Differential Inclusions and the Control of Forest Fires

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(Dedicated to Arrigo Cellina in the occasion of his 65-th birthday)

Abstract. This paper introduces a new class of variational problems for differential inclusions, motivated by the control of forest fires. The area burned by the fire at time $t > 0$ is modelled as the reachable set for a differential inclusion $\dot{x} \in F(x)$, starting from an initial set R_0. To block the fire, a wall can be constructed progressively in time, at a given speed. In this paper, we study the possibility of constructing a wall which completely encircles the fire. Moreover, we derive necessary conditions for an optimal strategy, which minimizes the total area burned by the fire.

1 - Introduction

Aim of this paper is to analyze a new type of mathematical problems, motivated by the control of forest fires. More generally, our model describes the spatial spreading of a contaminating agent, which a controller wishes to block by constructing a barrier, in real time.

At each time $t \geq 0$, we denote by $R(t) \subset I\!R^2$ the burned (or contaminated) region. If the controller takes no action, the time evolution of the set $R(t)$ will be modelled in terms of a differential inclusion. More precisely, consider a Lipschitz continuous multifunction $F : I\!R^2 \mapsto I\!R^2$ with compact, convex values, and a bounded set $R_0 \subset I\!R^2$. At any given time $t \geq 0$, the contaminated set $R(t)$ is defined as the reachable set for the differential inclusion

$$\dot{x} \in F(x) \qquad\qquad x(0) \in R_0\,, \tag{1.1}$$

where the upper dot denotes a derivative w.r.t. time. In other words,

$$R(t) = \Big\{ x(t)\,;\quad x(\cdot) \ \text{ absolutely continuous}\,,\quad x(0) \in R_0\,,\quad \dot{x}(\tau) \in F\big(x(\tau)\big) \ \text{ for a.e. } \tau \in [0, t] \Big\}\,.$$

For a comprehensive introduction to the theory of differential inclusions we refer to the classic monograph [1]. Throughout this paper, we shall assume that

$$0 \in F(x) \qquad\qquad \text{for all } x \in I\!R^2, \tag{1.2}$$

which implies

$$R(t_1) \subseteq R(t_2) \qquad\qquad \text{whenever} \quad t_1 \leq t_2\,. \tag{1.3}$$

We now assume that the spreading of the contamination can be controlled by erecting walls, or barriers. In the case of a forest fire, one may think of a thin strip of land which is either soaked with water poured from above (by an airplane or a helicopter), or cleared from all vegetation using a bulldozer. In any case, this will prevent the fire from crossing that particular strip of land.

We thus assume that the controller can construct a “wall”, i.e. a one-dimensional rectifiable curve, which blocks the spreading of the fire. In more precise mathematical terms, consider a continuous, strictly positive function $\psi : I\!R^2 \mapsto R_+$, and call $\gamma(t)$ the portion of the wall constructed within time $t \geq 0$.

Definition 1. A set valued map $t \mapsto \gamma(t) \subset I\!R^2$ is an **admissible strategy** if the following conditions hold.

(H1) For every $t_1 \leq t_2$ one has $\gamma(t_1) \subseteq \gamma(t_2)$.

(H2) Each $\gamma(t)$ is a rectifiable curve. Denoting by m_1 the one-dimensional Hausdorff measure, there holds

$$\int_{\gamma(t)} \psi \, dm_1 \leq t \qquad \text{for all } t \geq 0\,. \tag{1.4}$$

The assumption (H1) means that walls cannot be moved, once constructed. According to (1.4), we think of $1/\psi(x)$ as the speed at which a wall can be constructed, at the location x. In the special case where $\psi(x) \equiv 1/\sigma$ for some constant σ, the condition (1.4) simply means that

$$\big[\text{ total length of the wall } \gamma(t) \text{ constructed within time } t\,\big] \;\leq\; \sigma\, t\,. \tag{1.5}$$

Notice that we never require that the set $\gamma(t)$ be connected. For example, $\gamma(t)$ may be the union of countably many Lipschitz continuous arcs. In the case where $\gamma(t)$ consists of two arcs, the bound (1.5) is satisfied if, for example, the length of one arc increases at the rate $\sigma/3$ and the length of the other arc increases at the rate $2\sigma/3$ (length per unit time).

By constructing walls, the controller can now reduce the size of the contaminated set. Namely, the reachable set determined by the blocking strategy γ is defined as

$$R^\gamma(t) \doteq \Big\{ x(t)\,;\;\; x(\cdot) \text{ absolutely continuous}\,,\;\; x(0) \in R_0\,,\;\; \dot x(\tau) \in F\big(x(\tau)\big) \text{ for a.e. } \tau \in [0,t]\,, \\ x(\tau) \notin \gamma(\tau) \text{ for all } \tau \in [0,t] \Big\}\,. \tag{1.6}$$

In other words, $R^\gamma(t)$ is the set reached by trajectories of the differential inclusion (1.1) which, at any given time τ, do not cross the previously constructed walls.

To define an optimization problem, we now introduce a cost functional. In general, this should take into account:

- The value of the area destroyed by the fire.

- The cost of building the wall.

We thus consider two continuous, non-negative functions $\alpha, \beta : I\!R^2 \mapsto I\!R_+$ and define the following functional

$$J(\gamma) = \lim_{t\to\infty} \left\{ \int_{R^\gamma(t)} \alpha \, dm_2 + \int_{\gamma(t)} \beta \, dm_1 \right\}\,. \tag{1.7}$$

Here dm_2 indicates integration w.r.t. 2-dimensional Lebesgue measure, while dm_1 refers to 1-dimensional measure. In the case of a fire, $\alpha(x)$ is the value of a unit area of land around the point x, while $\beta(x)$ is the cost of building a unit length of wall near the point x. We recall that, for $s \leq t$, one has $R^\gamma(s) \subseteq R^\gamma(t)$ and $\gamma(s) \leq \gamma(t)$. Taking the limit as $t \to \infty$ is thus the same as taking the supremum over all $t \geq 0$. The right hand side of (1.7) accounts for the value of the entire burned area plus the total cost of building the walls. In a typical situation, we expect that at a finite time $T > 0$ the fire will be entirely encircled by walls. In this case, $R^\gamma(t) = R^\gamma(T)$ and $\gamma(t) = \gamma(T)$ for all $t > T$. This means that both fire propagation and wall construction will stop after time T.

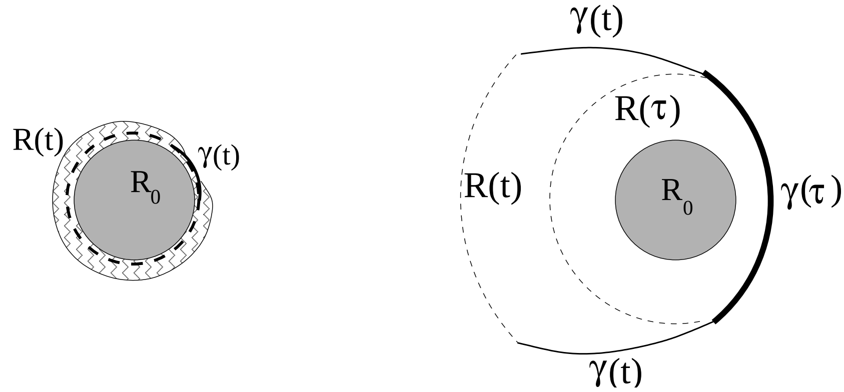

Figure 1: The wall has to be constructed at the same time as the contaminated set expands.

Remark 1. The fact that the wall has to be constructed "in real time" is an essential feature of the present model. For example, assume that the fire is initially burning on the unit disc $B_1 \subset I\!R^2$ and propagates with unit speed in all directions (fig. 1). Referring to (1.1), this means that $F(x) \equiv B_1$, $R_0 = B_1$. The shortest curve that entirely encircles the fire (at time $t = 0$) would be the circumference $\Gamma = \{x \in I\!R^2\,;\ |x| = 1\}$, as in fig. 1, left. However, this does not yield a good strategy. Before even a portion of this curve is constructed, the fire invades a larger disc, and it will no longer be confined by the wall. A better confinement strategy is illustrated in fig. 1, right. Here the fire is always confined to one side of the wall.

In the above setting, the following questions arise naturally:

1. Can the spreading of the fire be confined ?

Recalling (1.1) and (1.4), we thus seek conditions on the multifunction F and on the wall construction speed $1/\psi$ which imply the existence (or non-existence) of an admissible strategy $t \mapsto \gamma(t)$ such that

$$R^\gamma(t) \subseteq B_r \qquad \text{for all } t > 0\,, \tag{1.8}$$

for some fixed ball B_r centered at the origin with radius r.

2. Does there exists an optimal confinement strategy ?

This leads to the problem of proving the existence of an optimal solution γ^* to the minimization problem

$$\min_{\gamma \in \mathcal{S}} J(\gamma)\,, \tag{1.9}$$

where $\mathcal{S}$ is the set of all admissible strategies.

3. What are the properties of optimal strategies, and how can they be found ?
As in the case of optimal control problems, to answer this question one should study:
(i) Necessary conditions for the optimality of an admissible strategy $\gamma(\cdot)$.
(ii) Regularity conditions for the curves $\gamma(t)$ constructed by an optimal strategy.
(iii) Sufficient conditions for the optimality of a strategy $\gamma(\cdot)$.

We emphasize that the variational problem (1.9) is substantially different from standard problems in optimal control or in the Calculus of Variations. Indeed, assigning the curve which supports the wall is not sufficient to determine an optimal strategy. The order in which various portions of the wall are constructed, as time progresses, also plays an essential role. In particular, as admissible strategies we cannot just take functions $u : [0,T] \mapsto I\!R^N$, or curves $\gamma \subset I\!R^2$. Throughout the paper, the strategies that we consider will thus be set-valued functions $t \mapsto \gamma(t) \subset I\!R^2$, satisfying (H1)-(H2).

In the remainder of this paper we initiate the analysis of the above problems 1 and 3. A partial answer to the first question will be provided in Section 3, while in the last section we derive some necessary conditions for optimality. The existence of optimal solutions will be examined in a forthcoming paper.

