# Existence of Optima and Equilibria for Traffic Flow on Networks


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**Abstract**

This paper is concerned with a conservation law model of traffic flow on a network of roads, where each driver chooses his own departure time in order to minimize the sum of a departure cost and an arrival cost. The model includes various groups of drivers, with different origins and destinations and having different cost functions. Under a natural set of assumptions, two main results are proved: (i) the existence of a globally optimal solution, minimizing the sum of the costs to all drivers, and (ii) the existence of a Nash equilibrium solution, where no driver can lower his own cost by changing his departure time or the route taken to reach destination. In the case of Nash solutions, all departure rates are uniformly bounded and have compact support.


## 1 Introduction

Consider a model of traffic flow where drivers travel on a network of roads. We denote by $A_1, \ldots, A_m$ the nodes of the network, and by $\gamma_{ij}$ the arc connecting $A_i$ with $A_j$. Following the classical papers [13, 14], along each arc the flow of traffic will be modeled by the conservation law

$$\rho_t + [\rho\, v_{ij}(\rho)]_x \;=\; 0\,. \tag{1.1}$$

Here $t$ is time and $x \in [0, L_{ij}]$ is the space variable along the arc $\gamma_{ij}$. By $\rho = \rho(t,x)$ we denote the traffic density, while the map $\rho \mapsto v_{ij}(\rho)$ is the speed of cars as function of the density, along the arc $\gamma_{ij}$. We assume that $v_{ij}$ is a continuous, nonincreasing function. If $v_{ij}(0) > 0$ we say that the arc $\gamma_{ij}$ is *viable*. It is quite possible that two nodes $i, j$ are not directly linked by a road. This situation can be easily modeled by taking $v_{ij} \equiv 0$, so that the arc is not viable. The conservation laws (1.1) are supplemented by suitable boundary conditions at points of junctions, which will be discussed later.

We consider $n$ groups of drivers traveling on the network. Different groups are distinguished by the locations of departure and arrival, and by their cost functions. For $k \in \{1, \ldots, n\}$, let $G_k$ be the total number of drivers in the $k$-th group. All these drivers depart from a node $A_{d(k)}$ and arrive at a node $A_{a(k)}$, but can choose different paths to reach destination. Of course, we assume that there exists at least one chain of viable arcs

$$\Gamma \doteq \Big(\gamma_{i(0),i(1)}, \gamma_{i(1),i(2)}, \ldots, \gamma_{i(\nu-1),i(\nu)}\Big) \tag{1.2}$$

with $i(0) = d(k)$ and $i(\nu) = a(k)$, connecting the departure node $A_{d(k)}$ with the arrival node $A_{a(k)}$. We shall denote by

$$\mathcal{V} \doteq \Big\{\Gamma_1, \Gamma_2, \ldots, \Gamma_N\Big\}$$

the set of all viable chains (i.e. concatenations of viable arcs) which do not contain any closed loop. Since there are $m$ nodes, and each chain can visit each of them at most once, the cardinality of $\mathcal{V}$ is bounded by $(m+1)!$. For a given $k \in \{1, \ldots, n\}$, let $\mathcal{V}_k \subset \mathcal{V}$ be the set of all viable paths for the $k$-drivers, connecting $A_{d(k)}$ with $A_{a(k)}$.

By $U_{k,p}(t)$ we denote the total number of drivers of the $k$-th group, traveling along the viable path $\Gamma_p$, who have started their journey before time $t$.

**Definition 1.** *A* **departure distribution function** $t \mapsto U_{k,p}(t)$ *is a bounded, nondecreasing, left-continuous function, such that*

$$U_{k,p}(-\infty) \doteq \lim_{t\to-\infty} U_{k,p}(t) = 0\,.$$

*Given group sizes $G_1, \ldots, G_n \geq 0$, we say that a set of departure distribution functions $\{U_{k,p}\}$ is* **admissible** *if it satisfies the constraints*

$$\sum_p U_{k,p}(+\infty) = G_k \qquad\qquad k = 1, \ldots, n\,. \tag{1.3}$$

Since $G_k$ is the total number of drivers in the $k$-th group, the admissibility condition (1.3) means that, sooner or later, every driver of each group has to depart. If the function $U_{k,p}$ is absolutely continuous, its derivative will be denoted by

$$\overline{u}_{k,p}(t) = \frac{d}{dt}U_{k,p}(t)\,. \tag{1.4}$$

Clearly, $\bar{u}_{k,p}$ measures the rate of departures of $k$-drivers traveling along $\Gamma_p$.

The overall traffic pattern can be determined by (i) the departure distribution functions $U_{k,p}(\cdot)$, (ii) the conservation laws (1.1), and (iii) a suitable set of conditions at junctions, specifying the priorities assigned to drivers that wish to enter the same road.

In this paper we consider the simplest type of condition at junctions, where a separate queue can form at the entrance of each road. Drivers arriving at the node $A_i$ from all incoming roads, and who want to travel along the arc $\gamma_{ij}$, join a queue at the entrance of this outgoing arc. Their place in the queue is determined by the time at which they arrive at $A_i$, first-come first-serve. Some additional care is needed to handle the case where different groups of drivers depart from the same node. Indeed, if a positive amount of drivers initiate their journey exactly at the same time, some additional information is needed to determine their place in the queue. This can be achieved in terms of the *prioritizing functions* introduced in [3].

