\varepsilon \to 0$,

To complete the proof of (3.22), it only remains to observe that the point A can be connected to the unit disc R_0 by a segment of length $< |P_0| + \varepsilon_0$, without crossing Γ. Moreover, the segments $\overline{AA'}$ and $\overline{BB'}$ have lengths which approach zero as $\varepsilon_0, \varepsilon \to 0$.

In turn, (3.22) imples (3.21), completing the proof of the lemma. □

3.3 A star-shaped domain

As proved in Lemma 3.2, the set of points $P \in \Gamma$ lying in the shade can be expressed as the union of its connected components:

$$\Big\{\gamma(s)\,;\quad s\in\bigcup_{j\geq 1} I_j\Big\},$$

where the $I_j \subset [0,1]$ form a countable family of disjoint half-open intervals: either $I_j = [a_j, b_j[$, or $I_j =]a_j, b_j]$.

For $k \geq 0$, let Γ_k be the Jordan curve obtained by replacing the first k arcs $\{\gamma(s);\quad s\in I_j\}$, $1 \leq j \leq k$, by the segments

$$S_j \;\doteq\; \Big\{\theta\gamma(a_j)+(1-\theta)\gamma(b_j)\,;\qquad \theta\in[0,1]\Big\} \tag{3.26}$$

having the same endpoints. By induction on k, from Lemma 3.3 it follows that curve Γ_k is admissible. Moreover

$$m_1(\Gamma_{k+1})\;\leq\; m_1(\Gamma_k)\;\leq\; m_1(\Gamma)\,,\qquad\qquad \Omega_{k+1}\;\subseteq\;\Omega_k\;\subseteq\;\Omega \tag{3.27}$$

for every $k \geq 1$, with equalities holding if and only if the sets are equal. The Hausdorff distance between the above curves satisfies

$$\sum_{k\geq 1} d_H(\Gamma_k,\,\Gamma_{k-1})\;\leq\;\sum_{k\geq 1} m_1\Big(\{\gamma(s)\,;\quad s\in[a_k,b_k]\}\Big)\;\leq\; m_1(\Gamma)\,.$$

Therefore, the sequence of compact sets $(\Gamma_k)_{k\geq 0}$ is Cauchy w.r.t. the Hausdorff distance. Hence it admits a unique limit:

$$\Gamma_\infty \;\doteq\; \lim_{k\to\infty}\Gamma_k\,. \tag{3.28}$$

By Golab's theorem (see Theorem 3.18 in [14]), it follows

$$m_1(\Gamma_\infty)\;=\;\lim_{k\to\infty}\; m_1(\Gamma_k)\;\leq\; m_1(\Gamma)\,. \tag{3.29}$$

We will show that Γ_∞ coincides with the set

$$\Gamma' \;\doteq\; (\Gamma\cap\Omega)\;\cup\;\Big(\bigcup_{j\geq 1} S_j\Big). \tag{3.30}$$

In other words, Γ' is defined as the illuminated portion of Γ, together with all the segments S_j. A further characterization of $\Gamma\infty$ will be given in terms of the set

$$\Omega^o \;\doteq\; \Big\{x\in I\!R^2\,;\quad \lambda x\notin\Gamma\quad\text{for all } 0\leq\lambda<1\Big\}. \tag{3.31}$$

From the definition, it is clear that Ω^o is star shaped, and its closure contains the illuminated set Ω in (3.6). We observe that Ω^o is open. Indeed, assume that $x \in \Omega$ and the segment

$$S_x \;\doteq\; \big\{\lambda x\,;\quad \lambda\in[0,1]\big\} \tag{3.32}$$

does not intersect Γ. Then, by the compactness of S_x and Γ, there exists $\delta > 0$ such that every point $y \in S_x$ has distance $\geq \delta > 0$ from Γ. This implies $B(x,\delta) \subset \Omega^o$. We conclude that Ω^o is open, and coincides with the interior of Ω. The next lemma relates the above sets, defined at (3.28), (3.30), and (3.31).

Lemma 3.4. *One has the identities $\Gamma' = \Gamma_\infty = \partial\Omega^o$. Moreover, Γ' is an admissible, simple closed curve.*

Proof. 1. As a first step, we prove $\Gamma' \subseteq \Gamma_\infty$. Indeed, if x is a point in the illuminated portion of Γ, then $x \in \Gamma_k$ for every k, hence $x \in \Gamma_\infty$. On the other hand, if $x \in S_j$, then $x \in \Gamma_k$ for every $k \geq j$. This again implies $x \in \Gamma_\infty$.

