A VARIATIONAL CALCULUS FOR DISCONTINUOUS SOLUTIONS OF SYSTEMS OF CONSERVATION LAWS

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0. Introduction

This paper is concerned with the Cauchy problem for the perturbed system of conservation laws in a single space variable:

$$u_t + [F(u)]_x = h(t,x,u) \qquad\qquad u(0,x) = \bar u(x), \tag{1}$$

where $F : I\!R^n \mapsto I\!R^n$ and $h : [0,+\infty) \times I\!R \times I\!R^n \mapsto I\!R^n$ are smooth maps. We assume that the system (1) is strictly hyperbolic, and that each characteristic field is either linearly degenerate or genuinely nonlinear in the sense of Lax [8].

Given a piecewise Lipschitz continuous solution $u = u(t,x)$ of (1), we shall study the behavior of first-order variations of u, by defining a suitable class of "generalized tangent vectors" and determining their evolution in time. More precisely, consider a family of initial conditions $u(0,x) = \bar u^\varepsilon(x)$ depending on a small parameter ε, and assume that $\bar u^\varepsilon$ admits an expansion of the form

$$\bar u^\varepsilon(x) = \bar u(x) + \varepsilon \bar v(x) + o(\varepsilon). \tag{2}$$

with

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \|o(\varepsilon)\|_{\mathbf{L}^1} = 0.$$

As long as the solutions remain smooth, it is well known that (1) is equivalent to the quasilinear system

$$u_t + A(u)u_x = h(t,x,u) \tag{3}$$

where $A(u) = DF(u)$ denotes the $n \times n$ Jacobian matrix of F at u. Under suitable regularity assumptions, the solution of (3) with initial condition (2) can then be written as

$$u^\varepsilon(t,x) = u(t,x) + \varepsilon v(t,x) + o(\varepsilon), \tag{4}$$

where u is the solution of (1) and the first order perturbation v solves the linear Cauchy problem

$$v_t + A(u)v_x + \big[DA(u)\cdot v\big]u_x = \frac{\partial h}{\partial u}\cdot v, \qquad\qquad v(0,x) = \bar v(x). \tag{5}$$

However, if the solution u contains shocks, an expansion of the form (4) can no longer describe the perturbed solution u^ε with first order accuracy, in the standard $\mathbf{L}^1$ norm.

Example 1. For Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{6}$$

consider the one-parameter family of initial conditions

$$\bar{u}^\varepsilon(x) = (1+\varepsilon)x \cdot \chi_{[0,1]}(x), \tag{7}$$

where χ_I denotes the characteristic function of the interval I. Call u^ε the solution of (6) with initial condition (7). A direct calculation yields

$$u^\varepsilon(t,x) = \frac{(1+\varepsilon)x}{1+(1+\varepsilon)t} \cdot \chi_{[0,\ \sqrt{1+(1+\varepsilon)t}]}(x). \tag{8}$$

At $t = 0$ the "tangent vector"

$$\bar{v}(x) = \lim_{\varepsilon\to 0} \frac{\bar{u}^\varepsilon(x) - \bar{u}^0(x)}{\varepsilon} = x \cdot \chi_{[0,1]}(x)$$

is a well defined element of $\mathbf{L}^1$, but for any $t > 0$ the location of the jump in $u^\varepsilon(t,\cdot)$ depends on the parameter ε, and the expression

$$\lim_{\varepsilon\to 0} \frac{u^\varepsilon(t,\cdot) - u^0(t,\cdot)}{\varepsilon} \tag{9}$$

does not define any function in $\mathbf{L}^1$. Notice, however, that (9) remains meaningful for all t if interpreted as a weak limit in a space of measures. Indeed, by (8), the limit (9) yields a measure μ_t whose absolutely continuous part is given by

$$d\mu_t = \frac{x}{(1+t)^2} \cdot \chi_{[0,\ \sqrt{1+t}]}(x)\ dx,$$

and whose singular part consists of a point mass located at $x_1(t) \doteq \sqrt{1+t}$, with magnitude

$$\Big[u\big(t,\ x_1(t)-\big) \ - \ u\big(t,\ x_1(t)+\big)\Big] \cdot \frac{d}{d\varepsilon}\sqrt{1+(1+\varepsilon)t} = \frac{\sqrt{1+t}}{1+t} \cdot \frac{t}{2\sqrt{1+t}} = \frac{t}{2(1+t)}.$$

The previous simple example shows that, in order to achieve a description of the perturbed solution u^ε of (1) which is accurate up to first order as $\varepsilon \to 0$, one must consider "generalized tangent vectors", consisting of measures whose absolutely continuous part is some $\mathbf{L}^1$ function, and whose singular part is given by a number of point charges, one for each shock in the reference solution $u(t,\cdot)$. Aim of this paper is to develop a calculus for these first order variations, valid for a general class of piecewise Lipschitz continuous solutions of $n \times n$ systems of conservation laws.

The paper consists of three chapters, each divided into subsections. Chapter 1 collects the basic assumptions and notations concerning the hyperbolic system (1), and reviews the main properties of piecewise Lipschitz continuous solutions, obtained by the method of characteristics.

In Chapter 2 we introduce a family of "generalized first order tangent vectors" to a given solution u of (1). These vectors have the form $(v, \xi) \in \mathbf{L}^1 \times I\!R^N$, where the pointwise limit

$$v(t,x) = \lim_{\varepsilon \to 0+} \frac{u^\varepsilon(t,x) - u^0(t,x)}{\varepsilon} \tag{10}$$

yields the absolutely continuous part of the measure (9), while $\xi = (\xi_1, \ldots, \xi_N)$ describes the rates at which the locations $x_1^\epsilon(t) < \cdots < x_N^\epsilon(t)$ of the jumps in $u^\varepsilon(t, \cdot)$ are shifted as ε varies:

$$\xi_i(t) = \lim_{\varepsilon \to 0+} \frac{x_i^\varepsilon(t) - x_i^0(t)}{\varepsilon}. \tag{11}$$

This provides a complete description of the singular part of the measure (9), which thus consists of N point charges, located at the points $x_i(t)$, with magnitudes $\big[u(x_i(t)-) - u(x_i(t)+)\big]\xi_i(t)$.

In order to derive a linear evolution equation for the vector (v, ξ), we first define a class of variations $\varepsilon \to \bar{u}^\varepsilon$ of the initial condition $\bar{u}$ in (1), which can be described up to first order by a generalized tangent vector $(\bar{v}, \bar{\xi})$. Under suitable regularity assumptions, we show that, for $t > 0$, the corresponding solutions $u^\varepsilon(t, \cdot)$ of (1) with initial condition (2) can still be described with first order accuracy in terms of a vector $(v(t), \xi(t))$. As long as the shocks in u do not interact, we prove that (v, ξ) satisfies a linear hyperbolic system coupled with N ordinary differential equations, given at (2.16)–(2.18).

Chapter 3 is concerned with the behavior of tangent vectors at a time τ where two shocks in the reference solution u interact, so that the number of discontinuities in $u(t, \cdot)$ may change, say from N to N'. A solution of the generalized Riemann problem determined by the interaction is here constructed as the limit of a family u^θ of piecewise Lipschitz solutions, whose initial profiles are obtained by separating the shocks and rarefaction waves of $u(\tau+, \cdot)$ at a distance θ from each other. The convergence of approximate solutions is here guaranteed by a stability condition for the perturbed Riemann problem, quite similar to the one used in [10]. In essence, this assumption implies that the sizes of all waves, generated by successive reflections of a small perturbing wave, remain uniformly bounded.

The previous analysis on tangent vectors to solutions with isolated discontinuities allows us to prove the uniqueness and the Lipschitz continuous dependence (in the $\mathbf{L}^1$ norm) of all solutions constructed by our procedure. For a more general uniqueness result within the class of piecewise Lipschitz functions we refer to [6]. Alternative constructions can be found in [9, 12].

Our main result, stated as Theorem 3.10, shows that for $t > \tau$ the perturbed solutions $u^\varepsilon(t, \cdot)$ still admit a first order approximation in terms of some tangent vector $(v(t), \xi(t)) \in \mathbf{L}^1 \times I\!R^{N'}$. The equations (3.69), (3.70) are proved, relating the forward limit $(v, \xi)(\tau+)$ with the backward limit $(v, \xi)(\tau-)$.

The first-order variational calculus developed in this paper makes it possible to derive accurate estimates on the dependence of piecewise Lipschitz solutions of (1) on the initial data. As an application, the forthcoming paper [3] introduces a family of a weighted norms $\|(v, \xi)\|_u$ on the space of generalized tangent vectors, defined in terms of Glimm's wave interaction functional, which is nonincreasing in time, along all solutions of the linearized variational equation. The analysis indicates that the Riemann-type metric generated by these norms, equivalent to the standard $\mathbf{L}^1$-distance, is globally contractive w.r.t. the evolution semigroup generated by the system of conservation laws, on a set of functions with small total variation. For various classes of hyperbolic systems, the validity of this conjecture has recently been proved in [1, 2, 4].

A second application of the present results is concerned with optimization problems for controlled systems of conservation laws. Necessary conditions for an optimal distributed parameter control, in the form of a Pontryagin Maximum Principle, were recently derived by the authors in [5].

1. Piecewise Lipschitz Solutions

1.1 - Basic assumptions and notations.

In the following, $|\cdot|$ and $\langle\cdot\,,\cdot\rangle$ denote the euclidean norm and inner product, respectively. Consider an open convex set $\Omega \subseteq I\!R^n$, let $F : \Omega \mapsto I\!R^n$ be a $\mathcal{C}^2$ vector field and assume that $h : [0,T]\times I\!R\times\Omega \mapsto I\!R^n$ is a bounded continuous function, continuously differentiable w.r.t. x and u.

By a weak solution of the Cauchy problem

$$u_t + [F(u)]_x = h(t,x,u), \tag{1.1}$$

$$u(0,x) = \bar{u}(x) \tag{1.2}$$

on $[0,T]$ we mean a function $u \in \mathbf{L}^1_{loc}\big([0,T]\times I\!R;\ I\!R^n\big)$ such that

$$\begin{aligned}\int_0^T\int_{-\infty}^{+\infty}\Big[\big\langle u,\ \phi_t\big\rangle + \big\langle F(u),\ \phi_x\big\rangle\Big]\,dxdt + \int_{-\infty}^{+\infty}\big\langle \bar{u}(x),\ \phi(0,x)\big\rangle\,dx\\ = \int_{-\infty}^{\infty}\big\langle u(T,x),\ \phi(T,x)\big\rangle\,dx - \int_0^T\int_{-\infty}^{+\infty}\big\langle h\big(t,x,u(t,x)\big),\ \phi(t,x)\big\rangle\,dxdt,\end{aligned} \tag{1.3}$$

for every $\mathcal{C}^1$ function $\phi : I\!R\times I\!R \mapsto I\!R^n$ with compact support. If u is $\mathcal{C}^1$ or merely Lipschitz continuous, (1.1) is equivalent to the quasilinear system

$$u_t + A(u)u_x = h(t,x,u), \tag{1.4}$$

where $A(u) = DF(u)$ denotes the $n\times n$ Jacobian matrix of partial derivatives of F. We assume that the above system is strictly hyperbolic, i.e. each matrix $A(u)$ has n real distinct eigenvalues $\lambda_1 < \cdots < \lambda_n$. By $r_i(u)$, $l_i(u)$ we denote respectively the i-th right and left eigenvectors of $A(u)$, normalized so that

$$|r_i(u)| \equiv 1, \qquad \big\langle l_i(u),\ r_j(u)\big\rangle \equiv \delta_{ij}, \tag{1.5}$$

where δ_{ij} is the Kronecker symbol. For $u, v \in \Omega$, define the matrix

$$A(u,v) = \int_0^1 A\big(\theta u + (1-\theta)v\big)\,d\theta. \tag{1.6}$$

Clearly $A(u,v) = A(v,u)$ and $A(u,u) = A(u)$. For $i = 1,\dots,n$, the i-th eigenvalues and eigenvectors of $A(u,v)$ will be denoted by $\lambda_i(u,v)$, $r_i(u,v)$, $l_i(u,v)$. We assume that the ranges of the eigenvalues λ_i do not overlap, i.e. that there exist λ_i', $\lambda_i'' \in I\!R$ and a constant $\epsilon_1 > 0$ such that

$$\lambda_i(u,v) \in [\lambda_i', \lambda_i''] \qquad \forall u,v\in\Omega,\ i\in\{1,\dots,n\}, \tag{1.7}$$

$$\lambda_i'' + \epsilon_1 \le \lambda_{i+1}' \qquad \forall i = 1,\dots,n-1. \tag{1.8}$$

Because of the regularity of A, it is possible to choose $r_i,\ l_i$ to be $\mathcal{C}^1$ functions of $u,\ v$, normalized so that

$$\big|r_i(u,v)\big| \equiv 1, \qquad \big\langle l_i(u,v),\ r_j(u,v)\big\rangle \equiv \delta_{ij}. \tag{1.9}$$

We assume that there exist constants $\hat\lambda$, $\hat l$ such that

$$\big|\lambda_i(u,v)\big| \le \hat\lambda, \qquad \big|l_i(u,v)\big| \le \hat l \qquad i \in \{1,\dots,n\},\ u,v \in \Omega. \tag{1.10}$$

If ϕ is any function defined on Ω, its directional derivative along r_i at u will be denoted by

$$r_i \bullet \phi(u) \doteq [\nabla\phi(u)]\, r_i(u) = \lim_{\varepsilon\to 0} \frac{\phi\big(u+\varepsilon r_i(u)\big) - \phi(u)}{\varepsilon}.$$

Moreover, for $u^+, u^- \in \Omega$ we write

$$r_i^+ \bullet l_j(u^+,u^-) \doteq \lim_{\varepsilon\to 0} \frac{l_j\big(u^+ + \varepsilon r_i(u^+), u^-\big) - l_j(u^+,u^-)}{\varepsilon}, \tag{1.11}$$

$$r_i^- \bullet l_j(u^+,u^-) \doteq \lim_{\varepsilon\to 0} \frac{l_j\big(u^+, u^- + \varepsilon r_i(u^-)\big) - l_j(u^+,u^-)}{\varepsilon}. \tag{1.12}$$

For each $k \in \{1,\dots,n\}$, we assume that either the k-th characteristic field is genuinely nonlinear and

$$\min\Big\{|\lambda_k(u^+) - \lambda_k(u^+,u^-)|,\ |\lambda_k(u^-) - \lambda_k(u^+,u^-)|\Big\} \ge \epsilon_2 |u^+ - u^-| \tag{1.13}$$

for some $\epsilon_2 > 0$ and all $u^+, u^- \in \Omega$ connected by a shock of the k-th family, or else that the k-th characteristic field is linearly degenerate, so that $r_k \bullet \lambda_k(u) \equiv 0$ and

$$\lambda_k(u^+) = \lambda_k(u^+,u^-) = \lambda_k(u^-) \tag{1.14}$$

whenever u^+ and u^- are connected by a contact discontinuity of the k-th family.

For the differential of the i-th eigenvalue of the matrix A in (1.6) we use the notation

$$D\lambda_i(u^+,u^-)\cdot(v^+,v^-) \doteq \lim_{\varepsilon\to 0} \frac{\lambda_i(u^+ + \varepsilon v^+, u^- + \varepsilon v^-) - \lambda_i(u^+,u^-)}{\varepsilon}. \tag{1.15}$$

A similar notation will be used for the differentials $Dl_i(u^+,u^-),\ Dr_i(u^+,u^-)$ of the left and right eigenvectors of A.

1.2 - Weak solutions of semilinear systems.

This section is concerned with the definition and the basic properties of integral solutions for the semilinear hyperbolic system

$$u_t + A(t,x)u_x = h(t,x,u). \tag{1.16}$$

We assume that the entries of the matrix A are Lipschitz continuous and that h is measurable w.r.t. t, x and Lipschitz continuous w.r.t. u. As usual, λ_i, l_i, r_i denote the i-th eigenvalue and left, right eigenvectors of A, respectively.

Call $y_i(\cdot;\tau,\xi)$ the i-th characteristic line through the point (τ,ξ). Otherwise stated, let $t\mapsto y_i(t;\tau,\xi)$ be the solution of the Cauchy problem

$$\dot y(t)=\lambda_i\big(t,y(t)\big),\qquad\qquad y(\tau)=\xi. \tag{1.17}$$

Given an initial condition

$$u(0,x)=\bar u(x)\qquad\qquad x\in[a,b], \tag{1.18}$$

we say that a region $\mathcal{D}\subset I\!\!R_+\times I\!\!R$ is a *domain of determinacy* for the Cauchy problem (1.16), (1.18) provided that

$$y_i(0;\tau,\xi)\in[a,b],\qquad\qquad y_i(t;\tau,\xi)\in\mathcal{D}\quad\forall t\in[0,\tau], \tag{1.19}$$

for all $(\tau,\xi)\in\mathcal{D}$, $i=1,\ldots,n$. Observe that, if (1.10) holds and $\tau_0>0$, then

$$\mathcal{D}\doteq\big\{(t,x);\quad t\in[0,\tau_0],\quad a+\hat\lambda t\leq x\leq b-\hat\lambda t\big\}$$

is a domain of determinacy. For $i=1,\ldots,n$, define the components

$$u_i(t,x)\doteq\big\langle l_i(t,x),\ u(t,x)\big\rangle,\qquad\qquad \bar u_i(x)=\big\langle l_i(0,x),\ \bar u(x)\big\rangle, \tag{1.20}$$

Multiplying (1.16) on the left by $l_i(t,x)$, one obtains the system of n scalar equations

$$(u_i)_t+\lambda_i(u_i)_x=g_i(t,x,u),\qquad\qquad i=1,\ldots,n, \tag{1.21}$$

with

$$g_i=\langle l_i,\ h\rangle+\langle l_{i,t}+\lambda_i l_{i,x},\ u\rangle,\qquad\qquad u=\sum_{i=1}^n u_i r_i. \tag{1.22}$$

From (1.17) and (1.21) it follows

$$\frac{d}{dt}u_i\big(t,y_i(t;\tau,\xi)\big)=g_i\big(t,y_i(t;\tau,\xi),u(t,y_i(t;\tau,\xi))\big). \tag{1.23}$$

Given any (τ,ξ) in a domain of determinacy $\mathcal{D}$, integrating (1.23) on $[0,\tau]$ one obtains

$$u_i(\tau,\xi)=\bar u_i\big(y_i(0;\tau,\xi)\big)+\int_0^\tau g_i\big(t,y_i(t;\tau,\xi),u(t,y_i(t;\tau,\xi))\big)\ dt. \tag{1.24}$$

We say that a locally integrable function $u=\sum u_i r_i$, from $\mathcal{D}$ into $I\!\!R^n$, is a *broad solution* of the Cauchy problem (1.16), (1.18) provided that (1.24) holds for almost every $(\tau,\xi)\in\mathcal{D}$ and all $i\in\{1,\ldots,n\}$. By possibly redefining u on a set of measure zero, one can thus assume that each component u_i is absolutely continuous and satisfies (1.23), along almost every characteristic line $y_i(\cdot\,;\tau,\xi)$. For the local existence and uniqueness of broad solutions of semilinear systems we refer to [9].

1.3 - Piecewise Lipschitz solutions of conservative systems.

The purpose of this section is to review some basic properties of piecewise Lipschitz continuous solutions of systems of conservation laws, and derive some useful identities.

Let $u = u(t,x)$ be a piecewise Lipschitz continuous solution of (1.1). By Rademacher's theorem [13, p.50], the partial derivatives u_t, u_x exist almost everywhere. Consider the components

$$u_x^i(t,x) \doteq \big\langle l_i(u),\ u_x\big\rangle, \tag{1.25}$$

where $l_i(u)$ is the i-th left eigenvector of the matrix $A(u) = DF(u)$. With this notation, we have

$$u_x = \sum_{i=1}^n r_i(u)u_x^i, \qquad u_t + \sum_{i=1}^n \lambda_i(u)r_i(u)u_x^i = h(t,x,u) \qquad \text{a.e.} \tag{1.26}$$

Differentiating the first identity in (1.26) w.r.t. t, the second w.r.t. x and combining the results, one obtains the following semilinear system of n scalar equations:

$$(u_x^i)_t + \big(\lambda_i u_x^i\big)_x = \sum_{j<k} G_{ijk}u_x^j u_x^k + \sum_{j=1}^n B_{ij}u_x^j + E_i, \tag{1.27}$$

where

$$G_{ijk} = G_{ijk}(u) \doteq \big(\lambda_k - \lambda_j\big)\big\langle l_i,\ [r_k, r_j]\big\rangle,$$

$$B_{ij} \doteq B_{ij}(t,x,u) = \big\langle l_i,\ [r_j, h]\big\rangle, \qquad E_i = E_i(t,x,u) \doteq \big\langle l_i,\ h_x\big\rangle,$$

As usual, the Lie brackets are here defined according to

$$[r_k, r_j] \doteq \frac{\partial r_j}{\partial u}\cdot r_k - \frac{\partial r_k}{\partial u}\cdot r_j, \qquad [r_j, h] \doteq \frac{\partial h}{\partial u}\cdot r_j - \frac{\partial r_j}{\partial u}\cdot h.$$

Since u is piecewise Lipschitz, the bounded measurable functions u_x^i provide a broad solution to the system (1.27). More precisely, for every i one can redefine u_x^i on a set of measure zero so that, along almost all characteristic lines $x = x(t)$ of the i-th family, with $\dot x(t) = \lambda_i\big(u(t,x(t))\big)$, the following holds. The map $t \mapsto u_x^i\big(t,x(t)\big)$ is absolutely continuous and satisfies

$$\begin{aligned} u_x^i(t,x(t)) =& u_x^i(s,x(s)) + \sum_{j<k}\int_s^t G_{ijk}u_x^j u_x^k\ d\tau - \sum_j \int_s^t (r_j \bullet \lambda_i)u_x^j u_x^i\ d\tau \\ &+ \sum_j \int_s^t B_{ij}u_x^j\ d\tau + \int_s^t E_i\ d\tau, \end{aligned} \tag{1.28}$$

for $s < t$, as long as the characteristic does not cross any line where u is discontinuous. Of course, all functions in the integrals on the right hand side of (1.28) are evaluated at $(\tau, x(\tau))$.