As in [2, 3, 4], we consider a set of departure costs $\varphi_k(\cdot)$, and arrival costs $\psi_k(\cdot)$ for the various drivers. Namely, a driver of the $k$-th group departing at time $\tau^d$ and arriving at destination at time $\tau^a$ will incur in the total cost

$$\varphi_k(\tau^d) + \psi_k(\tau^a). \tag{1.5}$$

In this framework, the concepts of globally optimal solution and of Nash equilibrium solution considered in [2, 3] can be extended to traffic flows on a network of roads.

**Definition 2.** *An admissible family $\{U_{k,p}\}$ of departure distributions is* **globally optimal** *if it minimizes the sum of the total costs of all drivers.*

**Definition 3.** *An admissible family $\{U_{k,p}\}$ of departure distributions is a* **Nash equilibrium solution** *if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination.*

The main goal of this paper is to prove the existence of a globally optimal solution and of a Nash equilibrium solution, under natural assumptions on the costs and on the flux functions.

In the case of a single group of drivers traveling on a single road, the existence and uniqueness of such solutions were proved in [2]. We highlight the main features of the present analysis.

- Following the direct method of the Calculus of Variations, a globally optimal solution is constructed by taking the limit of a minimizing sequence $\{U_{k,p}^{(\nu)}\}_{\nu\geq 1}$ of admissible departure distributions. The existence of the limit is guaranteed by the "tightness" of the sequence of approximating measures. Namely, for each $\varepsilon > 0$ there exists $T > 0$ (independent of $n$) such that the total amount of drivers departing at times $t \notin [-T, T]$ is less than $\varepsilon$.

- Toward the existence of a Nash equilibrium, the results in [2, 3] used the assumption $\psi_k' \geq 0$, meaning that the arrival cost functions are nondecreasing. We now strengthen this assumption to $\psi_k' > 0$, so that the arrival costs are strictly increasing. This apparently minor change in the hypotheses has an important consequence. Namely, it allows us to prove a crucial a priori bound on all departure rates, in any Nash equilibrium solution.

- The proofs in [2, 3] relied on a monotonicity argument. Indeed, the departure distribution $U(\cdot)$ for a Nash equilibrium was obtained as the (unique) pointwise supremum of a family of admissible distributions, satisfying an additional constraint. On the other hand, the present existence result is proved by a fixed point argument. By its nature, this topological technique cannot yield information about uniqueness or continuous dependence of the Nash equilibrium.

The paper is organized as follows. In Section 2 we describe more carefully the traffic flow model, explaining how to compute the admissible solutions using the Lax-Hopf formula. In

Section 3 we prove the existence of a globally optimal solution, while Section 4 is devoted to the existence of a Nash equilibrium solution.

For the modeling of traffic flow we refer to [1, 6, 13, 14]. Traffic flow on networks has been the topic of an extensive literature, see for example [5, 8, 9, 10] and references therein. A different type of optimization problems for traffic flow was considered in [11].

## 2 Analysis of the traffic flow model

In our model, $x \in [0, L_{ij}]$ is the space variable, describing a point along the arc $\gamma_{ij}$. Here $L_{ij}$ measures the length of this arc. The basic assumptions on the flux functions $F_{ij}(\rho) = \rho\, v_{ij}(\rho)$ and on the cost functions $\varphi_k, \psi_k$ are as follows.

**(A1)** For every viable arc $\gamma_{ij}$, the flux function $\rho \mapsto F_{ij}(\rho) = \rho v_{ij}(\rho)$ is continuous, concave down, and non-negative on some interval $[0, \overline{\rho}_{ij}]$, with $F_{ij}(0) = F_{ij}(\overline{\rho}_{ij}) = 0$. We shall denote by $\rho^*_{ij} \in ]0, \overline{\rho}_{ij}[$ the unique value such that

$$F(\rho^*_{ij}) \;=\; F^{max}_{ij} \doteq \max_{\rho\in[0,\overline{\rho}_{ij}]} F_{ij}(\rho)\,, \qquad\qquad F'_{ij}(\rho) \;>\; 0 \qquad \text{for a.e.} \quad \rho \in [0, \rho^*_{ij}]\,. \tag{2.1}$$

**(A2)** For every $k \in \{1, \ldots, n\}$ the cost functions $\varphi_k, \psi_k$ are continuously differentiable and satisfy

$$\begin{cases} \varphi'_k(t) \;<\; 0\,, \\ \\ \psi'_k(t) \;>\; 0\,, \end{cases} \qquad\qquad \lim_{|t|\to\infty} \Big(\varphi_k(t) + \psi_k(t)\Big) \;=\; +\infty\,. \tag{2.2}$$