2. In this step we show that $\Gamma_\infty \subseteq \partial\Omega^o$. Indeed, if $x \in \Gamma_\infty$, there exists a sequence of ponts $x_k \in \Gamma_k$ such that $x_k \to x$ as $k \to \infty$. We consider two cases.

CASE 1: There exists an index j such that $x_k \in S_j$ for infinitely many k. In this case, $x \in S_j$, and hence $x \notin \Omega^o$. To fix the ideas, assume that the segment S_j has endpoints $\gamma(a_j)$, $\gamma(b_j)$, with Euclidean norm $1 \leq |\gamma(a_j)| < |\gamma(b_j)|$. Let $x = \lambda\gamma(b_j)$ for some $\lambda \in [0,1]$. Since the interval $]a_j, b_j]$ is a maximal interval in the shaded set, there exists a sequence of illuminated points $\gamma(s_n)$ with $s_n \to b_j$. For all $n \geq 1$, we now have

$$y_n \doteq \left(\lambda - \frac{1}{n}\right)\gamma(s_n) \in \Omega^o$$

Moreover, $y_n \to x$. Hence $x \in \partial\Omega^o$.

CASE 2: There exists a subsequence such that $x_k \notin S_1 \cup \cdots \cup S_{n(k)}$, with $n(k) \to \infty$ as $k \to \infty$. In this case, we claim that we can find a second sequence of illuminated points $y_k = \gamma(s_k) \in \Gamma \cap \Omega$ such that $|y_k - x_k| \to 0$. Indeed:

(i) If $x_k \in \Gamma \cap \Omega$, we simply take $y_k = x_k$.

(ii) If $x_k \in S_j$ for some $S_{n(k)} < j \leq k$, we take $y_k = \gamma(a_j)$ or $y_k = \gamma(b_j)$. By Lemma 3.2, exactly one of these two points is illuminated. Then $|y_k - x_k| \leq m_1(S_j)$. Since $j \to \infty$ as $k \to \infty$, the length of the segment S_j approaches zero.

(iii) If $x_k = \gamma(s)$ for some $s \in [a_j, b_j]$ with $j > k$, we again take $y_k = \gamma(a_j)$ or $y_k = \gamma(b_j)$. Then $|y_k - x_k| \leq m_1\big(\{\gamma(s);\ s \in [a_j, b_j]\}\big)$. Since $j \to \infty$ as $k \to \infty$, this length approaches zero.

Defining

$$z_k \doteq \frac{k-1}{k}\, y_k \in \Omega^o,$$

we now have

$$x = \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = \lim_{k\to\infty} z_k\,.$$

Hence $x \in \Gamma \cap \partial\Omega^0$.

3. We now show that $\partial\Omega^o \subseteq \Gamma'$. Let $x \in \partial\Omega^o$. Two cases will be considered.

CASE 1: x is an illuminated point, i.e., $x \in \Omega$. If $x \notin \Gamma$, then $x \in \Omega^o$, against the assumption. Therefore $x \in \Gamma \cap \Omega \subseteq \Gamma'$.

CASE 2: $x \notin \Omega$. In this case, taking $\lambda \doteq \min\{\lambda' ; \quad \lambda' x \in \Gamma\}$, we determine a point $y = \gamma(\bar s) \in \Gamma \cap \Omega$ such that $y = \lambda x$, for some $0 < \lambda < 1$.

As in Section 3, we denote by $(\rho(s), \theta(s))$ be the polar coordinates of the point $\gamma(s)$. If the map $s \mapsto \theta(s)$ does not have a local extremum at $s = \bar s$, then for every $\varepsilon > 0$ the set $\{\theta(s) ; \quad s \in [\bar s - \varepsilon, \, \bar s + \varepsilon]\}$ covers a whole neighborhood of $\theta(\bar s)$, hence there exists an entire neighborhood of x consisting of points in the shade. This contradicts the assumption $x \in \partial\Omega^o$. Therefore, $\bar\theta \doteq \theta(\bar s)$ is either a local minimum or a local maximum of the angular function $\theta(\cdot)$. Hence, $\bar s$ is in the boundary of the shaded set Z in (3.7). Using Lemma 3.1, we can assume that $\bar s = a_j$, where $]a_j, b_j]$ is a maximal interval contained in the shaded set Z, and $1 \leq |\gamma(a_j)| < |\gamma(s_j)|$, the other case being entirely similar.