Let now $x_\alpha(t)$, $\alpha = 1,\ldots,N$, describe the position of the α-th discontinuity of u at time t. If the jump at x_α occurs in the k_α-th characteristic family, the Rankine-Hugoniot and the entropy admissibility conditions [8, 12] imply

$$\big\langle l_i(u^-,u^+),\ u^+ - u^-\big\rangle = 0 \qquad \forall i \neq k_\alpha, \tag{1.29}$$

$$\dot x_\alpha = \lambda_{k_\alpha}(u^+, u^-), \tag{1.30}$$

$$\lambda_{k_\alpha}(u^+) \leq \lambda_{k_\alpha}(u^+,u^-) \leq \lambda_{k_\alpha}(u^-). \tag{1.31}$$

Here u^- and u^+ denote the left and right limits of $u(t,x)$ as x tends to $x_\alpha(t)$, respectively. Observe that, because of (1.28), the components of u_x:

$$u_x^{i^+} \doteq u_x^i(t, x_\alpha+), \qquad u_x^{i^-} \doteq u_x^i(t, x_\alpha-), \tag{1.32}$$

to the right and to the left of the discontinuity, can be defined pointwise for almost every t, except in the case where $i = k_\alpha$ and the k_α-th characteristic field is linearly degenerate. Differentiating (1.29) and using the identities in (1.26), one obtains

$$\begin{aligned} 0 = & -\left(\frac{\partial}{\partial t} + \dot{x}_\alpha \frac{\partial}{\partial x}\right) \Big\langle l_i(u^+, u^-),\ u^+ - u^- \Big\rangle \\ &= \sum_{j=1}^n \Big\langle Dl_i(u^+, u^-) \cdot \Big((\lambda_j^+ - \dot{x}_\alpha) r_j^+ u_x^{j^+} - h^+, (\lambda_j^- - \dot{x}_\alpha) r_j^- u_x^{j^-} - h^-\Big), u^+ - u^- \Big\rangle \\ &\quad + \sum_{j=1}^n \Big\langle l_i(u^+, u^-),\ \Big[(\lambda_j^+ - \dot{x}_\alpha) r_j^+ u_x^{j^+} - h^+\Big] - \Big[(\lambda_j^- - \dot{x}_\alpha) r_j^- u_x^{j^-} - h^-\Big] \Big\rangle, \end{aligned} \tag{1.33}$$

for almost every t and all $i \neq k_\alpha$. Equations of the form (1.33) will arise over again, so we study them in greater detail. For $u^+, u^- \in \Omega$, $w^+, w^- \in I\!R^m$, consider the mappings

$$\begin{aligned} \Phi_i(u^+, u^-, w^+, w^-) = & \sum_{j=1}^n \Big\langle Dl_i(u^+, u^-) \cdot (w_j^+ r_j^+,\ \ w_j^- r_j^-),\ \ u^+ - u^- \Big\rangle \\ & + \sum_{j=1}^n \Big\langle l_i(u^+, u^-),\ \ w_j^+ r_j^+ - w_j^- r_j^- \Big\rangle. \end{aligned} \tag{1.34}$$

For a fixed $\bar{k} \in \{1, \ldots, n\}$, define the sets $\mathcal{I}$ and $\mathcal{O}$ (incoming and outgoing) of signed indices

$$\begin{aligned} \mathcal{I} &= \{i^+;\ i \leq \bar{k}\} \cup \{i^-;\ \ i \geq \bar{k}\}, \\ \mathcal{O} &= \{j^-;\ j < \bar{k}\} \cup \{j^+;\ \ j > \bar{k}\}. \end{aligned} \tag{1.35}$$

Observe that the system of $n-1$ scalar equations

$$\Phi_i(u^-, u^+, w^-, w^+) = 0 \qquad (i \neq \bar{k}) \tag{1.36}$$

is linear homogeneous w.r.t. w^-, w^+, with coefficients which depend continuously on u^-, u^+. When $u^- = u^+$ one has

$$\frac{\partial \Phi_i}{\partial w_j^\pm} = \pm \delta_{ij}.$$

Therefore, if u^- and u^+ are sufficiently close to each other, one has

$$\det\left(\frac{\partial \Phi_i(u^-, u^+, w^-, w^+)}{\partial w_j^\pm}\right) \neq 0 \qquad (i \neq \bar{k},\ j^\pm \in \mathcal{O}). \tag{1.37}$$

In turn, when the $(n-1) \times (n-1)$ determinant in (1.37) does not vanish, one can solve (1.36) for the $n-1$ outgoing variables $w_j^\pm$, $j^\pm \in \mathcal{O}$:

$$w_j^\pm = W_j(u^-, u^+)(w_{\mathcal{I}}) \qquad j \neq \bar{k}. \tag{1.38}$$

Here $w_{\mathcal{I}}$ denotes the set of $n+1$ incoming variables $\{w_i^\pm;\ i^\pm \in \mathcal{I}\}$. In the special case where the $\bar{k}$-th characteristic field is linearly degenerate, one has

$$\frac{\partial \Phi_i}{\partial w_{\bar{k}}^\pm} \equiv 0, \tag{1.39}$$

hence all functions $W_{j^\pm}$ do not depend on $w_{\bar{k}}^+,\ w_{\bar{k}}^-$.

Looking back at the linear, non-homogeneous system (1.33), it is clear that, if (1.36) (with $\bar{k} = k_\alpha$) can be uniquely solved for the outgoing variables by (1.38), then also (1.33) can be uniquely solved for the outgoing waves $u_x^{j^\pm}$:

$$u_x^{j^\pm} = U_j(u^-, u^+)(u_x^{\mathcal{I}}) \qquad\qquad j^\pm \in \mathcal{O}, \tag{1.40}$$

with

$$\frac{\partial U_j}{\partial u_x^{i^\pm}} = \frac{\lambda_i^\pm - \dot{x}_\alpha}{\lambda_j^\pm - \dot{x}_\alpha} \cdot \frac{\partial W_j}{\partial w_{i^\pm}}. \tag{1.41}$$

Observe that in the linearly degenerate case the components $u_x^{k_\alpha^\pm}$ may not be defined. Yet, the functions $U_{j^\pm}$ are well defined for a.e. t because they no longer depend on $u_x^{k_\alpha^\pm}$.

2. Generalized Tangent Vectors

2.1 - First order variations.

Aim of this chapter is to study how a piecewise Lipschitz solution u of the hyperbolic system (1.1) varies, depending on the initial data. More precisely, given a family of initial conditions $\bar{u}^\varepsilon$, we seek a description of the corresponding solutions $u^\varepsilon(t,\cdot)$, for $t > 0$, which is first-order accurate as $\varepsilon \to 0$, with respect to the $\mathbf{L}^1$ norm. The remarks made in the Introduction motivate the following construction.

Let $u \in \mathbf{L}^1\big([a,b];\ I\!R^n\big)$ be any piecewise Lipschitz continuous function. Consider the family Σ_u of all continuous paths $\gamma : [0, \varepsilon_0] \mapsto \mathbf{L}^1$ with $\gamma(0) = u$. Here $\varepsilon_0 > 0$ may depend on γ. On Σ_u, define an equivalence relation by setting

$$\gamma \sim \gamma' \qquad \Longleftrightarrow \qquad \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \big\|\gamma(\varepsilon) - \gamma'(\varepsilon)\big\|_{\mathbf{L}^1} = 0. \tag{2.1}$$

We seek a first order approximation, as $\varepsilon \to 0$, for certain paths $\gamma \in \Sigma_u$. This can be achieved by introducing a suitable tangent space T_u.

Definition 1. Let $u : [a,b] \mapsto I\!R^n$ be piecewise Lipschitz continuous with jumps precisely at the points x_α, with $a < x_1 < x_2 < \cdots < x_N < b$. We then define the the space of *generalized tangent* *vectors* to u as $T_u \doteq \mathbf{L}^1\big([a,b];\ I\!R^n\big) \times I\!R^N$. A continuous path $\gamma \in \Sigma_u$ *generates* the tangent vector $(v,\xi) \in T_u$ if γ is equivalent to the path $\gamma_{(v,\xi;u)}$ defined by

$$\gamma_{(v,\xi;u)}(\varepsilon) \doteq u + \varepsilon v + \sum_{\xi_\alpha<0} \big(u(x_\alpha+) - u(x_\alpha-)\big)\chi_{[x_\alpha+\varepsilon\xi_\alpha,\ x_\alpha]} - \sum_{\xi_\alpha>0} \big(u(x_\alpha+) - u(x_\alpha-)\big)\chi_{[x_\alpha,\ x_\alpha+\varepsilon\xi_\alpha]}. \tag{2.2}$$

In other words, if γ generates the tangent vector (v,ξ), in first approximation one can obtain $\gamma(\varepsilon)$ starting with u, adding εv and then shifting by $\varepsilon\xi_\alpha$ each point x_α where u has a jump. It is clear that the tangent vector generated by γ, if it exists, is necessarily unique. Indeed, every $(v,\xi) \in T_u$ singles out precisely one equivalence class in Σ_u.

Our main goal is to derive a linearized evolution equation, describing how these generalized tangent vectors are transported along a given solution of (1.1). We begin by defining a special class of piecewise Lipschitz functions, related to the system (1.1). Some regularity conditions for a path γ will then be stated.

Definition 2. In connection with the system (1.1), we say that a function $u : [a,b] \mapsto I\!R^n$ is in the class PLSD of *Piecewise Lipschitz functions with Simple Discontinuities* if it satisfies the following conditions.

i) u has finitely many discontinuities, say at $a < x_1 < x_2 < \cdots < x_N < b$, and there exists a constant L such that
$$|u(x) - u(x')| \le L|x - x'| \tag{2.3}$$
whenever the interval $[x,x']$ does not contain any point x_α.

ii) Each jump of u consists of a contact discontinuity or of a single, stable shock. More precisely, for every $\alpha \in \{1,\ldots,N\}$, there exists $k_\alpha \in \{1,\ldots,n\}$ such that
$$\big\langle l_i(u^+,u^-),\ u^+ - u^-\big\rangle = 0 \qquad \forall i \ne k_\alpha, \tag{2.4}$$
$$u^+ \ne u^-, \qquad \lambda_{k_\alpha}(u^+) \le \lambda_{k_\alpha}(u^+,\ u^-) \le \lambda_{k_\alpha}(u^-), \tag{2.5}$$
where u^+, u^- denote respectively the right and left limits of $u(x)$ as $x \to x_\alpha$.

Definition 3. Let u be a PLSD function. A path $\gamma \in \Sigma_u$ is a *Regular Variation* (R.V.) for u if, for $\varepsilon \in [0,\ \varepsilon_0]$, all functions $u^\varepsilon \doteq \gamma(\varepsilon)$ are in PLSD, with jumps at points $x_1^\varepsilon < \ldots < x_N^\varepsilon$ depending continuously on ε. They all satisfy Definition 2 with a Lipschitz constant L independent of ε.

The above definition is motivated by the fact that regular variations are locally preserved by the evolution equation (1.1), while ordinary tangent vectors in $\mathbf{L}^1$ are not.

Returning to the example discussed in the introduction, for every $t > 0$ the path $\varepsilon \mapsto u^\varepsilon(t,\cdot)$, with
$$u^\varepsilon(t,x) = \frac{(1+\varepsilon)x}{1+(1+\varepsilon)t} \cdot \chi_{[0,\ \sqrt{1+(1+\varepsilon)t}]}(x),$$
is a Regular Variation for $u^0(t,\cdot)$. It generates the tangent vector $\big(v(t),\xi(t)\big) \in T_{u^0(t,\cdot)} = \mathbf{L}^1 \times I\!R$, with
$$v(t)(x) = \frac{x}{(1+t)^2} \cdot \chi_{[0,\ \sqrt{1+t}]}(x), \qquad \xi(t) = \frac{t}{2\sqrt{1+t}}.$$

The next result provides a characterization of certain Regular Variations.

Lemma 2.1. *Let $\varepsilon \mapsto \gamma(\varepsilon) = u^\varepsilon$ be a R.V. for u. Then γ generates the tangent vector $(v,\xi) \in \mathbf{L}^1 \times I\!R^N$ if and only if*

$$\lim_{\varepsilon\to 0+} \frac{x_\alpha^\varepsilon - x_\alpha}{\varepsilon} = \xi_\alpha \qquad\qquad \alpha = 1,\ldots,N, \tag{2.6}$$

and

$$\lim_{\varepsilon\to 0+} \int_{-\sigma}^{\eta} \left| \frac{u^\varepsilon(x_\alpha^\varepsilon + y) - u(x_\alpha + y)}{\varepsilon} - v(x_\alpha + y) - \xi_\alpha u_x(x_\alpha + y) \right| dy = 0 \tag{2.7}$$

whenever the closed interval $[x_\alpha - \sigma,\ x_\alpha + \eta]$ contains no point of discontinuity of u other than x_α.

Proof. **1.** Let $\gamma : \varepsilon \mapsto u^\varepsilon$ be a R.V. for u, generating the tangent vector (v,ξ). As a preliminary, we prove

$$\lim_{\varepsilon\to 0+} u^\varepsilon(x) = u(x) \qquad\qquad \forall x \notin \{x_1,\ldots,x_N\}. \tag{2.8}$$

Indeed, if (2.8) fails, then there exist $\delta > 0$, a point $\bar x$ where u is continuous and a sequence $\varepsilon_n \to 0$ such that

$$\big|u^{\varepsilon_n}(\bar x) - u(\bar x)\big| > \delta \qquad\qquad \forall n.$$

Choose $\rho \in (0, \delta/2L]$ so small that the interval $I = [\bar x,\ \bar x + \rho]$ does not contain any point x_α where u has a jump. On I, all functions u, u^ε are uniformly Lipschitz continuous with constant L. Therefore

$$\begin{aligned}
&\limsup_{\varepsilon\to 0+} \int_I \big|u^\varepsilon(x) - u(x) - \varepsilon v(x)\big|\, dx \\
&\qquad \geq \limsup_{n\to\infty} \int_{\bar x}^{\bar x+\rho} \Big\{ \big|u^{\varepsilon_n}(\bar x) - u(\bar x)\big| - 2L|x - \bar x| \Big\}\, dx - \limsup_{n\to\infty} \int_{\bar x}^{\bar x+\rho} \varepsilon_n \big|v(x)\big|\, dx \\
&\qquad \geq \int_{\bar x}^{\bar x+\rho} \big\{ \delta - 2L(x - \bar x) \big\}\, dx = \delta\rho - L\rho^2 \geq \frac{\delta\rho}{2}.
\end{aligned}$$

This contradicts the assumption that $\big\|u^\varepsilon - \gamma_{(v,\xi;u)}(\varepsilon)\big\|_{\mathbf{L}^1} \to 0$, thus proving (2.8).

2. In order to prove (2.6), we assume that it fails for some α, and derive a contradiction. To fix the ideas, assume

$$\xi_\alpha > 0, \qquad\qquad \xi_\alpha + \delta \leq \frac{x_\alpha^{\varepsilon_n} - x_\alpha}{\varepsilon_n} \tag{2.9}$$

for some sequence $\varepsilon_n \to 0$, the other cases being entirely similar. Choose $\rho > 0$ such that x_α is the only point of discontinuity of u inside the closed interval

$$J \doteq [x_\alpha - \rho,\ x_\alpha + \rho],$$

and such that

$$\big|u(x_\alpha+) - u(x_\alpha-)\big| \geq 4L\rho. \tag{2.10}$$

Using (2.9), (2.10), one obtains

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \big\| u^{\varepsilon_n} - \gamma_{(v,\xi;u)}(\varepsilon)\big\|_{\mathbf{L}^1(J)} \\
&\geq \liminf_{n\to\infty} \frac{1}{\varepsilon_n} \int_{x_\alpha+\varepsilon_n\xi_\alpha}^{x_\alpha^{\varepsilon_n}} \big|u^{\varepsilon_n}(x) - u(x) - \varepsilon_n v(x)\big|\, dx \\
&\geq \liminf_{n\to\infty} \frac{1}{\varepsilon_n} \int_{x_\alpha+\varepsilon_n\xi_\alpha}^{x_\alpha+\varepsilon_n(\xi_\alpha+\delta)} \big|u^{\varepsilon_n}(x) - u(x)\big|\, dx - \limsup_{n\to\infty} \int_{x_\alpha}^{x_\alpha^{\varepsilon_n}} \big|v(x)\big|\, dx \\
&\geq \lim_{n\to\infty} \frac{1}{\varepsilon_n} \int_{x_\alpha+\varepsilon_n\xi_\alpha}^{x_\alpha+\varepsilon_n(\xi_\alpha+\delta)} \big|u(x_\alpha+) - u(x_\alpha-)\big|\, dx - \limsup_{n\to\infty} \frac{1}{\varepsilon_n} \int_{x_\alpha+\varepsilon_n\xi_\alpha}^{x_\alpha+\varepsilon_n(\xi_\alpha+\delta)} \Big\{\big|u(x_\alpha-) - u(x_\alpha-\rho)\big| \\
&\qquad\qquad + \big|u(x_\alpha-\rho) - u^{\varepsilon_n}(x_\alpha-\rho)\big| + \big|u^{\varepsilon_n}(x_\alpha-\rho) - u^{\varepsilon_n}(x)\big| + \big|u(x) - u(x_\alpha+)\big|\Big\}dx \\
&\geq \delta\big|u(x_\alpha+) - u(x_\alpha-)\big| - \big\{\delta L\rho + 0 + 2\delta L\rho + 0\big\} \geq \delta L\rho > 0.
\end{aligned}
$$

This contradiction establishes (2.6)

3. Next, let $[x_\alpha - \sigma,\ x_\alpha + \eta]$ contain no discontinuity of u other than x_α. To fix the ideas, assume again $\xi_\alpha > 0$. The limit in (2.7) can be estimated by

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_{-\sigma}^{\eta} \Big| u^\varepsilon(x_\alpha^\varepsilon + y) - u(x_\alpha + y) - \varepsilon v(x_\alpha+y) - \varepsilon\xi_\alpha u_x(x_\alpha+y)\Big|\, dy \\
&\leq \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_{-\sigma}^{\eta} \Big| u^\varepsilon(x_\alpha^\varepsilon + y) - u(x_\alpha^\varepsilon + y) - \varepsilon v(x_\alpha^\varepsilon + y) \\
&\qquad\qquad + \big[u(x_\alpha+) - u(x_\alpha-)\big]\cdot \chi_{[x_\alpha,\ x_\alpha+\varepsilon\xi_\alpha]}(x_\alpha^\varepsilon + y)\Big|\, dy \\
&\quad + \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_{-\sigma}^{\eta} \Big| u(x_\alpha + \varepsilon\xi_\alpha + y) - u(x_\alpha + y) - \varepsilon\xi_\alpha u_x(x_\alpha + y) \\
&\qquad\qquad - \big[u(x_\alpha+) - u(x_\alpha-)\big]\cdot\chi_{[-\varepsilon\xi_\alpha,\ 0]}(y)\Big|\, dy \\
&\quad + \limsup_{\varepsilon\to 0+} \int_{-\sigma}^{\eta} \Big| v(x_\alpha^\varepsilon + y) - v(x_\alpha+y)\Big|\, dy \\
&\quad + \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_{-\sigma}^{\eta} \Big| u(x_\alpha^\varepsilon + y) - u(x_\alpha + \varepsilon\xi_\alpha + y)\Big| + \big|u(x_\alpha+) - u(x_\alpha-)\big|\cdot\chi_{I\diamondsuit J}(y)\, dy,
\end{aligned}
\tag{2.11}
$$

where

$$
I \doteq \big[x_\alpha - x_\alpha^\varepsilon,\ x_\alpha - x_\alpha^\varepsilon + \varepsilon\xi_\alpha\big], \qquad J \doteq \big[-\varepsilon\xi_\alpha,\ 0\big],
$$

and $I\diamondsuit J \doteq (I\setminus J)\cup(J\setminus I)$ is the symmetric difference. Observe that (2.6) implies

$$
\lim_{\varepsilon\to 0+} \varepsilon^{-1}\cdot meas(I\diamondsuit J) = 0. \tag{2.12}
$$

On the right hand side of (2.11), the second limit vanishes for every piecewise Lipschitz continuous function u, with a single jump at $x = x_\alpha$. The third limit vanishes for every $v \in \mathbf{L}^1$. The fourth limit vanishes because of (2.6) and (2.12). Finally, concerning the first limit, performing a change of coordinates we obtain

$$
\limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_{x_\alpha^\varepsilon - \sigma}^{x_\alpha^\varepsilon + \eta} \Big| u^\varepsilon(x) - u(x) - \varepsilon v(x) + \big[u(x_\alpha+) - u(x_\alpha-)\big]\cdot\chi_{[x_\alpha,\ x_\alpha+\varepsilon\xi_\alpha]}(x)\Big|\, dx = 0,
$$

because of the assumption that γ generates the tangent vector (v,ξ). This establishes (2.7).

