Measure Valued Solutions to a Harvesting Game with Several Players

Alberto Bressan* and Wen Shen

Abstract We consider Nash equilibrium solutions to a harvesting game in one-space dimension. At the equilibrium configuration, the population density is described by a second-order O.D.E. accounting for diffusion, reproduction, and harvesting. The optimization problem corresponds to a cost functional having sublinear growth, and the solutions in general can be found only within a space of measures. In this chapter, we derive necessary conditions for optimality, and provide an example where the optimal harvesting rate is indeed measure valued. We then consider the case of many players, each with the same payoff. As the number of players approaches infinity, we show that the population density approaches a well-defined limit, characterized as the solution of a variational inequality. In the last section, we consider the problem of optimally designing a marine park, where no harvesting is allowed, so that the total catch is maximized.

1 Introduction

We consider the noncooperative ~~non-cooperative~~ harvesting game introduced in [8]. Let $\phi(x)$ denote the density of a fish population, or some other marine resource, at the location x. As "players" we consider N fishing companies, whose strategies are described by measures $\mu_1, \ldots, \mu_N$. Here, μ_i describes the intensity of harvesting effort by the i-th player. In one space dimension, a steady-state ~~steady state~~ configuration is characterized as the solution to the two-point boundary value problem

$$\phi'' + g(x,\phi) = \sum_{i=1}^{N} \phi\, \mu_i\,, \tag{1}$$

Alberto Bressan and Wen Shen
Department of Mathematics, Penn State University, University Park, Pa. 16802, USA. e-mail: bressan@math.psu.edu, shen_w@math.psu.edu

with boundary conditions

$$\phi'(0) \;=\; \phi'(R) \;=\; 0\,, \tag{2}$$

where primes denote derivatives w.r.t. the space variable x. The first term accounts for diffusion, the nonlinear function g describes population growth, while $\phi\mu_i$ is the amount of fish harvested by the i-th fishing company. In general, $\mu_1,\ldots,\mu_N$ are positive Radon measures supported on the closed interval $[0,R]$. Notice that the conditions (2) imply that no flux occurs across the boundary.

The goal of the i-th player is to maximize his/her net payoff

$$J_i = \int_0^R (\phi - c_i(x))\,\mathrm{d}\mu_i\,,$$

where $c_i(\cdot)$ is a strictly positive function, accounting for the harvesting cost.

Under suitable assumptions, the existence of solutions to the noncooperative ~~non-cooperative~~ game was proved in [8], within the class of nonnegative ~~non-negative~~ Radon measures. In this chapter, ~~the present paper~~ we derive a set of necessary conditions satisfied by these solutions. Since the domain is one-dimensional, we can use a variable transformation that transforms the optimization problem into a standard optimal control problem, to which the Pontryagin maximum principle can then be applied. Our results are formulated, more generally, in the presence of a drift coefficient, and with a variable diffusion coefficient.

A couple of examples are worked out in more details. In particular, we show that if the cost function is discontinuous, then the optimal solution can be a measure containing Dirac masses. This is indeed the case when a marine park is present, i.e., there is an open subset $¶ \subset [0,R]$ such that $c_i(x) = +\infty$ for all $x \in ¶$.

We then study the density of the fish population in the case of a large number of fishermen, each with the same harvesting cost $c_i(x) = c(x)$. Calling ϕ_N the population density corresponding to a Nash equilibrium solution with N fishermen, as $N\to\infty$ we prove the convergence $\phi_N \to \phi_\infty$, where ϕ_∞ provides the largest subsolution to the boundary value problem $\phi'' + g(x,\phi) = 0$, $\phi'(0) = \phi'(R) = 0$ satisfying the pointwise constraint $\phi(x) \le c(x)$. Equivalently, ϕ_∞ can also be characterized as the unique solution to a variational inequality. Indeed, consider the family of functions

$$\begin{aligned}\mathscr{K}_c \doteq \Big\{\phi : [0,R] \mapsto I\!R\,, \quad &\phi \text{ is Lipschitz continuous with } \phi' \in BV,\\ &\phi'(0+) \ge 0, \quad \phi'(R-) \le 0\,, \quad \phi(x) \le c(x) \;\; \forall x \in [0,R]\Big\}\,,\end{aligned} \tag{3}$$

where the lower semicontinuous function $c(\cdot)$ plays the role of an obstacle. Then the limiting density ϕ_∞ of the fish population satisfies $\phi_\infty \in \mathscr{K}_c$ and

$$\int_0^R \phi'_\infty\,(\psi' - \phi'_\infty)\,\mathrm{d}x + \int_0^R g(x,\phi_\infty)\cdot(\phi_\infty - \psi)\,\mathrm{d}x \;\ge\; 0 \qquad \forall \psi \in \mathscr{K}_c\,. \tag{4}$$

Roughly speaking, our results show that ~~,~~ when the number of players becomes large, in a noncooperative ~~non-cooperative~~ game each single individual has no incentive to care for the environment: each player keeps harvesting until the fish population is so low that the profit is completely offset by the harvesting cost, at each point of the domain.

In the last section of the chapter, ~~paper~~ we consider the problem of the optimal designing of a marine park. Namely, we seek an open subset $¶ \subset [0,R]$ such that, imposing the new harvesting cost

$$c^{(¶)}(x) = \begin{cases} c(x) & \text{if } x \in [0,R] \setminus ¶, \\ +\infty & \text{if } x \in ¶, \end{cases}$$

the corresponding solution $\phi^{(¶)}$ to the variational inequality (3)–(4) maximizes the total harvest, measured by

$$H(¶) \doteq \int_0^R g\left(x, \phi^{(¶)}(x)\right) \, \mathrm{d}x. \tag{5}$$

We prove that, if the initial cost function $c(\cdot)$ is continuous, then there exists an open set $¶ \subset [0,R]$ for which the quantity in (5) is maximized. An example is explicitly worked out.

For more general results on the optimization of variational inequalities, we refer to [1, 2, 4–7].

2 Necessary Conditions for Measure-Valued Optimal Solutions

Given nonnegative ~~non-negative~~ cost function $c : [0,R] \mapsto I\!R_+ \cup \{+\infty\}$, we consider the optimal control problem

$$\text{maximize:} \ \ J(\sigma) \doteq \int_0^R \big(\phi(x) - c(x)\big) \, \mathrm{d}\sigma. \tag{6}$$

The maximum is sought over all pairs (σ, ϕ), where $\sigma \in \mathscr{M}_+\big([0,R]\big)$ is a nonnegative ~~non-negative~~ Radon measure on the closed interval $[0,R]$ and $\phi : [0,R] \mapsto I\!R_+$ provides a distributional solution to the second order boundary value problem

$$\big(\alpha(x)\phi'\big)' - \big(\beta(x)\phi\big)' + g(x,\phi) - \phi\nu = \phi\,\sigma, \tag{7}$$

with boundary conditions

$$\alpha(0)\phi'(0) - \beta(0)\phi(0) \ = \ \alpha(R)\phi'(R) - \beta(R)\phi(R) \ = \ 0. \tag{8}$$

Here and in the sequel, a prime denotes differentiation w.r.t. x. In connection with a noncooperative ~~non-cooperative~~ game, (6)–(8) describe the optimization problem faced by one of the players. The measure ν accounts for the combined fishing effort of all the other players. Notice that (7) describes a more general situation than (1), because we allow here a nonconstant ~~non-constant~~ diffusion coefficient, and the presence of a drift term. The boundary condition (8) formally implies that the flux through the boundary vanishes.

Denoting by f_ϕ the partial derivative of a function $f = f(x,\phi)$ w.r.t. ϕ, our basic assumptions will be

(A1) The functions α, β are of class $\mathscr{C}^1$, with $\alpha(x) > 0$ for all $x \in [0,R]$, while $\nu \in \mathscr{M}_+\big([0,R]\big)$ is a given, nonnegative ~~non-negative~~ Radon measure. The cost function c is lower semicontinuous and strictly positive, namely $c(x) \geq c_0 > 0$ for all $x \in [0,R]$.

(A2) The source term g can be written in the form $g(x,\phi) = f(x,\phi)\,\phi$, where the function $f = f(x,\phi)$ is continuous w.r.t. both variables and twice continuously differentiable w.r.t. ϕ. Moreover, for some continuous function $h = h(x)$ one has

$$f(x,0) > 0\,, \quad f_\phi(x,\phi) < 0\,, \quad f\big(x,h(x)\big) = 0 \quad \text{for all } \ x \in [0,R], \ \ \phi \geq 0\,. \tag{9}$$

One can think of $h(x)$ as the maximum population density supported by the environment at the location x. Since we need to consider not only classical solution of (7)–(8) but, more generally, measure-valued solutions, a precise definition is needed.