2 - Preliminary remarks

We begin with some easy but useful comparison results. Consider two functions $\psi, \tilde{\psi} > 0$ respectively. Call $\mathcal{S}, \widetilde{\mathcal{S}}$ the corresponding sets of admissible strategies $t \mapsto \gamma(t)$ satisfying (H1)-(H2). We then have the obvious implication

$$\psi \leq \tilde{\psi} \qquad \Longrightarrow \qquad \mathcal{S} \supseteq \widetilde{\mathcal{S}}\,. \tag{2.1}$$

Next, consider two multifunctions $F, \widetilde{F}$ and two initial sets R_0, $\widetilde{R}_0$. If $R_0 \subseteq \widetilde{R}_0$ and $F(x) \subseteq \widetilde{F}(x)$ for every x, then for every blocking strategy $t \mapsto \gamma(t)$ the corresponding contaminated sets satisfy

$$R^\gamma(t) \subseteq \widetilde{R}^\gamma(t) \qquad \text{for all } t \geq 0\,. \tag{2.2}$$

Finally, we observe that our problem has a natural group of rescalings. Namely, given $\lambda > 0$, define

$$\widetilde{F}(x) = \lambda\, F(x) = \big\{\lambda\, y\,;\;\; y \in F(x)\big\}\,, \qquad\qquad \tilde{\psi}(x) = \lambda^{-1}\psi(x)\,. \tag{2.3}$$

Referring to the condition (1.4), if a strategy $t \mapsto \gamma(t)$ is ψ-admissible, then the rescaled strategy

$$t \mapsto \tilde{\gamma}(t) = \gamma(\lambda t)$$

is $\tilde{\psi}$-admissible, with obvious meaning of notation. Moreover, the corresponding reachable sets are related as

$$\widetilde{R}^{\tilde{\gamma}}(t) = R^\gamma(\lambda t)\,. \tag{2.4}$$

The next result shows that every admissible strategy γ can be replaced by another strategy $\hat{\gamma}$ for which equality holds in (1.4).

Lemma 1. *Let a continuous function $\psi > 0$ be given, and let $t \mapsto \gamma(t)$ be an admissible strategy. Given any $T > 0$, there exists a second admissible strategy $\hat{\gamma}$ such that, for every $t \in [0, T]$,*

$$\gamma(t) \subseteq \hat{\gamma}(t)\,, \tag{2.3}$$

$$\int_{\hat{\gamma}(t)} \psi \, dm_1 = t\,. \tag{2.4}$$

Proof. The lemma will be established in three steps.
1. Consider the scalar function

$$h(t) \doteq \int_{\gamma(t)} \psi \, dm_1\,. \tag{2.5}$$

By assumption, $t \mapsto h(t)$ is non-decreasing and satisfies $h(t) \leq t$ for every t. As a preliminary, we observe that it is not restrictive to assume $h(T) = T$. Indeed, in the opposite case we can simply adjoin another smooth curve to the set $\gamma(T)$ and obtain a rectifiable set $\gamma^*(T) \supset \gamma(T)$ such that

$$\int_{\gamma^*(T)} \psi \, dm_1 = T\,.$$

Moreover, we can also assume that the map $t \mapsto h(t)$ is right continuous. Otherwise, we replace each set $\gamma(t)$ with

$$\gamma^*(t) \doteq \bigcap_{s>t} \gamma(s)\,.$$

2. As an intermediate step, given any $\varepsilon > 0$, we prove that there exists an admissible strategy γ^ε such that, for every $t \in [0, T]$,

$$\gamma(t) \subseteq \gamma^\varepsilon(t)\,, \tag{2.6}$$

$$t - \varepsilon \leq \int_{\gamma^\varepsilon(t)} \psi \, dm_1 \leq t\,. \tag{2.7}$$

The rectifiable sets $\gamma^\varepsilon(t)$ are constructed as follows. Choose times $0 = t_0 < t_1 < \cdots < t_N = T$, with $t_i - t_{i-1} \leq \varepsilon$ for every $i = 1, \ldots, N$. Moreover, define the function

$$\tau(t) \doteq \inf\ \big\{s \in [0, T]\,;\quad h(s) \geq t\big\}\,. \tag{2.9}$$

We can now choose N rectifiable sets $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \Gamma_1 \subset \cdots \subset \Gamma_{N-1}$ such that, for $i = 0, \ldots, N-1$,

$$\bigcup_{t<\tau(t_i)} \gamma\big(\tau(t)\big) \subseteq \Gamma_i \subseteq \gamma\big(\tau(t_i)\big)\,, \tag{2.10}$$

$$\int_{\Gamma_i} \psi \, dm_1 = t_i\,. \tag{2.11}$$

Defining the sets

$$\gamma^\varepsilon(t) \doteq \gamma(t) \cup \Gamma_i \qquad\qquad t \in [t_i\,,\ t_{i+1}\,[\,, \tag{2.12}$$

the conditions (2.6)-(2.7) are satisfied.

3. Fix any decreasing sequence $\varepsilon_n \to 0$. By induction, using the previous step we can construct a sequence of set-valued maps $t \mapsto \gamma_n(t)$ such that

$$\gamma(t) = \gamma_0(t) \subseteq \gamma_1(t) \subseteq \gamma_2(t) \subseteq \cdots,$$

and, for every $n \geq 1$,

$$t - \varepsilon_n \leq \int_{\gamma_n(t)} \psi \, dm_1 \leq t.$$

Taking

$$\hat{\gamma}(t) \doteq \bigcup_{n \geq 1} \gamma_n(t)$$

the conclusions of the lemma are clearly satisfied. □

3 - Existence of confining strategies

In this section we derive necessary conditions and sufficient conditions for the existence of an admissible strategy $t \mapsto \gamma(t)$ which confines the fire within a bounded set. We begin with the special case where

$$\psi(x) \equiv 1/\sigma, \qquad F(x) \equiv B_1, \qquad R_0 = B_1. \tag{3.1}$$

As usual, $B_1 \subset I\!R^2$ denotes the closed unit disc centered at the origin. In other words, the fire propagates with unit speed in all directions, and the wall can be constructed at speed σ.

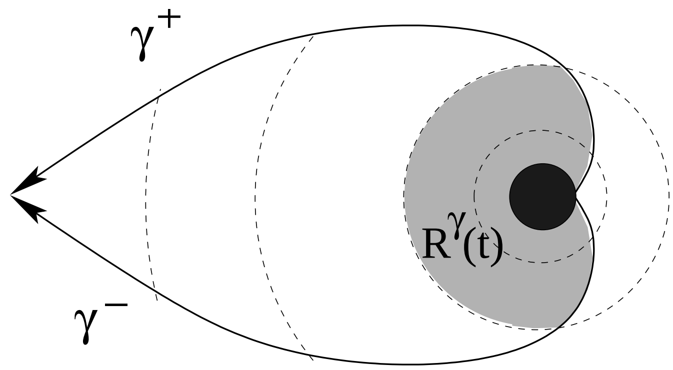

Figure 2: the contaminated set is enclosed by two arcs of logarithmic spirals.

Lemma 2. *Let (3.1) hold. If $\sigma > 2$, then there exists an admissible strategy $t \mapsto \gamma(t)$ which confines all sets $R^\gamma(t)$ within a bounded region.*

Proof. For each $t > 0$, the curve $\gamma(t)$ will be defined as the union of two arcs of logarithmic spirals (fig. 2). Introduce the positive constant

$$\lambda \doteq \left(\frac{\sigma^2}{4} - 1\right)^{-1/2},$$

so that

$$\sigma = \frac{2\sqrt{1+\lambda^2}}{\lambda}\,.$$

Using polar coordinates r, θ, we then define

$$\gamma(t) \doteq \gamma^+(t) \cup \gamma^-(t)\,, \tag{3.2}$$

$$\begin{aligned} \gamma^+(t) &\doteq \Big\{(r,\theta)\,;\quad r = e^{\lambda\theta}\,,\quad 1 \leq r \leq 1+t\Big\}\,,\\ \gamma^-(t) &\doteq \Big\{(r,\theta)\,;\quad r = e^{-\lambda\theta}\,,\quad 1 \leq r \leq 1+t\Big\}\,. \end{aligned} \tag{3.3}$$

We claim that both $\gamma^+(t)$ and $\gamma^-(t)$ have length $\sigma t/2$. Indeed, this length is computed by

$$\begin{aligned} \int_0^{\lambda^{-1}\cdot\ln(1+t)} \sqrt{r^2(\theta)+\dot r^2(\theta)}\,d\theta &= \int_0^{\lambda^{-1}\cdot\ln(1+t)} e^{\lambda\theta}\cdot\sqrt{1+\lambda^2}\,d\theta\\ &= \sqrt{1+\lambda^2}\cdot\frac{1}{\lambda}\Big[(1+t)-1\Big] = \frac{\sigma\,t}{2}\,. \end{aligned}$$

Therefore, the assignment $t \mapsto \gamma(t)$ is an admissible strategy.

From the definitions (3.2)-(3.3), it is clear that the contaminated sets $R^\gamma(t)$ remain always on the inner side of the spirals. When $\theta = \pm\pi$, i.e. at the time

$$T = e^{\lambda\pi} - 1\,,$$

the two arcs of spirals join together at the point $(r,\theta) = (1+T,\,\pi)$. Moreover, the union of the two arcs is the entire boundary of the contaminated set:

$$\gamma(T) = \gamma^+(T) \cup \gamma^-(T) = \partial R^\gamma(T)\,.$$

Hence $R^\gamma(t) = R^\gamma(T)$ for all $t \geq T$. This completes the proof. □

We now assume that $\sigma < 1$, meaning that the wall can be constructed at a speed smaller than the propagation speed of the fire. In this case, we prove that the fire cannot be confined to a bounded region, by any admissible strategy. At first sight, this result might seem trivial: if we construct a wall next to the burning region, the fire will instantly engulf both sides of the wall, making our effort completely useless. However, one can first construct a portion of the wall away from the fire, hoping to slow down its advance at a later time. Showing that even this second strategy cannot work is the main content of the following lemma.

Lemma 3. *Let (3.1) hold. If $\sigma < 1$, then there exists no admissible strategy $t \mapsto \gamma(t)$ which confines all reachable sets $R^\gamma(t)$ to a bounded region.*

Proof. We establish the lemma in several steps.

1. Let any admissible strategy $t \mapsto \gamma(t)$ be given. By Lemma 1, we can assume that $m_1\big(\gamma(t)\big) = \sigma t$ for all $t \geq 0$. To show that this strategy cannot confine the fire to a bounded region, we will construct a nondecreasing, continuous function $t \mapsto r(t)$, with

$$0 \leq \dot r(t) \leq 1\,, \qquad \lim_{t\to\infty} r(t) = \infty\,, \tag{3.4}$$

such that the following holds:

> For every $t \geq 0$, the portion of circumference of radius $r(t)$ which lies within the set $R^\gamma(t)$ reached by the fire has total length $\geq (5\pi/3)\, r(t)$. More precisely
>
> $$m_1\Big(\partial B_{r(t)} \cap R^\gamma(t)\Big) \geq \frac{5\pi}{3} r(t)\,. \tag{3.5}$$

Here ∂B_r denotes the boundary of the disc centered at the origin with radius r, while m_1 denotes 1-dimensional measure. To establish our claim, we introduce the non-negative functions

$$U(t) \doteq m_1\Big(\partial B_{r(t)} \setminus R^\gamma(t)\Big), \qquad\qquad W(t) \doteq m_1\Big(\gamma(t) \setminus B_{r(t)}\Big).$$

Notice that $U(t)$ is the length of the portion of the circumference $\partial B_{r(t)}$ still not reached by the fire at time t, while $W(t)$ is the total length of all walls in $\gamma(t)$ which lie outside $B_{r(t)}$.