4. To prove the converse, assume now that (2.6), (2.7) hold. Choose points $y_0 < \cdots < y_N$ such that

$$y_0 < -M, \quad y_N > M, \qquad y_{\alpha-1} < x_\alpha < y_\alpha \qquad \forall \alpha = 1,\ldots,N.$$

To prove that the path $\gamma : \varepsilon \mapsto u^\varepsilon$ generates the tangent vector (v,ξ), it suffices to show that, for all α,

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{y_{\alpha-1}}^{y_\alpha} \Big|u^\varepsilon(x) - \gamma_{(v,\xi;u)}(\varepsilon)(x)\Big|\,dx = 0. \tag{2.13}$$

Choose $\sigma,\eta > 0$ so that

$$x_{\alpha-1} < x_\alpha - \sigma < y_{\alpha-1} < y_\alpha < x_\alpha + \eta < x_{\alpha+1}.$$

For sake of definitness, assume $\xi_\alpha > 0$. The limit in (2.13) is then estimated by

$$\begin{aligned}
&\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{y_{\alpha-1}-x_\alpha^\varepsilon}^{y_\alpha - x_\alpha^\varepsilon} \Big|u^\varepsilon(x_\alpha^\varepsilon + y) - u(x_\alpha + y) - \varepsilon v(x_\alpha^\varepsilon + y) \\
&\qquad\qquad + \big[u(x_\alpha+) - u(x_\alpha-)\big]\cdot \chi_{[x_\alpha,\ x_\alpha+\varepsilon\xi_\alpha]}(x_\alpha^\varepsilon + y)\Big|\,dy \\
&\quad \le \limsup_{\varepsilon\to 0+} \int_{-\sigma}^{\eta} \left| \frac{u^\varepsilon(x_\alpha^\varepsilon + y) - u(x_\alpha + y)}{\varepsilon} - v(x_\alpha + y) - \xi_\alpha u_x(x_\alpha + y)\big]\right|\,dy \\
&\qquad + \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{-\sigma}^{\eta} \Big\{ \big|u(x_\alpha^\varepsilon + y) - u(x_\alpha^\varepsilon + \varepsilon\xi_\alpha + y)\big| + \big|u(x_\alpha+) - u(x_\alpha-)\big|\cdot \chi_{I\diamondsuit J}(y)\Big\}\,dy \\
&\qquad + \limsup_{\varepsilon\to 0+} \int_{-\sigma}^{\eta} \big|v(x_\alpha^\varepsilon + y) - v(x_\alpha + y)\big|\,dy \\
&\qquad + \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{-\sigma}^{\eta} \Big|u(x_\alpha + \varepsilon\xi_\alpha + y) - u(x_\alpha + y) - \varepsilon\xi_\alpha u_x(x_\alpha + y) \\
&\qquad\qquad - \big[u(x_\alpha+) - u(x_\alpha-)\big]\cdot \chi_{[-\varepsilon\xi_\alpha,\ 0]}(y)\Big|\,dy \\
&\quad = 0.
\end{aligned} \tag{2.14}$$

Indeed, the first limit on the right hand side of (2.14) vanishes because of (2.7), the second limit vanishes because of (2.6) and (2.12), while the other two vanish for every $v \in \mathbf{L}^1$ and every u piecewise Lipschitz with a single jump at the point x_α. This completes the proof of the lemma.

2.2 - A linearized evolution equation for tangent vectors.

In this section we prove that regular variations are locally preserved by the system (1.1) and derive a linearized evolution equation for the corrisponding tangent vectors.

For convenience, we recall here the basic assumptions concerning the system (1.1).

(H1) The vector field $F : \Omega \mapsto I\!R^n$ is $\mathcal{C}^2$. For each $u \in \Omega$, the matrix $A(u) = DF(u)$ has n real distinct eigenvalues. Its eigenvalues and eigenvectors λ_i, l_i, r_i, normalized as in (1.19), satisfy the uniform bounds (1.10). Each characteristic field is either linearly degenerate or genuinely nonlinear.

(H2) Solutions of (1.1) are considered in the domain

$$\mathcal{D} \doteq \Big\{(t,x);\quad 0\le t\le T,\quad x\in[a+\hat\lambda t,\ b-\hat\lambda t]\Big\},$$

where $\hat\lambda$ is the constant in (1.10). The function $h:\mathcal{D}\times\Omega\mapsto I\!\!R^n$ is bounded and continuously differentiable.

(H3) Whenever $u^+,u^-\in\Omega$ are connected by a shock or contact discontinuity, say of the $\bar k$-th characteristic family, the linear system (1.36) can be uniquely solved for the outgoing variables $w_{j^\pm}$, $j^\pm\in\mathcal{O}$. Moreover, the functions $W_{j^\pm}$ in (1.38) satisfy a bound of the form

$$\big|W_{j^\pm}(u^+,u^-,w_{\mathcal{I}})\big|\le K_0|w_{\mathcal{I}}|\qquad\qquad \forall z_{\mathcal{I}}\in I\!\!R^{n+1},\quad j^\pm\in\mathcal{O}.$$

Theorem 2.2. *Let the assumptions (H1)–(H3) hold. Let u be a piecewise Lipschitz continuous solution of (1.1), with $u(0,\cdot)=\bar u$ in the class PLSD. Let $(\bar v,\bar\xi)\in\mathbf{L}^1\times I\!\!R^N$ be a tangent vector to $\bar u$, generated by the R.V. $\varepsilon\mapsto\bar u^\varepsilon$, and call $u^\varepsilon=u^\varepsilon(t,x)$ the solution of (1.1) with initial condition $\bar u^\varepsilon$. Then there exists $\tau_0>0$ such that , for all $t\in[0,\tau_0]$, the path $\varepsilon\mapsto u^\varepsilon(t,\cdot)$ is a R.V. for $u(t,\cdot)$, generating the tangent vector $\big(v(t),\xi(t)\big)\in T_{u(t,\cdot)}$. The vector (v,ξ) is the unique broad solution of the initial boundary value problem:*

$$\xi(0)=\bar\xi,\qquad\qquad v(0,x)=\bar v(x),\tag{2.15}$$

$$v_t+A(u)v_x+[DA(u)v]u_x=h_u(t,x,u)v\tag{2.16}$$

outside the discontinuities of u, while, for $\alpha=1,\ldots,N$,

$$\begin{aligned}\Big\langle Dl_i(u^+,u^-)\cdot(v^++\xi_\alpha u_x^+,\ v^-+\xi_\alpha u_x^-),\quad u^+-u^-\Big\rangle&\\ +\Big\langle l_i(u^+,u^-),\quad v^++\xi_\alpha u_x^+-v^--\xi_\alpha u_x^-\Big\rangle=0&\end{aligned}\qquad \forall i\ne k_\alpha,\tag{2.17}$$

$$\dot\xi_\alpha=D\lambda_{k_\alpha}(u^+,u^-)\cdot(v^++\xi_\alpha u_x^+,\ v^-+\xi_\alpha u_x^-),\tag{2.18}$$

along each line $x=x_\alpha(t)$ where u suffers a discontinuity, in the k_α-th characteristic direction.

Proof. From standard theory, we can assume the existence of $\tau>0$ such that the functions u^ε are all defined on $[0,\tau]\times I\!\!R$, have exactly N non intersecting lines of discontinuity and satisfy a Lipschitz condition (outside the jumps) with a constant independent of ε. Due to the finite propagation speed, it suffices to prove the result locally in a neighborhood of any given point $\bar x\in I\!\!R$. We consider here the case where $\bar x=x_\alpha(0)$ is a point of jump for $\bar u$; in the case where $\bar u$ is continuous at $\bar x$ the analysis is much easier. For every ε, call $x_\alpha^\varepsilon(t)$ the location of the α-th jump in u^ε and introduce the new space variable $y=x-x_\alpha^\varepsilon(t)$, so that the α-th jump in u^ε always occurs at $y=0$. In the new variables (t,y), u^ε satisfies the system

$$u_t^\varepsilon+[A(u^\varepsilon)-\dot x_\alpha^\varepsilon]u_y^\varepsilon=h(t,y+x_\alpha^\varepsilon,u^\varepsilon),\tag{2.19}$$

$$\big\langle l_i(u^{\varepsilon+},u^{\varepsilon-}),\ u^{\varepsilon+}-u^{\varepsilon-}\big\rangle=0\qquad\qquad\forall i\ne k_\alpha,\tag{2.20}$$

$$\dot{x}^\varepsilon_\alpha = \lambda_{k_\alpha}(u^{\varepsilon+}, u^{\varepsilon-}), \tag{2.21}$$

$$u^\varepsilon(0,y) = \bar{u}^\varepsilon\big(y + x^\varepsilon_\alpha(0)\big). \tag{2.22}$$

Here $u^{\varepsilon+}$, $u^{\varepsilon-}$ denote the limits of $u^\varepsilon(t,y)$ as $y \to 0$ from the right and from the left, respectively. We study (2.19)-(2.22) in the domain

$$\mathcal{D} \doteq \big\{(t,y); \quad |y| \le \rho - 2\hat{\lambda}t, \quad 0 \le t \le \tau_0\big\}, \tag{2.23}$$

where $\hat{\lambda}$ was introduced at (1.10) and $\rho, \tau_0 > 0$ are small enough so that all functions $u^\varepsilon(t,\cdot)$ have a unique point of jump as y ranges inside the interval $[-\rho + \hat{\lambda}t, \quad \rho - \hat{\lambda}t]$, for all $t \in [0, \tau_0]$.

By Lemma 2.1, it suffices to show that, if $u^\varepsilon = u^\varepsilon(t,y)$ is the solution of (2.19)-(2.22) and if $(w, \xi) \doteq (v + \xi_\alpha u_x, \ \xi)$ solves the linearized problem:

$$w_t + [A(u) - \dot{x}_\alpha]w_y + [DA(u)w - \dot{\xi}_\alpha]u_y = h_x(t, y + x_\alpha, u)\xi_\alpha + h_u(t, y + x_\alpha, u)w, \tag{2.24}$$

$$\big\langle Dl_i(u^+, u^-)\cdot(w^+, w^-), \ u^+ - u^-\big\rangle + \big\langle l_i(u^+, u^-), \ w^+ - w^-\big\rangle = 0 \qquad \forall i \neq k_\alpha, \tag{2.25}$$

$$\dot{\xi}_\alpha = D\lambda_{k_\alpha}(u^+, u^-)\cdot(w^+, w^-), \tag{2.26}$$

$$\xi_\alpha(0) = \bar{\xi}_\alpha, \qquad\qquad w(0,y) = \bar{w}(y) \doteq \bar{v}\big(y + x_\alpha(0)\big) + \bar{\xi}_\alpha \bar{u}_x\big(y + x_\alpha(0)\big), \tag{2.27}$$

then the family of functions $u^\varepsilon(t,\cdot)$ is a R.V. for $u(t,\cdot)$, on the interval $[-\rho + 2\hat{\lambda}t, \ \rho - 2\hat{\lambda}t]$, and

$$\lim_{\varepsilon\to 0} \frac{x^\varepsilon_\alpha(t) - x^0_\alpha(t)}{\varepsilon} = \xi_\alpha(t), \tag{2.28}$$

$$\lim_{\varepsilon\to 0} \int_{-\rho+2\hat{\lambda}t}^{\rho-2\hat{\lambda}t} \left| \frac{u^\varepsilon(t,y) - u(t,y)}{\varepsilon} - w(t,y) \right| dy \ = \ 0, \tag{2.29}$$

for all $t \in [0, \tau_0]$. Define the rescaled variables

$$w^\varepsilon(t,y) = \frac{u^\varepsilon - u}{\varepsilon}, \qquad\qquad \xi^\varepsilon_\alpha(t) = \frac{x^\varepsilon_\alpha - x_\alpha}{\varepsilon}. \tag{2.30}$$

From (2.19)-(2.22) it follows that w^ε, ξ^ε_α satisfy

$$\begin{aligned}
w^\varepsilon_t &+ \big[A(u^\varepsilon) - \dot{x}^\varepsilon_\alpha\big]w^\varepsilon_y + \big[DA(u)w^\varepsilon - D\lambda_{k_\alpha}(u^+, u^-)\cdot(w^{\varepsilon+}, w^{\varepsilon-})\big]u_y \\
&\quad + \varepsilon^{-1}\big[A(u^\varepsilon) - A(u) - \varepsilon DA(u)\cdot(u^\varepsilon - u)\big]u_y \\
&\quad - \varepsilon^{-1}\big[\lambda_{k_\alpha}(u^{\varepsilon+}, u^{\varepsilon-}) - \lambda_{k_\alpha}(u^+, u^-) - D\lambda_{k_\alpha}(u^+, u^-)\cdot(u^{\varepsilon+} - u^+, \ u^{\varepsilon-} - u^-)\big]u_y \\
&= h_x(t, y + x_\alpha, u)\xi^\varepsilon_\alpha + h_u(t, y + x_\alpha, u)w^\varepsilon \\
&\quad + \varepsilon^{-1}\big[h(t, y + x^\varepsilon_\alpha, u^\varepsilon) - h(t, y + x_\alpha, u) - h_x(t, y + x_\alpha, u)(x^\varepsilon_\alpha - x_\alpha) - h_u(t, y + x_\alpha, u)(u^\varepsilon - u)\big],
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\big\langle l_i(u^{\varepsilon+}, u^{\varepsilon-}), \ w^{\varepsilon+} - w^{\varepsilon-}\big\rangle + \big\langle Dl_i(u^+, u^-)\cdot(w^{\varepsilon+}, w^{\varepsilon-}), \ u^+ - u^-\big\rangle \\
&\quad + \big\langle l_i(u^{\varepsilon+}, u^{\varepsilon-}) - l_i(u^+, u^-) - Dl_i(u^+, u^-)\cdot(u^{\varepsilon+} - u^+, u^{\varepsilon-} - u^-), \ u^+ - u^-\big\rangle = 0,
\end{aligned} \tag{2.32}$$

$$\begin{aligned}\dot\xi^\varepsilon_\alpha =& D\lambda_{k_\alpha}(u^+,u^-)\cdot(w^{\varepsilon+},w^{\varepsilon-})\\ &+\varepsilon^{-1}\big[\lambda_{k_\alpha}(u^{\varepsilon+},u^{\varepsilon-})-\lambda_{k_\alpha}(u^+,u^-)-D\lambda_{k_\alpha}(u^+,u^-)\cdot(u^{\varepsilon+}-u^+,u^{\varepsilon-}-u^-)\big],\end{aligned} \tag{2.33}$$

$$w^\varepsilon(0,y)=\bar w^\varepsilon(y)\doteq\frac{u^\varepsilon(0,y)-u(0,y)}{\varepsilon},\qquad\qquad \xi^\varepsilon_\alpha(0)=\bar\xi^\varepsilon_\alpha\doteq\frac{x^\varepsilon_\alpha(0)-x_\alpha(0)}{\varepsilon}. \tag{2.34}$$

The plan of the proof is the following. We introduce a Banach space Z and a family of contractions $\mathcal{T}^\varepsilon : Z\mapsto Z,\ \varepsilon\in[0,\varepsilon_0]$, such that, for $\varepsilon>0$, the unique fixed point of $\mathcal{T}^\varepsilon$ corresponds to the solution $(w^\varepsilon,\xi^\varepsilon_\alpha)$ of (2.31)–(2.34). Moreover, the fixed point of $\mathcal{T}^0$ yields the solution (w,ξ_α) of (2.24)–(2.27). By showing that $\mathcal{T}^\varepsilon(\tilde z)\to\mathcal{T}^0(\tilde z)$ for all $\tilde z\in Z$, as $\varepsilon\to0$, we then establish the limits

$$\lim_{\varepsilon\to0+}\sup_{t\in[0,\tau_0]}\int_{-\rho+\hat\lambda t}^{\rho-\hat\lambda t}\big|w^\varepsilon(t,y)-w(t,y)\big|\,dt=0,\qquad\qquad \lim_{\varepsilon\to0+}\sup_{t\in[0,\tau_0]}\big|\xi^\varepsilon_\alpha(t)-\xi_\alpha(t)\big|=0, \tag{2.35}$$

proving the theorem. All the analysis is carried out assuming that the k_α-th characteristic field is genuinely nonlinear. The minor modifications needed to cover the linearly degenerate case will be mentioned at the end.

We start by writing (2.31) in the more concise form

$$w_t+A^\varepsilon(t,y)w_y=B(t,y)w+C(t,y)\xi_\alpha+D^\varepsilon(t,y)\cdot(w^+,w^-)+E^\varepsilon(t,y), \tag{2.36}$$

where

$$A^\varepsilon=A(u^\varepsilon)-\dot x^\varepsilon_\alpha,\qquad\qquad Bw=-DA(u)w\cdot u_y+h_u(t,y+x_\alpha,u)w,$$

$$C=h_x(t,y+x_\alpha,u),\qquad\qquad D^\varepsilon\cdot(w^+,w^-)=D\lambda_{k_\alpha}(u^{\varepsilon+},u^{\varepsilon-})\cdot(w^+,w^-)u_y,$$

and E^ε collects all other terms in (2.31).

Next, observe that the boundary conditions (2.32) constitute a linear, non-homogeneous system of $n-1$ equations for the $2n$ variables $w_i^{\varepsilon+},w_i^{\varepsilon-}$ (denoting the components of the right and left limits of $w^\varepsilon(t,y)$ as $y\to0$). Because of the assumption (H3), this system can be solved for the outgoing variables

$$w_{j^\pm}=W^\varepsilon_{j^\pm}(t)\big(w_{\mathcal I}\big)\qquad\qquad j^\pm\in\mathcal{O}, \tag{2.37}$$

where $\mathcal{I},\mathcal{O}$ are the sets of indices introduced at (1.35), with $\bar k=k_\alpha$. From (H3) and the differentiability of l_i, one obtains a bound of the form

$$\big|w_{j^\pm}\big|\le K_1\big(1+|w_{\mathcal I}|\big)\qquad\qquad\forall t\in[0,\tau_0],\quad j^\pm\in\mathcal{O}. \tag{2.38}$$

As soon as the values w^+,w^- are known for a.e. $t\in[0,\tau_0]$, the function ξ^ε_α can be determined by simply integrating (2.33):

$$\xi^\varepsilon_\alpha(t)=\bar\xi^\varepsilon_\alpha+\int_0^t D\lambda_{k_\alpha}\big(u^{\varepsilon+}(s),u^{\varepsilon-}(s)\big)\cdot(w^+,w^-)+\phi^\varepsilon(s)\,ds, \tag{2.39}$$

where

$$\phi^\varepsilon = \varepsilon^{-1}\Big[\lambda_{k_\alpha}(u^{\varepsilon+},u^{\varepsilon-}) - \lambda_{k_\alpha}(u^+,u^-) - D\lambda_{k_\alpha}(u^+,u^-)\cdot(u^{\varepsilon+}-u^+,\ u^{\varepsilon-}-u^-)\Big]. \tag{2.40}$$

Consider the Banach space $Z \subset \mathcal{C}\big([0,\tau_0];\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)\big)\oplus\mathbf{L}^1\big([0,\tau_0];\mathbb{R}^{n+1}\big)$, consisting of all couples $(z,z_{\mathcal{I}})$, where $z:[0,\tau_0]\times\mathbb{R}\mapsto\mathbb{R}^n$ is a measurable function which vanishes outside the triangular domain $\mathcal{D}$ and such that the map $t\mapsto z(t,\cdot)$ is continuous in the $\mathbf{L}^1$ norm, and where $z_{\mathcal{I}}:[0,\tau_0]\mapsto\mathbb{R}^{n+1}$ is integrable. On Z we shall use the norm

$$\big\|(z,z_{\mathcal{I}})\big\|_\dagger \doteq \int_0^{\tau_0}\big|z_{\mathcal{I}}(\tau)\big|\,d\tau + \sup_{\tau\in[0,\tau_0]}\sum_{i=1}^n\int_{(\tau,\eta)\in\mathcal{D}}\big|z_i(\tau,\eta)\big|\,d\eta. \tag{2.41}$$

In order to define the transformation $\mathcal{T}$, let $(z,z_{\mathcal{I}})\in Z$ be given. First of all, using (2.37), from $z_{\mathcal{I}}(t)$ we determine the $n-1$ "outgoing" components $z_{j^\pm}(t)$, for $j^\pm\in\mathcal{O}$, $t\in[0,\tau_0]$. In analogy with (2.39), we then define the function ξ^ε_α by setting

$$\xi^\varepsilon_\alpha\big(z_{\mathcal{I}}\big)(t) \doteq \bar\xi^\varepsilon_\alpha + \int_0^t D\lambda_{k_\alpha}\big(u^{\varepsilon+}(s),u^{\varepsilon-}(s)\big)\cdot\big(z^+(s),z^-(s)\big) + \phi^\varepsilon(s)\,ds. \tag{2.42}$$

For $i=1,\dots,n$, call $y^\varepsilon_i(\cdot\,;\tau,\eta)$ the i-th characteristic line through (τ,η), i.e. the solution of

$$\dot y = \lambda^\varepsilon_i(t,y), \qquad\qquad y(\tau)=\eta.$$

Here and in the sequel, $\lambda^\varepsilon_i, l^\varepsilon_i, r^\varepsilon_i$ refer to the matrix $A^\varepsilon(t,y)\doteq A(u^\varepsilon)-\dot x^\varepsilon_\alpha$. Introduce the functions

$$g^\varepsilon_i = g^\varepsilon_i(t,y,z,z_{\mathcal{I}}) \doteq \Big\langle l^\varepsilon_{i,t}+\lambda^\varepsilon_i l^\varepsilon_{i,y},\ \sum_{j=1}^n r_jz_j\Big\rangle + \Big\langle l^\varepsilon_i,\ Bz+C\xi^\varepsilon_\alpha(z_{\mathcal{I}})+D^\varepsilon\Big(\sum z_j^+r_j^{\varepsilon+},\sum z_j^-r_j^{\varepsilon-}\Big)+E^\varepsilon\Big\rangle, \tag{2.43}$$

with ξ^ε_α given by (2.42). Observing that the functions $B,C,D^\varepsilon,E^\varepsilon$ in are uniformly bounded as well as l^ε_i, $l^\varepsilon_{i,t}$, $l^\varepsilon_{i,y}$, λ^ε_i, and recalling (2.38), (2.42), for g^ε_i we obtain a bound of the form

$$|g^\varepsilon_i| \le K_2\left(1+\sum_{j=1}^n|z_j| + \big|z_{\mathcal{I}}(t)\big| + \int_0^t\big|z_{\mathcal{I}}(s)\big|\,ds\right), \tag{2.44}$$

for some constant K_2. For $i=1,\dots,n$, define the subdomains

$$\mathcal{D}^-_i \doteq \Big\{(\tau,\eta)\in\mathcal{D};\quad \eta>0 \text{ and } i\le k_\alpha \ \text{ or }\ \eta<0 \text{ and } i\ge k_\alpha\Big\},$$
$$\mathcal{D}^+_i \doteq \Big\{(\tau,\eta)\in\mathcal{D};\quad \eta>0 \text{ and } i> k_\alpha \ \text{ or }\ \eta<0 \text{ and } i< k_\alpha\Big\}.$$

For each fixed $\varepsilon\in(0,\varepsilon_0]$, the transformation $\mathcal{T}^\varepsilon = \big(\mathcal{T}_1,\dots,\mathcal{T}_n,\mathcal{T}_{\mathcal{I}}\big)(z,z_{\mathcal{I}})$ can now be constructed in three steps.