Definition 1. By a solution of the boundary value problem (7)–(8), we mean a Lipschitz continuous map $x \mapsto \phi(x)$ such that

(i) The map $x \mapsto \phi'(x)$ has bounded variation and satisfies

$$\begin{cases} \alpha(0)\phi'(0+) - \Big[\beta(0) + \nu\big(\{0\}\big) + \sigma\big(\{0\}\big)\Big]\phi(0) \ = \ 0\,, \\ \alpha(R)\phi'(R-) - \Big[\beta(R) - \nu\big(\{R\}\big) - \sigma\big(\{R\}\big)\Big]\phi(R) \ = \ 0\,. \end{cases}$$

(ii) For every test function $\eta \in \mathscr{C}^1_c\big(]0,R[\big)$, in the space of continuously differentiable functions whose support is a compact subset of the open interval $]0,R[$, one has

$$\int_0^R \Big\{ -\alpha\phi'\eta' + \beta\phi\eta' + g(x,\phi)\,\eta \Big\}\,\mathrm{d}x - \int_0^R \phi\,\eta\,(\mathrm{d}\nu + \mathrm{d}\sigma) \ = \ 0\,.$$

If the measures ν, σ are absolutely continuous w.r.t. Lebesgue measure, and if the function ϕ has an absolutely continuous derivative, then $\nu\big(\{0\}\big) = \sigma\big(\{0\}\big) = \nu\big(\{R\}\big) = \sigma\big(\{R\}\big) = 0$ and the above definition coincides with the classical one. To see a consequence of Definition 1, call $\mathrm{d} \subset [0,R]$ the set of points where ϕ is differentiable. Since ϕ is Lipschitz continuous, $\mathrm{meas}(\mathrm{d}) = R$. For every $x_1, x_2 \in \mathrm{d}$ one has

$$\begin{aligned} &\alpha(x_2)\phi'(x_2) - \alpha(x_1)\phi'(x_1) \\ &\quad = \ \beta(x_2)\phi(x_2) - \beta(x_1)\phi(x_1) - \int_{x_1}^{x_2} g\big(x,\phi(x)\big)\,\mathrm{d}x + \int_{x_1}^{x_2} \phi(x)\,(\mathrm{d}\nu + \mathrm{d}\sigma)\,. \end{aligned}$$

Moreover, at any point $x \in]0,R[$, letting $x_1 \to x-$, $x_2 \to x+$, one checks that the left and right limits of the derivative $\phi'(x\pm)$ satisfy

$$\phi'(x+) - \phi'(x-) = \frac{\phi(x)}{\alpha(x)} \cdot (\nu + \sigma)\big(\{x\}\big)\,.$$

The following construction reduces the measure-valued optimization problem (6)–(8) to a standard optimal control problem, with control functions in $\mathbf{L}^\infty$. Fix any point $z \in [0,R]$ and let δ_z be the Dirac measure concentrating a unit mass at the point z. This will allow us to compare the optimal strategy σ with some other strategy containing a point mass at z. Introduce the variable

$$s(x) = x + \nu\big([0,x]\big) + \sigma\big([0,x]\big) + \delta_z\big([0,x]\big),$$

so that, as $x \in [0,R]$,

$$s(x) \in [0,S], \qquad\qquad S = R + \nu\big([0,R]\big) + \sigma\big([0,R]\big) + 1.$$

The function $x \mapsto s(x)$ admits a Lipschitz continuous inverse: $x = x(s)$, with

$$\theta^x(s) \doteq \frac{\mathrm{d}}{\mathrm{d}s} x(s) \in [0,1]. \tag{10}$$

Let $\tilde{\mu} \doteq \mathscr{L} + \nu + \sigma + \delta_z$ be the positive Radon measure on $[0,R]$ obtained as the sum of the Lebesgue measure $\mathscr{L}$ plus the three measures ν, σ, and δ_z. To ~~In order to~~ define the densities $\theta^\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\tilde{\mu}}$, $\theta^\sigma = \frac{\mathrm{d}\sigma}{\mathrm{d}\tilde{\mu}}$, and $\theta^z = \frac{\mathrm{d}\delta_z}{\mathrm{d}\tilde{\mu}}$ as functions of the variable s, we proceed as follows. For every $s \in [0,S]$, let

$$\begin{cases} s^+(s) \doteq \max\big\{\xi \in [0,S]\,;\ x(\xi) = x(s)\big\}, \\ s^-(s) \doteq \min\big\{\xi \in [0,S]\,;\ x(\xi) = x(s)\big\}. \end{cases} \tag{11}$$

Define the functions

$$\begin{cases} s \mapsto y^\nu(s) \doteq \nu\Big(\big[0, x(s)\big[\Big) + \nu\Big(\big\{x(s)\big\}\Big) \cdot \dfrac{s - s^-(s)}{s^+(s) - s^-(s)}, \\ s \mapsto y^\sigma(s) \doteq \sigma\Big(\big[0, x(s)\big[\Big) + \sigma\Big(\big\{x(s)\big\}\Big) \cdot \dfrac{s - s^-(s)}{s^+(s) - s^-(s)}, \\ s \mapsto y^z(s) \doteq \delta_z\Big(\big[0, x(s)\big[\Big) + \delta_z\Big(\big\{x(s)\big\}\Big) \cdot \dfrac{s - s^-(s)}{s^+(s) - s^-(s)}. \end{cases} \tag{12}$$

Observe that these functions are positive and nondecreasing~~non-decreasing~~. Moreover, (10) and (12) yield

$$x(s) + y^\nu(s) + y^\sigma(s) + y^z(s) = s \qquad\qquad \forall\, s \in [0,S]. \tag{13}$$

We can thus define

$$\theta^\nu(s) \doteq \frac{\mathrm{d}}{\mathrm{d}s} y^\nu(s), \qquad \theta^\sigma(s) \doteq \frac{\mathrm{d}}{\mathrm{d}s} y^\sigma(s), \qquad \theta^z(s) \doteq \frac{\mathrm{d}}{\mathrm{d}s} y^z(s).$$

Because of (13), the above definitions imply $\theta^x(s) + \theta^\nu(s) + \theta^\sigma(s) + \theta^z(s) = 1$ for a.e. $s \in [0,S]$.

For convenience, we shall now write $c(s) \doteq c\big(x(s)\big)$, and similarly for $\alpha(s)$ and $\beta(s)$. On the interval $[0,S]$, we also consider the functions

$$s \mapsto \phi(s) \doteq \phi\big(x(s)\big), \qquad s \mapsto \psi(s) \doteq \alpha\big(x(s)\big)\,\phi'\big(x(s)\big),$$

where ϕ' denotes the derivative of ϕ w.r.t. x. Notice that the map ϕ is well defined and continuous. However, if $\mu\big(\{x\}\big) > 0$, then ϕ' is discontinuous at x, hence it is not well defined as a function of the parameter s. To take care of points where μ has a point mass, and ϕ' thus has a jump, recalling (11) we define

$$\psi(s) \doteq \alpha\big(x(s)\big) \cdot \left(\frac{s-s^-}{s^+-s^-} \cdot \phi'\big(x(s)+\big) \;+\; \frac{s^+-s}{s^+-s^-} \cdot \phi'\big(x(s)-\big) \right).$$

By (7), the maps ϕ and $\psi = \phi'$ provide a solution to the system of O.D.E's

$$\begin{cases} \dfrac{d}{ds}\phi(s) = \theta^x(s)\,\dfrac{\psi(s)}{\alpha(s)}, \\ \dfrac{d}{ds}\psi(s) = \big(\theta^v(s) + \theta^\sigma(s)\big) \cdot \phi(s) \\ \qquad\qquad + \theta^x(s)\left[\beta'(s)\phi(s) + \dfrac{\beta(s)}{\alpha(s)}\psi(s) - g\big(x(s),\phi(s)\big)\right], \end{cases}$$

with boundary data

$$\psi(0) = \beta(0)\phi(0), \qquad \psi(S) = \beta(S)\phi(S).$$

From the optimality of the measure σ, it now follows that the control functions $u_1(s) \equiv 1$, $u_2(s) \equiv 0$ are optimal for the problem

$$\text{maximize:}\;\; J(u_1,u_2) \doteq \int_0^S \big(y_1(s) - c(s)\big)\Big[\theta^\sigma(s)u_1(s) + \theta^z(s)u_2(s)\Big]\,ds, \tag{14}$$

for the control system

$$\begin{cases} \dfrac{d}{ds}y_1 = \dfrac{\theta^x(s)}{\alpha(s)}\,y_2, \\ \dfrac{d}{ds}y_2 = \Big(\theta^v(s) + \theta^\sigma(s)u_1 + \theta^z u_2\Big)\,y_1 \\ \qquad\qquad + \theta^x(s)\left[\beta'(s)y_1 + \dfrac{\beta(s)}{\alpha(s)}y_2 - g\big(x(s),y_1\big)\right], \end{cases} \tag{15}$$

with boundary conditions

$$y_2(0) = \beta(0)y_1(0), \qquad y_2(S) = \beta(S)y_1(S). \tag{16}$$

The control $u = (u_1,u_2)$ in (14)–(16) ranges over all couples of nonnegative ~~non-negative~~ functions $u_1,u_2 : [0,S] \mapsto \mathbb{R}_+$. Notice that in this optimization problem all maps α,β,c,θ,x are given functions of $s \in [0,S]$ and do not depend on the particular choice of the controls u_1,u_2.