Since we already know that $\gamma(a_j) = \lambda x$ for some $\lambda < 1$, to prove that $x \in S_j$ it remains to show that $|x| \leq |\gamma(b_j)|$. To fix the ideas, let a_j be a point where the map $s \mapsto \theta(s)$ attains a local maximum; the case of a local minimum is entirely similar. Given $\varepsilon > 0$, there exists $\delta > 0$ such that all points $z = r(\cos\theta, \sin\theta)$ with $r > |\gamma(a_j)| + \varepsilon$ and $\theta \in [\bar\theta - \delta, \, \bar\theta]$ are in the shade. On the other hand, by the maximality of the interval $]a_j, b_j]$, there exist a sequence of illuminated points $\gamma(s_n) \in \Gamma \cap \Omega$, with $s_n \downarrow b_j$. By the previous analysis, we must have $\theta(s_n) > \bar\theta$. By continuity of the maps $s \mapsto (\rho(s), \theta(s))$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that all points $z = r(\cos\theta, \sin\theta)$ with $r > |\gamma(b_j)| + \varepsilon$ and $\theta \in [\bar\theta, \, \bar\theta + \delta]$ are in the shade. Hence, if $|x| > |\gamma(b_j)|$, we conclude that x has a whole neighborhood consisting of points in the shade, against the assumption $x \in \partial\Omega^o$.

4. Together, the three previous steps prove the identities $\Gamma' = \Gamma_\infty = \partial\Omega^o$. Being the boundary of the simply connected, star shaped set Ω^o, it is clear that $\Gamma' = \Gamma_\infty$ is a simple closed curve. It remains to prove that Γ' is admissible.

Call Ω_k the open region enclosed by the simple closed curve Γ_k, i.e. the bounded connected component of $I\!R^2 \setminus \Gamma_k$. Observe that $R^{\Gamma'}(t)$ is the set reached at time t by trajectories of the differential inclusion

$$|\dot x| \;\leq\; 1\,, \qquad\qquad |x(0)| \;<\; 1\,, \tag{3.33}$$

which remain inside Ω^o at all times. On the other hand, $R^{\Gamma_k}(t)$ is the set reached at time t by trajectories of (3.33) which remain inside Ω_k. Since $\Omega^o \subseteq \Omega_k$ for every k, we clearly have

$$R^{\Gamma'}(t) \;\subseteq\; R^{\Gamma_k}(t) \qquad\qquad \text{for all } t \geq 0.$$

For every $t \geq 0$ and $k \geq 1$, we can write

$$m_1(\Gamma' \cap \overline{R^{\Gamma'}(t)}) \;\;\leq\;\; m_1(\Gamma' \cap \overline{R^{\Gamma_k}(t)}) \;\;\leq\;\; m_1(\Gamma_k \cap \overline{R^{\Gamma_k}(t)}) + m_1(\Gamma' \setminus \Gamma_k)$$

$$\leq\;\; \sigma t + \sum_{j>k} m_1(S_j)\,.$$

Letting $k \to \infty$ the sum on the right hand side approaches zero, proving the admissibility of the closed curve Γ'. □

Being the boundary of a star-shaped set, the simple closed curve Γ' can be represented in polar coordinates as

$$\Gamma' \;=\; \Big\{(r\cos\theta\,,\; r\sin\theta)\,; \quad r^-(\theta) \leq r \leq r^+(\theta)\,, \quad \theta \in [-\pi,\, \pi]\Big\}, \tag{3.34}$$

for some suitable functions r^-, r^+. Here r^- is lower semicontinuous, while r^+ is upper semicontinuous. Moreover $r^-(\theta) = r^+(\theta)$ for all but countably many values of θ. The total variation of these functions on $[-\pi, \pi]$ satisfies

$$\text{Tot.Var.}\{r^-(\cdot)\} \;=\; \text{Tot.Var.}\{r^+(\cdot)\} \;\leq\; m_1(\Gamma'). \tag{3.35}$$

4 Symmetric rearrangement

Based on the analysis of the previous section, we can now assume that the curve $\Gamma = \Gamma'$ is already the boundary of a star-shaped, open domain

$$\Omega^o \;=\; \Big\{(r\cos\theta,\, r\sin\theta)\,;\quad 0 \;\leq\; r \;<\; r(\theta)\,,\quad \theta \in [-\pi,\, \pi]\Big\}\,,$$

for some lower semicontinuous BV function $\theta \mapsto r(\theta)$, with $r(-\pi) = r(\pi)$. To simplify the notation, we are here taking $r(\theta) \doteq r^-(\theta)$, dropping the superscript.