1. For every i and $(\tau,\eta)\in\mathcal{D}^-_i$, because of the bound (1.10) and the definition (2.23), the i-th characteristic $t\mapsto y^\varepsilon_i(t;\tau,\xi)$ remains inside $\mathcal{D}^-_i$ for $t\in[0,\tau]$. It is thus meaningful to define

$$\mathcal{T}_i\big(z,z_{\mathcal{I}}\big)(\tau,\eta) \doteq \bar w^\varepsilon_i\big(y^\varepsilon_i(0;\tau,\eta)\big) + \int_0^\tau g^\varepsilon_i\big(t,y^\varepsilon_i(t),z,z_{\mathcal{I}}\big)\,dt. \tag{2.45}$$

2. Next, we define the components $\mathcal{T}_{\mathcal{I}}(\tau)$ for a.e. $\tau \in [0, \tau_0]$. If $i \geq k_\alpha$, then there is a unique i-th characteristic line $t \mapsto y^\varepsilon_{i^-}(t)$ which impinges on the shock at $y = 0$ from the left, at time $t = \tau$. In this case, define

$$\mathcal{T}_{i^+}(z, z_{\mathcal{I}})(\tau) \doteq \bar{w}^\varepsilon_i\big(y_{i^-}(0)\big) + \int_0^\tau g^\varepsilon_i\big(t, y^\varepsilon_{i^-}(t), z, z_{\mathcal{I}}\big)\, dt. \tag{2.46}$$

Similarly, if $i \leq k_\alpha$, then there is a unique i-th characteristic $t \mapsto y^\varepsilon_{i^+}(t)$ which impinges on the shock at $y = 0$ from the right, at time $t = \tau$. In this case, define

$$\mathcal{T}_{i^+}(z, z_{\mathcal{I}})(t) \doteq \bar{w}^\varepsilon_i\big(y_{i^+}(0)\big) + \int_0^\tau g^\varepsilon_i\big(t, y_{i^+}(t), z, z_{\mathcal{I}}\big)\, dt. \tag{2.47}$$

Observe that, in turn, the $n+1$ values $\mathcal{T}_{\mathcal{I}}\big(z, z_{\mathcal{I}}\big)(t)$ determine the remaining $n-1$ "outgoing" components:

$$\hat{z}_{j^\pm}(t) \doteq W^\varepsilon_{j^\pm}(t)\big(\mathcal{T}_{\mathcal{I}}(z, z_{\mathcal{I}})\big) \qquad\qquad j^\pm \in \mathcal{O}, \tag{2.48}$$

where $W^\varepsilon_{j^\pm}$ are the linear, non homogeneus functions in (2.37), implicitly defined by the system (2.33)

3. Finally, for $i = 1, \ldots, n$, $(\tau, \eta) \in \mathcal{D}^+_i$, consider the backward i-th characteristic $y^\varepsilon_i(\cdot\,; \tau, \eta)$. If this line crosses the axis $y = 0$ at some time $\tau_i = \tau_i(\tau, \eta) > 0$, define

$$\mathcal{T}_i\big(z, z_{\mathcal{I}}\big)(\tau, \eta) \doteq \hat{z}_i(\tau_i) + \int_{t_0}^\tau g^\varepsilon_i\big(t, y^\varepsilon_i(t), w, w_{\mathcal{I}}\big)\, dt, \tag{2.49}$$

with $\hat{z}_i$ given at (2.48). On the other hand, if this i-th characteristic does not cross the line $y = 0$ for $t \in [0, \tau]$, then we define the value of $\mathcal{T}_i$ at (τ, η) again by (2.45).

It is easy to check that the transformation $\mathcal{T}$ maps Z into itself. In order to show that $\mathcal{T}$ is a strict contraction, assume $(z, z_{\mathcal{I}}), (v, v_{\mathcal{I}}) \in Z$ and

$$\big\|(z - v,\ z_{\mathcal{I}} - v_{\mathcal{I}})\big\|_\dagger = \delta > 0. \tag{2.50}$$

We claim that, if ρ and τ_0 in (2.23) are sufficiently small, then

$$\big\|\mathcal{T}(z, z_{\mathcal{I}}) - \mathcal{T}(v, v_{\mathcal{I}})\big\|_\dagger \leq \frac{\delta}{2}. \tag{2.51}$$

As a preliminary, observe that the Lipschitz continuity of the eigenvalues $\lambda_i = \lambda_i(t, y)$ implies that the maps $\eta \mapsto y^\varepsilon_i(t; \tau, \eta)$ are uniformly Lipschitz continuous. In particular, there exists a constant K_3 such that

$$\big|y^\varepsilon_i(t; \tau, \eta) - y^\varepsilon_i(t; \tau, \eta')\big| \leq K_3|\eta - \eta'|, \tag{2.52}$$

for all i and all t, τ, η, η', as long as the left hand side of (2.52) is defined.

Moreover, since all characteristics meet the shock at $y = 0$ at a nonzero angle, for some constant K_4 the following holds. If τ_i, τ'_i are the times at which two incoming characteristics $y^\varepsilon_i(\cdot\,; t, y)$, $y^\varepsilon_i(\cdot\,; t, y')$ reach the axis $y = 0$, then

$$\big|\tau_i - \tau'_i\big| \leq K_4|y - y'|; \tag{2.53}$$

in addition, if $0 \le t < t' \le \tau_0$, $j \neq k_\alpha$ and $y_j^\varepsilon(\cdot\,; t, 0)$, $y_j^\varepsilon(\cdot\,; t', 0)$ are the j-th characteristics which depart from the axis $y = 0$ at times t, t' respectively, then

$$\big|y_j^\varepsilon(\tau; t, 0) - y_j^\varepsilon(\tau; t', 0)\big| \le K_4|t - t'| \qquad \forall \tau \in [t', \tau_0]. \tag{2.54}$$

Toward a proof of (2.51), from the bounds (2.44) and the linearity of the functions g_i it follows

$$\big|g_i^\varepsilon(t, y, z, z_{\mathcal{I}}) - g_i^\varepsilon(t, y, v, v_{\mathcal{I}})\big| \le K_2\bigg(\sum_{j=1}^n \big|z_j - v_j\big|(t, y) + \big|z_{\mathcal{I}} - v_{\mathcal{I}}\big|(t) + \int_0^t \big|z_{\mathcal{I}} - v_{\mathcal{I}}\big|(s)\, ds\bigg). \tag{2.55}$$

For fixed $i \in \{1, \ldots, n\}$, $\tau \in [0, \tau_0]$, using (2.52) to change the variable of integration, from the bounds (2.55) and (2.50) one obtains

$$\begin{aligned}
&\int_{(\tau,\eta)\in\mathcal{D}_i^-} \big|\mathcal{T}_i(z, z_{\mathcal{I}}) - \mathcal{T}_i(v, v_{\mathcal{I}})\big|(\tau, \eta)\, d\eta \\
&\qquad \le \int_{(\tau,\eta)\in\mathcal{D}_i^-} \int_0^\tau K_2 \sum_j \big|z_j - v_j\big|\big(t, y_i^\varepsilon(t; \tau, \eta)\big)\, dt d\eta \\
&\qquad\quad + \int_{(\tau,\eta)\in\mathcal{D}_i^-} K_2 \int_0^\tau \bigg(\big|z_{\mathcal{I}} - v_{\mathcal{I}}\big|(t) + \int_0^t \big|z_{\mathcal{I}} - v_{\mathcal{I}}\big|(s) ds\bigg)\, dt d\eta \\
&\qquad \le \int_0^\tau K_3 \int_{(t,y)\in\mathcal{D}} K_2 \sum_{j=1}^n \big|z_j - v_j\big|(t, y)\, dy dt + K_2 \int_{(\tau,\eta)\in\mathcal{D}_i^-} (\delta + \tau\delta)\, d\eta \\
&\qquad \le K_2 K_3 \int_0^\tau \delta\, dt + K_2 \rho (1 + \tau_0)\delta \\
&\qquad \le \Big[K_2 K_3 \tau_0 + K_2 \rho(1 + \tau_0)\Big]\delta.
\end{aligned} \tag{2.56}$$

Next, if $i^\pm \in \mathcal{I}$, using (2.53) to change the variable of integration, from the bounds (2.55) and (2.50) we deduce

$$\begin{aligned}
&\int_0^{\tau_0} \Big|\mathcal{T}_{i^\pm}(z, z_{\mathcal{I}}) - \mathcal{T}_{i^\pm}(v, v_{\mathcal{I}})\Big|(t)\, dt \\
&\quad \le \int_0^{\tau_0} \int_0^t K_2 \Bigg\{\sum_{j=1}^n \big|z_j - v_j\big|\big(s, y_{i^\pm}^\varepsilon(s)\big) + \big|z_{\mathcal{I}} - v_{\mathcal{I}}\big|(s) + \bigg(\int_0^s \big|z_{\mathcal{I}} - v_{\mathcal{I}}\big|(\sigma)\, d\sigma\bigg)\Bigg\}\, ds dt \\
&\quad \le \int_0^{\tau_0} K_4 \int_{(t,y)\in\mathcal{D}} K_2 \sum_{j=1}^n \big|z_j - v_j\big|(t, y)\, dy dt + \int_0^{\tau_0} K_2(\delta + t\delta)\, dt \\
&\quad \le K_2 K_4 \int_0^{\tau_0} \delta\, dt + K_2(\tau_0 + \tau_0^2)\delta \\
&\quad = \Big[K_2 K_4 \tau_0 + K_2(\tau_0 + \tau_0^2)\Big]\delta.
\end{aligned} \tag{2.57}$$

Finally, using both (2.53) and (2.54) to change the variable of integration in the two cases encountered in the construction of $\mathcal{T}_i$ at step **3**, from (2.55) and (2.50) one obtains

$$\begin{aligned}
&\int_{(\tau,\eta)\in\mathcal{D}_i^+} \Big|\mathcal{T}_i(z,z_{\mathcal{I}})-\mathcal{T}_i(v,v_{\mathcal{I}})\Big|(\tau,\eta)\ d\eta \\
&\quad\le K_4\int_0^\tau \big|\hat z_i-\hat v_i\big|(t)\ dt+\int_0^\tau\int_{(t,y)\in\mathcal{D}} K_2\sum_{j=1}^n \big|z_j-v_j\big|(t,y)\ dtdy+K_2\int_{(\tau,\eta)\in\mathcal{D}_i^+}(\delta+\tau\delta)\ d\eta \\
&\quad\le K_4\int_0^{\tau_0} K_1\Big|\mathcal{T}_{\mathcal{I}}(z,z_{\mathcal{I}})-\mathcal{T}_{\mathcal{I}}(v,v_{\mathcal{I}})\Big|(t)\ dt+K_2K_3\int_0^\tau \delta\ dt+K_2\rho(1+\tau_0)\delta \\
&\quad\le \bigg\{K_1K_4(n+1)\Big(K_2K_4\tau_0+K_2(\tau_0+\tau_0^2)\Big)+K_2K_3\tau_0+K_2\rho(1+\tau_0)\bigg\}\delta.
\end{aligned}\tag{2.58}$$

The term $|\hat z_i-\hat v_i|$ was here estimated using (2.57) and (2.38), recalling that the functions $W^\varepsilon_{j\pm}$ in (2.37) are linear, non-homogeneous. Together, (2.56)–(2.58) yield

$$\begin{aligned}
\big\|\mathcal{T}(z,z_{\mathcal{I}})-\mathcal{T}(v,v_{\mathcal{I}})\big\|_\dagger \le \bigg\{ & n\Big[K_2K_3\tau_0+K_2\rho(1+\tau_0)\Big]+(n+1)\Big[K_2K_4\tau_0+K_2(\tau_0+\tau_0^2)\Big] \\
&+n\Big[K_1K_4(n+1)\Big(K_2K_4\tau_0+K_2(\tau_0+\tau_0^2)\Big)+K_2K_3\tau_0+K_2\rho(1+\tau_0)\Big]\bigg\}\delta.
\end{aligned}\tag{2.59}$$

By choosing ρ and τ_0 small enough, the right hand side of (2.59) can be made smaller than $\delta/2$. This establishes (2.51).

In the case where the k_α-th characteristic field is linearly degenerate, it suffices to define the set of indices of incoming waves as

$$\mathcal{I}\doteq\big\{i^+;\quad i<k_\alpha\big\}\cup\big\{i^-;\quad i<k_\alpha\big\},$$

and work with the space $Z\doteq\mathcal{C}\big([0,\tau_0];\mathbf{L}^1\big)\oplus\mathbf{L}^1\big([0,\tau_0];\mathbb{R}^{n-1}\big)$. Since the fuctions $W^\varepsilon_{j\pm}$ in (2.37) no longer depend on $w_{k_\alpha^\pm}$, all previous estimates remain valid.

The previous construction can be also performed in connection with the system (2.24)–(2.27), in which case it yields a transformation $\mathcal{T}^0: Z\mapsto Z$. Since all constants $K_0,\ldots,K_4$ occurring in our estimates can be chosen not depending on $\varepsilon\in[0,\varepsilon_0]$, all transformations $\mathcal{T}^\varepsilon$ are strict contractions w.r.t. the norm (2.41). By the contraction mapping principle, each $\mathcal{T}^\varepsilon$ thus has a unique fixed point $(z^\varepsilon,z^\varepsilon_{\mathcal{I}})\in Z$. From the definition of $\mathcal{T}^\varepsilon$, it is clear that the functions

$$w^\varepsilon(t,y)\doteq\sum_{i=1}^n r_i^\varepsilon z_i^\varepsilon(t,y),$$

$$\xi^\varepsilon_\alpha(t)\doteq\bar\xi^\varepsilon_\alpha+\int_0^t D\lambda_{k_\alpha}\big(u^{\varepsilon+}(s),u^{\varepsilon-}(s)\big)\cdot\Big(\sum_{j=1}^n r_j^{\varepsilon+}z_j^{\varepsilon+}(s),\ \sum_{j=1}^n r_j^{\varepsilon-}z_j^{\varepsilon-}(s)\Big)+\phi^\varepsilon(s)\ ds,$$

with ϕ^ε as in (2.40), provide a broad solution of the system (2.31)–(2.34). Hence, in particular,

$$\sum_{i=1}^n z_i^\varepsilon r_i^\varepsilon=\frac{u^\varepsilon-u}{\varepsilon},\qquad \xi^\varepsilon_\alpha=\frac{x^\varepsilon_\alpha-x_\alpha}{\varepsilon}\qquad\qquad \forall\varepsilon>0.$$

On the other hand, for $\varepsilon = 0$, the functions

$$w \doteq \sum_{i=1}^{n} r_i z_i^0, \quad \xi_\alpha(t) \doteq \bar{\xi}_\alpha + \int_0^t D\lambda_{k_\alpha}(u^+, u^-) \cdot \Big(\sum_j r_j^+ z_j^{0+}, \ \sum_j r_j^- z_j^{0-} \Big) \, ds$$

provide the unique broad solution of (2.24)–(2.27). In order to establish the limits in (2.35), it thus suffices to prove that the fixed point of $\mathcal{T}^\varepsilon$ converges to the fixed point of $\mathcal{T}^0$. This will certainly be the case if

$$\lim_{\varepsilon \to 0+} \mathcal{T}^\varepsilon(z, z_\mathcal{I}) = \mathcal{T}^0(z, z_\mathcal{I}) \tag{2.60}$$

for every $(z, z_\mathcal{I}) \in Z$. Since all maps $\mathcal{T}^\varepsilon$ are contractive, it actually suffices to prove (2.60) for all $(z, z_\mathcal{I})$ in a dense subset of Z, say on the subspace $Z' = \mathcal{C}\big(\mathcal{D}; I\!R^n\big) \oplus \mathcal{C}\big([0, \tau_0]; I\!R^{n+1}\big)$.

We now observe that, as $\varepsilon \to 0+$,

(i) The linear, nonhomogeneous functions $W_{j\pm}^\varepsilon(t)$ in (2.37) approach the linear homogeneous maps $W_{j\pm}^0$, implicitly defined by (2.25), uniformly for $t \in [0, \tau_0]$.

(ii) The functions $E^\varepsilon(t, y)$ in (2.36) converge to $E^0(t, y) \equiv 0$ uniformly on $\mathcal{D}$, while $D^\varepsilon = D\lambda_{k_\alpha}(u^{\varepsilon^+}, u^{\varepsilon-}) u_y$ uniformly converges to $D\lambda_{k_\alpha}(u^+, u^-) u_y$. Moreover, the function ϕ^ε in (2.40), (2.42) converges to zero uniformly on $[0, \tau_0]$.

(iii) The matrix-valued functions $A^\varepsilon = A(u^\varepsilon) - \dot{x}_\alpha^\varepsilon$ are uniformly Lipschitz continuous outside the axis $y = 0$. Calling $A^0 = A(u) - \dot{x}_\alpha$, we have the convergence

$$\lambda_i^\varepsilon \to \lambda_i^0, \qquad l_i^\varepsilon \to l_i^0, \qquad r_i^\varepsilon \to r_i^0,$$

uniformly on $\mathcal{D}$. As a consequence, the functions $(t; \tau, \eta) \mapsto y_i^\varepsilon(t; \tau, \eta)$, which define the characteristic curves, converge to the corresponding functions y_i^0 uniformly for $(\tau, \eta) \in \mathcal{D}$, $t \in [0, \tau]$.

Because of (iii), the convergence of the terms $\bar{w}_i^\varepsilon\big(y_i^\varepsilon(0; \tau, \eta)\big)$ in the definitions (2.45)–(2.47) of $\mathcal{T}^\varepsilon$ to the corresponding limit $\bar{w}_i\big(y_i^0(0; \tau, \eta)\big)$ is clear. Concerning the integral terms in (2.45)–(2.47), recalling (2.43) one has

$$\begin{aligned} \int_{t_1}^{t_2} g_i^\varepsilon\big(t, y_i^\varepsilon(t), z, z_\mathcal{I}\big) \, dt = \int_{t_1}^{t_2} \Big\langle \frac{d}{dt} l_i^\varepsilon, \ \sum_{j=1}^{n} r_j z_j \Big\rangle \, dt \\ + \int_{t_1}^{t_2} \Big\langle l_i^\varepsilon, \ Bz + C\xi_\alpha^\varepsilon(z_\mathcal{I}) + D^\varepsilon \Big(\sum z_j^+ r_j^{\varepsilon+}, \ \sum z_j^- r_j^{\varepsilon-} \Big) + E^\varepsilon \Big\rangle \, dt, \end{aligned} \tag{2.61}$$

where the integrals are computed along a characteristic curve $t \mapsto y_i^\varepsilon(t; \tau, \eta)$. When $z, z_\mathcal{I}$ are continuous, the properties (i)–(iii) imply that the second integral on the right hand side of (2.61) converges to

$$\int_{t_1}^{t_2} \Big\langle l_i^0, \ \ Bz + C\xi_\alpha^0(z_\mathcal{I}) + D^0 \Big(\sum z_j^+ r_j^+, \ \sum z_j^- r_j^- \Big) \Big\rangle \, dt.$$

To handle the first integral, fix $\sigma_0 > 0$ and choose a smooth function φ such that

$$\left| \varphi(t, y) - \sum_{j=1}^{n} r_i^0(t, y) z_i(t, y) \right| \leq \sigma_0 \qquad \forall (t, y) \in \mathcal{D}, \quad y \neq 0.$$

This is possible because r_i, z_i are continuous. An integration by parts now yields

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0+}\int_{t_1}^{t_2}\left|\Big\langle \frac{d}{dt}l_i^\varepsilon,\ \sum_j r_j^\varepsilon z_j\Big\rangle(t,y_i^\varepsilon(t)) - \Big\langle \frac{d}{dt}l_i^0,\ \sum_j r_j^0 z_j\Big\rangle(t,y_i^0(t))\right|\,dt\\
&\qquad\le \limsup_{\varepsilon\to 0+}\int_{t_1}^{t_2}\left\{\left|\frac{d}{dt}l_i^\varepsilon\right|\cdot\left|\sum_j r_j^\varepsilon z_j-\varphi\right|(t,y_i^\varepsilon(t)) + \left|\frac{d}{dt}l_i^0\right|\cdot\left|\sum_j r_j^0 z_j-\varphi\right|(t,y_i^0(t))\right\}\,dt\\
&\qquad\quad + \lim_{\varepsilon\to 0+}\int_{t_1}^{t_2}\left|\Big\langle l_i^\varepsilon,\ \frac{d}{dt}\varphi\Big\rangle(t,y_i^\varepsilon(t)) - \Big\langle l_i^0,\ \frac{d}{dt}\varphi\Big\rangle(t,y_i^0(t))\right|\,dt\\
&\qquad\quad + \lim_{\varepsilon\to 0+}\left[\big\langle l_i^\varepsilon,\ \varphi\big\rangle\Big|_{(t_1,y_i^\varepsilon(t_1))}^{(t_2,y_i^\varepsilon(t_2))} - \big\langle l_i^0,\ \varphi\big\rangle\Big|_{(t_1,y_i^0(t_1))}^{(t_2,y_i^0(t_2))}\right]\\
&\qquad\le 2C(t_2-t_1)\sigma_0 + 0 + 0,
\end{aligned}
\tag{2.62}
$$

for some constant C independent of σ_0. Since $\sigma > 0$ in (2.62) was arbitrary, this argument proves the convergence of the integrals in the definitions (2.45)–(2.47), and hence in (2.49) as well.

From (2.60), it follows the convergence of the fixed points of the transformations $\mathcal{T}^\varepsilon$ to the fixed point of $\mathcal{T}^0$. This establishes (2.35), proving the theorem.