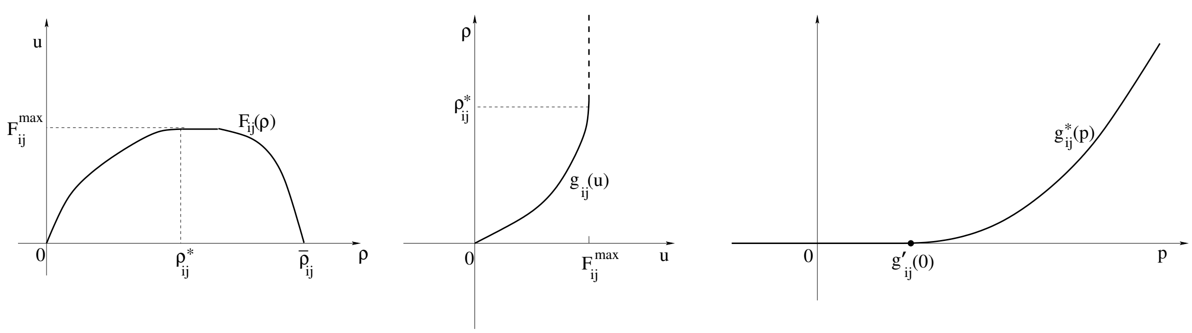


Figure 1: Left: the function $\rho \mapsto F_{ij}(\rho) = \rho\, v_{ij}(\rho)$ describing the flux of cars. Middle: the function $g_{ij}$, implicitly defined by $g_{ij}(\rho v_{ij}(\rho)) = \rho$. Right: the Legendre transform $g^*_{ij}$.

**Remark 2.1.** By (A1), the flux function $u = F_{ij}(\rho)$ is continuous, concave and strictly increasing on the interval $[0, \rho^*_{ij}]$. Therefore, it has a continuous inverse: $\rho = g_{ij}(u)$. As shown in fig. 1, the function $u \mapsto g_{ij}(u)$ is convex and maps the interval $[0, F^{max}_{ij}]$ onto $[0, \rho^*_{ij}]$.

**Remark 2.2.** According to (A2), the cost for early departure is strictly decreasing in time, while cost for late arrival is strictly increasing. The assumption that these costs tend to infinity as $t \to \pm\infty$ coincides with common sense and guarantees that in an equilibrium solution the departure rates are compactly supported.

**Remark 2.3.** In the engineering literature (see for example [9]) it is common to define the travel cost as the sum of the travel time plus a penalty if the arrival time does not coincide with the target time $T_A$ :

$$D(t) + \Psi\big(t + D(t) - T_A\big). \tag{2.3}$$

Here $D(t)$ denotes the total duration of the trip for a driver departing at time $t$, while $\Psi$ is a penalty function. Calling $\tau^a(t) = t + D(t)$ the arrival time of driver departing at $t$, the cost function (2.3) can be recast in the form (1.5). Indeed,

$$\begin{aligned} D(t) + \Psi\big(t + D(t) - T_A\big) &= \; -t + t + D(t) + \Psi\big(t + D(t) - T_A\big) \\ &= \; -t + \tau^a(t) + \Psi\big(\tau^a(t) - T_A\big) \\ &= \; \varphi(t) + \psi(\tau^a(t)) \,, \end{aligned}$$

where $\varphi(t) \doteq -t$, $\psi(\tau) \doteq \tau + \Psi\big(\tau - T_A\big)$. In order that the assumptions **(A2)** be satisfied, it suffices to require that the function $\Psi$ be continuously differentiable and

$$\Psi \;\geq\; 0, \qquad\qquad \Psi' \;>\; -1, \qquad\qquad \lim_{|t|\to\infty} \big(\Psi(t) - t\big) \;=\; +\infty\,. \tag{2.4}$$

## 2.1 Traffic flow with an absolutely continuous departure distribution.

We now describe more in detail how the traffic flow on the entire network can be uniquely determined, given the departure distributions $U_{k,p}$. As a first step, we consider the absolutely continuous case, so that (1.4) holds.

Along any arc $\gamma_{ij}$, the traffic density $\rho_{ij}$ satisfies a boundary value problem of the form

$$\begin{cases} \partial_t \rho_{ij}(t,\,x)_t + \partial_x F_{ij}\big(\rho_{ij}(t,\,x)\big) \;=\; 0 & (t,\,x) \in \mathbb{R} \times [0,\,L_{ij}]\,, \\ F_{ij}\big(\rho_{ij}(t,\,0)\big) \;=\; u_{ij}^-(t) & t \in \mathbb{R}\,, \end{cases} \tag{2.5}$$

where $u_{ij}^-(t)$ describes the incoming flux at $x = 0$. Following the approach in [2, 3], we switch the roles of the variables $t$, $x$, replacing the above boundary value problem (2.5) with a Cauchy problem for the conservation law describing the flux $u_{ij} = F_{ij}(\rho_{ij})$:

$$\begin{cases} \partial_x u_{ij}(t,\,x) + \partial_t g_{ij}\big(u_{ij}(t,\,x)\big) \;=\; 0 & (t,\,x) \in \mathbb{R} \times [0,\,L_{ij}]\,, \\ u_{ij}(t,\,0) \;=\; u_{ij}^-(t) & t \in \mathbb{R}\,. \end{cases} \tag{2.6}$$

Here $g_{ij} : [0, F_{ij}^{max}] \mapsto [0, \rho_{ij}^*]$ is the inverse of the function $F_{ij}$, as in Remark 2.1. Consider the integrated functions

$$U_{ij}(t,\,x) \;=\; \int_{-\infty}^{t} u_{ij}(s,\,x)\,ds, \qquad U_{ij}^-(t) \;=\; \int_{-\infty}^{t} u_{ij}^-(s)\,ds\,.$$