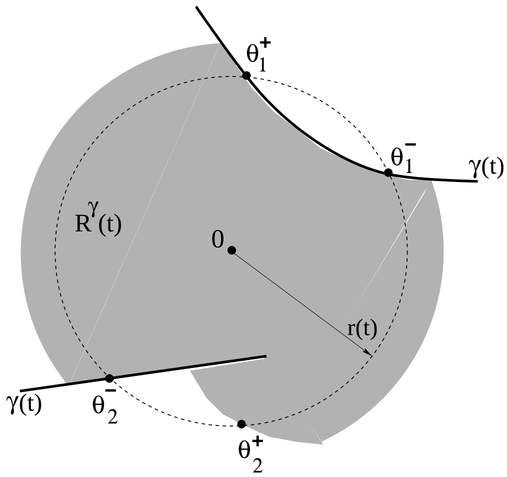

Figure 3: The set $\partial B_{r(t)} \setminus R^\gamma(t)$ here consists of two arcs.
The endpoints at angles θ_1^-, θ_1^+ and θ_2^- lie on portions of the wall $\gamma(t)$, while θ_2^+ is free.

The proof will be achieved by showing that, for every $t \geq 0$,

$$\Phi(t) \doteq \sin\frac{U(t)}{2r(t)} + \frac{W(t)}{2r(t)} \leq \frac{1}{2}\,. \tag{3.6}$$

Of course, this implies

$$\frac{U(t)}{r(t)} \leq 2 \arcsin\frac{1}{2} = \frac{\pi}{3}\,, \tag{3.7}$$

and hence

$$m_1\Big(\partial B_{r(t)} \cap R^\gamma(t)\Big) = 2\pi\, r(t) - U(t) \geq \Big(2\pi - \frac{\pi}{3}\Big) r(t)\,,$$

establishing (3.5).

2. In this step we prove that the function Φ is one-sided Lipschitz continuous, namely

$$\Phi(t+h)-\Phi(t)\leq h \qquad\qquad \text{for all} \quad t,h\geq 0\,. \tag{3.8}$$

First, observe that the total length of walls outside $B_{r(t)}$ will increase because of the construction of new walls, but decreases as some of the old walls fall inside the expanding disc $B_{r(t)}$. More precisely,

$$W(t+h)-W(t)\leq h-m_1\Big(\gamma(t)\cap\big(B_{r(t+h)}\setminus B_{r(t)}\big)\Big)\,. \tag{3.9}$$

Next, call $\Gamma_\theta\doteq\big\{(r,\theta)\,;\quad r>0\big\}$ the ray from the origin at an angle θ. Since $\dot r\leq 1$, if

$$\big(r(t),\theta\big)\in R^\gamma(t)\,, \qquad\qquad \big(r(t+h),\theta\big)\notin R^\gamma(t+h)\,,$$

then the ray Γ_θ must cross a wall, for some $r\in\big]r(t),r(t+h)\big]$. This implies

$$\begin{aligned}\frac{U(t+h)}{r(t+h)}-\frac{U(t)}{r(t)}&\leq m_1\Big(\theta;\quad\Gamma_\theta\cap\gamma(t+h)\cap\big(B_{r(t+h)}\setminus B_{r(t)}\big)\neq\emptyset\Big)\\ &\leq\frac{1}{r(t)}\cdot m_1\Big(\gamma(t+h)\cap\big(B_{r(t+h)}\setminus B_{r(t)}\big)\Big)\,.\end{aligned} \tag{3.10}$$

In the case where

$$\frac{U(t+h)}{r(t+h)}<\frac{U(t)}{r(t)}$$

the conclusion (3.8) follows easily from (3.9). In the opposite case, from (3.9) and (3.10) we obtain

$$\begin{aligned}\Phi(t+h)-\Phi(t)&\leq\frac{U(t+h)}{2r(t+h)}-\frac{U(t)}{2r(t)}+\frac{W(t+h)}{2r(t+h)}-\frac{W(t)}{2r(t)}\\ &\leq\frac{1}{2r(t)}\cdot m_1\Big(\gamma(t+h)\cap\big(B_{r(t+h)}\setminus B_{r(t)}\big)\Big)\\ &\qquad+\frac{1}{2r(t)}\cdot\Big(h-m_1\Big(\gamma(t)\cap\big(B_{r(t+h)}\setminus B_{r(t)}\big)\Big)\Big)\\ &\leq\ \frac{h}{r(t)}\ \leq\ h\,,\end{aligned}$$

proving (3.8). Thanks to this one-sided Lipschitz condition, to establish (3.6) it now suffices to prove a-priori estimates on the positive part of the time derivative $\dot\Phi$, valid at almost every time $t\geq 0$. In particular, these estimate can be proved assuming that the intersection $\gamma(t)\cap\partial B_{r(t)}$ consists of finitely many points.

3. At a given time $t>0$, assume that the uncontaminated portion of the circumference $\partial B_{r(t)}$ is described as the union of arcs (using polar coordinates)

$$\partial B_{r(t)}\setminus R^\gamma(T)=\Big\{\big(r(t),\theta\big)\,;\quad\theta\in\bigcup_{i=1}^N\ \big[\theta_i^-(t)\,,\ \theta_i^+(t)\big]\ \Big\}\,,$$

so that

$$U(t)=r(t)\cdot\sum_{i=1}^N\big(\theta_i^+(t)-\theta_i^-(t)\big)\,. \tag{3.11}$$

In order to define the time derivative $\dot r(t)$ and compute the corresponding derivative $\dot\Phi(t)$, we distinguish four cases.

CASE 1: There are at least two points $\big(r(t),\theta_i^\pm(t)\big)\in\partial B_{r(t)}\cap R^\gamma(t)$ that are not along the wall $\gamma(t)$.

Choose $\dot r(t)>0$ such that

$$\sqrt{1-\dot r^2}=\sigma<1\,. \tag{3.12}$$

Notice that, at every boundary point which is not protected by walls, an uncontaminated arc will shrink at least at the rate $\sqrt{1-\dot r^2}$. For example, if θ_i^- and θ_j^+ are not located along walls, then

$$r(t)\,\dot\theta_i^-(t)\geq\sqrt{1-\big(\dot r(t)\big)^2}\,,\qquad\qquad r(t)\,\dot\theta_i^+(t)\leq-\sqrt{1-\big(\dot r(t)\big)^2}\,. \tag{3.13}$$

Since there are at least two of these free endpoints, while new walls are created at speed $\sigma<1$, we obtain the estimate

$$\dot\Phi\;\leq\;\frac{\sigma}{2r}-\cos\frac{U}{2r}\cdot\frac{2\sqrt{1-\dot r^2}}{2r}\;\leq\;\frac{1}{2r}\left(\sigma-\frac{\sqrt3}{2}\cdot2\sqrt{1-\dot r^2}\right)\;<\;0\,, \tag{3.14}$$

as long as

$$\frac{U(t)}{2r(t)}\leq\frac{\pi}{6}\,. \tag{3.15}$$

CASE 2: The previous case does not hold, and there are at least two points $\big(r(t),\theta_i^\pm(t)\big)\in\partial B_{r(t)}\cap\partial R^\gamma(t)$ on the wall $\gamma(t)$.

In this case, we set $\dot r(t)=1$. To fix the ideas, let $\theta_i^\pm=\theta_i^\pm(r)$, $i=1,\ldots,N$ be the points on $\partial R^\gamma(t)\cap\partial B_{r(t)}$, and call $\mathcal W\subseteq\{1^+,1^-,2^+,2^-,\ldots,N^+,N^-\}$ the set of indices corresponding to points $\theta_i^\pm\in\gamma(t)$. The following inequalities then hold.

$$\dot W\leq\sigma-\sum_{i^\pm\in\mathcal W}\sqrt{1+(r\,\dot\theta_i^\pm)^2}\,, \tag{3.16}$$

$$\dot U\leq\sum_{i^+\in\mathcal W}r\,\dot\theta_i^+-\sum_{i^-\in\mathcal W}r\,\dot\theta_i^-+\sum_{i=1}^N(\theta_i^+-\theta_i^-)\,. \tag{3.17}$$

For sake of definiteness, in the following we assume that $1^+,1^-\in\mathcal W$, the other cases being entirely similar. Using (3.16)-(3.17) and recalling that $\dot r=1$, we obtain

$$\begin{aligned}\dot\Phi&=\cos\frac{U}{2r}\left(\frac{\dot U}{2r}-\frac{U}{2r^2}\,\dot r\right)+\left(\frac{\dot W}{2r}-\frac{W}{2r^2}\,\dot r\right)\\&\leq\cos\frac{U}{2r}\sum_{i^\pm\in\mathcal W}\frac{r\,|\dot\theta_i^\pm|}{2r}+\frac{1}{2r}\cdot\left(\sigma-\sum_{i^\pm\in\mathcal W}\sqrt{1+(r\,\dot\theta_i^\pm)^2}\right)-\frac{W}{2r^2}\\&\leq\frac{1}{2r}\cdot\Bigg\{\left(\cos\frac{U}{2r}\cdot r\,|\dot\theta_1^+|+\frac{\sigma-(W/r)}{2}-\sqrt{1+(r\,\dot\theta_1^+)^2}\right)\\&\qquad+\left(\cos\frac{U}{2r}\cdot r\,|\dot\theta_1^-|+\frac{\sigma-(W/r)}{2}-\sqrt{1+(r\,\dot\theta_1^-)^2}\right)\Bigg\}.\end{aligned} \tag{3.18}$$

We claim that the right hand side of (3.18) is strictly negative, whenever $\Phi(t)$ is sufficiently close to 1/2. Indeed, define for convenience

$$a \doteq r\,|\dot\theta_1^\pm|\,, \qquad\qquad c \doteq \cos\frac{U}{2r}\,.$$

Notice that, when $\Phi = 1/2$, we have

$$\frac{W}{2r} = \frac{1}{2} - \sqrt{1-c^2}\,.$$

Our claim can thus be proven by showing that

$$c\,a + \frac{\sigma}{2} < \frac{W}{2r} + \sqrt{1+a^2} = \frac{1}{2} - \sqrt{1-c^2} + \sqrt{1+a^2} \tag{3.19}$$

as long as $\sqrt{3}/2 \le c \le 1$. Equivalently

$$ac + \sqrt{1-c^2} \le \sqrt{1+a^2}\,,$$

$$a^2c^2 + 1 - c^2 + 2ac\sqrt{1-c^2} \le 1 + a^2\,.$$

Consider the quadratic polynomial

$$\phi(a) \doteq a^2(1-c^2) + c^2 - 2ac\sqrt{1-c^2}\,.$$

For a fixed value of c, this function achieves the minimum value when

$$a = a_c \doteq \frac{c}{\sqrt{1-c^2}}\,.$$

Hence

$$\phi(a) \ge \phi(a_c) \;=\; \frac{c^2}{(1-c^2)}\,(1-c^2) + c^2 - 2\frac{c}{\sqrt{1-c^2}}\,c\sqrt{1-c^2} \;=\; 0\,.$$

This elementary computation proves (3.19). Hence in (3.18) we have

$$\dot\Phi(t) < 0\,, \tag{3.20}$$

whenever $\Phi(t)$ is sufficiently close to 1/2.