Since $(u_1,u_2) \equiv (1,0)$ is optimal, by Pontryagin's maximum principle there exists an adjoint vector $p = (p_1,p_2)$ such that the following equations hold.

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} y_1 = \frac{\theta^x}{\alpha} y_2 \,, \\ \frac{\mathrm{d}}{\mathrm{d}s} y_2 = (\theta^\nu + \theta^\sigma) \cdot y_1 + \theta^x \cdot \left[\beta' y_1 + \frac{\beta}{\alpha} y_2 - g(x, y_1)\right] , \end{cases} \tag{17}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} p_1 = -\Big[(\theta^\nu + \theta^\sigma) + \theta^x \big(\beta' - g_\phi(x, y_1)\big)\Big] p_2 - \theta^\sigma \,, \\ \frac{\mathrm{d}}{\mathrm{d}s} p_2 = -\frac{\theta^x}{\alpha} (p_1 + \beta\, p_2) \,, \end{cases} \tag{18}$$

together with the boundary conditions

$$\begin{cases} y_2(0) - \beta(0) y_1(0) = 0 \,, \\ y_2(S) - \beta(S) y_1(S) = 0 \,, \end{cases} \qquad \begin{cases} p_1(0) + \beta(0) p_2(0) = 0 \,, \\ p_1(S) + \beta(S) p_2(S) = 0 \,, \end{cases} \tag{19}$$

and moreover, for almost every $s \in [0, S]$ the following maximality condition holds:

$$\begin{aligned} &\theta^\sigma \big[(p_2 + 1) y_1 - c\big] \\ &\quad = \max_{\omega_1, \omega_2 \geq 0} \Big\{ \theta^\sigma \big[(p_2 + 1) y_1 - c\big] \omega_1 + \theta^z \big[(p_2 + 1) y_1 - c\big] \omega_2 \Big\} . \end{aligned} \tag{20}$$

Notice that (20) is equivalent to the two conditions

$$\theta^\sigma \cdot \big[(p_2 + 1) y_1 - c\big] = 0 \,, \qquad \theta^z \cdot \big[(p_2 + 1) y_1 - c\big] \leq 0 \,, \tag{21}$$

for a.e. $s \in [0, S]$.

It is convenient to rewrite the above conditions in terms of the original space variable x. Recall that $y_1 = \phi$, $y_2 = \alpha \phi'$, and set $q = p_2$. Observing that

$$\frac{\mathrm{d}(\alpha\phi')}{\mathrm{d}x} = \frac{\mathrm{d}(\alpha\phi')}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}x} = \frac{1}{\theta^x} \frac{\mathrm{d}y_2}{\mathrm{d}s} \,, \qquad q' = \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{1}{\theta^x} \frac{\mathrm{d}p_2}{\mathrm{d}s} = -\frac{p_1 + \beta p_2}{\alpha} \,,$$

from (17)–(19), we obtain the second-order ~~second order~~ equations

$$\begin{cases} (\alpha\phi')' + (\beta\phi)' + g(x, \phi) = \phi(\nu + \sigma), \\ (\alpha q')' + \beta q' + g_\phi(x, \phi) q = q(\nu + \sigma) + \sigma \,, \end{cases} \tag{22}$$

with boundary conditions

$$\begin{cases} \alpha(0)\phi'(0) - \beta(0)\phi(0) = 0 \,, \\ \alpha(R)\phi'(R) - \beta(R)\phi(R) = 0 \,, \end{cases} \qquad \begin{cases} q'(0) = 0 . \\ q'(R) = 0 . \end{cases} \tag{23}$$

By first identity in (21) there exists a set $\mathcal{N} \subset [0, R]$ with $\sigma(\mathcal{N}) = 0$ such that $(q(x) + 1)\phi(x) - c(x) = 0$ at every point $x \notin \mathcal{N}$. Moreover, at the particular point z, the second inequality in (21) implies $(q(z) + 1)\phi(z) - c(z) \leq 0$. We now observe that the previous construction can be performed with an arbitrary choice of the point $z \in [0, R]$. Our analysis can thus be summarized as follows.

Theorem 1. *Assume that the couple (σ, ϕ) provides an optimal solution to the optimization problem (6)–(8), where σ ranges within the class of all nonnegative*

~~non-negative~~ Radon measures on the interval $[0,R]$*, and* ϕ *is a corresponding solution of (7)–(8). Then there exists an adjoint function* $q : [0,R] \mapsto \mathbb{R}$ *such that the boundary value problem (22)–(23) is satisfied, in the sense of Definition 1. Moreover, one has the optimality conditions*

$$(q(x)+1)\phi(x) - c(x) = 0 \qquad \sigma\text{-a.e. on } [0,R], \tag{24}$$

$$(q(x)+1)\phi(x) - c(x) \le 0 \qquad \forall x \in [0,R]. \tag{25}$$

In the special case $\alpha(x) \equiv 1$, $\beta(x) \equiv 0$, the (22)–(25) reduce to

$$\begin{cases} \phi'' + g(x,\phi) = \phi\,(\nu+\sigma), \\ q'' + g_\phi(x,\phi) q = q(\nu+\sigma) + \sigma\,, \end{cases} \tag{26}$$

with boundary conditions

$$\phi'(0) = \phi'(R) = 0\,, \qquad q'(0) = q'(R) = 0\,, \tag{27}$$

The optimality conditions are still given by (24)–(25).

We remark that for a multidimensional ~~, for a multi-dimensional~~ optimization problem related to a linear elliptic PDE, necessary conditions of a similar type were derived in [9].

3 Examples

Relying on the necessary conditions established in the previous section, we now examine more in detail the solution to the optimization problem

$$\text{maximize: } J(\sigma) \doteq \int_0^R (\phi - c)\, d\sigma\,.$$

subject to

$$\phi'' + g(x,\phi) = \phi\,\sigma\,, \qquad \phi'(0) = \phi'(R) = 0\,. \tag{28}$$

Example 1. Assume $g(\phi) = (2-\phi)\phi$, $c(x) \equiv 1$. This corresponds to a space-homogeneous optimal control problem. As already observed in [12], in this case the optimal strategy σ is the measure having constant density $1/2$ w.r.t. Lebesgue measure. The corresponding optimal solution is $\phi(x) = 3/2$. The optimality conditions (24)–(27) are satisfied taking as adjoint function $q(x) = -1/3$.

Example 2. We now show that ~~,~~ if the cost function c is discontinuous, the optimal strategy can be a measure σ containing point masses. On the interval $[0,2]$, consider the functions

$$g(x,\phi) = (2-\phi)\phi\,, \qquad c(x) = \begin{cases} 1 & \text{if} \quad x \le 1\,, \\ 3 & \text{if} \quad x > 1\,. \end{cases} \tag{29}$$

Since the cost function $c(\cdot)$ is lower semicontinuous, the existence of an optimal solution (σ,ϕ) to (6)–(8) is provided by Theorem 1 in [8]. We observe that σ must be a nonzero ~~non-zero~~ measure whose support satisfies

$$\mathrm{Supp}(\sigma)\subset\Gamma\doteq\Big\{x\in[0,2]\,;\ \ \phi(x)>c(x)\Big\}\subseteq[0,1]\,. \tag{30}$$

Otherwise, the alternative measure $\tilde\sigma$, defined as $\tilde\sigma(A)\doteq\sigma(A\cap\Gamma)$ for every Borel set A, would achieve a strictly better payoff.

As shown in Lemma 1 in [8], if the fish density ϕ vanishes at some point $x_0\in[0,2]$, then ϕ is identically zero. In this ~~the present~~ case, we claim that

$$1\le\phi(x)\le2\qquad\qquad\forall x\in[0,2]\,.$$

Indeed, for any positive measure σ, every nonnegative ~~non-negative~~ solution of (28) satisfies $\phi\le2$. On the other hand, if the open set $S\doteq\big\{x\in]0,2[\,;\ \ \phi(x)<1\big\}$ is nonempty, consider a maximal open interval $]a,b[\ \subseteq S$. Observing that

$$0<\phi(x)<1,\qquad \phi''\ =\ -(2-\phi)\phi\ <\ 0\qquad\forall x\in]a,b[\,,$$

$$\begin{cases}\phi'(a)=0 & \text{if } a=0,\\ \phi(a)=1 & \text{if } a>0,\end{cases}\qquad\qquad\begin{cases}\phi'(b)=0 & \text{if } b=2,\\ \phi(b)=1 & \text{if } b<2,\end{cases}$$

an application of the maximum principle for parabolic equations yields $\phi(x)\ge1$ for all $x\in]a,b[$.