Then $U_{ij}(t,\,x)$ provides a solution to the Hamilton-Jacobi equation

$$\begin{cases} \partial_x U_{ij}(t,\,x) + g_{ij}\big(\partial_t U_{ij}(t,\,x)\big) \;=\; 0\,, \\ U_{ij}(t,\,0) \;=\; U_{ij}^-(t)\,. \end{cases} \tag{2.7}$$

The viscosity solution to the above Cauchy problem is given by the Lax-Hopf formula [7]

$$U_{ij}(t,\, x) \;=\; \min_{\tau} \left\{ U^-_{ij}(\tau) + x\, g^*_{ij}\Big(\frac{t-\tau}{x}\Big) \right\}, \qquad x \in [0,\, L_{ij}]\,. \tag{2.8}$$

Here $g^*_{ij}$ denotes the Legendre transform of $g_{ij}$, namely

$$g^*_{ij}(p) \;\doteq\; \max_{u\in[0,\, F^{max}_{ij}]} \Big(pu - g_{ij}(u)\Big). \tag{2.9}$$

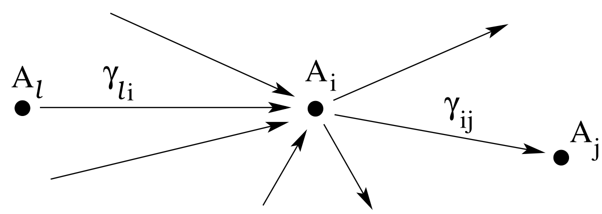


Figure 2: A generic node of the network.

For any given node $A_i$, in general there will be several incoming arcs $\gamma_{\ell,i}$, $\ell \in \mathcal{I}(i)$ and several outgoing arcs $\gamma_{i,j}$, $j \in \mathcal{O}(i)$ (see fig. 2). To determine the flux at the entrance of the arc $\gamma_{ij}$, one needs to know how many drivers, after reaching the node $A_i$, actually want to take the road $\gamma_{ij}$. For this purpose, we need to introduce the distribution functions

- $U^-_{kp,ij}(t) \;\doteq\;$ *total number of drivers of the k-th group, traveling along the path $\Gamma_p$, that have entered the arc $\gamma_{ij}$ (possibly joining a queue at the entrance) within time t.*

- $U^+_{kp,ij}(t) \;\doteq\;$ *total number of drivers of the k-th group, traveling along the path $\Gamma_p$, that have exited from the arc $\gamma_{ij}$ (reaching the node $A_j$) within time t.*

Of course, $U^{\pm}_{kp,ij} \equiv 0$ if the path $\Gamma_p$ does not contain the arc $\gamma_{ij}$.

The entire flux along the network is entirely determined by the functions $U^{\pm}_{kp,ij}$. To recursively compute these functions, observe that the total number of drivers who have entered the arc $\gamma_{ij}$ (possibly joining a queue) within time $t$ is determined by

$$U^-_{ij}(t) \;=\; \sum_{\ell\in\mathcal{I}(i)} \sum_{k} \sum_{\gamma_{\ell i},\gamma_{ij}\in\Gamma_p} U^+_{kp,\ell i}(t) \;+\; \sum_{d(k)=i} \sum_{\gamma_{ij}\in\Gamma_p} U_{kp}(t)\,. \tag{2.10}$$

Notice that the first sum accounts for the drivers that transit through the node $A_i$, while the second sum accounts for the drivers that initiate their journey from the node $A_i$. Using the solution formula (2.8) with $x = L_{ij}$ we obtain

$$U^+_{ij}(t) \;\doteq U_{ij}(t, L_{ij}) \;=\; \min_{\tau} \left\{ U^-_{ij}(\tau) + L_{ij}\, g^*_{ij}\Big(\frac{t-\tau}{L_{ij}}\Big) \right\}. \tag{2.11}$$

Notice that the function $U^+_{ij}$ is nondecreasing and Lipschitz continuous. Indeed, the rate at which drivers arrive at the end of the arc $\gamma_{ij}$ is computed by

$$u^+_{ij}(t) \;=\; \frac{d}{dt} U^+_{ij}(t) \;\in\; [0,\, F^{max}_{ij}] \qquad\qquad \text{for a.e. } t\,.$$

Among the drivers who reach the end of the arc $\gamma_{ij}$ within time $t$, we still need to compute how many belong to the various groups.

In the case where all departure rates $U_{k,p}$ with $d(k) = i$ are absolutely continuous, the functions $U^+_{kp,ij}$ can be computed as follows. For a.e. $t$ there exists a unique time $\tau^{enter}(t)$ such that

$$U^-_{ij}(\tau^{enter}(t)) \;=\; U^+_{ij}(t).$$

A driver entering the arc $\gamma_{ij}$ at time $\tau^{enter}(t)$ thus reaches the end of the arc at time $t$. The first-come first-serve assumption now implies

$$u^+_{kp,ij}(t) \;=\; \frac{u^+_{kp,\ell i}(\tau^{enter}(t))}{u^-_{ij}(\tau^{enter}(t))} \qquad \text{if} \quad \ell \in \mathcal{I}(i)\,, \quad \gamma_{\ell i}, \gamma_{ij} \in \Gamma_p\,, \tag{2.12}$$

$$u^+_{kp,ij}(t) \;=\; \frac{u_{kp}(\tau^{enter}(t))}{u^-_{ij}(\tau^{enter}(t))} \qquad \text{if} \quad \gamma_{ij} \in \Gamma_p\,, \quad i = d(k)\,. \tag{2.13}$$