Remark 2.3. The previous result is local in time. By a prolongation argument, the existence of the tangent vector $(v,\xi) \in \mathbf{L}^1 \times I\!R^n$ generated by the family of solutions u^ε, and the validity of the equations (2.16)–(2.18) can be proved up to the the first time $\tau \le T$ such that, as $t \to \tau-$, one of the following cases occurs.

(i) The values of the $u = u(t,x)$ approach the boundary of Ω.

(ii) The Lipschitz constant of the solutions u^ε tends to infinity.

(iii) The location of one of the jumps of u approaches the boundary of the domain $\mathcal{D}$.

(iv) Two or more shocks of u interact.

In the cases (i), (ii), the solution u itself may fail to exist, in the class PLSD, beyond time τ. In case (iii), some tangent vector still exists beyond time τ. Of course, it will be an element of a space $\mathbf{L}^1 \times I\!R^{N'}$ with $N' < N$, because the restriction of the solution to the domain $\mathcal{D}$ will contain a smaller number of discontinuities. The case (iv) is the most interesting, and will be examined in detail in the next chapter.

Remark 2.4. If the functions u^ε are solutions of

$$u_t^\varepsilon + \big[F(u^\varepsilon)\big]_x = h(t,x,u,\varepsilon),$$

where the perturbation h also depends on the parameter ε in a continuously differentiable way, an entirely similar analysis shows that the conclusion of Theorem 2.2 remains valid, with (2.16) replaced by

$$v_t + A(u)v_x + \big[DA(u)v\big]u_x = h_u(t,x,u,0)\cdot v + h_\varepsilon(t,x,u,0). \tag{2.63}$$

For future applications, it is convenient to derive a version of (2.16) involving the components $u_x^i = \langle l_i(u), u_x \rangle, \;\; v_i = \langle l_i(u), v \rangle$. Differentiating w.r.t. ε the equation

$$A(u+\varepsilon v)u_x = \sum_{i=1}^n \lambda_i(u+\varepsilon v)\langle l_i(u+\varepsilon v), \; u_x\rangle r_i(u+\varepsilon v),$$

one obtains

$$\big[DA(u)\cdot v\big]u_x = \sum_{i,j}(r_j \bullet \lambda_i)u_x^i v_j r_i + \sum_{i,j}\lambda_i\langle r_j\bullet l_i, \; u_x\rangle v_j r_i + \sum_{i,j}\lambda_i u_x^i (r_j\bullet r_i)v_j. \tag{2.64}$$

Using (2.64) together with the relations

$$l_{i,t} = \sum_j (r_j\bullet l_i)\Big(-\lambda_j u_x^j + \langle l_j, h\rangle\Big),$$

$$l_{i,x} = \sum_j (r_j\bullet l_i)u_x^j, \qquad\qquad \lambda_{i,x} = \sum_j (r_j\bullet\lambda_i)u_x^j,$$

$$\big\langle r_j\bullet l_i, \; r_k\big\rangle + \big\langle l_i, \; r_j\bullet r_k\big\rangle = r_j\bullet\big\langle l_i, \; r_k\big\rangle \equiv 0,$$

multiplying (2.16) on the left by l_i we find

$$\begin{aligned} &(v_i)_t + \big(\lambda_i v_i\big)_x + \sum_{k\neq i}(r_k\bullet\lambda_i)\big\{u_x^i v_k - u_x^k v_i\big\} + \sum_{j\neq k}\big\langle l_i, \; [r_j, r_k]\big\rangle(\lambda_i-\lambda_j)u_x^j v_k \\ &\quad = \; -\sum_{j,k}\big\langle l_i, \; r_j\bullet r_k\big\rangle\cdot\big\langle l_j, \; h\big\rangle v_k + \sum_k \big\langle l_i, \; r_k\bullet h\big\rangle v_k \qquad\qquad (i=1,\ldots,n). \end{aligned} \tag{2.65}$$

2.3 - Length of regular paths.

For a continuous path $\gamma : [\theta^-, \; \theta^+] \mapsto \mathbf{L}^1$, the length of γ is defined as

$$\|\gamma\| \doteq \sup\left\{\sum_{i=1}^{\nu}\big\|\gamma(\theta_i)-\gamma(\theta_{i-1})\big\|_{\mathbf{L}^1}\;; \qquad \nu\geq 1, \quad \theta^-\leq\theta_0<\cdots<\theta_\nu\leq\theta^+\right\}.$$

In the case of paths whose values are piecewise Lipschitz continuous functions and which admit a differential in a generalized sense, we will show that the length can be computed by integrating the norm of the corresponding tangent vector.

Results of this kind are particularly useful in connection with Theorem 2.2 concerning the existence and the evolution of tangent vectors. Indeed, let $\theta\mapsto\bar u^\theta \doteq \gamma^0(\theta)$ be a one-parameter family of initial conditions, and let $\theta\mapsto u^\theta(t,\cdot)\doteq\gamma^t(\theta)$ represent the corresponding solutions of (1.1) at time t. Using Theorem 2.2, one can estimate the length of the path γ^t by studying the size of the corresponding tangent vectors. This can often yield a proof of the uniform Lipschitz continuous dependence of solutions of (1.1) upon the initial data.

Definition 4. A map $\gamma : \theta \mapsto u^\theta$ from an open interval $]\theta^-, \theta^+[$ into $\mathbf{L}^1\big([a,b]; I\!R^n\big)$ is a *regular path* if

(i) each function u^θ is piecewise Lipschitz continuous with a fixed number of discontinuities located at points x_α^θ, with $a < x_1^\theta < \cdots < x_N^\theta < b$.

(ii) There exists a continuous map $\theta \mapsto (v^\theta, \xi^\theta)$ with values in $\mathbf{L}^1 \times I\!R^N$ such that, for every θ one has

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \big\| u^{\theta+\varepsilon} - \gamma_{(v^\theta,\xi^\theta;u^\theta)}(\varepsilon)\big\|_{\mathbf{L}^1} = 0, \tag{2.66}$$

where $\gamma_{(v,\xi;u)}$ was defined at (2.2).

A continuous map $\gamma : \theta \mapsto u^\theta$ from a closed interval $[\theta^-, \theta^+]$ into $\mathbf{L}^1$ is a *piecewise regular path* if there exist finitely many values $\theta^- = \theta_0 < \cdots < \theta_p = \theta^+$ such that the restriction of γ to each open subinterval $]\theta_{i-1}, \theta_i[$ is a regular path.

Theorem 2.5 *The length of a piecewise regular path $\gamma : [\theta^-, \theta^+] \mapsto \mathbf{L}^1$ is computed by*

$$\|\gamma\| = \int_{\theta^-}^{\theta^+} \big\|(v^\theta, \xi^\theta)\big\|_{u^\theta}\, d\theta, \tag{2.67}$$

where (v^θ, ξ^θ) is the corresponding tangent vector at $u^\theta \doteq \gamma(\theta)$, with norm

$$\big\|(v,\xi)\big\|_u \doteq \int_a^b \big|v(x)\big|\, dx + \sum_\alpha \big|u(x_\alpha+) - u(x_\alpha-)\big| \cdot |\xi_\alpha|. \tag{2.68}$$

Proof. It is not restrictive to assume that the restriction of γ to $]\theta^-, \theta^+[$ is a regular path. Observing that

$$\|\gamma\| = \sup_{\varepsilon>0} \|\gamma_\varepsilon\|,$$

where γ_ε denotes the restriction of γ to the closed subinterval $[\theta^- + \varepsilon,\ \theta^+ - \varepsilon]$, it suffices to prove the equality (2.67) for each γ_ε. Equivalently, it suffices to establish (2.67) under the additional hypothesis that γ can be extended to a regular path defined on $]\theta^- - \varepsilon',\ \theta^+ + \varepsilon'[$, for some $\varepsilon' > 0$.

Call I the right hand side of (2.67). Our previous additional assumption implies that I is finite, because the integrand is continuous on the compact interval $[\theta^-, \theta^+]$. If $\|\gamma\| > I$, there exist $\theta_0 = \theta_- < \theta_1 < \cdots < \theta_\nu = \theta^+$ and $\sigma_0 > 0$ such that

$$\sum_{i=1}^{\nu} \big\|\gamma(\theta_i) - \gamma(\theta_{i-1})\big\|_{\mathbf{L}^1} > \int_{\theta^-}^{\theta^+} \Big(\big\|(v^\theta, \xi^\theta)\big\|_{\gamma(\theta)} + \sigma_0\Big)\, d\theta.$$

Therefore, for some $j \in \{0, \ldots, \nu - 1\}$ one has

$$\big\|\gamma(\theta_{j+1}) - \gamma(\theta_j)\big\|_{\mathbf{L}^1} > \int_{\theta_j}^{\theta_{j+1}} \Big(\big\|(v^\theta, \xi^\theta)\big\|_{\gamma(\theta)} + \sigma_0\Big)\, d\theta. \tag{2.69}$$

To derive a contradiction, define

$$\vartheta \doteq \max\left\{\theta' \in [\theta_j, \theta_{j+1}]; \qquad \big\|\gamma(\theta') - \gamma(\theta_j)\big\|_{\mathbf{L}^1} \le \int_{\theta_j}^{\theta'} \Big(\big\|(v^\theta, \xi^\theta)\big\|_{\gamma(\theta)} + \sigma_0\Big)\, d\theta\right\}. \tag{2.70}$$

Observe that the maximum in (2.70) is certainly attained, because of the continuity of both sides of the inequality, as functions of θ'; moreover, at $\theta' = \vartheta$ these two sides must be equal.

If $\vartheta < \theta_{j+1}$, from (2.66) and the continuity of the integrand in (2.67) it follows

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left[\big\|\gamma(\vartheta+\varepsilon)-\gamma(\vartheta)\big\|_{\mathbf{L}^1}-\int_\vartheta^{\vartheta+\varepsilon}\Big(\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}+\sigma_0\Big)\,d\theta\right]\\
&\qquad\le \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left[\big\|\gamma_{(v^\vartheta,\xi^\vartheta;u^\vartheta)}(\varepsilon)-u^\vartheta\big\|_{\mathbf{L}^1}-\int_\vartheta^{\vartheta+\varepsilon}\Big(\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}+\sigma_0\Big)\,d\theta\right]\\
&\qquad= -\sigma_0
\end{aligned}$$

Therefore, for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned}
&\big\|\gamma(\vartheta+\varepsilon)-\gamma(\theta_j)\big\|_{\mathbf{L}^1}-\int_{\theta_j}^{\vartheta+\varepsilon}\Big(\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}+\sigma_0\Big)\,d\theta\\
&\qquad\qquad\le \big\|\gamma(\vartheta+\varepsilon)-\gamma(\vartheta)\big\|_{\mathbf{L}^1}-\int_{\vartheta}^{\vartheta+\varepsilon}\Big(\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}+\sigma_0\Big)\,d\theta<0.
\end{aligned}$$

This contradicts the maximality of ϑ, proving the inequality $\|\gamma\| \le \mathcal{I}$.

To prove the converse inequality, let $\sigma_0 > 0$ be given. For every θ there exists $\varepsilon(\theta) > 0$ such that

$$\big\|\gamma(\theta+\varepsilon)-\gamma(\theta)\big\|_{\mathbf{L}^1}>\varepsilon\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}-\varepsilon\sigma_0, \tag{2.71}$$

$$\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}>\big\|(v^{\theta+\varepsilon},\xi^{\theta+\varepsilon})\big\|_{\gamma(\theta+\varepsilon)}-\sigma_0, \tag{2.72}$$

for all $\varepsilon \in \big[0, \varepsilon(\theta)\big]$. Using Vitali's covering theorem, we can extract finitely many disjoint intervals $[\theta_i,\ \theta_i+\varepsilon_i]$, with $\varepsilon_i \in \big[0, \varepsilon(\theta_i)\big]$, such that

$$\sum_i\int_{\theta_i}^{\theta_i+\varepsilon_i}\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}\,d\theta\ >\ \int_{\theta^-}^{\theta^+}\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}\,d\theta-\sigma_0. \tag{2.73}$$

Together, (2.71)–(2.73) imply

$$\begin{aligned}
\|\gamma\| &\ge \sum_i\big\|\gamma(\theta_i+\varepsilon_i)-\gamma(\theta_i)\big\|_{\mathbf{L}^1}>\sum_i\varepsilon_i\Big(\big\|(v^{\theta_i},\xi^{\theta_i})\big\|_{\gamma(\theta_i)}-\sigma_0\Big)\\
&>\sum_i\int_{\theta_i}^{\theta_i+\varepsilon_i}\Big(\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}-2\sigma_0\Big)\,d\theta>\int_{\theta^-}^{\theta^+}\big\|(v^\theta,\xi^\theta)\big\|_{\gamma(\theta)}\,d\theta-\sigma_0\big[2(\theta^+-\theta^-)+1\big].
\end{aligned}$$

Since $\sigma_0 > 0$ was arbitrary, this completes the proof.

3. The Case of Interacting Shocks

3.1 - Preliminaries.

This chapter is concerned with the behavior of tangent vectors at a time τ when two shocks interact. As usual, we assume that $u = u(t,x)$ is a piecewise Lipschitz continuous solution of the system of conservation laws (1.1). Let $x_\alpha(t)$, $x_\beta(t)$ denote points where u has a jump, say in the k_α-th, k_β-th characteristic family, respectively. To fix the ideas, let $k_\alpha \geq k_\beta$ and let $x_\alpha(t) < x_\beta(t)$ before the interaction time τ. Call $\eta = x_\alpha(\tau) = x_\beta(\tau)$ the place where the interaction occurs. Define

$$u_* = \lim_{x\to\eta-} u(\tau,x), \qquad\qquad u^* = \lim_{x\to\eta+} u(\tau,x), \tag{3.1}$$

and let $u = \varphi(t,x)$ be the self-similar solution of the standard Riemann problem

$$u_t + \big[F(u)\big]_x = 0, \tag{3.2}$$

$$u(0,-x) = u_*, \qquad u(0,x) = u^* \qquad\qquad \forall x > 0. \tag{3.3}$$

It is well known [8, 11] that this solution consists of $n+1$ constant states $\omega_0 = u_*,\ \omega_1,\ldots,\omega_n = u^*$, each couple (ω_{i-1},ω_i) being connected by a shock or contact discontinuity of the i-th characteristic family $\big($if $\lambda_i(\omega_i) \leq \lambda_i(\omega_{i-1})\big)$, or by a centered rarefaction wave (if $\lambda_i(\omega_i) > \lambda_i(\omega_{i-1})\big)$. Call $\mathcal{S}$ the set of indices $k \in \{1,\ldots,n\}$ corresponding to a nontrivial shock or contact discontinuity, and let $\mathcal{R}$ be the set of indices which correspond to a rarefaction wave (including the trivial case $\omega_{k-1} = \omega_k$). If, for $t < \tau$, $u(t,\cdot)$ has N points of jump, then the number of discontinuities in $u(t,\cdot)$ after the interaction takes place is $N' = N - 2 + |\mathcal{S}|$.

For $t < \tau$, let the parametrized family $\varepsilon \to u^\varepsilon(t,\cdot)$, of solutions of (1.1) be a Regular Variation of u, generating the tangent vector $(v,\xi) \in \mathbf{L}^1 \times I\!R^N$. We assume that, before their respective interaction times τ^ε, all functions u^ε are uniformly Lipschitz continuous outside the corresponding shock lines $x = x^\varepsilon_\gamma(t)$, $\gamma = 1,\ldots,N$. The purpose of this chapter is to establish the existence of a tangent vector $(v,\xi) \in \mathbf{L}^1 \times I\!R^{N'}$, generated by the family of solutions u^ε, also beyond the time of interaction. We will also derive a set of equations relating the forward limit $(v,\xi)(0+)$ with the backward limit $(v,\xi)(0-)$.

In the following, $\dot x^-_\alpha$, $\dot x^-_\beta$ denote the limits of the shock speeds $\dot x_\alpha(t)$, $\dot x_\beta(t)$ as $t \to \tau-$. Similarly, we write ξ^-_α, ξ^-_β for the limits of the components of $\xi(t)$ as $t \to 0-$. The assumption of uniform Lipschitz continuity of the solutions u^ε implies that these limits exist. An easy computation now yields

Lemma 3.1. *Let τ^ε, η^ε be the time and location of the shock interaction, in the solution u^ε. With the above notation one has*

$$\lim_{\varepsilon\to 0+} \frac{\tau^\varepsilon - \tau}{\varepsilon} = \frac{\xi^-_\beta - \xi^-_\alpha}{\dot x^-_\alpha - \dot x^-_\beta}, \tag{3.4}$$

$$\lim_{\varepsilon\to 0+} \frac{\eta^\varepsilon - \eta}{\varepsilon} = \frac{\dot x^-_\alpha \xi^-_\beta - \dot x^-_\beta \xi^-_\alpha}{\dot x^-_\alpha - \dot x^-_\beta}. \tag{3.5}$$

Indeed, by Lemma 2.1, for every fixed $t < \tau$ we have

$$\lim_{\varepsilon\to 0+} \frac{x^\varepsilon_\alpha(t) - x_\alpha(t)}{\varepsilon} = \xi_\alpha(t), \qquad\qquad \lim_{\varepsilon\to 0+} \frac{x^\varepsilon_\beta(t) - x_\beta(t)}{\varepsilon} = \xi_\beta(t). \tag{3.6}$$

Moreover, the uniform Lipschitz continuity of the functions u^ε outside the shocks implies

$$\lim_{\substack{\varepsilon\to 0\\ t\to\tau}} \dot x^\varepsilon_\alpha(t) = \dot x^-_\alpha, \qquad\qquad \lim_{\substack{\varepsilon\to 0\\ t\to\tau}} \dot x^\varepsilon_\beta(t) = \dot x^-_\beta, \tag{3.7}$$

the limits being taken over the region where $t < \tau^\varepsilon$. Together, (3.6) and (3.7) imply

$$x_\alpha^\varepsilon(\tau^\varepsilon) - x_\alpha(\tau) = \varepsilon\xi_\alpha^- + (\tau^\varepsilon - \tau)\dot{x}_\alpha^- + o(\varepsilon) = x_\beta(\tau^\varepsilon) - x_\beta(\tau) = \varepsilon\xi_\beta^- + (\tau^\varepsilon - \tau)\dot{x}_\beta^- + o(\varepsilon),$$

$$\xi_\alpha^- - \xi_\beta^- + \frac{\tau^\varepsilon}{\varepsilon}(\dot{x}_\alpha^- - \dot{x}_\beta^-) + \frac{o(\varepsilon)}{\varepsilon} = 0 \tag{3.8}$$

where the Landau notation $o(\varepsilon)$ has been used to indicate infinitesimals of higher order w.r.t. ε. Letting $\varepsilon \to 0$ in (3.8) we obtain (3.4), and hence (3.5) because

$$\eta^\varepsilon = x_\alpha^\varepsilon(\tau^\varepsilon) = x_\alpha(\tau) + \varepsilon\xi_\alpha^- + \varepsilon\dot{x}_\alpha^- \cdot \frac{d\tau^\varepsilon}{d\varepsilon} + o(\varepsilon) = \eta + \varepsilon\left(\xi_\alpha^- + \dot{x}_\alpha \frac{\xi_\beta^- - \xi_\alpha^-}{\dot{x}_\alpha - \dot{x}_\beta}\right) + o(\varepsilon).$$

3.2 - Estimates on the gradients of solutions.

When two shocks collide, a piecewise Lipschitz solution u of (1.1) can be prolonged further in time by solving the generalized Riemann problem determined by the interaction. In order to prove the existence of a tangent vector generated by the family of solutions u^ε beyond the time of interaction, some basic assumptions are needed. Roughly speaking, one should require that

(i) The generalized Riemann problems for the functions u^ε are well posed; i.e. they admit a unique local entropic solution, satisfying a uniform Lipschitz condition outside the shocks and the centered rarefaction waves.

(ii) The linear system (2.16)–(2.18), satisfied by a generalized tangent vector to the reference solution u, is well posed; i.e. the norm of a solution $(v, \xi) \in \mathbf{L}^1 \times I\!R^{N'}$ of (2.16)–(2.18) at times $t > \tau$ can be uniformly bounded in terms of the norm of (v, ξ) at the interaction time $t = \tau$.

The conditions (i), (ii) amount to the existence of a-priori bounds on the $\mathbf{L}^\infty$ norm of the components $u_x^i = \langle l_i, u_x\rangle$ of the spatial gradient of u, and on the $\mathbf{L}^1$ norm of the components $v_i = \langle l_i, v\rangle$ of a tangent vector. From the basic equations (1.27), (1.33), it follows that the components u_x^i are absolutely continuous along a.e. characteristic line $t \mapsto y_i(t)$, except at points where this characteristic crosses a shock. In order to estimate the $\mathbf{L}^\infty$ norm of u_x, for a solution of a generalized Riemann problem, it is thus convenient to work with a weighted norm of the form

$$\big\|u_x(t,\cdot)\big\|_\diamond \doteq \max_i \ \operatorname*{ess\,sup}_x P_i(t,x)\big|u_x^i(t,x)\big|, \tag{3.9}$$

where the weight functions P_i depend on the location of the point (t,x) relative to the n shocks or centered rarefaction fans emanating from the origin. More precisely, given a suitable family of constants $P_{i,k}$, we shall define $P_i(t,x) = P_{i,k}$ if the point (t,x) is located between the k-th and the $(k+1)$-th shock (or rarefaction fan). In the interior of the k-th rarefaction fan, the weights $P_{i,k-1}$, $P_{i,k}$ will be interpolated smoothly.