By Theorem 1, there exists an adjoint function q such that

$$\begin{cases}\phi''=-(2-\phi)\phi+\phi\sigma\,,\\ q''=(2\phi-2)q+(1+q)\sigma\,,\end{cases}\qquad\qquad\begin{cases}\phi'(0)\ =\ \phi'(2)\ =\ 0\,,\\ q'(0)\ =\ q'(2)\ =\ 0\,,\end{cases} \tag{31}$$

$$\begin{cases}(1+q)\phi\ =\ 1 & \sigma\text{-a.e.}\\ (1+q)\phi\ \le\ c(x) & \text{for a.e. } x\in[0,2]\,.\end{cases} \tag{32}$$

Observing that the functions ϕ,q are Lipschitz continuous, (32) can be rewritten as

$$\begin{cases}(1+q)\phi=1 & x\in\mathrm{Supp}(\sigma)\\ (1+q)\phi\le1 & x\in[0,1]\,,\\ (1+q)\phi\le3 & x\in]1,2]\,.\end{cases} \tag{33}$$

Recalling (29), (30), and observing that $g(\phi)\le0$ for $\phi\ge2$, we conclude that the function ϕ must satisfy the strict inequalities

$$1<\phi(x)<2\qquad\qquad\forall x\in[0,2]\,. \tag{34}$$

Since q is continuous and $\phi\in[1,2]$, the first equality in (33) implies

$$q\ =\ \frac{1}{\phi}-1\ \in\ \Big]-\frac12\,,\,0\Big[\qquad\qquad\forall x\in\mathrm{Supp}(\sigma).$$

Next, we prove the following claim:

(C) The measure σ is absolutely continuous w.r.t. Lebesgue measure on the half-open interval $[0,1[$, but contains a positive mass at the point $x = 1$. Moreover, $0 \in \text{Supp}(\sigma)$.

Indeed, by (34) and (30) it is clear that $\text{Supp}(\sigma) \subseteq [0,1]$. To prove that $0 \in \text{Supp}(\sigma)$, assume that, on the contrary, $a \doteq \min\{x;\ x \in \text{Supp}(\sigma)\} > 0$. Then, from (31) it follows that $\phi''(x) < 0$, $q''(x) < 0$ for all $x \in]0,a[$, and hence also $q'(x) < 0$ and $p'(x) < 0$ for all $x \in]0,a[$. Calling $\Theta(x) = \big(1+q(x)\big)\phi(x)$ the switching function, its derivative satisfies

$$\Theta' \;=\; (1+q)\phi' + q'\phi \;<\; 0 \qquad\qquad \forall x \in]0,a[\,.$$

In turn, this implies $\Theta(x) > \Theta(a) = 1$, $\forall x \in]0,a[$, providing a contradiction with the second inequality in (33).

On the other hand, if $b \doteq \max\{x;\ x \in \text{Supp}(\sigma)\} < 1$, a contradiction is obtained by an entirely similar argument. For $x \in]b,2[$, we have $\phi''(x) < 0$, $q''(x) < 0$, and hence $q'(x) > 0$ and $p'(x) > 0$. This implies

$$\Theta'(x) > 0 \qquad\qquad \forall x \in]b,2[\,.$$

In turn, this implies $1 = \Theta(b) < \Theta(x)$, $\forall x \in]b,2[$, providing a contradiction with the second inequality in (33).

Next, assume $\sigma\big(\{1\}\big) = 0$. Since $1 \in \text{Supp}(\sigma)$ and the functions ϕ, q are continuous, there must exist an increasing sequence of points $x_n \in \text{Supp}(\sigma)$, with $x_n \to 1$. By a nonsmooth ~~non-smooth~~ version of the intermediate value theorem, there exist a sequence of points $y_n \to 1$ such that $0 \in \partial\Theta(y_n)$ for every $n \geq 1$. Here, $\partial\Theta$ denotes the Clarke generalized gradient of Θ. Since $\Theta'(1+) > 0$, this shows that $\sigma\big([y_n, 1]\big) \geq c_0$ for some constant $c_0 > 0$ and all $n \geq 1$. Hence $\sigma\big(\{1\}\big) \geq c_0$, proving that σ must contain a Dirac mass at $x = 1$.

Finally, if the restriction of σ to $[0,1[$ is not absolutely continuous w.r.t. Lebesge measure, we could find a sequence of intervals $[a_n,b_n] \subset [0,1[$ with $a_n, b_n \in \text{Supp}(\sigma)$ and $\sigma\big([a_n,b_n]\big) \geq n(b_n - a_n) > 0$, $\forall n \geq 1$. Observing that

$$\begin{aligned}\Theta'' &= (1+q)\phi'' + 2q'\phi' + q''\phi \\ &= (1+q)\big[(2-\phi)\phi + \phi\sigma\big] + 2q'\phi' + \big[(2\phi - 2)q + (1+q)\sigma\big]\phi\end{aligned}$$

we conclude that $\Theta'(b_n+) > \Theta'(a_n-)$ for all n sufficiently large. This provides a contradiction with the assumptions $\Theta(a_n) = \Theta(b_n) = 1$ and $\Theta(x) \leq 1$ for all x in a neighborhood of the interval $[a_n,b_n]$. This completes the proof of our claim **(C)**.

Relying on the necessary conditions (31)–(32), to construct an optimal solution we proceed as follows. Assuming that $\text{Supp}(\sigma) = [0,1]$, for $x \in [0,1[$ the optimality conditions yield

$$\begin{cases} \phi'' = (\phi - 2 + u)\phi\,, \\ q'' = (2\phi - 2 + u)q + u\,, \end{cases} \tag{35}$$

and

$$\begin{cases} \phi(1+q) = 1\,, \\ \phi'(1+q) + \phi q' = 0\,, \\ \phi''(1+q) + 2\phi' q' + \phi q'' = 0\,. \end{cases} \tag{36}$$

In turn these imply

$$u = \frac{\phi''}{\phi} - \phi + 2\,, \qquad\qquad q = \frac{1}{\phi} - 1\,. \tag{37}$$

From the third equation in (36), using (37), the second equation in (36) and then the second equation in (35), one gets $\frac{\phi''}{\phi} + 2\phi' \left(-\frac{\phi'}{\phi}(1+q)\right) + \phi(2\phi - 2 + u)q + \phi u = 0$ and finally

$$\frac{\phi''}{\phi} + 2\phi' \left(-\frac{\phi'}{\phi}\frac{1}{\phi}\right) + \phi\left(2\phi - 2 + \frac{\phi''}{\phi} - \phi + 2\right)\left(\frac{1}{\phi} - 1\right) + \phi\left(\frac{\phi''}{\phi} - \phi + 2\right) = 0\,,$$

which leads to

$$\phi'' = \frac{(\phi')^2}{\phi} + \left(\phi - \frac{3}{2}\right)\phi^2\,.$$

Combine this with the first equation in (35), we get

$$u = \left(\frac{\phi'}{\phi}\right)^2 + \phi^2 - \frac{5}{2}\phi + 2 \geq \left(\frac{\phi'}{\phi}\right)^2 + \frac{7}{16} > 0\,.$$

To construct the optimal solution, we seek a continuous function $\phi : [0,2] \mapsto [1,2]$ such that

$$\phi'' = \frac{(\phi')^2}{\phi} + \left(\phi - \frac{3}{2}\right)\phi^2 \qquad x \in]0,1[\,, \tag{38}$$

$$\phi'' = (\phi - 2)\phi \qquad\qquad \text{if} \quad x \in]1,2[\,, \tag{39}$$

and satisfies the boundary conditions

$$\phi'(0) = 0\,, \qquad\qquad \phi'(2) = 0. \tag{40}$$

Notice that ϕ is Lipschitz continuous but ϕ' is expected to have a discontinuity at $x = 1$.

In addition, we seek a solution q to

$$q'' = (2\phi - 2)q \qquad\qquad x \in]1,2[\tag{41}$$

with boundary conditions

$$q(1) = \frac{1}{\phi(1)} - 1\,, \qquad q'(2) = 0\,, \quad \frac{q'(1+)}{1+q(1)} = \frac{\phi'(1+) - 2\phi'(1-)}{\phi(1)}. \tag{42}$$

Notice that the first identity in (40) is derived from (36), while the last one follows from the jump conditions

$$\begin{cases} \phi'(1+) - \phi'(1-) = \phi(1)\,\sigma\big(\{1\}\big)\,, \\ q'(1+) - q'(1-) = \big(1+q(1)\big)\,\sigma\big(\{1\}\big)\,, \end{cases}$$

observing that (36) implies $\frac{q'(1-)}{1+q(1)} = -\frac{\phi'(1-)}{\phi(1)}$. The optimal solution $\phi(\cdot)$ is now determined by solving the three-second ~~three second~~ order O.D.E's (38), (39), and (41), together with six boundary conditions, namely (40), (42), and the trivial continuity relation $\phi(1+) = \phi(1-)$. A numerical solution to this problem is given in Fig. 1. A posteriori, we check that the assumption $\text{Supp}(\sigma) = [0,1]$ is satisfied by the numerically computed solution. We note that the derivative ϕ' of the fish population density has an upward jump at $x = 1$, corresponding to a point mass in the measure σ, which is equal to 0.4075 in this case. Moreover, on the subinterval $[0,1[$ the optimal harvesting effort σ has density very close to 0.5. This is the optimal density in the spatially independent setting considered in Example 1.