CASE 3: The set $\partial B_{r(t)} \setminus R^\gamma(t)$ consists of exactly one arc $\big[\theta^-(t)\,,\ \theta^+(t)\big]$. Moreover, one of the endpoints lies on $\gamma(t)$ while the other is free.

To fix the ideas, assume $\theta^+(t) \in \gamma(t)$. We then choose $\dot r(t)$ so that

$$\sqrt{\dot r^2 + (r\dot\theta^+)^2} = 1\,.$$

Or, more precisely, so that

$$m_1\Big(B_{r(t+h)} \cap \gamma(t+h)\Big) - m_1\Big(B_{r(t)} \cap \gamma(t)\Big) = h\,, \tag{3.21}$$

for $h > 0$ small enough.

Because of (3.21), along the circumference $\partial B_{r(t)}$ the free endpoint can move according to

$$\dot\theta^-(t) \geq \dot\theta^+(t) \tag{3.22}$$

In this case one finds

$$\dot W = \sigma - \sqrt{\dot r^2 - (r\,\dot\theta^+)^2} = \sigma - 1 < 0\,,$$

$$\dot U = r(\dot\theta^+ - \dot\theta^-) + \dot r(\theta^+ - \theta^-)\,,$$

Recalling (3.22), we thus have

$$\begin{aligned}\dot\Phi &= \cos\frac{U}{2r}\left(\frac{\dot U}{2r} - \frac{U}{2r^2}\,\dot r\right) + \left(\frac{\dot W}{2r} - \frac{W}{2r^2}\,\dot r\right)\\ &\leq \cos\frac{U}{2r}\left(\frac{\dot r(\theta^+ - \theta^-)}{2r} - \frac{r\,(\theta^+ - \theta^-)}{2r^2}\,\dot r\right) + \frac{\sigma - 1}{2r}\ <\ 0\,.\end{aligned} \tag{3.23}$$

CASE 4: The set $\partial B_{r(t)} \setminus R^\gamma(t)$ is empty.

We then set $\dot r(t) = 1$ and compute

$$\dot U = 0\,,\qquad\qquad \dot W \leq \sigma\,,\qquad\qquad W = 2r\,\Phi\,.$$

Therefore, for $\Phi(t)$ sufficiently close to $1/2$ we have

$$\dot\Phi\ =\ \frac{\dot W}{2r} - \frac{W}{2r^2}\,\dot r \leq \frac{\sigma}{2r} - \frac{2r\,\Phi}{2r^2} < 0\,. \tag{3.24}$$

Together, the four inequalities (3.14), (3.20), (3.23), and (3.24), show that the function $\Phi(t)$ can never become larger than $1/2$. This establishes (3.6), for all $t \geq 0$.

To prove the limit in (3.4), we observe that, in Case 3, $\dot W \leq \sigma - 1 < 0$. Therefore, the set of times t where Case 3 does not hold must have infinite measure. Since $\dot r(t) > 0$ is uniformly positive in all the remaining Cases 1,2 and 4, we conclude that $r(t) \to \infty$. □

Remark 2. The above argument actually yields a stronger result. If $\sigma < 1$, not only the fire cannot be confined to a bounded set, but for every ball B_ρ the measure of the subset eventually burned by the fire satisfies the estimate

$$m_2\Big(R^\gamma(t) \cap B_\rho\Big) \geq \frac{5}{6}\,m_2\big(B_\rho\big)\,, \tag{3.25}$$

for every t sufficiently large.

Indeed, by (3.5) and (3.4), the inequality (3.25) holds for all t such that $r(t) > \rho$.

By a straightforward comparison argument, from the two above lemmas we deduce

Theorem 1. *For the system described at (1.1)–(1.4), assume*

$$F(x) \subseteq B_\rho\,,\qquad\qquad \psi(x) \leq \frac{1}{\rho'}\,,$$

for some $\rho' > 2\rho$ *and every* $x \in I\!\!R^2$. *Then, for every bounded initial set* R_0*, there exists* $r > 0$ *and an admissible strategy* γ *such that* $R^\gamma(t) \subseteq B_r$ *for all* $t \geq 0$.

Theorem 2. *For the system described at (1.1)–(1.4), assume*

$$F(x) \supseteq B_\rho\,, \qquad \psi(x) > \frac{1}{\rho}\,,$$

for some $\rho > 0$ *and every* $x \in I\!\!R^2$. *Then, starting from any nonempty open set* R_0*, for every admissible strategy* γ *one has*

$$\lim_{t\to\infty} m_2\big(R^\gamma(t)\big) = \infty\,.$$

Remark 3. Consider again the special case described in (3.1). By an obvious comparison argument, there must be a critical speed σ^* such that the following holds:

(*) Let σ be the speed at which the wall can be constructed. If $\sigma > \sigma^*$, then there exists an admissible strategy that confines the fire to a bounded set. If $\sigma < \sigma^*$, no such strategy exists.

The previous results show that $\sigma^* \in [1, 2]$. However, determining the precise value of σ^* remains an interesting open problem.

4 - A classification of optimal arcs

In the remainder of this paper we study the optimization problem defined at (1.7), (1.9). Assume that an optimal blocking strategy $t \mapsto \gamma(t)$ exists, in the form of finitely many regular arcs. By a **regular arc** we mean a $\mathcal{C}^1$, simply connected, one-dimensional embedded manifold $\Gamma \subset I\!\!R^2$. Our eventual goal is to derive a set of O.D.E's determining the various portions of the wall.

To simplify the analysis, we make the following assumptions:

(H) The functions in (1.4) and (1.7) satisfy $\beta \equiv 0$, $\psi \equiv 1$, while $\alpha > 0$ is a smooth function. Moreover, we assume that at a final time T the contaminated (or burned) region is completely encircled by walls, and no new walls need to be constructed after time T. This means

$$\gamma(t) = \gamma(T)\,, \qquad R^\gamma(t) = R^\gamma(T) \qquad t \geq T\,. \tag{4.1}$$

Some further notations will be useful. For $0 < t < T$ we call

$$\partial\gamma(t) = \bigcap_{\tau > t} \overline{\gamma(\tau) \setminus \gamma(t)} \doteq \big\{x_1(t), \ldots, x_\nu(t)\big\} \tag{4.2}$$

the points along the set of walls $\gamma(t)$ where new construction is taking place. Arcs in $\gamma(T)$ are classified in two groups: $\mathcal{F}$ and $\mathcal{B}$ (Free arcs and Boundary arcs).

- Free arcs are those which, at the time when they are constructed, lie away from the contaminated region. More precisely, we say that a regular arc $\Gamma_j \subseteq \gamma(T)$ is a **free arc** if

$$\Gamma_j \cap \partial\gamma(t) \cap \overline{R^\gamma(t)} \;=\; \emptyset \qquad \text{for all } t \in [0, T]\,. \tag{4.4}$$

- Boundary arcs are those which, at the time when they are constructed, lie on the boundary the contaminated region. More precisely, we say that a regular arc $\Gamma_j \subseteq \gamma(T)$ is a **boundary arc** if

$$\Big(\Gamma_j \cap \partial\gamma(t)\Big) \subset \overline{R^\gamma(t)} \qquad\qquad \text{for all } t \in [0,T]\,. \tag{4.5}$$

We write $\Gamma_j \in \mathcal{F}$ or $\Gamma_j \in \mathcal{B}$ in the case where (4.4) or (4.5) holds, respectively.

In addition, for each point x on the arc Γ_j we define three (possibly different) times:

$$\tau_0(x) \doteq \inf \big\{t\,; \quad x \in \gamma(t)\big\}$$

is the time when the wall passing through x is constructed,

$$\tau_1(x) \doteq \inf \big\{t\,; \quad x \in \overline{R^\gamma(t)}\big\}$$

is the first time when the contamination touches x from one side of the wall,

$$\tau_2(x) \doteq \inf \big\{t\,; \quad x \in \ \text{int}\,\overline{R^\gamma(t)}\big\}$$

is the first time when the contamination surrounds x from both sides of the wall.

We can further split the sets $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, calling $\mathcal{F}_1, \mathcal{B}_1$ the family of arcs which touch the contaminated area only on one side, while $\mathcal{F}_2, \mathcal{B}_2$ are the sets of arcs who are eventually surrounded by the contaminated area from both sides. More precisely:

$$\Gamma_j \in \mathcal{F}_1 \quad \text{if and only if} \quad \tau_0(x) < \tau_1(x) < \tau_2(x) = +\infty \qquad \text{for all } x \in \Gamma_j\,,$$

$$\Gamma_j \in \mathcal{F}_2 \quad \text{if and only if} \quad \tau_0(x) < \tau_1(x) < \tau_2(x) \leq T \qquad \text{for all } x \in \Gamma_j\,,$$

$$\Gamma_j \in \mathcal{B}_1 \quad \text{if and only if} \quad \tau_0(x) = \tau_1(x) < \tau_2(x) = +\infty \qquad \text{for all } x \in \Gamma_j\,,$$

$$\Gamma_j \in \mathcal{B}_2 \quad \text{if and only if} \quad \tau_0(x) = \tau_1(x) < \tau_2(x) \leq T \qquad \text{for all } x \in \Gamma_j\,,$$

Remark 4. It is clear that it is useless to construct an arc with $\tau_1 = \tau_2 = +\infty$, so that it never touches the contaminated set. Constructing an arc with $\tau_0 = \tau_1 = \tau_2$, so that it is immediately surrounded by the contaminant, is equally useless. We thus expect that an optimal strategy will consist only of arcs of type $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_2$.

Remark 5. Given an arc $\Gamma_j \in \mathcal{F}$, the time order in which its portions are constructed is irrelevant. This leads to infinitely many equivalent optimal strategies. For sake of definiteness, we shall henceforth assume that each smooth arc is constructed by adding portions of wall only along one, or both of its endpoints, say

$$\Gamma_j \cap \partial\gamma(t) = \big\{x_j^-(t)\,,\ x_j^+(t)\big\} \qquad \text{or} \qquad \Gamma_j \cap \partial\gamma(t) = \big\{x_j(t)\big\}\,. \tag{4.6}$$

Remark 6. Consider a wall $\Gamma_j \in \mathcal{B}$. Then the speed at which its length increases must be strictly larger that the local speed at which the fire propagates. In other words, calling $x(t)$ the moving endpoint of the wall, we must have

$$\big|\dot{x}_j(t)\big| > \sup \Big\{r\,; \quad F\big(x_j(t)\big) \supseteq B_r\Big\}. \tag{4.7}$$

The assumption $\psi \equiv 1$ now implies

$$\sum_j \big|\dot x_j(t)\big| \leq 1\,. \tag{4.8}$$

If the right hand side of (4.7) is strictly positive, this puts an upper bound on the number of boundary arcs that can be constructed at any given time.