In the following definitions, we call $\omega_0, \ldots, \omega_n$ the constant states present in the self-similar solution of (3.2), (3.3). If $k \in \mathcal{R}$, i.e. if the couple (ω_{k-1}, ω_k) is connected by a rarefaction fan, for

$\theta \in [0,1]$ we define the intermediate state $\omega_{k,\theta}$ as the value at time $s=\theta$ of the solution of the Cauchy problem

$$\dot\omega(s) = \frac{\lambda_k(\omega_k)-\lambda_k(\omega_{k-1})}{r_k \bullet \lambda_k\big(\omega(s)\big)} \cdot r_k\big(\omega(s)\big), \qquad\qquad \omega(0)=\omega_{k-1}. \tag{3.10}$$

Observe that the above definition implies

$$\omega_{k,0}=\omega_{k-1}, \quad \omega_{k,1}=\omega_k, \qquad\qquad \lambda_k(\omega_{k,\theta}) = \theta\lambda_k(\omega_k)+(1-\theta)\lambda_k(\omega_{k-1}) \qquad \forall\theta\in[0,1]. \tag{3.11}$$

In the case of a shock, for convenience we define $\omega_{k,\theta} \doteq \omega_k$ for all $\theta \in [0,1]$. Given weights $P_{i,k-1}$, $P_{i,k}$, we shall also use the intermediate values

$$P_{i,k,\theta} \doteq \theta P_{i,k} + (1-\theta)P_{i,k-1} \qquad\qquad \theta\in[0,1]. \tag{3.12}$$

If $k \in \mathcal{S}$, i.e. if the couple (ω_{k-1},ω_k) is connected by a shock or by a contact discontinuity, we call

$$W_j^k = W_j(\omega_k,\omega_{k-1})(w_{\mathcal{I}}), \qquad\qquad j\neq k, \tag{3.13}$$

the corresponding $n-1$ linear functions in (1.38), implicitly defined by the linear homogeneous system

$$\begin{aligned}\sum_{j=1}^n \Big\langle Dl_i(\omega_k,\omega_{k-1})\cdot\big(r_j(\omega_k)w_{j^+},\ r_j(\omega_{k-1})w_{j^-}\big),\ \omega_k-\omega_{k-1}\Big\rangle \\ +\sum_{j=1}^n \Big\langle l_i(\omega_k,\omega_{k-1}),\ r_j(\omega_k)w_{j^+}+r_j(\omega_{k-1})w_{j^-}\Big\rangle = 0\end{aligned} \qquad \forall i\neq k. \tag{3.14}$$

Let now $\bar u$ be a piecewise Lipschitz continuous function having a single jump at $x=0$, with $u_*=\bar u(0-)$, $u^*=\bar u(0+)$. In this paper, we shall always consider local solutions of the generalized Riemann problem

$$u_t+\big[F(u)\big]_x = h(t,x,u), \qquad\qquad u(0,x)=\bar u(x), \tag{3.15}$$

which are obtained as limits for $\epsilon\to 0+$ of solutions of (1.1) with initial conditions

$$u(0,x)=\Psi_\epsilon(\bar u)(x) \doteq \begin{cases}\bar u(x) & \text{if} \quad x<0,\\ \omega_{k,\theta} & \text{if} \quad x=\epsilon(k-1+\theta), \qquad k=1,\dots,n, \quad \theta\in[0,1],\\ \bar u(x-\epsilon n) & \text{if} \quad x>\epsilon n.\end{cases} \tag{3.16}$$

Here the states $\omega_{k,\theta}$ are those defined by (3.10). They are precisely the values taken by the solution of the standard Riemann problem (3.2), (3.3) determined by the values $\bar u(0+)$, $\bar u(0-)$.

Observe that, for $\epsilon>0$, the initial condition $\bar u^\epsilon \doteq \Psi_\epsilon(\bar u)$ is in the class PLSD, because all discontinuities have been shifted apart at a distance ϵ from each other. The local existence and uniqueness of a solution of (1.1) with initial condition (3.16) is well known. Moreover, the behavior of first variations for such solutions was studied in Chapter 2.

We shall introduce a set of assumptions which implies the well-posedness of any generalized Riemann problem sufficiently close to the standard problem (3.2), (3.3). The first hypotheses will be used to derive a-priori bounds on the gradients of solutions to (1.1), (3.16), uniformly valid as $\epsilon\to 0$.

(H4) There exists a family of weights $P_{i,k} > 0$, $i = 1, \ldots, n$, $k = 0, \ldots, n$ such that, if $k \in \mathcal{S}$, then

$$\sum_{i \geq k} \big|\lambda_i(\omega_{k-1}) - \lambda_k(\omega_{k-1}, \omega_k)\big| \left|\frac{\partial W_j^k}{\partial w_{i^-}}\right| P_{i,k-1}^{-1} + \sum_{i \leq k} \big|\lambda_i(\omega_{k-1}) - \lambda_k(\omega_{k-1}, \omega_k)\big| \left|\frac{\partial W_j^k}{\partial w_{i^+}}\right| P_{i,k}^{-1}$$
$$< \begin{cases} \big|\lambda_j(\omega_{k-1}) - \lambda_k(\omega_{k-1}, \omega_k)\big| \cdot P_{j,k-1}^{-1} & \text{if} \quad j < k, \\ \big|\lambda_j(\omega_k) - \lambda_k(\omega_{k-1}, \omega_k)\big| \cdot P_{j,k}^{-1} & \text{if} \quad j > k, \end{cases} \tag{3.17}$$

while, if $k \in \mathcal{R}$, then

$$\frac{P_{j,k} - P_{j,k-1}}{\lambda_k(\omega_k) - \lambda_k(\omega_{k-1})} P_{j,k,\theta}^{-1} \cdot (\lambda_j - \lambda_k) r_k \bullet \lambda_j(\omega_{k,\theta}) - r_k \bullet \lambda_j(\omega_{k,\theta})$$
$$+ P_{j,k,\theta} \sum_{i \neq k} \Big|(\lambda_k - \lambda_i)\big\langle l_j,\ [r_k, r_i]\big\rangle(\omega_{k,\theta})\Big| P_{i,k,\theta}^{-1} \ < \ 0 \qquad \forall j \neq k, \quad \theta \in [0,1]. \tag{3.18}$$

In the degenerate case $\omega_k = \omega_{k-1}$, (3.18) is meant to be replaced simply by $\big(P_{j,k} - P_{j,k-1}\big) \cdot (j - k) < 0$.

Remark 3.2 The assumption (H4) is certainly satisfied by the weights

$$P_{i,k} \doteq e^{|k-i+(1/2)|},$$

whenever the states $u_*,\ u^*$ in the Riemann problem (3.2), (3.3) are sufficiently close.

Indeed, $u^* \to u_*$ implies $\omega_k \to u_*$ for all k. Therefore, the partial derivatives of the functions W_j^k in (3.13) satisfy

$$\lim_{u^* \to u_*} \left|\frac{\partial W_j(\omega_k, \omega_{k-1})}{\partial w_{i^\pm}}\right| = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Hence, for $j > k$ the left and the right hand side of (3.15) approach

$$(\lambda_j - \lambda_k)(u_*) \cdot e^{-|(k-1)-j+(1/2)|}, \qquad\qquad (\lambda_j - \lambda_k)(u_*) \cdot e^{-|k-j+(1/2)|},$$

respectively. Since $k \leq j - 1$, $\lambda_j > \lambda_k$, the inequality (3.17) is clear. The case $j < k$ is entirely similar.

Concerning (3.18), to fix the ideas assume again $j > k$, so that $P_{j,k} < P_{j,k-1}$. As $|u^* - u_*| \to 0$ we also have $\lambda_k(\omega_k) - \lambda_k(\omega_{k-1}) \to 0$. The first term on the left hand side of (3.18) thus approaches $-\infty$, while the other two terms remain bounded. Therefore, a strict inequality holds whenever u^* is sufficiently close to u_*.

Proposition 3.3. *In connection with the standard Riemann problem (3.2), (3.3), let (H4) hold. Then, for every Lipschitz constant L, there exist $\rho, \delta, \epsilon_0 > 0$ for which the following holds. If the initial condition $\bar{u}$ is piecewise Lipschitz continuous with constant L and has a single jump at $x = 0$ with limits $\bar{u}(0-) = u_*$, $\bar{u}(0+) = u^*$, then for every $\epsilon \in (0, \epsilon_0]$ the problem (1.1), (3.16) has a solution $u = u(t,x)$ defined on a domain*

$$\mathcal{D} = \big\{(t,x); \qquad t \in [0, \delta], \quad |x| \leq \rho - \hat{\lambda} t\big\} \tag{3.19}$$

independent of ϵ. Outside the rarefaction fans, the components u_x^i of the gradient of u remain uniformly bounded as $\epsilon \to 0$.

Proof. For every $k \in \mathcal{S}$, by continuity (3.17) implies the existence of $\delta_1, \delta_2 > 0$ such that, if

$$|u^+ - \omega_k| < \delta_1, \qquad |u^- - \omega_{k-1}| < \delta_1 \tag{3.20}$$

and $\dot y_k = \lambda_k(u^-, u^+)$, then

$$\begin{aligned}\sum_{i \geq k} \big|\lambda_i(u^-) - \dot y_k\big| \left|\frac{\partial W_j(u^+, u^-)}{\partial w_{i^-}}\right| P_{i,k-1}^{-1} + \sum_{i \leq k} \big|\lambda_i(u^+) - \dot y_k\big| \left|\frac{\partial W_j(u^+, u^-)}{\partial w_{i^+}}\right| P_{i,k}^{-1} \\ < \begin{cases} \big|\lambda_j(u^-) - \dot y_k\big|\big(P_{j,k-1}^{-1} - \delta_2\big) & \text{if} \quad j < k, \\ \big|\lambda_j(u^+) - \dot y_k\big|\big(P_{j,k}^{-1} - \delta_2\big) & \text{if} \quad j > k. \end{cases}\end{aligned} \tag{3.21}$$

Moreover, for each $k \in \mathcal{R}$, there exist $\hat P_{j,k-1}$, $\hat P_{j,k}$ and $\delta_3, \delta_4 > 0$, with

$$\begin{aligned} &\hat P_{j,k-1} < P_{j,k-1} \qquad \hat P_{j,k} > P_{j,k} \qquad \text{if} \quad j > k, \\ &\hat P_{j,k-1} > P_{j,k-1} \qquad \hat P_{j,k} < P_{j,k} \qquad \text{if} \quad j < k, \end{aligned} \tag{3.22}$$

such that, defining

$$\widetilde P_{j,k}(u) \doteq \hat P_{j,k-1} + \frac{\lambda_k(u) - \lambda_k(\omega_{k-1})}{\lambda_k(\omega_k) - \lambda_k(\omega_{k-1})} \cdot \big(\hat P_{j,k} - \hat P_{j,k-1}\big), \tag{3.23}$$

one has

$$\begin{aligned} &\widetilde P_{j,k}^{-1}(u)\cdot(\lambda_j - \lambda_k)(r_k \bullet \widetilde P_{j,k})(u) - r_k \bullet \lambda_j(u) \\ &\qquad + \widetilde P_{j,k}(u) \sum_{i \neq k} \Big|(\lambda_k - \lambda_i) \cdot \big\langle l_j,\ [r_k, r_i]\big\rangle(u)\Big| \widetilde P_{i,k}^{-1}(u) \ < \ -\delta_4 \end{aligned} \tag{3.24}$$

for all $j \neq k$ and all states u such that

$$\big|u - \omega_{k,\theta}\big| \leq \delta_3 \qquad \text{for some} \quad \theta \in [0,1]. \tag{3.25}$$

Indeed, if we were to choose $\hat P_{j,k} = P_{j,k}$, $\hat P_{j,k-1} = P_{j,k-1}$, then from (3.11), (3.12) and (3.23) it would follow that $\widetilde P_{j,k}(\omega_{k,\theta}) = P_{j,k,\theta}$. With this choice, (3.24) would be a straightforward consequence of (3.18). By continuity and compactness we can now slightly modify the values of $\hat P_{j,k-1}$, $\hat P_{j,k}$ so that the additional inequalities (3.22) also hold.

Observe that, by (3.22), (3.23), there exists $\delta_5 > 0$ such that

$$|u - \omega_{k-1}| < \delta_5, \qquad |u' - \omega_k| < \delta_5 \tag{3.26}$$

imply

$$\begin{aligned} &\widetilde P_{j,k}(u) < P_{j,k-1}, \qquad \widetilde P_{j,k}(u') > P_{j,k} \qquad \text{if} \quad j > k, \\ &\widetilde P_{j,k}(u) < P_{j,k-1}, \qquad \widetilde P_{j,k}(u') < P_{j,k} \qquad \text{if} \quad j < k. \end{aligned} \tag{3.27}$$

Now let u be a local piecewise Lipschitz solution of (1.1), (3.16). To estimate the components u_x^i of its gradient we introduce the weights $P_j = P_j(t,x)$ as follows. If $k \in \mathcal{S}$, let

$$y_k(t) = y_k^-(t) = y_k^+(t) \tag{3.28}$$

be the location of the k-th shock at time t. If $k \in \mathcal{R}$, let y_k^-, y_k^+ denote the characteristic lines bounding the k-th rarefaction fan from the left and from the right, respectively:

$$y_k^-(t) \doteq y_k(t;0,\epsilon(k-1)), \qquad y_k^+(t) \doteq y_k(t;0,\epsilon k). \tag{3.29}$$

For convenience, denote the boundaries of the domain $\mathcal{D}$ in (3.19) as

$$y_0^+(t) \doteq -\rho + \hat{\lambda}t, \qquad y_{n+1}^-(t) \doteq \rho - \hat{\lambda}t. \tag{3.30}$$

For $j = 1, \ldots, n$, we now define the weights

$$P_j(t,x) = \begin{cases} P_{j,k} & \text{if} \quad y_k^+(t) < x < y_{k+1}^-(t), \\ \widetilde{P}_{j,k}\big(u(t,x)\big) & \text{if} \quad y_k^-(t) < x < y_k^+(t). \end{cases} \tag{3.31}$$

Recalling the equation (1.27), we introduce the constants

$$\begin{aligned} C_0 &\doteq \sup_{i,j,k,t,x,u} \big\{ |G_{ijk}|, |B_{ij}|, |D_i|, |r_i \bullet \lambda_j| \big\}, \\ \overline{P} &\doteq \max_{i,k} \big\{ P_{i,k}, \hat{P}_{i,k}, P_{i,k}^{-1}, \hat{P}_{i,k}^{-1} \big\}. \end{aligned} \tag{3.32}$$

In order to derive a differential inequality for the weighted norm in (3.9), consider a generic characteristic of the j-th family, say $t \mapsto y_j(t)$. Along y_j, the quantity u_x^j evolves according to (1.27), while (1.33) holds at times when the characteristic crosses a shock.

We first examine what happens when y_i crosses the k-th shock, say at time τ, located at $y_k(\tau)$. To fix the ideas, let $j > k$ and call u^+, u^- the right and left limits of $u(\tau, \cdot)$ across the shock. Assume that the component $u_x^j \doteq \langle l_j(u), u_x \rangle$ is well defined along y_k and that its "outgoing" limit for $t \to \tau+$ can be determined by the linear nonhomogenous system (1.33) in terms of the $n+1$ components $u_x^{i^\pm}$ of the incoming waves. The linearity of the equations (1.33) implies a bound of the form

$$\big|u_x^{j^\pm}\big| \le \sum_{i^\pm \in \mathcal{I}} \left| \frac{\lambda_i(u^\pm) - \dot{y}_k}{\lambda_j(u^+) - \dot{y}_k} \right| \cdot \left| \frac{\partial W_j(u^+, u^-)}{\partial w_{i^\pm}} \right| \big|u_x^{i^\pm}\big| + C_1, \tag{3.33}$$

where the constant C_1 depends only on the supremum of the function h. Choose a constant Z_1 so large that

$$C_1 < \delta_2 Z_1. \tag{3.34}$$

Assume that all incoming waves satisfy the bounds

$$P_{i,k-1}\big|u_x^{i^-}\big| \le Z(\tau) \qquad \forall i \ge k, \qquad\qquad P_{i,k}\big|u_x^{i^+}\big| \le Z(\tau) \quad \forall i \le k,$$

for some value $Z(\tau) \ge Z_1$. Then, as long as (3.20) holds, from (3.33), (3.21), (3.34) we deduce

$$\begin{aligned} \big|u_x^{j^+}\big| &\le Z(\tau) \Bigg\{ \sum_{i \ge k} \left| \frac{\lambda_i(u^-) - \dot{y}_k}{\lambda_j(u^+) - \dot{y}_k} \right| \cdot \left| \frac{\partial W_j(u^+, u^-)}{\partial w_{i^-}} \right| P_{i,k-1}^{-1} \\ &\qquad\qquad + \sum_{i \le k} \left| \frac{\lambda_i(u^+) - \dot{y}_k}{\lambda_j(u^+) - \dot{y}_k} \right| \cdot \left| \frac{\partial W_j(u^+, u^-)}{\partial w_{i^+}} \right| P_{i,k}^{-1} + \frac{C_1}{Z(\tau)} \Bigg\} \\ &< Z(\tau) \left\{ \left(P_{j,k}^{-1} - \delta_2 \right) + \frac{C_1}{Z_1} \right\} \\ &< Z(\tau) P_{j,k}^{-1}. \end{aligned} \tag{3.35}$$

The above computation shows that the crossing of a shock does not produce any increase in the weighted norm (3.9), whenever this norm is already bigger than Z_1.

Outside the shocks and rarefaction fans, the weight $P_j = P_j(t,x)$ is locally constant. From (1.27) and the choice of the constant C_0 in (3.32) it thus follows

$$\frac{d}{dt}\Big\{P_j|u_x^j|(t,y_j(t))\Big\} \le \overline{P}\cdot\Big\{(n+n^2)C_0\Big(\max_i|u_x^i|\Big)^2 + nC_0\max_i|u_x^i| + C_0\Big\}. \tag{3.36}$$

Finally, we study what happens inside a rarefaction fan. To fix the ideas, consider the case $k\in\mathcal{R}$, $j>k$. At the time τ when the characteristic $y_j(\cdot)$ enters in the k-th rarefaction fan, the component u_x^j is continuous, while the weight $P_j(t,y_j(t))$ jumps downward, because of (3.27), as long as $|u(\tau,y_j(\tau))-\omega_{k-1}|<\delta_5$. Therefore, the product $P_j|u_x^j|$ can only decrease at time τ. The same happens at the time τ' when y_j exits from the rarefaction fan, provided that $\big|u(\tau',y_j(\tau'))-\omega_k\big|<\delta_5$.

When $(t,y_j(t))$ lies in the interior of the k-th rarefaction fan, by (1.26), (1.27) and (3.23) one has

$$\begin{aligned}
\frac{d}{dt}\Big\{P_j|u_x^j|(t,y_j(t))\Big\} &= \frac{dP_j}{dt}\cdot|u_x^j| + P_j\cdot\frac{d}{dt}|u_x^j|\\
&\le(\lambda_j-\lambda_k)\big(r_k\bullet\widetilde{P}_{j,k}(u)\big)u_x^k|u_x^j| + \sum_{i\ne k}(\lambda_j-\lambda_i)\big(r_i\bullet\widetilde{P}_{j,k}\big)u_x^i|u_x^j| + \sum_i\big(r_i\bullet\widetilde{P}_{j,k}(u)\big)\langle l_i,\ h\rangle|u_x^j|\\
&\quad-\widetilde{P}_{j,k}(u)\Big[(r_k\bullet\lambda_j)u_x^k|u_x^j| + \sum_{i\ne k}(r_i\bullet\lambda_j)u_x^i|u_x^j|\Big]\\
&\quad+\widetilde{P}_{j,k}(u)\sum_{i\ne k}\Big|(\lambda_k-\lambda_i)\big\langle l_j,\ [r_k,r_i]\big\rangle(u)\Big|\cdot u_x^k|u_x^i|\\
&\quad+\widetilde{P}_{j,k}(u)\sum_{i,h\ne k}\Big|(\lambda_h-\lambda_i)\big\langle l_j,\ [r_h,r_i]\big\rangle(u)\Big|\cdot|u_x^i||u_x^h|\\
&\quad+\widetilde{P}_{j,k}(u)\Big[\big|\big\langle l_j,\ [r_k,h]\big\rangle\big|u_x^k + \sum_{i\ne k}\big|\big\langle l_j,\ [r_i,h]\big\rangle u_x^i\big| + \big|\big\langle l_j,\ h_x\big\rangle\big|\Big].
\end{aligned} \tag{3.37}$$

Notice that in (3.37) we have singled out all terms containing the positive component u_x^k, which may be arbitrarily large inside the rarefaction fan. Choose a constant Z_2 so large that

$$\widetilde{P}_{j,k}(u)\big|\big\langle l_j,\ [r_k,h]\big\rangle\big| < Z_2\delta_4 \qquad \forall u\in\Omega,\quad \forall j,k. \tag{3.38}$$

Assume that, for some value $Z(t)\ge Z_2$, we have

$$\widetilde{P}_{j,k}(u)\cdot|u_x^j| = Z(t), \qquad \widetilde{P}_{i,k}(u)\cdot|u_x^i|\le Z(t) \qquad \forall i\ne k.$$

Then, using (3.24) and (3.38), for some constant C_2 one obtains the estimate

$$\begin{aligned}
\frac{d}{dt}\Big\{P_j|u_x^j|(t,y_j(t))\Big\} &\le Z(t)u_x^k\Big\{(\lambda_j-\lambda_k)\big(r_k\bullet\widetilde{P}_{j,k}\big)\widetilde{P}_{j,k}^{-1} - r_k\bullet\lambda_j\\
&\quad+\widetilde{P}_{j,k}\sum_{i\ne k}\Big|(\lambda_k-\lambda_i)\big\langle l_j,\ [r_k,r_i]\big\rangle\Big|\widetilde{P}_{i,k}^{-1} + \widetilde{P}_{j,k}\Big|\big\langle l_j,\ [r_k,h]\big\rangle\Big|Z^{-1}(t)\Big\} + C_2\big[1+Z^2(t)\big]\\
&\le C_2\big[1+Z^2(t)\big],
\end{aligned} \tag{3.39}$$

as long as (3.25) holds.