In this section, we consider the new domain

$$\widetilde{\Omega} \;=\; \Big\{(r\cos\theta,\, r\sin\theta)\,;\quad 0 \;\leq\; r \;<\; \tilde{r}(\theta)\Big\}\,,$$

where $\tilde{r}(\cdot)$ is the symmetric, nondecreasing rearrangement of the function $r^-(\cdot)$. In other words, $\tilde{r} : [-\pi, \pi] \mapsto I\!\!R_+$ is the unique lower semicontinuous function with the properties

$$meas\Big(\{\theta\,;\;\; \tilde{r}(\theta) < \rho\}\Big) \;=\; meas\Big(\{\theta\,;\;\; r^-(\theta) < \rho\}\Big) \qquad \text{for all } \rho > 0\,, \tag{4.1}$$

$$\tilde{r}(-\theta) \;=\; \tilde{r}(\theta)\,, \qquad \tilde{r}(\theta) \;\leq\; \tilde{r}(\theta') \qquad \text{for all} \quad 0 \leq \theta \leq \theta' \leq \pi. \tag{4.2}$$

From (4.1) one immediately obtains

$$m_2(\widetilde{\Omega}) \;=\; \int_0^\infty \rho\cdot meas\Big(\{\theta\,;\;\; \tilde{r}(\theta) > \rho\}\Big)\, d\rho \;=\; \int_0^\infty \rho\cdot meas\Big(\{\theta\,;\;\; r^-(\theta) > \rho\}\Big)\, d\rho \;=\; m_2(\Omega^o)\,. \tag{4.3}$$

To prove that $\widetilde{\Gamma} \doteq \partial\widetilde{\Omega}$ is an admissible curve, we need to show that, for every $t > 0$ one has

$$m_1\Big(\widetilde{\Gamma} \cap \overline{B}(0, 1+t)\Big) \;\leq\; m_1\Big(\Gamma \cap \overline{B}(0, 1+t)\Big) \;\leq\; \sigma t\,. \tag{4.4}$$

Notice that the second inequality in (4.4) follows from the assumption that Γ is admissible. The first inequality apparently cannot be deduced directly from general theorems on symmetric rearrangements in [11, 15]. However, the ideas involved in the proof are quite similar. We start with an elementary inequality.

Lemma 4.1. *Given any numbers $b_1, \ldots, b_n$, with $n \geq 2$, one has*

$$\sum_{i=1}^n \sqrt{1 + b_i^2} \;\;\geq\;\; 2\sqrt{1 + \left(\frac{\sum_{i=1}^n b_i}{2}\right)^2}\,. \tag{4.5}$$

Proof. Applying Jensen's inequality to the convex function $\varphi(s) = \sqrt{1+s^2}$ we obtain

$$\sum_{i=1}^n \sqrt{1 + b_i^2} \;\;\geq\;\; n\sqrt{1 + \left(\frac{\sum_{i=1}^n b_i}{n}\right)^2} \;\;\geq\;\; m\sqrt{1 + \left(\frac{\sum_{i=1}^n b_i}{m}\right)^2} \tag{4.6}$$

whenever $n \geq m$. Taking $m = 2$, one obtains (4.5). □

Using the lemma, the first inequality in (4.4) will be proved first in the case where $r(\cdot)$ is a piecewise affine function, then in the general case, using an approximation argument.