Fig. 1 Numerically computed solutions: the fish population density ϕ (*left*) and the fishing intensity density u (*right*) for Example 1 (*dotted line*) and Example 2 (*solid line*). In the second case, the fishing intensity contains a point mass of 0.4075 at $x = 1$

4 Necessary Conditions for the Differential Game

Consider a differential game for N equal players, where each one wishes to maximize his/her payoff

$$\text{maximize:}\ \ J(\sigma) \doteq \int_0^R \big(\phi(x) - c(x)\big)\,\mathrm{d}\sigma\,. \tag{43}$$

The maximum is sought over all pairs (σ,ϕ), where $\sigma \in \mathcal{M}_+\big([0,R]\big)$ is a nonnegative ~~non-negative~~ Radon measure on the closed interval $[0,R]$ and $\phi : [0,R] \mapsto I\!R_+$ provides a distributional solution to the second-order ~~second order~~ boundary value problem

$$\phi'' + g(x,\phi) - \phi\,\nu = \phi\,\sigma\,, \tag{44}$$

with boundary conditions

$$\phi'(0) \ = \ \phi'(R) \ = \ 0\,. \tag{45}$$

If the couple (σ,ϕ) provides an optimal solution of the above problem, in connection with the measure $\nu = (N-1)\sigma$, then we say that (σ,ϕ) a *symmetric Nash* *equilibrium solution* to the N-players, noncooperative non-cooperative differential game.

Applying (26)–(27), with $\nu = (N-1)\sigma$, we obtain the existence of an adjoint function q such that

$$\begin{cases} \phi'' + g(x,\phi) = N\phi\sigma, \\ q'' + g_\phi(x,\phi)q = (Nq+1)\sigma, \end{cases} \tag{46}$$

with boundary conditions

$$\phi'(0) = \phi'(R) = 0, \qquad q'(0) = q'(R) = 0. \tag{47}$$

Moreover, the conditions in (24)–(25) remain the same

$$\begin{aligned} (q(x)+1)\phi(x) - c(x) &= 0 \qquad && \sigma\text{-a.e. on } [0,R], \\ (q(x)+1)\phi(x) - c(x) &\le 0 \qquad && \forall x \in [0,R]. \end{aligned} \tag{48}$$

Example 3. For the problem (43)–(45), assume that $c(x) \equiv \gamma > 0$, while $g(x,\phi) = (2-\phi)\phi$. Then a spatially independent symmetric Nash equilibrium solution is found by solving the algebraic system

$$\begin{cases} (2-\phi) = Nu, \\ (2-2\phi)q = (Nq+1)u, \\ (q+1)\phi = \gamma. \end{cases}$$

This yields

$$\begin{aligned} q = \frac{\gamma}{\phi} - 1, \qquad\qquad & u = \frac{2-\phi}{N}, \\ (2-2\phi)\left(\frac{\gamma}{\phi}-1\right) &= \left\{N\left(\frac{\gamma}{\phi}-1\right)+1\right\}\frac{2-\phi}{N}, \\ N(2-2\phi)(\gamma-\phi) &= (N\gamma - N\phi + \phi)(2-\phi), \\ (N+1)\phi^2 - (N\gamma+2)\phi &= 0, \\ \phi &= \frac{N\gamma+2}{N+1}. \end{aligned}$$

On an interval of length R, the total catch is $Nu\phi R = g(\phi)R = \left(2 - \frac{N\gamma+2}{N+1}\right)\frac{N\gamma+2}{N+1}R$, while the total payoff is

$$Nu(\phi-\gamma)R = \left(2 - \frac{N\gamma+2}{N+1}\right)\left(\frac{N\gamma+2}{N+1} - \gamma\right)R.$$

Notice that , as $N \to \infty$, the fish density satisfies $\phi_N \to \gamma$. The total catch approaches $(2-\gamma)\gamma R$, while the total payoff approaches zero.

Example 4. We now modify Example 3, assuming that a marine park is created on the open domain $]\xi,R]$. The new cost function thus takes the form

$$c(x) = \begin{cases} \gamma & \text{if} \quad x \le \xi\,, \\ +\infty & \text{if} \quad x > \xi\,. \end{cases}$$

Given an integer $N \ge 1$, a symmetric Nash equilibrium solution with N players can be computed by the same techniques used in Example 2.

Assuming that $\mathrm{Supp}(\sigma) = [0,\xi]$, for $x \in [0,\xi[$ the optimality conditions yield

$$\begin{cases} \phi'' = (\phi - 2 + Nu)\phi\,, \\ q'' = (2\phi - 2 + Nu)q + u\,, \end{cases} \tag{49}$$

$$\begin{cases} \phi(1+q) = \gamma, \\ \phi'(1+q) + \phi q' = 0\,, \\ \phi''(1+q) + 2\phi' q' + \phi q'' = 0\,. \end{cases} \tag{50}$$

In turn these imply

$$Nu = \frac{\phi''}{\phi} - \phi + 2\,, \qquad\qquad q = \frac{\gamma}{\phi} - 1\,. \tag{51}$$

From the third equation in (50), using (51), the second equation in (50) and then the second equation in (49), one gets

$$\frac{\gamma\phi''}{\phi} + 2\phi'\left(-\frac{\phi'}{\phi}(1+q)\right) + \phi(2\phi - 2 + Nu)q + \phi u = 0\,,$$

and finally

$$\frac{\gamma\phi''}{\phi} + 2\phi'\left(-\frac{\phi'}{\phi}\frac{\gamma}{\phi}\right) + \phi\left(2\phi - 2 + \frac{\phi''}{\phi} - \phi + 2\right)\left(\frac{\gamma}{\phi} - 1\right) + \frac{\phi}{N}\left(\frac{\phi''}{\phi} - \phi + 2\right) = 0\,,$$

which gives

$$\left(\frac{2\gamma}{\phi} - \frac{N-1}{N}\right)\phi'' - 2\gamma\left(\frac{\phi'}{\phi}\right)^2 + \left(\gamma + \frac{2}{N}\right)\phi - \frac{N+1}{N}\phi^2 = 0\,.$$

To construct the optimal solution, we seek a continuous function $\phi : [0,R] \mapsto [\gamma, 2]$ such that

$$\phi'' = \left(\frac{2\gamma}{\phi} - \frac{N-1}{N}\right)^{-1}\left[2\gamma\left(\frac{\phi'}{\phi}\right)^2 - \left(\gamma + \frac{2}{N}\right)\phi + \frac{N+1}{N}\phi^2\right] \quad x \in]0,\xi[\,, \tag{52}$$

$$\phi'' = (\phi - 2)\phi \qquad\qquad x \in]\xi, R[\,, \tag{53}$$

and satisfies the boundary conditions

$$\phi'(0) = 0\,, \qquad\qquad \phi'(R) = 0. \tag{54}$$

Notice that ϕ is Lipschitz continuous but ϕ' is expected to have a discontinuity at $x = \xi$.

In addition, we seek a solution q to

$$q'' = (2\phi - 2)q \qquad\qquad x \in]\xi, R[\tag{55}$$

with boundary conditions

$$\begin{aligned} q(\xi) &= \frac{\gamma}{\phi(\xi)} - 1, \quad q'(R) = 0, \\ q'(\xi+) &= \frac{1+Nq}{N\phi}\,\phi'(\xi+) - \frac{1+N+2Nq}{N\phi}\,\phi'(\xi-). \end{aligned} \tag{56}$$

Notice that the first identity in (56) is derived from (50), while the last one follows from the jump conditions

$$\begin{cases} \phi'(\xi+) - \phi'(\xi-) = N\phi(\xi)\,\sigma(\{\xi\}), \\ q'(\xi+) - q'(\xi-) = \big(1 + Nq(\xi)\big)\,\sigma(\{\xi\}), \end{cases}$$

observing that the second equation in (50) implies $\frac{q'(\xi-)}{1+q(\xi)} = -\frac{\phi'(\xi-)}{\phi(\xi)}$. The optimal solution $\phi(\cdot)$ can now be determined by solving the three second-order second order O.D.E's (52), (53), and (55), together with six boundary conditions, namely (54), (56), and the trivial continuity relation $\phi(\xi+) = \phi(\xi-)$.