In other words, among all drivers who exit from the arc $\gamma_{ij}$ at time $t$, the percentage of $(k,p)$-drivers must be equal to the percentage of $(k,p)$-drivers that enter $\gamma_{ij}$ at time $\tau^{enter}(t)$. In turn, the arrival distributions are computed by

$$U^+_{kp,ij}(t) \;=\; \int_{-\infty}^{t} u^+_{kp,ij}(t)\,dt\,. \tag{2.14}$$

**Remark 2.4.** The above equations can be solved iteratively in time. Namely, let

$$\Delta_{min} \;\doteq\; \min_{ij} \frac{L_{ij}}{v_{ij}(0)}$$

be the minimum time needed to travel along any viable arc of the network. Given the departure rates $\bar u_{k,p}$, if the functions $U^\pm_{kp,ij}(t)$ are known for all $t \le \tau$, by the above equations (2.10)–(2.14) one can uniquely determine the values of $U^\pm_{kp,ij}(t)$ also for $t \le \tau + \Delta_{min}$.

Let $G_{k,p}$ be the total number of drivers of the $k$-th group who travel along the path $\Gamma_p$. The admissibility condition implies $G_{k,1} + \cdots + G_{k,N} = G_k$. We use the Lagrangian variable $\beta \in [0, G_{k,p}]$ to label a particular driver in the subgroup $\mathcal{G}_{k,p}$ of $k$-drivers traveling along the path $\Gamma_p$. The departure and arrival time of this driver will be denoted by $\tau^d_{k,p}(\beta)$ and $\tau^a_{k,p}(\beta)$, respectively. Let $U^{depart}_{k,p}(t) = U_{k,p}(t)$ denote the amount of drivers of the subgroup $\mathcal{G}_{k,p}$ who have departed before time $t$. Similarly, let $U^{arrive}_{k,p}(t)$ be the amount of $(k,p)$-drivers who have arrived at destination before time $t$. For a.e. $\beta \in [0, G_{k,p}]$ we then have

$$\tau^d_{k,p}(\beta) \;=\; \inf\Big\{\tau\,;\; U^{depart}_{k,p}(t) \;\ge\; \beta\Big\}\,, \qquad \tau^a_{k,p}(\beta) \;=\; \inf\Big\{\tau\,;\; U^{arrive}_{k,p}(t) \;\ge\; \beta\Big\}\,. \tag{2.15}$$

With this notation, the definition of globally optimal and of Nash equilibrium solution can be more precisely formulated.

**Definition 2′.** *An admissible family of departure distributions* $\{U_{k,p}\}$ *is a* **globally optimal solution** *if it provides a global minimum to the functional*

$$J \;\doteq\; \sum_{k,p} \int_0^{G_{k,p}} \Big(\varphi_k(\tau^d_{k,p}(\beta)) + \psi_k(\tau^a_{k,p}(\beta))\Big)\,d\beta\,. \tag{2.16}$$

**Definition 3′.** *An admissible family of departure distributions* $\{U_{k,p}\}$ *is a* **Nash equilibrium solution** *if there exist constants* $c_1, \ldots, c_n$ *such that:*

*(i) For almost every* $\beta \in [0, G_{k,p}]$ *one has*

$$\varphi_k(\tau^q_{k,p}(\beta)) + \psi_k(\tau^a_{k,p}(\beta)) \;=\; c_k\,. \tag{2.17}$$

*(ii) For all* $\tau \in \mathbb{R}$*, there holds*

$$\varphi_k(\tau) + \psi_k(A_{k,p}(\tau)) \;\geq\; c_k\,. \tag{2.18}$$

*Here* $A_{k,p}(\tau)$ *is the arrival time of a driver that starts at time* $\tau$ *from the node* $A_{d(k)}$ *and reaches the node* $A_{a(k)}$ *traveling along the path* $\Gamma_p$*.*

In other words, condition (i) states that all $k$-drivers bear the same cost $c_k$. Condition (ii) means that, regardless of the starting time $x$, no $k$-driver can achieve a cost $< c_k$.

## 2.2 Traffic flow with general departure distribution

If all departure distributions $U_{k,p}$ are absolutely continuous, the previous analysis shows that the first-come first-serve assumption on the queues completely determines the traffic pattern. This is no longer true if a positive amount of drivers of different groups initiate their journey exactly at the same time. For example, assume that $i = d(k) = d(k')$ is the departure node for both $k$-drivers and $k'$-drivers. Let $\gamma_{ij}$ be the first arc in the paths $\Gamma_p$ and $\Gamma_{p'}$, and assume that, at the instant $t_0$, a positive amount of drivers in the subgroups $\mathcal{G}_{k,p}$ and $\mathcal{G}_{k',p'}$ initiate their journey. Since all these drivers join the queue at the entrance of the arc $\gamma_{ij}$ at the same time, additional information must be provided to determine their relative position in the queue. For this purpose, we follow the approach introduced in [3].