To complete the proof, for $\tau \geq 0$ define

$$Z(\tau) = \max\Big\{Z_1,\ Z_2,\ \operatorname*{esssup}_{j,x} P_j(\tau,x)\big|u_x^j(\tau,x)\big|\Big\},$$

recalling that (τ,x) range in the domain $\mathcal{D}$ defined at (3.19), so that $|x| \leq \rho - \hat{\lambda}\tau$. From (3.35), (3.36) and (3.39) it follows

$$\frac{d^+Z(\tau)}{d\tau} \doteq \limsup_{\varepsilon\to 0+} \frac{Z(\tau+\varepsilon)-Z(\tau)}{\varepsilon} \leq C_3\big[Z^2(\tau)+1\big],$$

$$Z(\tau) \leq \tan\big[\arctan Z(0) + C_3 t\big], \tag{3.40}$$

for some constant C_3, as long as the bounds (3.20), (3.25), (3.26) hold. This is certainly the case on a suitably small interval $[0,\tau_0]$, because of the estimate

$$\Big|u(\tau,y_i(\tau)) - u(0,y_i(0))\Big| \leq \int_0^\tau \Big\{\sum_{j\neq i}\big|\lambda_j u_x^j(t,y_i(t))\big| + \big|h(t,y_i(t))\big|\Big\}\,dt,$$

valid along every characteristic $y_i(\cdot)$ of the i-th family which does not cross any shock on $[0,\tau]$.

The estimates (3.40) provide an a-priori bound on Z, and hence on all components u_x^i of the gradient of u outside the rarefaction fans, proving Proposition 3.3.

Remark 3.4. By (1.27), along a characteristic $y_k(\cdot)$ contained in the k-th rarefaction fan, the large, positive component u_x^k satisfies a differential inequality of the form

$$\begin{aligned}\frac{d}{dt}u_x^k\big(t,y_k(t)\big) &\leq -(r_k\bullet\lambda_k)\cdot(u_x^k)^2 + C_4\bigg\{\Big(\max_{i\neq k}|u_x^i|+1\Big)u_x^k + \Big(\max_{i\neq k}|u_x^i|\Big)^2 + 1\bigg\}\\ &\leq -\epsilon_2(u_x^k)^2 + C_5\big(u_x^k+1\big),\end{aligned} \tag{3.41}$$

for some constants $C_4, C_5, \epsilon_2 > 0$, as long as the bounds stated in Proposition 3.3 hold. For each fixed $\tau > 0$, the component u_x^k is thus uniformly bounded also in the interior of the k-th rarefaction fan. In particular, the solution $u^\epsilon(\tau,\cdot)$ of (1.1), (3.16) is piecewise continuous, with a Lipschitz constant which depends on τ but not on ϵ.

3.3 - Estimates on generalized tangent vectors.

We now introduce an additional hypothesis, quite similar to (H4), which will guarantee the well posedness of the linearized variational system (2.16)–(2.18), in the case where u is any solution of (1.1), (3.16).

(H5) There exists a family of weights $Q_{i,k} > 0$, $i = 1,\ldots,n$, $k = 0,\ldots,n$ such that, if $k \in \mathcal{S}$, setting $\dot{x}_k = \lambda(\omega_{k-1},\omega_k)$ one has

$$\begin{aligned}Q_{i,k-1}\big(\lambda_i(\omega_{k-1})-\dot{x}_k\big) &> \sum_{j>k} Q_{j,k}\big(\lambda_j(\omega_k)-\dot{x}_k\big)\left|\frac{\partial W_j^k}{\partial w_{i^-}}\right|\\ &\quad+\sum_{j<k} Q_{j,k-1}\big(\lambda_j(\omega_{k-1})-\dot{x}_k\big)\left|\frac{\partial W_j^k}{\partial w_{i^-}}\right| \qquad \text{for}\ \ i\geq k\ \ [i>k],\end{aligned} \tag{3.42}$$

$$Q_{i,k}\big(\dot x_k-\lambda_i(\omega_k)\big)>\sum_{j>k}Q_{j,k}\big(\lambda_j(\omega_k)-\dot x_k\big)\left|\frac{\partial W_j^k}{\partial w_{i^+}}\right|$$
$$+\sum_{j<k}Q_{j,k-1}\big(\lambda_j(\omega_{k-1})-\dot x_k\big)\left|\frac{\partial W_j^k}{\partial w_{i^+}}\right| \text{for } \ i\le k \ \ [i<k], \tag{3.43}$$

in the genuinely nonlinear [resp. linearly degenerate] case. Moreover, if $k\in\mathcal{R}$, one has

$$\begin{aligned}\frac{Q_{i,k}-Q_{i,k-1}}{\lambda_k(\omega_k)-\lambda_k(\omega_{k-1})}(\lambda_i-\lambda_k)\big(r_k\bullet\lambda_k(\omega_{k,\theta})\big)+Q_{i,k,\theta}\big(r_k\bullet\lambda_i(\omega_{k,\theta})\big)\\ +\sum_{j\neq k}Q_{j,k,\theta}\Big|(\lambda_j-\lambda_k)\big\langle l_j,\ [r_k,r_i]\big\rangle(\omega_{k,\theta})\Big|<0\end{aligned}\qquad \forall i\neq k,\quad \theta\in[0,1]. \tag{3.44}$$

In the degenerate case $\omega_k=\omega_{k-1}$, (3.44) is meant to be replaced simply by

$$\big(Q_{i,k}-Q_{i,k-1}\big)\cdot(i-k)<0.$$

The values $\omega_k,\omega_{k,\theta}$ are here the same as in (H4), while

$$Q_{i,k,\theta}\doteq\theta Q_{i,k}+(1-\theta)Q_{i,k-1}\qquad\qquad \theta\in[0,1].$$

Remark 3.5. The assumption (H5) is satisfied by the weights

$$Q_{i,k}\doteq e^{|k-i+(1/2)|}$$

whenever the states u_*,u^* are sufficiently close.

Proposition 3.6. *In connection with the standard Riemann problem (3.2), (3.3), let (H4), (H5) hold. Let the initial condition $\bar u$ be piecewise Lipschitz continuous, having a single jump at $x=0$ with limits $\bar u(0+)=u^*$, $\bar u(0-)=u_*$, and for $\epsilon\in(0,\epsilon_0]$ let u^ϵ be the piecewise Lipschitz solution of (1.1), (3.16), defined on the domain $\mathcal{D}$ in (3.19). Then there exist constants $K,\tau_1>0$, independent of ϵ, such that any solution (v,ξ) of the linear variational system (2.16)–(2.18) corresponding to u^ϵ satisfies the estimate*

$$\big\|v(t,\cdot)\big\|_{\mathbf{L}^1([-\rho+t\hat\lambda,\ \rho-t\hat\lambda])}+\big|\xi(t)\big|\le K\Big(\big\|v(0,\cdot)\big\|_{\mathbf{L}^1([-\rho,\rho])}+\big|\xi(0)\big|\Big)\qquad\qquad \forall t\in[0,\tau_1]. \tag{3.45}$$

Proof. For each $k\in\mathcal{S}$, denote by

$$J_k(t)\doteq\big|u(y_k(t)-)-u(y_k(t)+)\big|$$

the strength of the k-th shock. By Proposition 3.3, the components u_x^i of the gradient of u remain uniformly bounded in a neighborhood of the shock lines. This fact, together with the equations (2.17), (2.18), implies bounds of the form

$$\big|\dot J_k\big|\le C_6J_k, \tag{3.46}$$

$$\left|v_{j^\pm}-\sum_{i^\pm\in\mathcal{I}}\frac{\partial W_j}{\partial w_i^\pm}\cdot v_{i^\pm}\right|\le C_7J_k|\xi_k|\qquad\qquad \forall j^\pm\in\mathcal{O}, \tag{3.47}$$

$$|\dot{\xi}_k| \leq C_8|\xi_k| + \left| D\lambda_k(u^+, u^-) \cdot \Big(\sum_{i=1}^n v_{i^+} r_i(u^+),\ \sum_{i=1}^n v_{i^-} r_i(u^-) \Big) \right|, \tag{3.48}$$

for suitable constants $C_6, \ldots, C_8$ and all $k \in \mathcal{S}$. As usual, $v_i \doteq \langle l_i(u), v \rangle$ denotes the i-th component of the tangent vector v, while, for a given k, the sets of indices $\mathcal{I}, \mathcal{O}$ of incoming and outgoing waves are defined as in (1.35).

By continuity, (3.42), (3.42) imply the existence of $\delta_6, \delta_7 > 0$ such that, if

$$\big|u^+ - \omega_k\big| < \delta_6, \qquad \big|u^- - \omega_{k-1}\big| < \delta_6 \tag{3.49}$$

and $\dot{y}_k = \lambda_k(u^-, u^+)$, then

$$\begin{aligned} \big(Q_{i,k-1} - \delta_7\big)\big(\lambda_i(u^-) - \dot{y}_k\big) &> \sum_{j>k} Q_{j,k}\big(\lambda_j(u^+) - \dot{y}_k\big) \left| \frac{\partial W_j(u^-, u^+)}{\partial w_{i^-}} \right| \\ &\quad + \sum_{j<k} Q_{j,k-1}\big(\lambda_j(u^-) - \dot{y}_k\big) \left| \frac{\partial W_j(u^-, u^+)}{\partial w_{i^-}} \right| \qquad \text{for} \quad i \geq k \quad [i > k], \end{aligned} \tag{3.50}$$

$$\begin{aligned} \big(Q_{i,k} - \delta_7\big)\big(\dot{y}_k - \lambda_i(u^+)\big) &> \sum_{j>k} Q_{j,k}\big(\lambda_j(u^+) - \dot{y}_k\big) \left| \frac{\partial W_j(u^-, u^+)}{\partial w_{i^+}} \right| \\ &\quad + \sum_{j<k} Q_{j,k-1}\big(\lambda_j(u^-) - \dot{y}_k\big) \left| \frac{\partial W_j(u^-, u^+)}{\partial w_{i^+}} \right| \qquad \text{for} \quad i \leq k \quad [i < k]. \end{aligned} \tag{3.51}$$

in the genuinely nonlinear [linearly degenerate] case, respectively. On the other hand, for each $k \in \mathcal{R}$, by continuity and compactness from (3.44) we deduce the existence of positive constants δ_8 and $\hat{Q}_{j,k}$ $(j \neq k)$ with

$$\begin{array}{lll} \hat{Q}_{j,k-1} < Q_{j,k-1} & \hat{Q}_{j,k} > Q_{j,k} & \text{if } j > k, \\ \hat{Q}_{j,k-1} > Q_{j,k-1} & \hat{Q}_{j,k} < Q_{j,k} & \text{if } j < k, \end{array} \tag{3.52}$$

such that, defining

$$\widetilde{Q}_{j,k}(u) \doteq \hat{Q}_{j,k-1} + \frac{\lambda_k(u) - \lambda_k(\omega_{k-1})}{\lambda_k(\omega_k) - \lambda_k(\omega_{k-1})} \cdot \big(\hat{Q}_{j,k} - \hat{Q}_{j,k-1}\big),$$

one has

$$\begin{aligned} (\lambda_i - \lambda_k)\big(r_k \bullet \widetilde{Q}_{i,k}(u)\big) &+ \widetilde{Q}_{i,k}(u)\big(r_k \bullet \lambda_i(u)\big) \\ &+ \sum_{j\neq k} \widetilde{Q}_{j,k}(u) \Big| (\lambda_j - \lambda_k)\langle l_j,\ [r_k, r_i]\rangle(u) \Big| < 0, \end{aligned} \tag{3.53}$$

for all $j \neq k$ and all states u such that

$$\big|u - \omega_{k,\theta}\big| < \delta_8 \qquad \text{for some } \theta \in [0,1]. \tag{3.54}$$

Let the characteristic lines $y_k^\pm$ be as in (3.28)-(3.30). In analogy with (3.31), introduce the weights

$$Q_j(t,x) \doteq \begin{cases} Q_{j,k} & \text{if} \quad y_k^+(t) < x < y_{k+1}^-(t), \\ \widetilde{Q}_{j,k}\big(u(t,x)\big) & \text{if} \quad y_k^-(t) < x < y_k^+(t). \end{cases} \tag{3.55}$$

Observe that there exists a constant C_9 such that

$$\begin{aligned}\left| D\lambda_k(u^+,u^-)\cdot\Big(\sum_{i=1}^n w_i^+ r_i(u^+),\ \sum_{i=1}^n w_i^- r_i(u^-)\Big)\right|\cdot J_k \\ \le C_9\left[\sum_{i\ge k}\big(\lambda_i(u^-)-\lambda_k(u^+,u^-)\big)|w_i^-| + \sum_{i\le k}\big(\lambda_k(u^+,u^-)-\lambda_i(u^+)\big)|w_i^+|\right]\end{aligned} \tag{3.56}$$

for all k, all states u^-, u^+ connected by a contact discontinuity or by an admissible shock of the k-th family, and all w_i^+, w_i^- satisfying (1.36). Recalling (3.50), (3.51), consider the constant

$$\mu_0 \doteq \delta_7/C_9. \tag{3.57}$$

Define the weighted norm of a generalized tangent vector (v,ξ) as

$$\Big\|\big(v(t,\cdot),\xi(t)\big)\Big\|_{u(t)} \doteq \sum_{j=1}^n \int_{|x|\le\rho-\hat\lambda t} Q_j(t,x)\big|v_j(t,x)\big|\ dx + \sum_{k\in\mathcal{S}}\mu_0 J_k(t)\big|\xi_k(t)\big|. \tag{3.58}$$

Integrating along characteristics, we now estimate

$$\begin{aligned}\frac{d}{dt}\big\|(v,\xi)\big\|_{u(t)} \le & \sum_{k\in\mathcal{S}}\mu_0\Big\{\dot J_k(t)\big|\xi_k(t)\big| + J_k(t)\dot\xi_k(t)\mathrm{sign}\big(\xi_k(t)\big)\Big\} \\ & + \sum_i\int_{|x|<\rho-\hat\lambda t}\big|v_i(t,x)\big|\cdot\left(\frac{\partial}{\partial t}+\lambda_i(u)\frac{\partial}{\partial x}\right)Q_i(t,x)\ dx \\ & + \sum_i\int_{|x|<\rho-\hat\lambda t} Q_i(t,x)\cdot\mathrm{sign}(v_i)\big[(v_i)_t+(\lambda_i v_i)_x\big]\ dx \\ & + \sum_{k\in\mathcal{S}}\sum_i\Big\{Q_{i,k}\big(\lambda_i(u(y_k+))-\dot y_k\big)\big|v_i(y_k+)\big| - Q_{i,k-1}\big(\lambda_i(u(y_k-))-\dot y_k\big)\big|v_i(y_k-)\big|\Big\} \\ \doteq & E_1+E_2+E_3+E_4.\end{aligned} \tag{3.59}$$

Using (3.46), (3.48) and (3.56), one obtains

$$E_1 \le (C_6+C_8)\sum_{k\in\mathcal{S}}\mu_0 J_k(t)\big|\xi_k(t)\big| + C_9\sum_{k\in\mathcal{S}}\sum_{i^\pm\in\mathcal{I}}\mu_0\big|\lambda_i(u^\pm)-\dot y_k\big||v_{i^\pm}|. \tag{3.60}$$

From the equations (2.65), singling out the terms containing the large gradients u_x^k inside the rarefaction fans, we obtain an estimate of the form

$$\begin{aligned}E_2+E_3 \le & C_{10}\int_{|x|\le\rho-\hat\lambda t}\big|v_i(t,x)\big|\ dx + \sum_{k\in\mathcal{R}}\sum_{i\ne k}u_x^k\int_{y_k^-(t)}^{y_k^+(t)}\Big\{(\lambda_i-\lambda_k)\big(r_k\bullet\widetilde Q_{i,k}(u)\big)|v_i| \\ & + \widetilde Q_{i,k}(u)\big(r_k\bullet\lambda_i(u)\big)|v_i| + \sum_{j\ne k}\widetilde Q_{i,k}(u)\Big|\big\langle l_i,\ [r_k,r_j]\big\rangle(\lambda_i-\lambda_k)\Big||v_j|\Big\}\ dx \\ \le & C_{10}\int_{|x|\le\rho-\hat\lambda t}\big|v_i(t,x)\big|\ dx.\end{aligned} \tag{3.61}$$

Indeed, using (3.53) and reversing the order of the summations over i, j, one checks that the integrals over the rarefaction fans have negative sum.

Finally, using (3.47), (3.50), (3.51), we obtain

$$E_4 \leq \sum_{k \in \mathcal{S}} \Big[C_{11} J_k |\xi_k| - \delta_7 \sum_{i^\pm \in \mathcal{I}} \big|\lambda_i(u^\pm) - \dot{y}_k\big| |v_{i^\pm}| \Big]. \tag{3.62}$$

Summing together the estimates (3.60)–(3.62), and recalling (3.57)-(3.58), we obtain

$$\frac{d^+}{dt} \big\|(v, \xi)(t)\big\|_{u(t)} \doteq \limsup_{t' \to t+} \frac{\big\|(v, \xi)(t')\big\|_{u(t')} - \big\|(v, \xi)(t)\big\|_{u(t)}}{t' - t} \leq C_{12} \big\|(v, \xi)(t)\big\|_{u(t)} \tag{3.63}$$

for a suitable constant C_{12}, as long as the bounds (3.49), (3.54) hold. This is certainly the case for $t \geq 0$ sufficiently small, uniformly w.r.t. ϵ. Proposition 3.6 is thus proved.

Remark 3.7. By continuity, the assumptions (H4)-(H5), and hence the uniform estimates obtained in Propositions 3.3 and 3.6, remain valid for the solution of any initial value problem of the form (1.1), (3.16), as long as

$$\big|\bar{u}(0+) - u^*\big| < \delta_9, \qquad \big|\bar{u}(0-) - u_*\big| < \delta_9, \tag{3.64}$$

for some constant $\delta_9 > 0$ sufficiently small.

Theorem 3.8. *In connection with the standard Riemann problem (3.2), (3.3), let (H4), (H5) hold. Then, for every Lipschitz constant L, there exist $\delta_9 > 0$ and a domain $\mathcal{D}$ as in (3.19) such that the following holds. Given any piecewise continuous initial condition $\bar{u}$ with Lipschitz constant L and a single jump at $x = 0$ with limits $u(0-), u(0+)$ satisfying (3.64), the initial value problem (3.15) has a solution defined in $\mathcal{D}$, which depends continuously on the initial data. More precisely, there exists a constant K such that, for every couple of solutions u, w one has*

$$\int_{-\rho+\hat{\lambda}t}^{\rho-\hat{\lambda}t} \big|u(t,x) - w(t,x)\big|\, dx \leq K \int_{-\rho}^{\rho} \big|u(0,x) - w(0,x)\big|\, dx \qquad \forall t \in [0, \tau_0]. \tag{3.65}$$

Proof. Choose the domain $\mathcal{D}$ and the constant δ_9 according to Propositions 3.3, 3.6 and Remark 3.7. For a given $\bar{u}$ satisfying the assumptions of the theorem, we first show that, for every $t \in [0, \tau_0]$, the $\mathbf{L}^1$-limit of the solutions $u^\epsilon(t, \cdot)$ of the perturbed problem (1.1), (3.15) exists as $\epsilon \to 0$. Indeed, for every fixed $\hat{\epsilon}$, the assignments

$$\gamma_+ : \varepsilon \mapsto \bar{u}^{\hat{\epsilon}+\varepsilon}, \qquad \gamma_- : \varepsilon \mapsto \bar{u}^{\hat{\epsilon}-\varepsilon}$$

are Regular Variations of $u^{\hat{\epsilon}}$, generating the tangent vectors $(\bar{v}_\pm^{\hat{\epsilon}}, \bar{\xi}_\pm^{\hat{\epsilon}})$, with

$$\bar{v}_\pm^{\hat{\epsilon}}(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \pm n \bar{u}_x^{\hat{\epsilon}} & \text{if} \quad x > \hat{\epsilon} n, \\ \pm \big(x/\hat{\epsilon}\big) \bar{u}_x^{\hat{\epsilon}} & \text{if} \quad 0 < x < \hat{\epsilon} n, \end{cases}$$

$$\bar{\xi}^{\hat{\epsilon}}_{\pm} = (\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_N),$$

where $\alpha_1, \cdots, \alpha_N \in \{1, \ldots, n\}$ are the indices which correspond to a nontrivial shock or contact discontinuity in the solution of the Riemann problem. Since the total variation of $\bar{u}^{\hat{\epsilon}}$ remains uniformly bounded as $\hat{\epsilon} \to 0$, so does the norm $\|\bar{v}^{\hat{\epsilon}}_{\pm}\|_{\mathbf{L}^1} + |\bar{\xi}^{\hat{\epsilon}}_{\pm}|$. From Theorem 2.2 and Proposition 3.3 it follows that at time $t \in [0, \tau_0]$, the curves $\gamma_{\pm}(\varepsilon) \doteq u^{\hat{\epsilon}\pm\varepsilon}(t, \cdot)$ are R.V. of $u^{\hat{\epsilon}}(t, \cdot)$. Therefore, the well posedness of the linearized system (2.16)–(2.18) for $u = u^{\hat{\epsilon}}$ proved in Proposition 3.6, together with Theorem 2.5, implies the existence of a constant C_{13} such that, for each $\epsilon \in (0, \epsilon_0]$,

$$\|u^{\hat{\epsilon}+\varepsilon}(t, \cdot) - u^{\hat{\epsilon}}(t, \cdot)\|_{\mathbf{L}^1} < C_{13} \cdot \varepsilon$$

for $|\varepsilon|$ sufficiently small. This implies the Lipschitz continuity of the map $\epsilon \mapsto u^{\epsilon}(t, \cdot)$. Hence, its $\mathbf{L}^1$-limit for $\epsilon \to 0+$ is a well defined solution of (1.1).

The proof of the Lipschitz continuous dependence of the solution on the initial data relies on the same technique. Given two initial conditions $\bar{u}, \bar{w}$ satisfying the assumptions of the theorem, for $\varepsilon_1 > 0$ sufficiently small consider the path of initial conditions joining $\bar{u}$ with $\bar{w}$:

$$\gamma(\sigma) = \begin{cases} \Psi_{\sigma\varepsilon_1}(\bar{u}) & \text{if} \quad \sigma \in [0, 1], \\ \Psi_{\varepsilon_1}\big((\sigma - 1)\bar{u} + (2 - \sigma)\bar{w}\big) & \text{if} \quad \sigma \in [1, 2], \\ \Psi_{(3-\sigma)\varepsilon_1}(\bar{w}) & \text{if} \quad \sigma \in [2, 3], \end{cases}$$

where Ψ_ε is the operator defined at (3.16). Observe that γ is a piecewise regular path. Moreover, there exists some constant C_{14} such that, for every couple $\bar{u}, \bar{w}$, the length of the path γ satisfies

$$\|\gamma\| \leq C_{14} \|\bar{u} - \bar{w}\|_{\mathbf{L}^1}, \tag{3.66}$$

provided that ε_1 is chosen suitably small (depending on $\bar{u}, \bar{w}$).