Figure 2 shows the plots of total catch as functions of ξ, where $¶ =]\xi, 2]$ is the location of the marine park where no fishing is allowed. Here, the domain is $[0, R] = [0, 2]$ and the number of fishermen is $N = 40$. Two different fishing costs are considered: $\gamma = 0.5$ and $\gamma = 0.3$. For the smaller fishing cost $\gamma = 0.3$, we see that the marine park yielding the largest catch is $¶ \approx]0.45, 2]$. On the other hand, for the larger cost $\gamma = 0.5$, the optimal marine reserve is smaller, namely $¶ \approx]0.75, 2]$.

Figure 3 shows the total catch and total payoff as functions of the number of fishermen. Here, the marine park is $¶ =]1, 2]$, while the fishing cost outside the park is $\gamma = 0.3$. We see that the total catch decreases to a nonzero non-zero limit as N becomes large, with the maximum value reached with 2 fishermen. But for total payoff, it is a decreasing function and goes to 0 as N increases, with the maximum value reached with 1 fisherman.

Fig. 2 Plots of total catch depending on the location of marine reserve ξ, with $N = 40$ fishermen. On the left the cost of fishing (outside the park) is $\gamma = 0.5$, on the right $\gamma = 0.3$

Fig. 3 The total catch (marked with 'o') and total payoff (marked with '*') depending on the number of fishermen. Here ¶ $=]1,2]$ and $\gamma = 0.3$

5 The Fish Population for a Large Number of Fishermen

Let the assumptions (A1)–(A2) hold. Assume that ~~,~~ for each $N \geq 1$, the noncooperative ~~non-cooperative~~ game (43)–(45) admits a symmetric Nash equilibrium solution, say (σ_N, ϕ_N).

As the number N of fishermen grows without bound, a natural problem is to study the limiting density of the fish population. In this section, we will prove that the limit

$$\phi_\infty(x) \doteq \lim_{N\to\infty} \phi_N(x)$$

indeed exists, and can be characterized as the largest subsolution to

$$\phi'' + g(x,\phi) = 0, \qquad \phi'(0) = \phi'(R) = 0, \tag{57}$$

which satisfies the additional constraint

$$\phi(x) \leq c(x) \qquad \forall x \in [0,R]. \tag{58}$$

Equivalently, the limit ϕ_∞ can also be characterized as the unique strictly positive solution to a variational inequality. To state these results more precisely, we begin with some definitions.

Definition 2. By a subsolution ~~sub-solution~~ of the boundary value problem (57), we mean a Lipschitz continuous map $\phi : [0,T] \mapsto I\!R$ such that

(i) The map $x \mapsto \phi'(x)$ has bounded variation and satisfies

$$\phi'(0+) \geq 0, \qquad \phi'(R-) \leq 0.$$

(ii) For every nonnegative ~~non-negative~~ test function $\eta \in \mathscr{C}^1_c\big(]0,R[\big)$, one has

$$\int_0^R \Big\{ -\phi'\eta' + g(x,\phi)\,\eta \Big\}\,\mathrm{d}x \;\geq\; 0\,.$$

We remark that the largest subsolution of (57) satisfying the constraint (58) can be characterized as the solution to a variational inequality. Indeed, calling BV the space of functions with bounded variation, consider the family of functions

$$\mathscr{K}_c \doteq \Big\{ \phi : [0,R] \mapsto I\!R\,, \qquad \phi \text{ is Lipschitz continuous with } \phi' \in BV, \\ \phi'(0+) \geq 0, \quad \phi'(R-) \leq 0\,, \quad \phi(x) \leq c(x) \;\; \forall x \in [0,R] \Big\}\,.$$

Here, the lower semicontinuous function $c(\cdot)$ plays the role of an obstacle. Then the limiting density ϕ_∞ of the fish population satisfies $\phi_\infty \in \mathscr{K}_c$ and

$$\int_0^R \phi_\infty'(\psi' - \phi_\infty')\,dx + \int_0^R g(x,\phi_\infty)\cdot(\phi_\infty - \psi)\,dx \;\geq\; 0 \quad \forall \psi \in \mathscr{K}_c\,.$$

In particular, this implies

$$\begin{aligned} &\phi_\infty'' + g(x,\phi_\infty) \geq 0 && \text{for all } \; x \in]0,R[\,, \\ &\phi_\infty'' + g(x,\phi_\infty) = 0 && \text{on the open set where } \; \phi_\infty(x) < c(x)\,. \end{aligned}$$

Theorem 2. *Let the function g and c satisfy the assumptions in (A1)–(A2). For each $N \geq 1$, let (ϕ_N, σ_N) be a symmetric Nash equilibrium solution to the N-players noncooperative ~~non-cooperative~~ differential game (43)–(45), with $\phi_N > 0$. Then, as $N \to \infty$, one has the uniform convergence $\phi_N \to \phi_\infty$ in $\mathscr{C}^0\big([0,R]\big)$, where ϕ_∞ provides the largest positive subsolution to (57) which satisfies the additional constraint (58).*

Proof. We divide the argument in several steps.

(i). We first observe that the sequence of nonnegative ~~non-negative~~ Radon measures $N\sigma_N$ remains uniformly bounded. Indeed, since the support of each measure σ_N is contained in the region where $\phi_N(x) \geq c(x) \geq c_0 > 0$, we have

$$\int_{[0,R]} c_0 N\,d\sigma_N \leq \int_{[0,R]} \phi_N\, N\,d\sigma_N = \int_0^R g\big(x,\phi_N(x)\big)\,dx \leq \int_0^R \Big(\max_\xi g(x,\xi)\Big)\,dx\,.$$

In turn, the boundedness of the measures $N\sigma_N$ implies that the positive functions ϕ_N are uniformly Lipschitz continuous. By possibly taking a subsequence, we thus obtain the existence of a Lipschitz map ϕ_∞ such that $\phi_N(x) \to \phi_\infty(x)$ as $N \to \infty$, uniformly on $[0,R]$.

(ii). Recalling the assumptions in (A1), introduce the constants

$$h_{\min} \doteq \min_{x\in[0,R]} h(x) \;>\; 0\,, \qquad\qquad h_{\max} \doteq \max_{x\in[0,R]} h(x)\,.$$

A comparison argument now shows that the positive functions ϕ_N satisfy the uniform bounds

$$0 \;<\; \min\{c_0\,, h_{\min}\} \;\leq\; \phi_N(x) \;\leq\; h_{\max}\,. \tag{59}$$

In turn, this implies

$$0 \;<\; \min\{c_0\,, h_{\min}\} \;\leq\; \phi_\infty(x) \;\leq\; h_{\max}\,.$$

Next, define the largest subsolution ϕ^* by setting

$$\phi^*(x) \;\doteq\; \sup\Big\{\phi(x);\;\; \phi \;\text{ is a subsolution of (57)},\;\; 0 \leq \phi(y) \leq c(y) \;\; \forall y \in [0,R]\Big\}. \tag{60}$$

Since the cost function $c(\cdot)$ is lower semicontinuous, it is clear that ϕ^* is a Lipschitz continuous subsolution of (57) and satisfies the constraint (58). In the remainder of the proof, we will establish the equality $\phi_\infty = \phi^*$. In particular, this will show that the limit ϕ_∞ is independent of the choice of the subsequence. Hence, the entire sequence $(\phi_N)_{N\geq 1}$ converges to the same limit.

(iii). The inequality $\phi_\infty \geq \phi^*$ will be proved by showing that, for every $N \geq 1$,

$$\phi_N(x) \geq \phi^*(x) \qquad \text{for all } x \in [0,R]. \tag{61}$$

To prove (61), we first observe that the measure σ_N is supported on the closed set where $\phi_N(x) \geq c(x)$. Otherwise, any player could choose the alternative strategy $\tilde{\sigma}_N \doteq \sigma_N \cdot \chi_{\{\phi_N \geq c\}}$ and achieve a strictly better payoff.

If now $\phi_N(y) < \phi^*(y)$ at some point $y \in [0,R]$, define $\lambda \doteq \max_{x\in[0,R]} \frac{\phi^*(x)}{\phi_N(x)} > 1$. We then have $\lambda\phi_N(\bar{x}) = \phi^*(\bar{x})$ at some point $\bar{x}$, while $\lambda\phi_N(x) \geq \phi^*(x)$ for all $x \in [0,R]$. Introducing the function

$$\varphi(x) \doteq \lambda\phi_N(x) - \phi^*(x) \geq 0, \tag{62}$$

a contradiction is obtained as follows.