Given an arc $\gamma_{ij}$, call

$$\mathcal{G}_{ij} \;=\; \bigcup_{d(k)=i,\,\gamma_{ij}\in\Gamma_p} \mathcal{G}_{k,p} \tag{2.19}$$

the family of all drivers that initiate their journey from the node $A_i$, traveling along $\gamma_{ij}$ as first leg of their journey. The total number of these drivers is

$$G_{ij} \;=\; \sum_{d(k)=i,\,\gamma_{ij}\in\Gamma_p} G_{k,p}\,.$$

The *cumulative departure distribution* $U^{depart}_{ij} : \mathbb{R} \mapsto [0, G_{ij}]$ is defined as

$$U^{depart}_{ij}(t) \;\doteq\; \sum_{d(k)=i,\,\gamma_{ij}\in\Gamma_p} U_{k,p}(t)\,. \tag{2.20}$$

In other words, $U^{depart}_{ij}(t)$ is the total number of drivers in the family $\mathcal{G}_{ij}$ that depart before time $t$. The relative position of these drivers in the queue at the entrance of the arc $\gamma_{ij}$ will be determined by a set of prioritizing functions.

**Definition 4.** *A set of* **prioritizing functions** *for departures on the arc* $\gamma_{ij}$ *is a family of nondecreasing maps*

$$\mathcal{B}_{k,p} : [0, G_{ij}] \;\mapsto\; [0, G_{k,p}]\,, \qquad\qquad \mathcal{G}_{k,p} \subseteq \mathcal{G}_{ij}\,, \tag{2.21}$$

*satisfying*

$$\sum_{\mathcal{G}_{k,p}\subseteq\mathcal{G}_{ij}} \mathcal{B}_{k,p}(\beta) \;=\; \beta \qquad\qquad \text{for all } \beta \in [0, G_{ij}]\,, \tag{2.22}$$

$$\mathcal{B}_{k,p}(U_{ij}^{depart}(t)) \;=\; U_{k,p}(t) \qquad\qquad \text{for a.e. } \; t \in \mathbb{R}\,. \tag{2.23}$$

Otherwise stated, among the first $\beta$ drivers that depart and choose $\gamma_{ij}$ as the first arc of their journey, $\mathcal{B}_{k,p}(\beta)$ counts how many belong to the subgroup $\mathcal{G}_{k,p}$. If all functions $U_{k,p}$ are continuous, then the conditions (2.22)-(2.23) uniquely determine the prioritizing functions $\mathcal{B}_{k,p}$. In this case, there is actually no need to introduce this concept. On the other hand, if two or more functions $U_{k,p}$ have an upward jump at a time $t_0$, then different prioritizing functions are possible.

We now show that, by choosing one set of prioritizing functions, the traffic flow on the entire network is entirely determined. Indeed, for all $k, p, i, j$ the values $U^+_{kp,ij}(t)$, determining how many drivers of the subgroup $\mathcal{G}_{k,p}$ reach the end of the arc $\gamma_{ij}$ within time $t$, are computed as follows.

CASE 1: The only drivers traveling on the arc $\gamma_{ij}$ are those who started their journey from the node $A_i$. In this case, as in (2.11) the total number of drivers arriving at the end of the arc $\gamma_{ij}$ within time $t$ is

$$U^+_{ij}(t) \;=\; \sum_{k,p} U^+_{kp,ij}(t) \;=\; \min_{\tau} \left\{ U^{depart}_{ij}(\tau) + L_{ij}\, g^*_{ij}\Big(\frac{t-\tau}{L_{ij}}\Big) \right\}. \tag{2.24}$$

Given the prioritizing functions $\mathcal{B}_{k,p}$, the values $U^+_{kp,ij}(t)$ are immediately obtained by the formula

$$U^+_{kp,ij}(t) \;=\; \mathcal{B}_{k,p}\left(U^+_{ij}(t)\right). \tag{2.25}$$

CASE 2: The arc $\gamma_{ij}$ is also traveled by drivers who transit through the node $A_i$, departing from other nodes. In this case, drivers originating from $A_i$ have to merge with drivers in transit, coming from other nodes. The cumulative distribution function, accounting for the total number of drivers that have entered the arc $\gamma_{ij}$ within time $t$ is

$$U^-_{ij}(t) \;=\; U^{depart}_{ij}(t) + U^{transit}_{ij}(t) \;=\; \sum_{d(k)=i,\gamma_{ij}\in\Gamma_p} U^-_{kp,ij}(t) + \sum_{d(k)\neq i,\gamma_{ij}\in\Gamma_p} U^-_{kp,ij}(t)\,. \tag{2.26}$$

As before, the total number of drivers who have exited from the arc $\gamma_{ij}$ before time $t$ is given by (2.11). To determine how many of these drivers belong to each subgroup $\mathcal{G}_{k,p}$, we proceed as follows.

Consider the driver who exits from the arc $\gamma_{ij}$ at time $t$. This driver will have entered the arc $\gamma_{ij}$ at an earlier time $\tau = \tau^{enter}(t)$, such that

$$\lim_{s\to\tau-} U^-_{ij}(s) \;\leq\; U^+_{ij}(t) \;\leq\; \lim_{s\to\tau+} U^-_{ij}(s)\,. \tag{2.27}$$

Notice that $\tau^{enter}(t)$ is uniquely determined, for all but countably many times $t$. For all subgroups of drivers in transit, the first-come first-serve priority assumption implies

$$U^+_{kp,ij}(t) \;=\; U^-_{kp,ij}(\tau^{enter}(t)). \tag{2.28}$$

Here the right hand side of (2.28) is uniquely determined, because the distribution function $U^-_{kp,ij}$ in (2.26) is Lipschitz continuous.