For $t \in [0, \tau_0]$, let $\gamma^t(\sigma)$ be the value at time t of the solution of (1.1) with initial condition $\gamma(\sigma)$. By Proposition 3.3, the path γ^t is still piecewise regular. By Proposition 3.6 and Theorem 2.5, its length satisfies

$$\|\gamma^t\| \leq C_{15} \|\gamma\| \tag{3.67}$$

for some uniform constant C_{15}. Together, (3.66) and (3.67) imply the conclusion of Theorem 3.8.

Remark 3.9. The proof of the above theorem provides an example of a general technique for establishing the Lipschitz continuous dependence of solutions of (1.1) on their initial data. Indeed, if $\mathcal{F}$ is a given family of initial conditions, relying on Theorems 2.2 and 2.5 it suffices to implement the following three steps:

1. Any couple $\bar{u}, \bar{w} \in \mathcal{F}$, can be connected by a piecewise regular path γ inside $\mathcal{F}$ whose length satisfies a uniform estimate of the form (3.66).
2. For $t \in [0, \tau_0]$, the corresponding path γ^t remains piecewise regular.
3. The time derivative of the size of tangent vectors satisfies a uniform estimate of the form (3.63).

In particular, the above argument can be applied to the family $\mathcal{F}$ of all initial conditions $\bar{u} : [-\rho, \rho] \mapsto \mathbb{R}^n$ such that, for suitable points $y_j = y'_j = y''_j$, $j \in \mathcal{S}$ and intervals $[y'_j, y''_j]$, $j \in \mathcal{R}$, the following assumptions hold.

(i) $y''_j \leq y'_k$ whenever $j < k$,

(ii) $\bar{u}$ is piecewise Lipschitz continuous, with jumps only at points y_j, $j \in \mathcal{S}$, where it suffers an admissible shock or a contact discontinuity in the j-th characteristic field.

(iii) For some constant L and all $j = 1, \ldots, n$ one has

$$\left|\bar{u}_x^j(x)\right| \leq L$$

for a.e. x, except possibly on the interval $[y_j', y_j'']$, where $u_x^j > 0$.

(iv) For some constant $\delta_{10} > 0$ sufficiently small and all j, one has

$$\lim_{x \to y_j'-} \left|\bar{u}(x) - \omega_{j-1}\right| < \delta_{10}, \qquad \lim_{x \to y_j''+} \left|\bar{u}(x) - \omega_j\right| < \delta_{10},$$

where $\omega_0, \ldots, \omega_n$ are the constant states in the solution of the standard Riemann problem (3.2)-(3.3).

Indeed, for these initial conditions, the corresponding solutions of (1.1) have locally the same structure as the solution of the Riemann problem (3.2)-(3.3), hence the estimates proved in Propositions 3.3 and 3.6 still hold, provided that δ_{10} and the domain $\mathcal{S}$ are chosen suitably small (depending on L).

3.4 - The main theorem.

We are now in the position to prove the main result of this chapter, showing that generalized tangent vectors continue to exist beyond the time of interaction of two shocks. Due to the finite speed of propagation, it is not restrictive to assume that our reference solution u of (1.1) has exactly two discontinuities, located at $x_1(t) < x_2(t)$, with jumps occurring in the k_1-th, k_2-th characteristic fields ($k_1 \geq k_2$). For simplicity, let the discontinuities interact when $t = 0$, at $x_1(0) = x_2(0) = 0$. Let a family $\{u^\varepsilon;\ \varepsilon \in [0, \varepsilon_0]\}$ of solutions of (1.1) be given, and assume that for $t < 0$ the map $\varepsilon \mapsto u^\varepsilon(t, \cdot)$ is a Regular Variation of $u(t, \cdot)$, generating the tangent vector $\big(v(t), \xi(t)\big) \in \mathbf{L}^1 \times I\!R^2$.

At the time of interaction $t = 0$, call u^*, u_* respectively the limits of $u(0, x)$ as $x \to 0$ from the right and from the left. Let $\varphi(t, x)$ be the self-similar solution of the standard Riemann problem (3.2), (3.3), with constant states $\omega_0, \ldots, \omega_n$; call $\mathcal{S} \subseteq \{1, \ldots, n\}$ the set of indices k such that ω_{k-1} and ω_k are connected by a nontrivial shock or contact discontinuity, and let $\mathcal{R}$ be the set of indices corresponding to a rarefaction fan.

For $t > 0$, a generalized tangent vector to the reference solution u of (1.1) will thus have the form $(v, \xi) \in \mathbf{L}^1 \times I\!R^N$, with $N = |\mathcal{S}|$, $\xi = (\xi_j)_{j \in \mathcal{S}}$. In the following, we write $v^-, \xi_1^-, \xi_2^-, \dot{x}_1^-, \dot{x}_2^-$ respectively for the limits as $t \to 0-$ of $v, \xi_1, \xi_2, \dot{x}_1, \dot{x}_2$, while $\dot{y}_j^+ = \lambda_j(\omega_{j-1}, \omega_j)$ denotes the speed of the j-th discontinuity in the solution φ of the standard Riemann problem (3.2), (3.3), for $j \in \mathcal{S}$. Call $\tau^\varepsilon, \eta^\varepsilon$ the time and location of the interaction in the solution u^ε, and define the function

$$V(x) \doteq \limsup_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left|u^\varepsilon(\tau^\varepsilon, x + \eta^\varepsilon) - u(0, x)\right|. \tag{3.69}$$

We recall that Lemma 3.1 yields

$$\left.\frac{d\tau^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\xi_2^- - \xi_1^-}{\dot{x}_1^- - \dot{x}_2^-} \doteq \alpha, \qquad \left.\frac{d\eta^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\dot{x}_1^- \xi_2^- - \dot{x}_2^- \xi_1^-}{\dot{x}_1^- - \dot{x}_2^-} \doteq \beta. \tag{3.70}$$

Theorem 3.10. *In the above setting, let the assumptions (H4), (H5) hold. Assume that the limits v^-, ξ_1^-, ξ_2^- exist, that all solutions $u^\varepsilon(t,\cdot)$ are uniformly Lipschitz continuous before their respective interaction times τ^ε and that the function V in (3.69) is integrable in a neighborhood of the origin. Then there exists a domain $\mathcal{D}$ as in (3.19) such that the family $u^\varepsilon(t,\cdot)$ is a R.V. of $u(t,\cdot)$ on $\mathcal{D}$, generating a generalized tangent vector $(v,\xi) \in \mathbf{L}^1 \times I\!R^N$. This vector provides the unique solution to the linear system (2.16)–(2.18) with initial conditions*

$$\lim_{t\to 0+} \int_{|x|\le \rho-\hat\lambda t} \left| v(t,x) - v^-(x) + \frac{\xi_2^- - \xi_1^-}{\dot x_1^- - \dot x_2^-}\cdot \varphi_t(t,x) + \frac{\dot x_1^- \xi_2^- - \dot x_2^- \xi_1^-}{\dot x_1^- - \dot x_2^-}\cdot \varphi_x(t,x)\right| \, dx = 0, \quad (3.71)$$

$$\lim_{t\to 0+} \xi_j(t) \;=\; -\frac{\xi_2^- - \xi_1^-}{\dot x_1^- - \dot x_2^-}\cdot \dot y_j^+ + \frac{\dot x_1^- \xi_2^- - \dot x_2^- \xi_1^-}{\dot x_1^- - \dot x_2^-} \qquad \forall j \in \mathcal{S}. \quad (3.72)$$

Proof. By Propositions 3.3 and 3.6 and by Remark 3.7, we can assume that all solutions u^ε are defined on a common domain $\mathcal{D}$ of the form (3.19). For every fixed $t > 0$ sufficiently small, by Remark 3.4 we can also assume that all functions $u^\varepsilon(t,\cdot)$ have exactly N jumps, say at $y_j^\varepsilon(t)$, $j \in \mathcal{S}$, and are uniformly Lipschitz continuous outside these points; hence the family $u^\varepsilon(t,\cdot)$ is a Regular Variation of $u(t,\cdot)$.

For convenience, instead of (1.5), when $k \in \mathcal{R}$ the eigenvectors r_k will be here renormalized according to

$$r_k \bullet \lambda_k(u) \equiv 1. \quad (3.73)$$

For $k \in \mathcal{R}$, let $y_k'(t) \le y_k''(t)$ be the k-th characteristics through the origin which bound the k-th rarefaction fan from the left and from the right. From the uniform bounds on the gradients u_x^i outside the corresponding rarefaction fans we deduce

$$\big|\dot y_k'(t) - \lambda_k(\omega_{k-1})\big| < C_1 t, \qquad \big|\dot y_k''(t) - \lambda_k(\omega_k)\big| < C_1 t, \quad (3.74)$$

$$\big|y_k'(t) - t\lambda_k(\omega_{k-1})\big| < C_1 t^2, \qquad \big|y_k''(t) - t\lambda_k(\omega_k)\big| < C_1 t^2, \quad (3.75)$$

for a suitable constant C_1. Moreover, the equations (1.27) and the normalization (3.73) imply

$$\big(u_x^k\big)_t + \lambda_k \big(u_x^k\big)_x = -\big(u_x^k\big)^2 + \sigma(t,x,u,u_x) \qquad x \in \big[y_k'(t),\ y_k''(t)\big],$$

with

$$\big|\sigma(t,x,u,u_x)\big| < C_2\big(|u_x^k| + 1\big).$$

This yields the estimate

$$\left| u_x^k(t,x) - \frac{1}{t}\right| < C_3 \qquad \text{for a.e.} \quad x \in \big[y_k'(t),\ y_k''(t)\big], \quad (3.76)$$

valid for some constant C_3, uniformly as $t \to 0+$.

Because of the uniform well-posedness of the linear system of equations (2.16)–(2.18), proved in Proposition 3.6, there can be at most one generalized tangent vector $(v,\xi) : [0,\tau_0] \mapsto \mathbf{L}^1 \times I\!R^N$ which satisfies these equations for $t > 0$, together with the boundary conditions (3.71)-(3.72). To prove that one such solution exists, we shall construct it in the form

$$(v,\xi) = (v^1,\xi^1) + (v^2,\xi^2) + (v^3,\xi^3), \quad (3.77)$$

where the vectors (v^ℓ, ξ^ℓ), $\ell = 1, 2, 3$ are defined as follows. First, with α, β as in (3.69), set

$$v^1(t,x) \doteq -\alpha u_t(t,x) - \beta u_x(t,x), \qquad \xi^1_j(t) \doteq -\alpha \dot y_j(t) + \beta \qquad \forall j \in \mathcal{S}. \tag{3.78}$$

Observe that (v^1, ξ^1) provides a solution to the linear nonhomogeneous system

$$\big(v^1\big)_t + \big[DA(u)\cdot v^1\big]u_x + A(u)v^1_x = h_u \cdot v^1 - \big(\alpha h_t + \beta h_x\big)$$

with boundary conditions (2.17)-(2.18). Moreover, using (3.75) and (3.76) we deduce

$$\lim_{t\to 0+} \int_{|x|\le \rho - \hat\lambda t} \Big| v^1(t,x) + \alpha\varphi_t(t,x) + \beta\varphi_x(t,x) + \alpha u_t(0,x) + \beta u_x(0,x)\Big|\, dx = 0. \tag{3.79}$$

$$\lim_{t\to 0+} \xi^1_j(t) \;=\; -\alpha \dot y_j^+ + \beta \qquad \forall j \in \mathcal{S}. \tag{3.80}$$

Next, let (v^2, ξ^2) be the unique solution of (2.16)–(2.18) with initial conditions

$$v^2(0,x) = v^-(x) + \alpha u_t(0,x) + \beta u_x(0,x), \qquad \xi^2_j(0) = 0 \qquad \forall j \in \mathcal{S}.$$

This solution is well defined. Indeed, by the well-posedness of the linear system, proved in Proposition 3.6, (v^2, ξ^2) can be obtained as the unique limit as $\varepsilon \to 0+$ of the solutions $\big(v^{2,\varepsilon}, \xi^{2,\varepsilon}\big)$ of (2.16)–(2.18) with initial conditions

$$v^{2,\varepsilon}(0,x) = \big[v^-(x) + \alpha u_t(0,x) + \beta u_x(0,x)\big]\cdot\Big(1 - \chi_{[-\varepsilon,\varepsilon]}(x)\Big), \qquad \xi^{2,\varepsilon}_j(0) = 0 \qquad \forall j \in \mathcal{S}.$$

Finally, let (v^3, ξ^3) be the solution of the linear nonhomogeneous problem

$$v^3_t + \big[DA(u)\cdot v^3\big]u_x + A(u)v^3_x = h_u \cdot v^3 + \big(\alpha h_t + \beta h_x\big),$$

with boundary conditions (2.17)-(2.18) and initial conditions

$$v^3(0,x) = 0, \qquad \xi^3_j(0) = 0 \qquad \forall j \in \mathcal{S}.$$

By Proposition 3.6 and the regularity of the partial derivatives h_t, h_x, such solution exists and is unique.

By linearity, the sum in (3.77) thus provides a solution to (2.16)–(2.18). Moreover, the limits (3.79)-(3.80) and the choice of the initial data for $(v^2, \xi^2), (v^3, \xi^3)$ imply that the conditions (3.71)-(3.72) hold as well.

In the next step, for this solution (v, ξ) we prove the intermediate estimate

$$\begin{aligned}
\limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_{|x|\le\rho-\hat\lambda t} & \Big| u^\varepsilon(t,x) - \gamma_{(v(t),\xi(t);u(t))}(\varepsilon)(x)\Big|\, dx \\
&= \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{-\hat\lambda t}^{\hat\lambda t} \Big| u^\varepsilon(t,x) - \gamma_{(v(t),\xi(t);u(t))}(\varepsilon)(x)\Big|\, dx \\
&= \sigma(t),
\end{aligned} \tag{3.81}$$

with $\sigma(t) \to 0$ as $t \to 0+$. Indeed, the first equality in (3.81) is a trivial consequence of Theorem 2.2 and of the fact that all propagation speeds are bounded by $\hat\lambda$. To estimate $\sigma(t)$, let the constant

K be as in Theorem 3.8 and define the auxiliary tangent vector (v^1, ξ^1) again by (3.78). When $t > 0$ is suitably small one has

$$\begin{aligned}
\sigma(t) &\leq \limsup_{\varepsilon\to0+} \frac{1}{\varepsilon}\int_{-\hat\lambda t}^{\hat\lambda t} \Big\{ \big|u^\varepsilon(t,x) - u(t-\tau^\varepsilon, x-\eta^\varepsilon)\big| + \big|u(t-\tau^\varepsilon, x-\eta^\varepsilon) - \gamma_{(v(t),\xi(t);u(t))}(\varepsilon)(x)\big|\Big\}\, dx \\
&\leq \limsup_{\varepsilon\to0+} \frac{K}{\varepsilon}\int_{-3\hat\lambda t}^{3\hat\lambda t} \big|u^\varepsilon(\tau^\varepsilon, x+\eta^\varepsilon) - u(0,x)\big|\, dx \\
&\quad + \limsup_{\varepsilon\to0+} \frac{1}{\varepsilon}\int_{-\hat\lambda t}^{\hat\lambda t} \Big|u(t-\tau^\varepsilon, x-\eta^\varepsilon) - \gamma_{(v^1(t),\xi^1(t);u(t))}(\varepsilon)(x)\Big|\, dx \\
&\quad + \limsup_{\varepsilon\to0+} \frac{1}{\varepsilon}\int_{-\hat\lambda t}^{\hat\lambda t} \Big|\gamma_{(v^1(t),\xi^1(t);u(t))}(\varepsilon)(x) - \gamma_{(v(t),\xi(t);u(t))}(\varepsilon)(x)\Big|\, dx \\
&\doteq \sigma_1(t) + \sigma_2(t) + \sigma_3(t).
\end{aligned}$$

As $t \to 0+$, the assumption on the function V in (3.69) now implies $\sigma_1(t) \to 0$. Moreover, $\sigma_2(t) \equiv 0$ for all t and $\sigma_3(t) \to 0$ because of the definitions of the functions (v^ℓ, ξ^ℓ) in (3.77). This establishes our claim.

To complete the proof, fix any $\sigma_0 > 0$. Choose $t_0 > 0$ so small that $\sigma(t_0) < \sigma_0$. Let $\big\{w^\varepsilon;\ \ \varepsilon \in [0, \varepsilon_0]\big\}$ be a set of solutions of (1.1), defined on the domain $\big\{(t,x) \in \mathcal{D};\ \ t \geq t_0\big\}$, such that the path $\varepsilon \mapsto w^\varepsilon(t_0, \cdot)$ is a R.V. of $u(t_0,\cdot)$ generating the tangent vector $\big(v(t_0), \xi(t_0)\big)$, and such that all functions $w^\varepsilon(t_0,\cdot)$ satisfy the conditions (i)–(iv) in Remark 3.9. This last assumption guarantees that the values $w^\varepsilon(t,\cdot)$ depend in a Lipschitz continuous way on their initial data $w^\varepsilon(t_0,\cdot)$.

We observe that such a family can be constructed as follows. Choose a sequence of $\mathcal{C}^1$ functions ϕ^ν and then a sequence of numbers ε_ν strictly decreasing to zero, such that

$$\lim_{\nu\to\infty} \big\|\phi^\nu - v(t_0,\cdot)\big\|_{\mathbf{L}^1} = 0, \qquad\qquad \lim_{\nu\to\infty} \sup_x \varepsilon_\nu \big|\phi^\nu_x(x)\big| = 0.$$

Then define

$$\begin{aligned}
\phi^\varepsilon &\doteq \theta\phi^\nu + (1-\theta)\phi^{\nu-1} \qquad \text{if} \quad \varepsilon = \theta\varepsilon_{\nu+1} + (1-\theta)\varepsilon_\nu, \quad \theta \in [0,1], \\
w^\varepsilon(t_0,x) &\doteq u(t_0,x) + \varepsilon\phi^\varepsilon(x) + \sum_{\xi_j(t_0)<0} \Delta u\big(t_0, y_j(t_0)\big)\chi_{[y_j(t_0)+\varepsilon\xi_j(t_0),\ y_j(t_0)]}(x) \\
&\qquad - \sum_{\xi_j(t_0)>0} \Delta u\big(t_0, y_j(t_0)\big)\chi_{[y_j(t_0),\ y_j(t_0)+\varepsilon\xi_j(t_0)]}(x),
\end{aligned}$$

where, for $j \in \mathcal{S}$, $\Delta u\big(t_0, y_j(t_0)\big)$ denotes the jump of $u(t_0,\cdot)$ at the point $y_j(t_0)$.

For $t > t_0$, using (3.81), Theorem 2.2 and the Lipschitz continuous dependence of solutions on the initial data at $t = t_0$, we now obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to0+} \frac{1}{\varepsilon}\int_{|x|\leq\rho-\hat\lambda t} \Big|u^\varepsilon(t,x) - \gamma_{(v(t),\xi(t),u(t))}(\varepsilon)(x)\Big|\, dx \\
&\qquad\leq \limsup_{\varepsilon\to0+} \frac{1}{\varepsilon}\int_{|x|\leq\rho-\hat\lambda t} \Big\{\big|u^\varepsilon(t,x) - w^\varepsilon(t,x)\big| + \big|w^\varepsilon(t,x) - \gamma_{(v(t),\xi(t),u(t))}(\varepsilon)(x)\big|\Big\}\, dx \\
&\qquad\leq \limsup_{\varepsilon\to0+} \frac{K}{\varepsilon}\int_{|x|\leq\rho-\hat\lambda t_0} \big|u^\varepsilon(t_0,x) - w^\varepsilon(t_0,x)\big|\, dx \\
&\qquad= K\sigma(t_0) < K\sigma_0.
\end{aligned}$$

Since σ_0 was arbitrary and the Lipschitz constant K is independent of t, t_0, the theorem is proved.

References

[1] A. Bressan, Contractive metrics for nonlinear hyperbolic systems, *Indiana Univ. Math. J.* **37** (1988), 409-421.

[2] A. Bressan, A contractive metric for systems of conservation laws with coinciding shock and rarefaction waves, *J. Differential Equations* **106** (1993), 332-366.

[3] A. Bressan, A locally contractive metric for systems of conservation laws, *Ann. Scuola Normale Sup. Pisa*, Serie IV, Vol. XXII (1995), 109-135.

[4] A. Bressan and R. M. Colombo, The semigroup generated by 2×2 conservation laws, *Arch. Rational Mech. Anal.*, to appear.

[5] A.Bressan and A. Marson, A maximum principle for optimally controlled systems of conservation laws, preprint.

[6] R. DiPerna, Uniqueness of solutions to hyperbolic conservation laws, *Indiana Univ. Math. J.* **28** (1979), 244-257.

[7] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 95-105.

[8] P. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), 537-567.

[9] B. L. Rozdestvenskii and N. Yanenko, "Systems of quasilinear equations", *A.M.S. Translations of Mathematical Monographs*, Vol. 55, 1983.

[10] S. Schochet, Sufficient conditions for local existence via Glimm's scheme for large BV data, *J. Differential Equations* **89** (1991), 317-354.

[11] Li Ta-Tsien and Yu Wen-ci, "Boundary Value Problems for Quasilinear Hyperbolic Systems", Duke University, Durham, 1985.

[12] J. Smoller, "Shock Waves and Reaction-Diffusion Equations", Springer-Verlag, New York, 1983

[13] W. P. Ziemer, "Weakly Differentiable Functions", Springer-Verlag, New York, 1989.