1. Assume that the function $\theta \mapsto r(\theta)$ is continuous, piecewise affine, and satisfies

$$r(-\pi) \;=\; r(\pi)\,, \qquad\qquad \frac{dr}{d\theta}(\theta) \;\neq\; 0 \quad \text{for a.e. } \theta\,. \tag{4.7}$$

For each $\rho > 0$, consider the sets

$$\Theta(\rho) \;\doteq\; \Big\{\theta \in [-\pi,\pi]\,;\;\; r(\theta) = \rho\Big\}, \qquad\qquad \widetilde\Theta(\rho) \;\doteq\; \Big\{\theta \in [-\pi,\pi]\,;\;\; \tilde r(\theta) = \rho\Big\}.$$

By the assumption (4.7), every set $\widetilde\Theta(\rho)$ contains at most two elements, say $\tilde\theta(\rho)$, $-\tilde\theta(\rho)$, while $\Theta(\rho)$ contains at most finitely many elements, say $\theta_1(\rho),\ldots,\theta_N(\rho)$, with N also depending on ρ. Writing

$$\frac{d\theta}{dr}(\theta) \;=\; \left(\frac{dr}{d\theta}(\theta)\right)^{-1}, \qquad\qquad \frac{d\theta}{d\tilde r}(\theta) \;=\; \left(\frac{d\tilde r}{d\theta}(\theta)\right)^{-1},$$

the assumption that $\tilde r(\cdot)$ is a rearrangement of $r(\cdot)$ implies

$$\left|\frac{d\theta}{d\tilde r}(\tilde\theta)\right| \;=\; \left|\frac{d\theta}{d\tilde r}(-\tilde\theta)\right| \;=\; \frac{1}{2}\sum_{i=1}^{N}\left|\frac{d\theta}{dr}(\theta_i)\right|. \tag{4.8}$$

Using (4.5) with $b_i \doteq \rho\left|\frac{d\theta}{dr}(\theta_i)\right|$, the two expressions in (4.4) can now be estimated as

$$\begin{aligned}
m_1\Big(\widetilde\Gamma\cap B(0,1+t)\Big) \;&=\; \int_0^{1+t}\sum_{\theta\in\widetilde\Theta(\rho)}\sqrt{1+\rho^2\left|\frac{d\theta}{d\tilde r}(\theta)\right|^2}\,d\rho \;=\; \int_0^{1+t} 2\sqrt{1+\rho^2\left|\frac{d\theta}{d\tilde r}(\tilde\theta(\rho))\right|^2}\,d\rho\\
&\leq\; \int_0^{1+t}\sum_{i=1}^{N(\rho)}\sqrt{1+\rho^2\left|\frac{d\theta}{dr}(\theta_i(\rho))\right|^2}\,d\rho \;\leq\; \int_0^{1+t}\sum_{\theta\in\Theta(\rho)}\sqrt{1+\rho^2\left|\frac{dr}{d\theta}(\theta(\rho))\right|^2}\,d\rho\\
&=\; m_1\Big(\Gamma\cap\overline B(0,1+t)\Big) \;\leq\; \sigma t\,.
\end{aligned} \tag{4.9}$$

2. To cover the general case, let $\theta\mapsto r(\theta)$ be a lower semicontinuous function with bounded variation. Then there exists a sequence of piecewise affine functions $r_n(\cdot)$ converging to $r(\cdot)$ at a.e. $\theta\in[-\pi,\pi]$ and such that, calling

$$\Gamma_n \;\doteq\; \Big\{r_n(\theta)(\cos\theta,\,\sin\theta)\,;\;\; \theta\in[-\pi,\,\pi]\Big\}, \tag{4.10}$$

there holds

$$m_1\Big(\Gamma_n\cap\overline B(0,\,1+t)\Big) \;\leq\; \sigma t + \frac{1}{n}\,, \tag{4.11}$$

for every $t\geq 0$ and $n\geq 1$. Call $\tilde r_n(\cdot)$ the non-decreasing symmetric rearrangement of $r_n(\cdot)$, and consider the curve $\widetilde\Gamma_n$, defined as in (4.10) with r_n replaced by $\tilde r_n$. By the previous step we have

$$m_1\Big(\widetilde\Gamma_n\cap\overline B(0,\,1+t)\Big) \;\leq\; \sigma t + \frac{1}{n}\,, \tag{4.12}$$

for every $t \geq 0$ and $n \geq 1$.