On the other hand, for drivers who initiate their journey at $A_i$, the distribution function $U^-_{kp,ij}$ can have a jump at $\tau^{enter}(t)$, in which case the formula (2.28) is not meaningful. Given a set of prioritizing functions $\mathcal{B}_{k,p}$, the distribution functions $U^+_{kp,ij}$ can be determined by setting

$$\beta \;\doteq\; U^+_{ij}(t) - U^{transit}_{ij}(\tau^{enter}(t))\,, \tag{2.29}$$

$$U^+_{kp,ij} \;=\; \mathcal{B}_{k,p}(\beta)\,. \tag{2.30}$$

Indeed, $U^+_{ij}(t)$ is the total number of drivers that reach the end of the arc $\gamma_{ij}$ before time $t$, while $\beta$ counts how many of these drivers start their journey from the node $A_i$. In turn, $\mathcal{B}_{k,p}(\beta)$ determines how many belong to the subgroup $\mathcal{G}_{k,p}$.

## 3 Globally optimal solutions

In this section we establish the existence of a globally optimal solution. The proof follows the direct method of the Calculus of Variations, constructing a minimizing sequence of solutions and showing that a subsequence converges to the optimal one.

**Theorem 1 (existence of a globally optimal solution).** *Let the flux functions $F_{ij}$ and the cost functions $\varphi_k, \psi_k$ satisfy the assumptions (A1)-(A2). Then, for any $n$-tuple $(G_1, \ldots, G_n)$ of nonnegative numbers, there exists an admissible set of departure distributions $U_{k,p}$ and prioritizing functions $\mathcal{B}_{k,p}$ which yield a globally optimal solution of the traffic flow problem.*

**Proof. 1.** By (A2), all functions $\varphi_k + \psi_k$ are bounded below. By possibly adding a constant, it is not restrictive to assume that $\varphi_k(t)+\psi_k(t) \geq 0$ for every time $t$. Calling $m_0$ the infimum of all total costs in (2.16), taken among all admissible departure distributions $\{U_{k,p}\}$, this implies $m_0 \geq 0$. In addition, it is clearly not restrictive to assume $G_k > 0$ for all $k \in \{1, \ldots, n\}$.

Recalling Definitions 1 and 4, consider a minimizing sequence of departure distributions $U^\nu_{k,p}$ and prioritizing functions $\mathcal{B}^\nu_{k,p} : [0, G^\nu_{ij}] \mapsto [0, G^\nu_{kp}]$. Here

$$G^\nu_{k,p} \;=\; U^\nu_{k,p}(+\infty).$$

By choosing a subsequence, we can assume

$$\lim_{\nu\to\infty} G^\nu_{k,p} \;=\; G_{k,p} \qquad \text{with} \qquad \sum_p G_{k,p} \;=\; G_k\,. \tag{3.1}$$

Moreover, by Helly's compactness theorem we can assume that, as $\nu \to \infty$, one has the pointwise convergence

$$U^\nu_{k,p}(t) \;\to\; U_{k,p}(t) \qquad \text{for a.e. } t \in \mathbb{R}\,, \tag{3.2}$$

while Ascoli's theorem yields the uniform convergence

$$\mathcal{B}^{\nu}_{k,p}(\beta) \ \to\ \mathcal{B}_{k,p}(\beta) \qquad\qquad \beta\in\mathbb{R}\,. \tag{3.3}$$

In (3.3), we have extended the prioritizing functions to the entire real line by setting

$$\mathcal{B}^{\nu}_{k,p}(\beta) \ =\ \begin{cases} 0 & \text{if} \quad \beta\leq 0\,,\\ G^{\nu}_{k,p} & \text{if} \quad \beta\geq G^{\nu}_{ij}\,.\end{cases}$$

In the remainder of the proof we show that the set of departure distributions $U_{k,p}$ together with the prioritizing functions $\mathcal{B}_{k,p}$ yield a globally optimal solution.

**2.** Let $\varepsilon>0$ be given. We claim that there exists a large enough constant $T$, independent of $\nu$ such that

$$\begin{cases} \displaystyle\sum_p U^{\nu}_{k,p}(t) \ \leq\ \varepsilon & \text{for} \ \ t\leq -T\,,\\[2mm] \displaystyle\sum_p U^{\nu}_{k,p}(t) \ \geq\ G_k-\varepsilon & \text{for} \ \ t\geq T\,,\end{cases} \tag{3.4}$$

for all $\nu$ sufficiently large. Indeed, by (A2) there exists $T$ such that

$$\varphi_k(t)+\psi_k(t) \ >\ \frac{m_0+1}{\varepsilon} \qquad\qquad \text{for} \quad |t|\geq T\,,\ k\in\{1,\ldots,n\}\,.$$

If any one of the two conditions in (3.4) fails, then the total cost would be $> m_0+1$. Since by assumption as $\nu\to\infty$ the total cost approaches the infimum $m_0$, this proves our claim.