5 - Necessary conditions for optimality

In this section we derive a set of differential equations for the arcs Γ_j constructed by an optimal strategy $t \mapsto \gamma^*(t) \subset I\!R^2$. The form of these equations will be very different, depending on the type of arcs. We thus need to consider various cases.

1. A free arc: $\Gamma \in \mathcal{F}_1$ **.**

Let $s \mapsto \Gamma(s)$, $0 < s < \bar s$, be a parametrization of the curve Γ in terms of arc-length. Call $\mathbf{n}(s)$ the unit outer normal to the reachable set $R^\gamma(T)$ at the point $\Gamma(s)$. We choose the orientation so that the unit tangent vector $\mathbf{t}(s) = d\Gamma(s)/ds$ satisfies

$$\mathbf{t}(s) \wedge \mathbf{n}(s) \equiv 1\,.$$

Fix any given point $P_0 = \Gamma(s_0)$ and define (x, y) to be the coordinates of the point

$$P_0 + x\,\mathbf{t}(s_0) + y\,\mathbf{n}(s_0)\,.$$

In this system of orthogonal coordinates, the curve Γ will locally have the expression $y = \phi(x)$, for some $\mathcal{C}^1$ function ϕ with $\phi(0) = \phi'(0) = 0$. For a sufficiently small neighborhood $[a, b]$ of the origin, the assumption that the strategy γ^* is optimal implies that the function $\phi : [a, b] \mapsto I\!R$ solves the following iso-perimetric problem

$$\min_{w(\cdot)} \int_a^b A\big(x,\, w(x)\big)\,dx\,, \tag{5.1}$$

subject to the constraints

$$\int_a^b \sqrt{1 + \big[w'(x)\big]^2}\,dx = \int_a^b \sqrt{1 + \big[\phi'(x)\big]^2}\,dx\,, \qquad w(a) = \phi(a)\,, \quad w(b) = \phi(b)\,. \tag{5.2}$$

Here the primes denote differentiations w.r.t. x, while the function A is defined as

$$A(x, w) \doteq \int_0^w \alpha(x, y)\,dy\,.$$

The solution of the above problem is a standard exercise in the Calculus of Variations.

If the curve ϕ is a straight line, then there is no other curve satisfying the constraints (5.2) and we are done. Otherwise, for some Lagrange multiplier $\lambda \geq 0$, setting $L(x, w, w') \doteq A(x, w) + \lambda\sqrt{1 + (w')^2}$ one has

$$\frac{d}{dx}\frac{\partial L}{\partial w'} = \frac{\partial L}{\partial w}\,. \tag{5.3}$$

Therefore

$$\lambda\cdot\frac{d}{dx}\frac{w'(x)}{\sqrt{1+\big[w'(x)\big]^2}}~=~\lambda\cdot\frac{\big[w''(x)\big]^2}{\Big[1+\big(w'(x)\big)^2\Big]^{3/2}}~=~\alpha\big(x,w(x)\big)\,. \tag{5.4}$$

Since we are assuming that the land value α is strictly positive, we must have $\lambda>0$. Observe that the unit tangent vector to the curve Γ is

$$\mathbf{t}(x)=\left(\frac{1}{\sqrt{1+\big[w'(x)\big]^2}}\,,~\frac{w'(x)}{\sqrt{1+\big[w'(x)\big]^2}}\right).$$

Moreover

$$\left|\frac{dx}{ds}\right|=\frac{1}{\sqrt{1+\big[w'(x)\big]^2}}$$

$$\begin{aligned}\left|\frac{d\mathbf{t}}{ds}\right|^2=\left|\frac{d\mathbf{t}}{dx}\right|^2\cdot\left|\frac{dx}{ds}\right|^2&=\left\{\left(\frac{w'w''}{\big(1+(w')^2\big)^{3/2}}\right)^2+\left(\frac{w''}{\big(1+(w')^2\big)^{3/2}}\right)^2\right\}\cdot\frac{1}{1+(w')^2}\\&=\frac{(w'')^2}{\big(1+(w')^2\big)^3}\,.\end{aligned} \tag{5.5}$$

Comparing (5.4) with (5.5) we conclude

$$\left|\frac{d}{ds}\,\mathbf{t}(s)\right|=\frac{\alpha\big(\Gamma(s)\big)}{\lambda}\,. \tag{5.6}$$

According to (5.6), on a neighborhood of the point $\Gamma(s_0)$ the curvature of Γ at each point $\Gamma(s)$ is proportional to the local value of the land $\alpha\big(\Gamma(s)\big)$. Since we are assuming that the arc Γ is connected, this constant λ of proportionality must be the same over the whole arc Γ.

Remark 7. One can think of the Lagrange multiplier λ as the cost for relaxing the integral constraint in (5.2). Equivalently, λ can be interpreted as the **value of a unit length of wall**, during the construction of the free arc Γ. Indeed, let the construction of Γ occur during the time interval $[t_1\,,~t_2]$. Assume that, sometimes within this interval $[t_1\,,~t_2]$, the controller is allowed to construct an additional portion of wall, of length ε. The constraints in (5.2) would then be replaced by

$$\int_a^b\sqrt{1+\big[w'(x)\big]^2}\,dx=\int_a^b\sqrt{1+\big[\phi'(x)\big]^2}\,dx+\varepsilon\,,\qquad\qquad w(a)=\phi(a)\,,\quad w(b)=\phi(b)\,.$$

In this way, the controller could construct a slightly longer wall, confining the contamination to a smaller region. Calling J_ε this new cost, according to (5.3) we have

$$\lim_{\varepsilon\to 0}\frac{J_\varepsilon-J(\gamma^*)}{\varepsilon}~=~-\lambda\,. \tag{5.7}$$

By the assumption (H) at the beginning of Section 4, the wall is constructed at unit speed. In this case, the value of a unit of time is equivalent to the value of a unit length of wall. The above analysis can thus be summarized as follows.

Theorem 3. *Let the strategy γ^* be optimal. Then, at every point of a free arc $\Gamma \in \mathcal{F}_1$, the vector $d\mathbf{t}/ds$ is oriented in the direction of the outer normal to $\partial R^{\gamma^*}(T)$, and the curvature $|d\mathbf{t}/ds|$ is proportional to the cost α. The constant of proportionality λ in (5.6) corresponds to the instantaneous value of time.*

2. A single boundary arc: $\Gamma \in \mathcal{B}_1$.

Assume that, during an open interval of time $t \in]a, b[$, wall construction occurs along one single boundary arc Γ. Recalling the notation at (4.2), this means $\partial\gamma(t) = \{\eta(t)\}$, with $\eta(t) \in \Gamma$.

This arc can be determined by the two conditions

$$\eta(t) \in \partial R^{\gamma^*}(t) \qquad \text{for all } t \in [t_1, t_2]\,, \tag{5.8}$$

$$\big|\dot\eta(t)\big| \equiv 1\,. \tag{5.9}$$

Notice that the boundary of the reachable set can be determined by solving a Hamilton-Jacobi equation for the minimum time function

$$u(x) \doteq \inf\big\{t\,;\quad x \in R^{\gamma^*}(t)\big\}\,, \tag{5.10}$$

outside the walls. By (1.1), this equation takes the form

$$\max_{y\in F(x)} \big\langle \nabla u(x)\,,\ y\big\rangle = 1\,. \tag{5.11}$$

In this case, we expect

$$\partial R^{\gamma^*}(t) = \big\{x\,;\quad u(x) = t\big\}\,.$$

To make further progress, we need to impose some regularity conditions on the minimum time problem (5.10).

(H′) The time-optimal trajectories for the problem (5.10) can be extended to regular field of extremals, defined on a whole neighborhood $\mathcal{N}$ of the arc Γ. All these trajectories cross the arc Γ transversally.

In particular, this implies that the minimum time function u, which is actually defined only on one side of the wall Γ, can be uniquely extended to a $\mathcal{C}^1$ solution of (5.11) defined on the entire neighborhood $\mathcal{N}$. In the following, we define the unit vector

$$\mathbf{n}(x) \doteq \frac{\nabla u(x)}{\big|\nabla u(x)\big|} \tag{5.12}$$

For $t = u(x)$, the vector $\mathbf{n}(x)$ is the unit outer normal to the reachable set $R(t)$ at the point x.

Under the assumptions (H′), a point $x \in \mathcal{N}$ can be identified by the two coordinates (t, s). Here $t = u(x)$ is the time when $x \in \partial R(t)$. As second coordinate, we take s to be the signed length of the arc joining $\eta(t)$ with x, along the arc $\partial R(t)$. We choose the orientation so that $s > 0$ corresponds to points outside the sets $R^{\gamma^*}(t)$, see fig. 4.

Recalling (1.1), from the condition (5.8) we now deduce

$$\big\langle \dot\eta(t)\,,\ \mathbf{n}(\eta(t))\big\rangle = \sup_{y\in F(\eta(t))} \big\langle y\,,\ \mathbf{n}(\eta(t))\big\rangle\,. \tag{5.13}$$

Example 1. Assume that the fire propagates in all directions with a speed depending on x, say

$$F(x) = B_{\rho(x)} \qquad\qquad 0 < \rho(x) < 1\,. \tag{5.14}$$

Call $\theta(t)$ the angle between the wall and the boundary of the contaminated region, at the point $\eta(t)$. Then (5.13) and (5.9) together imply the identity

$$\sin\theta(t) = \rho\big(\eta(t)\big)\,. \tag{5.15}$$

3. Two or more boundary arcs, constructed simultaneously.

We now consider a more general situation where the boundary arcs $\Gamma_1, \Gamma_2, \ldots, \Gamma_\nu \in \mathcal{B}_1$ are simultaneously constructed, on a time interval $t \in]a, b[\,$.