To pass to the limit as $n \to \infty$, we proceed as follows. Let $f : [-\pi, \pi] \mapsto I\!R_+$ be a lower semicontinuous function, with $f(-\pi) = f(\pi)$. Let $Df = \mu^{ac} + \mu^s$ be its distributional derivative, decomposed as the sum of an absolutely continuous measure with density f' w.r.t. Lebesgue measure, plus a singular part. Consider the functional

$$\Phi(f) \doteq \int_{-\pi}^{\pi} \sqrt{f^2(\theta) + (f')^2(\theta)}\, d\theta + |\mu^s|\Big(]-\pi, \pi]\Big). \tag{4.13}$$

The second term on the right hand side of (4.13) denotes the total mass of the singular measure μ^s. Observe that, if $r = f(\theta)$ is the polar coordinate representation of a smooth curve $\Gamma \subset I\!R^2$, then $\Phi(f) = m_1(\Gamma)$ yields the total length of the curve. More generally, if f is a lower semicontinuous BV function and we choose $r^-(\theta) = f(\theta)$, $r^+(\theta) = \limsup_{\theta' \to \theta} f(\theta')$, then $\Phi(f)$ yields the length of the curve Γ' in (3.34).

For a given radius $\rho > 0$, the portion of Γ' contained inside the open ball $B(0, \rho)$ can be expressed as

$$m_1(\Gamma' \cap B(0, \rho)) = \Phi(f \wedge \rho) - \rho \cdot m_1\Big(\{\theta\,;\;\; f(\theta) \geq \rho\}\Big).$$

We use here the truncated function $(f \wedge \rho)(\theta) \doteq \min\{f(\theta), \rho\}$. In particular, for any $t \geq 0$ and $\varepsilon > 0$ one has

$$\begin{aligned} m_1\Big(\widetilde{\Gamma} \cap \overline{B}(0, 1+t)\Big) &\leq m_1\Big(\widetilde{\Gamma} \cap B(0, 1+t+\varepsilon)\Big) \\ &= \Phi\Big(\tilde{r} \wedge (1+t+\varepsilon)\Big) - (1+t+\varepsilon) \cdot m_1\Big(\{\theta\,;\;\; \tilde{r}(\theta) \geq 1+t+\varepsilon\}\Big), \end{aligned} \tag{4.14}$$

By the lower semicontinuity of the integral functional Φ (see [12] or section 5.5 in [1]), it follows

$$\Phi\Big(\tilde{r} \wedge (1+t+\varepsilon)\Big) \leq \liminf_{n \to \infty} \Phi\Big(\tilde{r}_n \wedge (1+t+\varepsilon)\Big) \leq \liminf_{n \to \infty} \Phi\Big(\tilde{r}_n \wedge (1+t+2\varepsilon)\Big). \tag{4.15}$$

By construction, for every n, t, ε we have

$$\begin{aligned} &\Phi\Big(\tilde{r}_n \wedge (1+t+2\varepsilon)\Big) - (1+t+2\varepsilon) \cdot m_1\Big(\{\theta\,;\;\; \tilde{r}_n(\theta) \geq 1+t+2\varepsilon\}\Big) \\ &= m_1\Big(\Gamma_n \cap B(0,\, 1+t+2\varepsilon)\Big) \leq m_1\Big(\Gamma_n \cap \overline{B}(0,\, 1+t+2\varepsilon)\Big) \leq \sigma(t+2\varepsilon) + \frac{1}{n}. \end{aligned} \tag{4.16}$$

Using (4.14)–(4.16) and the pointwise a.e. convergence $\tilde{r}_n(\theta) \to \tilde{r}(\theta)$, we conclude

$$\begin{aligned} m_1\Big(\widetilde{\Gamma} \cap \overline{B}(0, 1+t)\Big) \leq \liminf_{n \to \infty} &\left(\sigma(t+2\varepsilon) + \frac{1}{n}\right) \\ &+ (1+t+2\varepsilon) \cdot \limsup_{n \to \infty} m_1\Big(\{\theta\,;\;\; \tilde{r}_n(\theta) \geq 1+t+2\varepsilon\}\Big) \\ &- (1+t+\varepsilon) \cdot m_1\Big(\{\theta\,;\;\; \tilde{r}(\theta) \geq 1+t+\varepsilon\}\Big) \\ \leq\; &\sigma(t+2\varepsilon) + \varepsilon \cdot 2\pi. \end{aligned} \tag{4.17}$$

Since $\varepsilon > 0$ was arbitrary, this proves the admissibility of the curve $\widetilde{\Gamma}$. □