By continuity, $\phi_N(x) < \phi^*(x) \leq c(x)$ for all x in an open neighborhood $\mathcal{N}_{\bar{x}}$ of the point $\bar{x}$. Hence, recalling the assumption (A2) on the source term,

$$(\phi^*)'' + f(x,\phi^*)\phi^* \geq 0, \qquad \phi_N'' + f(x,\phi_N)\phi_N = 0 \qquad x \in \mathcal{N}_{\bar{x}}.$$

In turn, this yields

$$\begin{aligned} (\varphi)'' + f\big(x,\phi_N(x)\big)\,\varphi &= -(\phi^*)'' - f(x,\phi_N)\phi^* \\ &\leq \Big[f\big(x,\phi^*(x)\big) - f\big(x,\phi_N(x)\big)\Big]\,\phi^*(x) < 0. \end{aligned} \tag{63}$$

Indeed, $\phi^*(x) > \phi_N(x) > 0$ and by the assumption (9) the map $\phi \mapsto f(x,\phi)$ is strictly decreasing.

Since φ is continuous and $\varphi(\bar{x}) = 0$, (63) implies that $\varphi'' < 0$ in a neighborhood of $\bar{x}$. Three cases must be considered.

If $0 < \bar{x} < R$, we immediately obtain a contradiction with the inequality in (62).

If $\bar{x} = 0$, since $\sigma_N(\{\bar{x}\}) = 0$, we have $(\phi^*)'(0+) \geq 0$, $\phi_N'(0+) = \phi_N(0)\sigma_N(\{0\}) = 0$. In a neighborhood of the origin, the inequalities

$$\varphi(0) = 0, \qquad \varphi'(0+) = \lambda\phi_N'(0+) - (\phi^*)'(0+) \leq 0, \qquad \varphi''(x) < 0$$

clearly yield a contradiction with (62).

In a similar way, if $\bar{x} = R$, we deduce

$$\varphi(R) = 0, \qquad \varphi'(R-) = \lambda\phi_N'(R-) - (\phi^*)'(R-) \geq 0, \qquad \varphi''(x) < 0,$$

reaching again a contradiction with (62).

Since (61) holds for every $N \geq 1$, this establishes the inequality $\phi_\infty \geq \phi^*$.

(iv). In the next two steps, we work toward the converse inequality $\phi_\infty \leq \phi^*$.

For each N, call (ϕ_N, q_N) the solution to the corresponding boundary value problem (46)–(47). Aim of this step is to prove that

$$\lim_{N\to\infty} \|q_N\|_{\mathscr{C}^0} \;=\; 0\,. \tag{64}$$

Indeed, q_N provides a solution to the linear, nonhomogeneous non-homogeneous boundary value problem

$$q_N'' + \Big(g_\phi(x,\phi_N) - N\sigma_N\Big)\, q_N = \sigma_N\,, \qquad q_N'(0) = q_N'(R) = 0\,.$$

Hence, we have a representation $q_N(x) = \int_{[0,R]} K_N(x,y)\, d\sigma_N(y)$, where K_N is the Green kernel for the linear operator

$$\Lambda\,\psi \doteq \psi'' + \Big(g_\phi(x,\phi_N) - N\sigma_N\Big)\, \psi\,,$$

whose domain consists of functions satisfying $\psi'(0) = \psi'(R) = 0$.
By step **1**, as $N \to \infty$, the total mass of the measure σ_N approaches zero: $\sigma_N([0,R]) \to 0$. To establish the limit (64) it thus suffices to prove that all the kernels $K_N(\cdot,\cdot)$ are uniformly bounded. Toward this goal, fix $y \in [0,R]$ and call $\psi(x) \doteq K_N(x,y)$. Then ψ can be characterized as the solution to

$$\psi'' + \Big(g_\phi(x,\phi_N) - N\sigma_N\Big)\, \psi = \delta_y\,, \qquad \psi'(0) = \psi'(R) = 0\,,$$

where δ_y denotes the Dirac measure concentrating a unit mass at the point y. We now recall that $g_\phi(x,\phi_N) = f(x,\phi_N) + f_\phi(x,\phi_N)\phi_N$. Moreover, the functions ϕ_N satisfy the uniform bounds (59) and, for some constant c_f, the assumption (9) yields

$$f_\phi(x,\phi_N) \;\leq\; -c_f \;<\; 0 \qquad\qquad \forall x \in [0,R]\,.$$

Introduce the functions

$$z_N \doteq \frac{\phi_N'}{\phi_N}\,, \qquad\qquad z = \frac{\psi'}{\psi}\,.$$

Observe that z_N has bounded total variation, uniformly w.r.t. N, and provides a measurable solution to the boundary value problem

$$z_N' + z_N^2 + f(x,\phi_N) = N\sigma_N\,, \qquad\qquad z_N(0) = z_N(R) = 0\,. \tag{65}$$

On the other hand, the function $z = \psi'/\psi$ satisfies $z(0) = z(R) = 0$, together with

$$z' + z^2 + f(x,\phi_N) \;=\; N\sigma_N - f_\phi(x,\phi_N)\phi_N \;>\; N\sigma_N + c_f\,, \tag{66}$$

separately on the subintervals $[0,y[$ and $]y,R]$. Comparing (66) with (65), we conclude that

$$z(y-) - z(y+) \;\geq\; c_0 \;>\; 0\,, \tag{67}$$

for some positive constant c, independent of N,y.
The function $\psi(\cdot) = K_N(\cdot,y)$ can now obtained as

$$\psi(x) \;=\; \begin{cases} A\cdot \exp\left(\int_0^x z(s)\,ds\right) & \text{if } x\in[0,y]\,,\\ B\cdot \exp\left(-\int_x^R z(s)\,ds\right) & \text{if } x\in[y,R]\,,\end{cases}$$

choosing the constants A,B so that $\psi(y+) = \psi(y-)$, $\psi'(y+) - \psi'(y-) = 1$. This leads to the linear algebraic system

$$\begin{cases} B\cdot \exp\left(-\int_x^R z(s)\,ds\right) - A\cdot \exp\left(\int_0^x z(s)\,ds\right) \;=\; 0\,,\\ B\cdot \exp\left(-\int_x^R z(s)\,ds\right) z(y+) - A\cdot \exp\left(\int_0^x z(s)\,ds\right) z(y-) \;=\; 1\,.\end{cases} \tag{68}$$

From the uniform bounds on z, and the lower bound (67), we conclude that the constants A,B in (68) remain uniformly bounded, for all N,y. This establishes the uniform bound on K_N, proving our claim. In turn, this implies (64).

(v). From the optimality conditions (48), we deduce $(q_N+1)\phi_N - c \le 0$ for all $x\in[0,R]$. Letting $N\to\infty$ we conclude

$$\phi_\infty(x) \;\le\; \limsup_{N\to\infty} \phi_N(x) \;\le\; \limsup_{N\to\infty} \Big(c(x) - q_N(x)\phi_N(x)\Big) \;=\; c(x)$$

for all $x\in[0,R]$. We have thus shown that ϕ_∞ is a subsolution of (57) which satisfies $0<\phi_\infty(x)\le c(x)$ for all $x\in[0,R]$. By the definition of ϕ^* at (60), this trivially implies that $\phi_\infty\le\phi^*$, completing the proof.

□

6 Optimizing a Variational Inequality

Motivated by the above result, we now consider the problem of optimally designing a marine park, where $c=+\infty$, in such a way that the total catch is maximized. Calling ¶ $\subset [0,R]$ the open set where the park is located, it will be convenient to work with the complement $\Sigma \doteq [0,R]\setminus$ ¶. Given a closed set $\Sigma\subseteq[0,R]$, we thus consider the cost function

$$c(x,\Sigma) = \begin{cases} c(x) & \text{if} \quad x\in\Sigma\,,\\ +\infty & \text{if} \quad x\notin\Sigma\,,\end{cases}$$

and the domain

$$\mathscr{K}_\Sigma \doteq \Big\{\phi:[0,R]\mapsto I\!R\,, \qquad \phi \text{ is Lipschitz continuous with } \phi'\in BV,$$
$$\phi'(0+)\ge 0, \qquad \phi'(R-)\le 0\,, \qquad \phi(x)\le c(x,\Sigma)\ \ \forall x\in[0,R]\Big\}.$$

We now seek an optimal pair (ϕ,Σ), such that the integral $\int_0^R g(x,\phi)\,dx$ is maximized. Here, Σ ranges over all closed subsets of $[0,R]$, while $\phi \in \mathscr{K}_\Sigma$ provides a solution to the corresponding variational inequality

$$\int_0^R \phi'_\infty(\psi' - \phi'_\infty)\,dx + \int_0^R g(x,\phi_\infty)\cdot(\phi_\infty - \psi)\,dx \geq 0 \qquad \forall \psi \in \mathscr{K}_\Sigma\,. \tag{69}$$

Theorem 3. *In addition to the assumptions (A1)–(A2), let the cost function* c : $[0,R] \mapsto \mathbb{R}_+$ *be continuous. Then the one-dimensional optimization problem for the variational inequality has an optimal solution* (ϕ,Σ).