Some additional notation will be useful. Fix a particular arc Γ_j and let

$$\big\{\eta_j(t)\big\} = \partial\gamma^*(t) \cap \Gamma_j$$

be the point along the wall Γ_j where construction occurs at time t. The map $t \mapsto \eta_j(t) \in \partial R^{\gamma^*}(t)$ thus provides a parametrization of the boundary arc Γ_j, based on the time of construction. The fact that construction is simultaneously occurring along all arcs means that

$$\big|\dot\eta_j(t)\big| > 0 \qquad\qquad t \in]a, b[\,, \quad j = 1, \ldots, \nu\,. \tag{5.16}$$

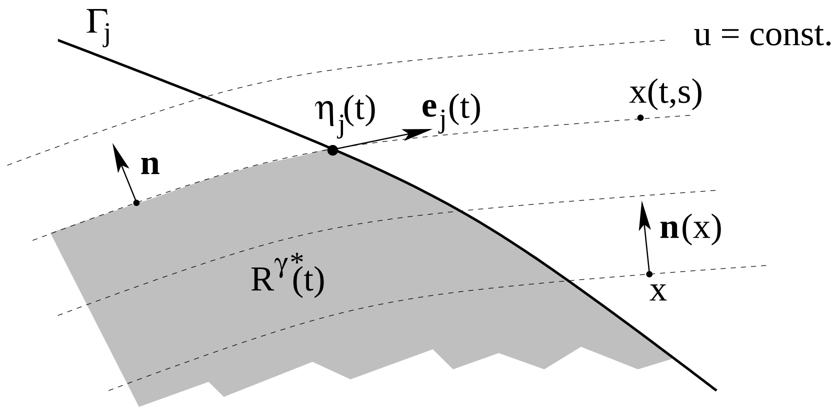

Figure 4. Testing the optimality of boundary arcs.

As in the previous case, we assume that the condition (H′) holds. In particular, for every arc Γ_j there will be a neigborhood $\mathcal{N}_j \supset \Gamma_j$ and a $\mathcal{C}^1$ minimal time function $u : \mathcal{N}_j \mapsto [a, b]$ which satisfies the Hamilton-Jacobi equation (5.11), such that

$$u(x) = \min\big\{t\,;\ \ x \in R^{\gamma^*}(t)\big\} \qquad\qquad \text{for all } x \in \mathcal{N}_j \cap R^{\gamma^*}(T)\,.$$

This provides us with a field of unit normal vectors

$$\mathbf{n}_j(x) \doteq \frac{\nabla u(x)}{\big|\nabla u(x)\big|} \qquad\qquad x \in \mathcal{N}_j \tag{5.17}$$

defined on a neighborhood of the arc Γ_j, see fig. 4. In particular, $\mathbf{n}_j\big(\eta_j(t)\big)$ is a unit outer normal to the reachable set

$$R(t) \doteq \big\{x\,;\quad u(x) \leq t\big\} \tag{5.18}$$

at the point $\eta_j(t)$. A point $x \in \mathcal{N}_j \subset I\!\!R^2$ will be identified by the two coordinates (t, s). Here $t = u(x)$ is the minimum time function, while s is the signed length of the arc joining $\eta_j(t)$ with x along the boundary of the reachable set

$$\partial R(t) = \big\{x\,;\quad u(x) = t\big\}\,.$$

Next, consider the speed at which the reachable set is locally expanding:

$$h(x) \doteq \sup_{y\in F(x)} \quad \big\langle y\,,\ \mathbf{n}_j(x)\big\rangle\,. \tag{5.19}$$

Moreover, denote by $\mathbf{e}_j(t)$ the unit vector which is perpendicular to $\mathbf{n}\big(\eta_j(t)\big)$ (and hence tangent to the boundary $\partial R(t)$) at the point $\eta_j(t)$, oriented toward the outer part of the wall. Calling $x_j(t, s) \in \mathcal{N}_j$ the point whose coordinates are (t, s), we then have

$$x_j(t, 0) = \eta_j(t)\,, \qquad\qquad x_j(t, s) = \eta_j(t) + s\,\mathbf{e}_j(t) + \mathcal{O}(s^2)\,. \tag{5.20}$$

We are now ready to state our necessary conditions for optimality, in the case where several arcs are constructed simultaneously.

Theorem 4. *Let $\Gamma_1, \ldots, \Gamma_\nu \in \mathcal{B}_1$ be boundary arcs simultaneously constructed by an optimal strategy γ^*, and assume that each of these arcs satisfies the regularity condition (H'). Let $t \mapsto \eta_j(t)$ be the parametrization of the arc Γ_j based on construction time $t \in\,]a, b[\,$, and call $\mathbf{e}_j(t)$ be the unit vector tangent to the boundary of the reachable set $R(t)$ at the point $\eta_j(t)$, oriented toward the outside of the wall. Then, for each $j = 1, \ldots \nu$, there exists a constant $\lambda_0 \geq 0$ and a nontrivial solution of the adjoint equation*

$$\dot p_j(t) = \frac{\big\langle \dot\eta_j(t)\,,\ \dot{\mathbf{e}}_j(t)\big\rangle}{\big\langle \dot\eta_j(t)\,,\ \mathbf{e}_j(t)\big\rangle}\, p_j(t)\ - \lambda_0\, h\big(\eta_j(t)\big)\, \alpha\big(\eta_j(t)\big) \tag{5.21}$$

such that the functions

$$V_j(t) \doteq \left\langle \frac{\dot\eta_j(t)}{\big|\dot\eta_j(t)\big|}\ ,\ \mathbf{e}_j(t)\right\rangle^{-1} \cdot p_j(t) \tag{5.22}$$

all coincide, at each time t.

Proof. To prove the theorem, we reformulate the minimization problem (1.9) as a problem of optimal control with running cost and terminal constraints. The necessary conditions stated at (5.21)-(5.22) will then follow from a direct application of the Pontryagin maximum principle [2]. Consider the control system on $I\!\!R^\nu$

$$\dot s_j(t) = f_j\big(t,\ s_j(t),\ w_j(t)\big)\,, \qquad\qquad j = 1, \ldots, \nu\,, \tag{5.23}$$

where, for $x_j = x(t, s_j) \in \mathcal{N}_j$ and $w_j > h(x_j)$, the vector

$$\mathbf{f}_j = \frac{\partial}{\partial t} + f_j(t, s, w_j)\,\frac{\partial}{\partial s} \in I\!\!R^2$$

is determined by the conditions

$$|\mathbf{f}_j| = w_j\,, \qquad\qquad \big\langle \mathbf{f}_j\,,\ \mathbf{n}(x_j)\big\rangle = h(x_j) \doteq \sup_{y\in F(x_j)} \big\langle y\,,\ \mathbf{n}(x_j)\big\rangle\,. \tag{5.24}$$

As in (5.13), the above condition guarantees that the point $x_j\big(t, s_j(t)\big)$ always remains on the boundary of the reachable set $R(t)$. We notice that, for $w_j < h(x_j)$ there is no vector $\mathbf{f}_j$ satisfying both conditions in (5.24), while for $w_j > h(x_j)$ there are two. This ambiguity can be easily removed, choosing $\mathbf{f}_j$ to be a continuous function of its arguments, which coincides with $\dot\eta_j(t)$ in case $x_j = \eta_j(t)$, $w_j = \big|\dot\eta_j(t)\big|$. Equivalently, $f_j(t, s, w_j)$ is a continuous function which vanishes when $s = 0$, $w_j = \big|\dot\eta_j(t)\big|$.

The set of admissible controls is defined as

$$\mathcal{W} \doteq \Big\{ w = (w_1, \ldots, w_\nu) : [a, b] \mapsto I\!R^\nu\,, \qquad w_j(t) = \big|\dot\eta_j(t)\big| + \phi_j(t) > h(x_j)\,, \quad \sum_{j=1}^{\nu} \phi_j(t) \le 0\Big\}. \tag{5.25}$$

The control system (5.23) is supplemented by the initial and terminal constraints

$$s_j(a) = s_j(b) = 0\,, \qquad\qquad j = i, \ldots, \nu\,. \tag{5.26}$$

This corresponds to the constraints

$$x_j\big(a,\, s_j(a)\big) = \eta_j(a) \qquad\qquad x_j\big(b,\, s_j(b)\big) = \eta_j(b)\,, \qquad\qquad j = 1, \ldots, \nu\,,$$

Indeed, we wish to perturb the middle sections of the arcs Γ_j, but not their endpoints.

We now consider the optimization problem

$$\text{minimize :} \qquad \Lambda(w) \doteq \sum_{j=1}^{\nu} \int_a^b A_j\big(t, s_j(t)\big)\, dt\,, \tag{5.27}$$

where A_j accounts for the value of the additional land burned by the fire, if strategy w is adopted. More precisely,

$$A_j(t, s) \doteq \int_0^s h\big(x_j(t, s)\big)\, \alpha\big(x_j(t, s)\big)\, ds \tag{5.28}$$

The functional at (5.27) should be minimized among all controls $w : [a, b] \mapsto I\!R^\nu$ in the admissible set (5.25), such that the corresponding trajectory satisfies the boundary conditions (5.26).

By assumption, the control $w_j^*(t) = \big|\dot\eta_j(t)\big|$, corresponding to the trajectory $s_1(t) = \cdots = s_\nu(t) = 0$ is optimal. By the Pontryagin maximum principle, there exist a nontrivial adjoint function $p(t) = (p_1, \ldots, p_\nu)(t)$, and a constant $\lambda_0 \ge 0$ such that

$$\dot p_i(t) = -p_i(t) \cdot \frac{\partial f_i}{\partial s}\big(t, 0, w_i^*(t)\big) - \lambda_0 \frac{\partial A_j}{\partial s}(t, 0)\,, \qquad\qquad i = 1, \ldots, \nu\,, \tag{5.29}$$

$$\sum_{i=1}^{\nu} p_i(t) \cdot f_i\big(t, 0,\, w_i^*(t)\big) = \min_{w\in W} \sum_{i=1}^{\nu} p_i(t) \cdot f_i\big(t, 0,\, w_i\big)\,. \tag{5.30}$$

Since each map $w_j \mapsto f_j(t, 0, w_j)$ is monotone decreasing, from (5.30) it follows

$$p_i(t) > 0 \qquad\qquad i = 1, \ldots, \nu\,, \qquad t \in \,]a, b[\,. \tag{5.31}$$

From (5.30) we also deduce

$$p_1(t)\cdot f_1\big(t,0,\,w_1^*(t)\big) = \;\cdots\; = p_\nu(t)\cdot f_\nu\big(t,0,\,w_\nu^*(t)\big)\,. \tag{5.32}$$

Otherwise, one could increase one of the controls w_i^* and decrease another control w_j^* by the same amount, achieving a smaller value on the right hand side of (5.30).