5 The optimal barrier

In this final section we consider an admissible curve $\Gamma = \partial\Omega$, enclosing a set of the form

$$\Omega \doteq \Big\{(\rho\cos\theta,\, \rho\sin\theta)\,;\quad 0\leq\rho<r(\theta)\,,\quad \theta\in[-\pi,\pi]\Big\}.$$

We assume that the map $\theta\mapsto r(\theta)$ satisfies

$$r(-\theta) \;=\; r(\theta)\,, \qquad r(\theta)\;\leq\; r(\theta') \qquad \text{for all}\quad 0\leq\theta\leq\theta'\leq\pi. \tag{5.1}$$

We claim that, if Γ is optimal, i.e. if it minimizes the cost functional (1.8) among all admissible curves, then Γ must consist of the concatenation of an arc of circumference and two arcs of logarithmic spirals.

Indeed, consider the set of times where the admissibility condition (1.5) is satisfied as an equality:

$$\mathcal{S} \;\doteq\; \Big\{t\geq 0\,;\quad m_1(\Gamma\cap\overline{B}(0,\,1+t)) = \sigma t\Big\}. \tag{5.2}$$

By Lemma 2.1, this set is closed.

Assume that the open interval $]\tau_a,\,\tau_b[$ is a maximal connected set contained in the complement of $\mathcal{S}$, with $t_a>0$. This means that

$$m_1(\Gamma\cap\overline{B}(0,\,1+\tau_a)) \;=\; \sigma\tau_a\,, \tag{5.3}$$

$$m_1(\Gamma\cap\overline{B}(0,\,1+\tau_b)) \;=\; \sigma\tau_b, \tag{5.4}$$

$$m_1(\Gamma\cap\overline{B}(0,\,1+t)) \;<\; \sigma t \qquad \text{for all } t\in\,]\tau_a,\,\tau_b[\,. \tag{5.5}$$

By the necessary conditions for optimality proved in [4, 9], the restriction of Γ to the set of points

$$\{x\,;\quad T^\Gamma(x)\in\,]\tau_a,\,\tau_b[\} \;=\; \{x\,;\quad 1+\tau_a<|x|<1+\tau_b\}$$

consists of two arcs of circumferences, say with endpoints A_1, B_1 and A_2, B_2, as shown in Fig. 8.

However, this is impossible. Indeed, call Q the point where the circumference is tangent to a ray from the origin. From (5.3) and (5.5) it follows that, at the point A_1, the angle α between the arc $\widehat{A_1B_1}$ and the circumference $\{x\in I\!R^2\,;\quad |x|=|A_1|\}$ must satisfy

$$\frac{\sigma}{2}\sin\alpha \;\geq\; 1. \tag{5.6}$$

However, if (5.6) holds, there exists no point B_1 along the arc $\widehat{A_1Q}$ where (5.4) is satisfied.

The previous analysis shows that, for the optimal barrier Γ, the set $\mathcal{S}$ in (5.2) is connected. Therefore, Γ must be obtained as the concatenation of an arc of circumference, and two logarithmic spirals.

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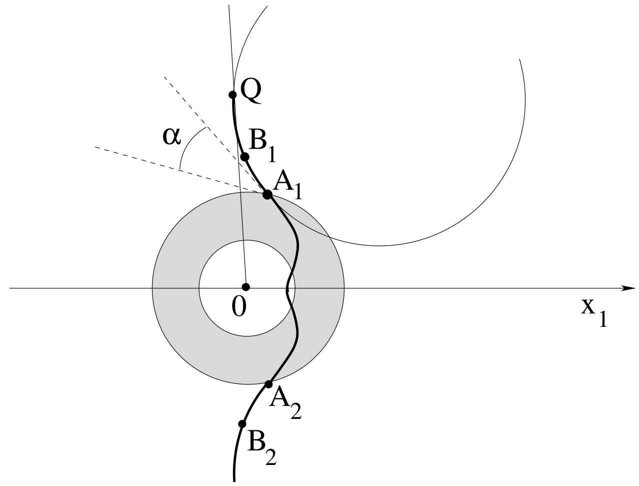

Figure 8: The shaded area corresponds to points $P \in I\!R^2$ with $|P| - 1 \in \mathcal{S}$. If Γ is an optimal curve, then the set $\mathcal{S} \subset [0, \infty[$ of times where the constraint is saturated must be an interval.

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