Proof. Let $\Sigma_n \subset [0,R]$ be a maximizing sequence of compact sets, and let ϕ_n be the corresponding solutions to the variational inequality, for $n \geq 1$. Since all the ϕ_n are uniformly Lipschitz continuous, by taking a subsequence we can assume that $\phi_n \to \phi$ uniformly on $[0,R]$. Moreover, we can assume that $\Sigma_n \to \Sigma$ in the Hausdorff metric [3], for some compact set $\Sigma \subseteq [0,R]$.

The uniform convergence $\phi_n \to \phi$ implies

$$\int_0^R g(x,\phi_n(x))\,dt \to \int_0^R g(x,\phi(x))\,dx \qquad \text{as } n \to \infty.$$

To conclude the proof, it suffices to show that ϕ provides the solution to the variational inequality (69), corresponding to the compact set $\Sigma \subseteq [0,R]$. Equivalently, we need to show that

(i) ϕ is a subsolution of (57)
(ii) $\phi(x) \leq c(x,\Sigma)$ for all $x \in [0,R]$
(iii) On the open set where $\phi(x) < c(x,\Sigma)$, the function ϕ satisfies

$$\phi''(x) + g(x,\phi(x)) = 0. \tag{70}$$

The property (i) follows from the uniform convergence $\phi_n \to \phi$, because each ϕ_n is a subsolution of (57).

To prove (ii), assume first $x \in \Sigma$. By the Hausdorff convergence $\Sigma_n \to \Sigma$, we can select points $x_n \in \Sigma_n$ such that $x_n \to x$ as $n \to \infty$. By the uniform Lipschitz continuity of the functions ϕ_n, and by the continuity of the cost function $c(\cdot)$, this yields

$$\phi(x) \;=\; \lim_{n\to\infty} \phi_n(x_n) \;\leq\; \lim_{n\to\infty} c(x_n) \;=\; c(x).$$

On the other hand, if $x \notin \Sigma$, one trivially has $\phi(x) \;<\; c(x,\Sigma) = +\infty$. This establishes (ii).

To prove (iii), assume $\phi(y) < c(y,\Sigma)$. We consider two cases. If $y \in \Sigma$, then $\phi(y) < c(y)$. By the uniform continuity and uniform convergence properties, we deduce $\phi_n(x) < c(x)$ for all n sufficiently large and every x in an open neighborhood $\mathscr{N}_y$ of y. Restricted to $\mathscr{N}_y$, all functions ϕ_n are solutions to the same equation (70). By the uniform convergence $\phi_n \to \phi$ we conclude that, on the open set $\mathscr{N}_y$, ϕ satisfies (70) as well.

On the other hand, if $y \notin \Sigma$, then we can find an open neighborhood $\mathscr{N}_y$ such that $\mathscr{N}_y \cap \Sigma_n = \emptyset$ for all n sufficiently large. In this case, each function ϕ_n satisfies (70) on $\mathscr{N}_y$, for n large enough. By the uniform convergence $\phi_n \to \phi$, we again conclude that, on the open set $\mathscr{N}_y$, ϕ satisfies (70) as well. This completes the proof. □

Example 5. On the domain $[0,R] \doteq [0,1]$, assume $g(x,\phi) = (2-\phi)\phi$ and consider the cost function

$$c(x) \doteq \left(1 - \frac{x}{2}\right)\gamma \qquad\qquad x \in [0,1]\,. \tag{71}$$

We claim that, if the constant $\gamma > 0$ is sufficiently small, then the choice $\Sigma = \{0\}$ is the unique optimal one. Indeed, it is clear that $\Sigma = \emptyset$ yields zero total catch, and cannot be optimal. If now $y \in \Sigma$, the corresponding solution ϕ^Σ of the variational inequality (69) will satisfy $\phi^\Sigma(x) \leq \phi^y(x)$, $y \in [0,1]$, where ϕ^y denotes the solution to

$$\begin{aligned} \phi'' + g(x,\phi) &= 0 \;\forall x \in]0,y[\cup]y,1[, \\ \phi(y) &= c(y) \;\; \phi'(0) = \phi'(1) = 0. \end{aligned}$$

Notice that, by choosing the constant γ sufficiently small, we can achieve $0 < \phi^y(x) < 1$ for all $x, y \in [0,1]$.

If Σ contains not only y but also additional points, then $0 < \phi^\Sigma(x) < \phi^y(x) < 1$ for all $x \neq y$. Hence

$$\int_0^1 \big(2 - \phi^\Sigma(x)\big)\, \phi^\Sigma(x)\, \mathrm{d}x \;<\; \int_0^1 \big(2 - \phi^y(x)\big)\, \phi^y(x)\, \mathrm{d}x,$$

showing that Σ cannot be optimal. By the above remarks, the optimal choice is restricted to singletons: $\Sigma = \{y\}$ for some $y \in [0,1]$. The particular form of the cost function $c(\cdot)$ implies that $\Sigma = \{0\}$ is the unique optimal strategy. This means that , if γ in (71) is sufficiently small, in order to maximize the total catch the marine park should be ¶ $=]0,1]$, and fishing should be allowed only at the point $x = 0$.

Notice that the same conclusion remains valid for every strictly decreasing cost function $c : [0,1] \mapsto I\!R_+$, provided that $c(0)$ is sufficiently small.

Remark 1. In Theorem 3, the continuity assumption on the cost function $c(\cdot)$ is essential. Otherwise, a counterexample could be constructed as follows. With reference to Example 5 above, let us replace $c(\cdot)$ in (71) with the lower semicontinuous cost function

$$\tilde{c}(x) \doteq \begin{cases} \left(1 - \dfrac{x}{2}\right)\gamma & \text{if} \quad 0 < x \leq 1\,, \\[2ex] \dfrac{\gamma}{3} & \text{if} \quad x = 0\,. \end{cases} \tag{72}$$

Then, choosing

$$\Sigma_n \doteq \left\{\frac{1}{n}\right\}$$

we obtain a maximizing sequence where the total catch converges to the same maximum achieved in Example 5. However, when the cost is given by (72), the set $\Sigma = \{0\}$ is not optimal. In this case, the variational problem does not have any optimal solution.

Remark 2. In this chapter, ~~the present paper~~ we analyzed only problems in one-space ~~one space~~ dimension. However, we expect that Theorems 1 and 2 can be extended to multidimensional ~~multi-dimensional~~ problems. On the other hand, Theorem 3 cannot have a direct counterpart valid in dimension $n \geq 2$. Indeed, the results in [10, 11] indicate that the multidimensional ~~multi-dimensional~~ optimization problem is not well posed. To ~~In order to~~ have the existence of an optimal solution, an additional cost term is needed. For example, one could add a penalization term proportional to the total length of the boundary ∂¶ of the marine park.

Acknowledgements The work of the first author was partially supported by NSF, with grant DMS-0807420. "The work of the second author was partially supported by NSF grant DMS-0908047. This paper was written as part of the international research program on Nonlinear Partial Differential Equations at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during the academic year 2008–2009.

References

1. Adams, D.R., Lenhart, S.: An obstacle control problem with a source term. Appl. Math. Optim. **47**, 79–95 (2003)
2. Adams, D.R., Lenhart, S., Yong, J.: Optimal control of the obstacle for an elliptic variational inequality. Appl. Math. Optim. **38**, 121–140 (1998)
3. Aubin, J.P., Cellina, A.: Differential Inclusions. Springer, Berlin (1984)
4. Bergounioux, M.: Optimal control of semilinear Elliptic Obstacle Problem. J. Nonlinear Convex Anal. **3** (2002)
5. Bergounioux, M., Lenhart, S.: Optimal control of the obstacle in semilinear variational inequalities. Positivity **8**, 229–242 (2004)
6. Bergounioux, M., Mignot, F.: Optimal control of obstacle problems: existence of Lagrange multipliers. ESAIM Control Optim. Calc. Var. **5**, 45–70 (2000)
7. Bergounioux, M., Zidani, H.: Pontryagin maximum principle for optimal control of variational inequalities. SIAM J. Control Optim. **37**, 1273–1290 (1999)
8. Bressan, A., Shen, W.: Measure valued solutions for a differential game related to fish harvesting. SIAM J. Control Optim. **47**, 3118–3137 (2009)
9. Buttazzo, G., Dal Maso, G.: Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions. Appl. Math. Optim. **23**, 17–49 (1991)
10. Cioranescu, D., Murat, F.: A strange term coming from nowhere. In: Topics in the mathematical modelling of composite materials. Progr. Nonlinear Differential Equations Appl. **31**, 45–93. Birkhäuser, Boston (1997)
11. Dal Maso, G., Murat, F., Orsina, L., Prignet, A.: Renormalized solutions of elliptic equations with general measure data. Ann. Sc. Norm. Super Pisa Cl. Sci. **28**, 741–808 (1999)
12. Neubert, M.G.: Marine reserves and optimal harvesting. Ecol. Lett. **6**, 843–849(2003)