To complete the proof, it remains to examine the meaning of the conditions (5.29)–(5.32), computing the partial derivatives in (5.29). By (5.28) it follows

$$\frac{\partial A_j}{\partial s}(t,0) \;=\; h\big(x_j(t,0)\big)\,\alpha\big(x_j(t,0)\big) \;=\; h\big(\eta_j(t)\big)\,\alpha\big(\eta_j(t)\big)\,. \tag{5.33}$$

The computation of $\partial f_j/\partial s$ requires more work. Consider a family of perturbed trajectories, having the form

$$t\mapsto \eta_j^\varepsilon(t) \;=\; x_j\big(t\,,\;\varepsilon\zeta(t)+\mathcal{O}(\varepsilon^2)\big) \;=\; \eta_j(t)+\varepsilon\zeta(t)\cdot\mathbf{e}_j(t)+\mathcal{O}(\varepsilon^2)\,, \tag{5.34}$$

such that, for all ε, t,

$$\big|\dot\eta_j^\varepsilon(t)\big| = \big|\dot\eta_j(t)\big| = w_j^*(t)\,. \tag{5.35}$$

From (5.34)-(5.35) one obtains

$$0 = \big\langle \dot\eta_j^\varepsilon\,,\;\dot\eta_j^\varepsilon\big\rangle - \big\langle \dot\eta_j\,,\;\dot\eta_j\big\rangle = 2\varepsilon\big\langle \dot\eta_j\,,\;\dot\zeta\,\mathbf{e}_j+\zeta\,\dot{\mathbf{e}}_j\big\rangle+\mathcal{O}(\varepsilon^2)\,.$$

Therefore, ζ satisfies the following linear, first order, homogeneous O.D.E.:

$$\dot\zeta(t) = -\frac{\big\langle \dot\eta_j(t)\,,\;\dot{\mathbf{e}}_j(t)\big\rangle}{\big\langle \dot\eta_j(t)\,,\;\mathbf{e}_j(t)\big\rangle}\,\zeta\,. \tag{5.36}$$

On the other hand, we know that the scalar functions $s_j^\varepsilon(t)=\varepsilon\zeta(t)+\mathcal{O}(\varepsilon^2)$ are all solutions to the same O.D.E.

$$\dot s_j^\varepsilon(t) = f_j\big(t, s_j^\varepsilon(t),\,w^*(t)\big)\,,$$

with possibly different initial data. Hence the first order term $\zeta(\cdot)$ in the expansion (5.34) provides a solution to the linear equation

$$\dot\zeta(t) = \frac{\partial f_j}{\partial s}\big(t,0,w^*(t)\big)\,. \tag{5.37}$$

Comparing (5.36) with (5.37) we conclude

$$\frac{\partial f_j}{\partial s}\big(t,0,w^*(t)\big) = -\frac{\big\langle \dot\eta_j(t)\,,\;\dot{\mathbf{e}}_j(t)\big\rangle}{\big\langle \dot\eta_j(t)\,,\;\mathbf{e}_j(t)\big\rangle}\,. \tag{5.38}$$

Together, (5.33) and (5.38) confirm that the equations satisfied by the adjoint variables p_j are indeed given by (5.21).

Finally, from the maximality condition (5.30) we deduce

$$p_i\cdot\frac{\partial}{\partial w_i}f_i\big(t,0,w_i^*(t)\big) = p_j\cdot\frac{\partial}{\partial w_j}f_j\big(t,0,w_j^*(t)\big) \qquad i,j\in\{1,\ldots,\nu\}\,. \tag{5.39}$$

To compute the partial derivative $\partial f_j/\partial s$ we observe that, when $s=0$, one has

$$\mathbf{f}_j = \dot\eta_j(t) + f_j(t,0,w_j)\cdot \mathbf{e}_j(t)\,. \tag{5.40}$$

Differentiating the identity

$$|\mathbf{f}_j|^2 = \big\langle \dot\eta_j + f_j\mathbf{e}_j\,,\ \dot\eta_j + f_j\mathbf{e}_j\big\rangle = w_j^2$$

w.r.t. w_j, we obtain

$$2w_j = 2\big\langle \dot\eta_j\,,\ \mathbf{e}_j\big\rangle\,\frac{\partial f_j}{\partial w_j} + 2f_j\,\frac{\partial f_j}{\partial w_j}$$

At $w_j = w_j^*(t)$, since $f_j\big(t,0,w^*(t)\big)=0$ one obtains

$$\frac{\partial f_j}{\partial w_j}\big(t,0,w_j^*(t)\big) \ =\ \frac{w_j^*(t)}{\big\langle \dot\eta_j(t)\,,\ \mathbf{e}_j(t)\big\rangle}\ <\ 0\,. \tag{5.41}$$

Observing that $w_j^*(t) = \big|\dot\eta_j(t)\big|$, the above identity can be rewritten as

$$\frac{\partial f_j}{\partial w_j}\big(t,0,w_j^*(t)\big) = \left\langle \frac{\dot\eta_j(t)}{\big|\dot\eta_j(t)\big|}\ ,\ \mathbf{e}_j(t)\right\rangle^{-1}. \tag{5.42}$$

Together, (5.42) and (5.39) yield the necessary condition stated at (5.22), proving the theorem.

Remark 8. Up to a constant factor, the function $V(t)\doteq V_1(t)=\cdots=V_\nu(t)$ in (5.22) can be interpreted as the **instantaneous value of time**. Intuitively, this function should be non-increasing. We recall that, in the case of a free arc, by (5.6) the value of time is the constant

$$\lambda = \alpha\big(\eta(s)\big)\cdot r\big(\eta(s)\big)\,, \tag{5.43}$$

i.e. the local value of the land multiplied by the radius of curvature of the wall $r = |d\mathbf{t}/ds|^{-1}$.

Since in general the functions V_j in (5.22) are strictly decreasing in time, this indicates that free arcs and boundary arcs cannot be constructed simultaneously.

In the previous analysis, we only considered necessary conditions derived from local perturbations, which leave unchanged the initial and terminal point of each arc. A better understanding of the "value of time" could be provided by a global analysis of the optimal strategy γ^*. This we leave as a topic for future research.

6 - Necessary conditions at junctions between arcs

In this section we consider an strategy γ^* which constructs a pair of adjacent arcs Γ_1,Γ_2. In a couple of significant cases, we shall derive necessary conditions for the strategy γ^* to be optimal. Our basic assumptions are as follows.

(H'') The arcs Γ_1,Γ_2 are C^1, and touch at a point P. The minimum time function u at (5.10) can be extended to a $\mathcal{C}^1$ solution of the Hamilton-Jacobi equation (5.11) on a whole neighborhood $\mathcal{N}$ of the point P. Moreover, all characteristic curves cross both arcs Γ_1,Γ_2 transversally.

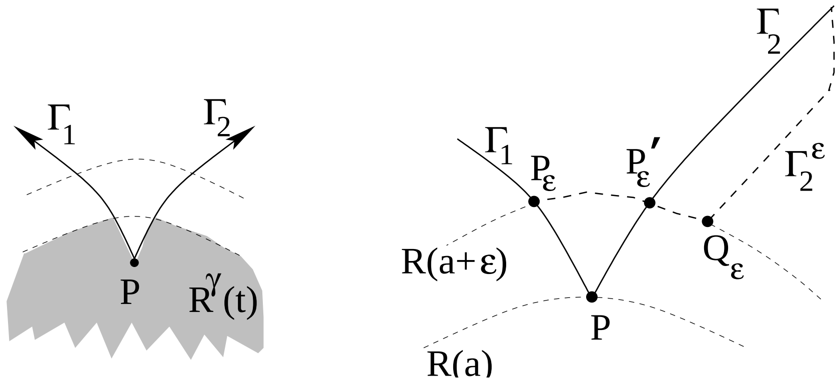

Figure 5. Two boundary arcs originating at the same point are not optimal.

We study two cases, namely: (i) The junction of two boundary arcs. (ii) The junction of a free arc and a boundary arc.

Theorem 5. *Consider a strategy γ^* which constructs two boundary arcs $\Gamma_1, \Gamma_2 \in \mathcal{B}_1$ originating from the same point P in opposite directions w.r.t. the front of the fire (fig. 5). If the assumptions (H'') hold, then the strategy γ^* is not optimal.*

Proof. A family of strategies γ^ε which achieve a lower cost is illustrated in (fig. 5), right. Let $t = a$ be the time when the fire reaches the point P and both walls start to be constructed. Assume that Γ_2 is continuously constructed during an entire interval $[a, b]$, and choose an intermediate time $a < b' < b$.

For every $\varepsilon > 0$, the strategy γ^ε is defined as follows. By the transversality assumption, if no walls are constructed along the curves Γ_1, Γ_2, at time $t = a + \varepsilon$ the boundary of the reachable set $R(a + \varepsilon)$ will contain an arc crossing both Γ_1 and Γ_2 transversally. Referring to fig. 5, right, consider the points P_ε at the intersection $\Gamma_1 \cap \partial R(a+\varepsilon)$, and P'_ε at the intersection $\Gamma_2 \cap \partial R(a+\varepsilon)$. Moreover, let Q_ε be the point along the boundary $\partial R(a + \varepsilon)$ such that the length of the arc $P_\varepsilon, Q_\varepsilon$ is exactly the same as the combined length of the arcs P_ε, P and P, P'_ε.

The strategy γ^ε is defined by replacing the two boundary arcs P_ε, P and P, P'_ε by the single free arc $P_\varepsilon, Q_\varepsilon$, during the interval $[a\,,\ a+\varepsilon]$. During the subsequent interval $[a+\varepsilon,\ b']$ we construct the boundary wall Γ_1 as before (of course, starting from P_ε in place of P). Moreover, we construct a boundary wall Γ_2^ε at the same speed as the previous wall Γ_2. During the remaining part of the time interval, i.e. for $t \in [b',\ b]$, we construct the boundary wall Γ_2^ε at a rate somewhat slower than Γ_2, in such a way that the two walls eventually coincide at $t = b$.

It is clear that the new strategy γ^ε is also admissible. For $\varepsilon > 0$ sufficiently small, we claim that γ^ε yields a strictly smaller cost than γ^*. Indeed, the small triangular region between the points $P, P_\varepsilon, P'_\varepsilon$ has area $\mathcal{O}(\varepsilon^2)$. This is the additional region burned by the fire if the strategy γ^ε is adopted. On the other hand, the value of the region between the two curves Γ_2, Γ_2^ε and the two boundaries $\partial R(a + \varepsilon)$, $\partial R(b')$ is $\geq c_0\,\varepsilon$, for some constant $c_0 > 0$. This is a region saved from the flames, thanks to the new strategy γ^ε. For $\varepsilon > 0$ small, it is now clear that $J(\gamma^\varepsilon) < J(\gamma^*)$. Hence γ^* is not optimal. □

Theorem 6. *Consider a strategy γ^* which constructs a free arc $\Gamma_1 \in \mathcal{F}_1$ during a time interval $[a, b]$, and then a boundary arc $\Gamma_2 \in \mathcal{B}_1$ for $t \in [b, c]$, where the two arcs have one end-point P in common. Moreover, let the assumptions (H'') hold.*

If the arcs Γ_1, Γ_2 form an inward corner at P, then the strategy γ^* is not optimal.

Proof. As shown in fig. 6, let $\mathbf{t}_1, \mathbf{t}_2$ be the unit tangent vectors to Γ_1 and Γ_2 at the point P, respectively. Consider the intermediate direction

$$\mathbf{t} = \frac{\mathbf{t}_1 + \mathbf{t}_2}{|\mathbf{t}_1 + \mathbf{t}_2|}\,.$$

For each $\varepsilon > 0$, an alternative strategy is defined as follows. Let P_ε be the point on the free arc Γ_1 such that the sub-arc P_ε, P has length ε. To fix the ideas, let $t_\varepsilon^- < b$ be the time when the wall Γ_1 reaches P_ε.

Construct a boundary arc Γ_ε, along the segment starting form P_ε parallel to $\mathbf{t}$. Notice that, because of the inward-pointing assumption, the rate of construction of this new arc Γ_ε will be

- strictly slower than the rate of construction of the free arc Γ_1, for $t < b$,

- strictly faster than the rate of construction of the boundary arc Γ_2, for $t > b$.

Therefore, there will be a unique time $t_\varepsilon > b$ such that the length of the arc Γ_ε constructed by the new strategy γ^ε during the time interval $[t_\varepsilon^-\,,\,t_\varepsilon]$ is the same as the combined lengths of the arcs P_ε, P on Γ_1 and P, P'_ε on Γ_2, constructed by the old strategy γ^* during the same time interval $[t_\varepsilon^-\,,\,t_\varepsilon]$.

Choose a time b' independent of ε, with $b < b' < c$. Call Q_ε the point on the segment Γ_ε reached by the new wall at time t_ε. Notice that Q_ε lies in the interior of the region R^{γ^*} previously reached by the fire. During the time interval $[t_\varepsilon, b']$ we construct a boundary wall Γ_2^ε at the same speed as the previous wall Γ_2. Finally, during the remaining time interval $[b', c]$ we we construct the boundary wall Γ_2^ε at a rate somewhat slower than Γ_2, in such a way that the two walls eventually coincide when $t = c$.

It is now clear that the new strategy γ^ε is admissible. Comparing γ^ε with γ^* we see that

- An additional triangular region between the points P_ε, P and the segment Γ_ε is now burned by the fire. However, this is small: its area is $\mathcal{O}(\varepsilon^2)$.

- The region between the arcs Γ_2, Γ_2^ε and the boundaries $\partial R(b)$, $\partial R(t_\varepsilon)$ is now saved from the flames. The value of this region is $\geq c_0\,\varepsilon$, for some constant $c_0 > 0$.

We conclude that the new admissible strategy γ^ε is strictly better than γ^*.

□

Remark 9. In the same setting as above, one might wonder if the result remains valid when the arcs Γ_1, Γ_2 form an outward pointing corner at P. In principle, one might try again to "cut the corner", constructing a segment in the intermediate direction $\mathbf{t}$. However, this would require a faster construction speed on $[t_\varepsilon^-, b]$, and a slower speed on $[b, t_\varepsilon]$. This strategy will not be admissible, in general.

On the positive side, we notice that, if the construction speeds of Γ_1 and Γ_2 around P are the same, an outward corner simply cannot occur. Indeed, recalling (5.19), consider the normal velocity at which the reachable set expands:

$$h(x) = \sup_{y \in F(x)} \big\langle y\,,\ \mathbf{n}(x)\big\rangle\,.$$

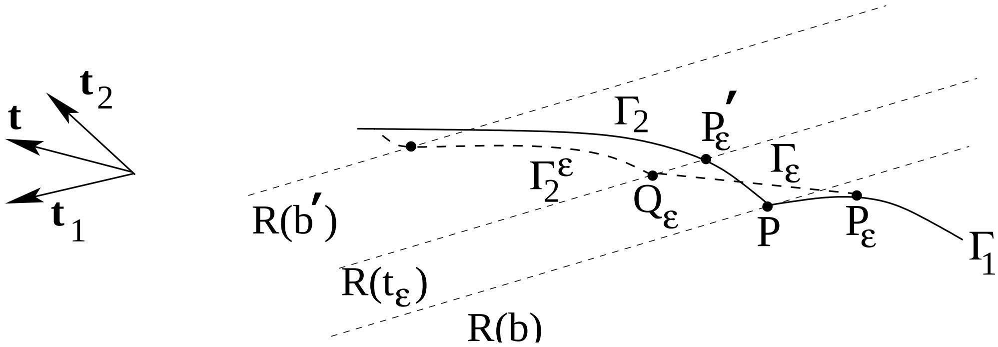

Figure 6: A non-parallel junction between a free arc and a boundary arc is not optimal.

Let σ be the common speed at which the walls Γ_1, Γ_2 are constructed, near P. The assumption that Γ_2 is a boundary arc implies

$$\big\langle \mathbf{t}_2\,,\ \mathbf{n}(P)\big\rangle\,\sigma = h(P)\,.$$

In addition, the outward corner assumption now implies

$$\big\langle \mathbf{t}_1\,,\ \mathbf{n}(P)\big\rangle\,\sigma > h(P)\,.$$

As a result, all points in Γ_1 are constructed away from the fire, including the endpoint P. This contradicts the assumption $P \in \partial R^{\gamma^*}(b)$, proving our claim.

7 - An example

We consider here the case of enclosing a region of minimal area, if walls constructed at a constant speed $\sigma > 2$. The initial contaminated area is $R_0 = B_1$, and $F(x) = B_1$ for all $x \in I\!R^2$, so that the infection spreads at unit speed in all directions.

We consider here four different strategies which block the contamination within a set of minimal area.

STRATEGY 1: Assuming $\sigma > 2\pi$, construct a circumference Γ centered at the origin with radius

$$r = \left(1 - \frac{2\pi}{\sigma}\right)^{-1}.$$

This takes a time $T = 2\pi r/\sigma = r - 1$ to construct, which is exactly the time needed by the contamination to reach the wall. In this case we have $\Gamma \in \mathcal{B}_1$, i.e. the entire circumference is a free arc. This strategy is not optimal, because the curvature vector $d\mathbf{t}/ds$ has the wrong direction, pointing toward the interior of the contaminated set. This violates Thoerem 3. One can easily see that, by replacing a small arc of the circumference by a straight segment, the area of the contaminated region can be reduced.

STRATEGY 2: Assume $\sigma > 2$ and consider the bounding strategy γ in terms of two logarithmic spirals, defined at (3.2)-(3.3). These two spirals originate from the same initial point $(r, \theta) = (1, 0)$, in different directions. According to Theorem 5, this strategy is not optimal.

STRATEGY 3: Start from the point Q_0 on the unit circumference with polar coordinates $(r, \theta) = (1, 0)$, and keep constructing a boundary wall in the counter-clockwise direction. This can be done as soon as $\sigma > 1$. The first portion of this wall will be an arc of logarithmic spiral, namely

$$r = e^{\lambda\theta} \qquad\qquad \theta \in [0, 2\pi],$$

with $\lambda = 1/\sqrt{\sigma^2 - 1}$.

The further portions of the spiral-like curve will be tighter, because, to reach a point (r, θ) starting from R_0 and avoiding the walls, the shortest possible path must go around the arc of the spiral γ already constructed. If σ is large enough, in finite time the spiral will close on itself stopping any further spread of the contamination. This strategy is illustrated in figure 7. We can divide the wall in two sections: an inner arc Γ_1, between Q_0 and Q_2, and an outer arc Γ_2 which begins and ends at Q_2. Notice that the portion of Γ_1 between Q_0 and Q_1 is exactly a logarithmic spiral. In this case, we have

$$\Gamma_1 \in \mathcal{B}_2 \qquad\qquad \Gamma_2 \in \mathcal{B}_1 .$$

Indeed, all arcs are constructed along the boundary of the set $R^\gamma(t)$. At a later time, points on Γ_1 are completely surrounded by the contamination, while Γ_2 coincides with the boundary of the contaminated set $R^\gamma(T)$, at the final time T. This strategy satisfies (trivially) our necessary conditions for optimality, although it might not be optimal.

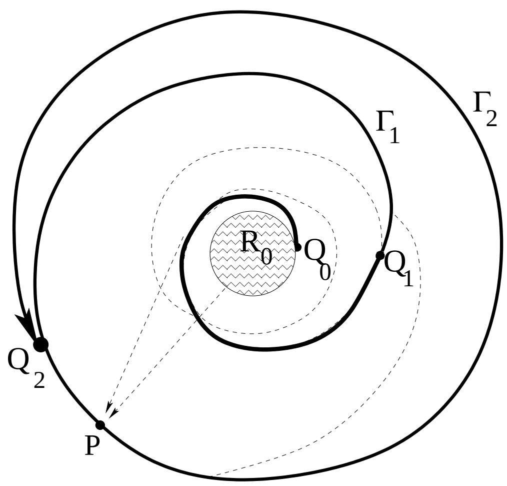

Figure 7. The contaminated set is enclosed by a spiral-like curve.

STRATEGY 4: First construct an arc of circumference Γ_1. Then construct two arcs of logarithmic spirals Γ_2, Γ_3 along the boundary of the contaminated set (see fig. 8). In this case, $\Gamma_1 \in \mathcal{F}_1$, while $\Gamma_2, \Gamma_3 \in \mathcal{B}_1$. Here the length of the arc Γ_1 should satisfy

$$m_1(\Gamma_1) = \sigma \cdot d(Q_1, R_0) = \sigma \cdot d(Q_2, R_0),$$

so that the two end-points Q_1, Q_2 are reached by the boundary of the contaminated set $R^\gamma(\tau)$ exactly at the time τ when the construction of the arc Γ_1 is completed. Moreover, by Theorem 5 and Remark 9 in Section 6, the junctions at Q_1 and at Q_2 must be C^1, i.e. the two arcs must have the same unit tangent vector. For each time τ, the above conditions determine a unique strategy

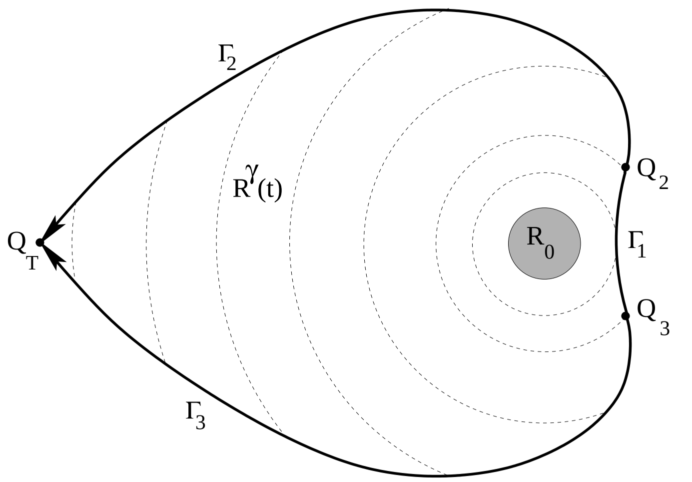

Figure 8: The contaminated set is enclosed by a circular arc Γ_1 and by two arcs of logarithmic spirals Γ_2, Γ_3.

$\gamma^{(\tau)}$. This reduces the problem to an optimization problem over the scalar parameter τ. A more detailed analysis will be needed to decide whether this "double spiral" strategy is actually better that the "single spiral" strategy described earlier.

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