By (3.4) it follows that the limit functions satisfy

$$U_{k,p}(-\infty) \ =\ 0, \qquad\qquad U_{k,p}(+\infty) \ =\ G_{k,p}\,.$$

In particular, the limit departure distribution is admissible.

**3.** For $\beta\in[0,G^{\nu}_{k,p}]$ let

$$\beta \ \mapsto\ \tau^{d,\nu}_{k,p}(\beta) \qquad \text{and} \qquad \beta \ \mapsto\ \tau^{a,\nu}_{k,p}(\beta) \tag{3.5}$$

describe the departure and arrival time of the $\beta$-driver, in the subgroup $\mathcal{G}_{k,p}$. We claim that, by possibly extracting a further subsequence, one has the pointwise convergence

$$\tau^{d,\nu}_{k,p}(\beta) \ \to\ \tau^{d}_{k,p}(\beta)\,, \qquad\qquad \tau^{a,\nu}_{k,p}(\beta) \ \to\ \tau^{a}_{k,p}(\beta) \qquad\qquad \text{for a.e. } \beta\in[0,G_{k,p}]. \tag{3.6}$$

Indeed, the maps in (3.5) are nondecreasing. Moreover, for any $\varepsilon>0$, by (3.4) these maps are uniformly bounded when restricted to the subinterval $[\varepsilon,\,G_{k,p}-\varepsilon]$, uniformly w.r.t. $\nu$. By Helly's compactness theorem, we can find a subsequence that converges pointwise on $[\varepsilon,\,G_{k,p}-\varepsilon]$. Since $\varepsilon>0$ was arbitrary, a standard argument proves our claim.

**4.** It remains to prove that the limit departure distribution is optimal. Recall that, without loss of generality, we are assuming $\varphi_k(t)+\psi_k(t)\geq 0$ for all $k,t$. The total cost $\bar J$ determined by the set of departure distributions $\{U_{k,p}\}$ and prioritizing functions $\{\mathcal{B}_{k,p}\}$ can be estimated by

$$\begin{aligned} \bar{J} &\doteq \sum_{k,p} \int_0^{G_{k,p}} \Big(\varphi_k(\tau^d_{k,p}(\beta)) + \psi_k(\tau^a_{k,p}(\beta))\Big) d\beta \\ &= \sup_{\varepsilon>0} \sum_{k,p} \int_\varepsilon^{G_{k,p}-\varepsilon} \Big(\varphi_k(\tau^d_{k,p}(\beta)) + \psi_k(\tau^a_{k,p}(\beta))\Big) d\beta\,. \end{aligned} \tag{3.7}$$

Fix $\varepsilon > 0$. By (3.4) all departures and arrivals of $\beta$-drivers with $\beta \in [\varepsilon,\, G_{k,p} - \varepsilon]$ take place in a uniformly bounded interval of time, say $[-T, T']$. On this interval, all functions $\varphi_k, \psi_k$ are uniformly continuous. Hence the pointwise convergence (3.6) yields

$$\begin{aligned} &\sum_{k,p} \int_\varepsilon^{G_{k,p}-\varepsilon} \Big(\varphi_k(\tau^d_{k,p}(\beta)) + \psi_k(\tau^a_{k,p}(\beta))\Big) d\beta \\ &\qquad = \sum_{k,p} \lim_{\nu\to\infty} \int_\varepsilon^{G_{k,p}-\varepsilon} \Big(\varphi_k(\tau^{d,\nu}_{k,p}(\beta)) + \psi_k(\tau^{a,\nu}_{k,p}(\beta))\Big) d\beta \\ &\qquad \leq \lim_{\nu\to\infty} \sum_{k,p} \int_0^{G_{k,p}} \Big(\varphi_k(\tau^{d,\nu}_{k,p}(\beta)) + \psi_k(\tau^{a,\nu}_{k,p}(\beta))\Big) d\beta \;=\; m_0\,. \end{aligned}$$

Together with (3.7), this implies $\bar{J} \leq m_0$, completing the proof. □

**Remark 3.1.** The above theorem remains valid if in assumption (A2) we only require $\varphi_k' \leq 0$, $\psi_k' \geq 0$.

**Remark 3.2.** A natural conjecture is that, in a globally optimal solution, all departure rates $\bar{u}_{k,p} = \frac{d}{dt} U_{k,p}$ are uniformly bounded and have compact support. Hence the corresponding solution should be uniquely determined without need of prioritizing functions. In the case of one group of drivers traveling on a single road, this fact was proved in [2]. In the general case, a proof of the above conjecture will likely require a more detailed study of the globally optimal solution, establishing necessary conditions for optimality.

## 4 Nash equilibria

In this section we prove the existence of a Nash equilibrium solution for traffic flow on a network. For our model, it turns out that in a Nash equilibrium all departure rates must be uniformly bounded and have compact support. As a consequence, the corresponding solution is uniquely determined without need of prioritizing functions.

**Theorem 2 (existence of a Nash equilibrium).** *Let the flux functions $F_{ij}$ and the cost functions $\varphi_k, \psi_k$ satisfy the assumptions (A1)-(A2).*

*(i) For any $n$-tuple $(G_1, \ldots, G_n)$ of nonnegative numbers, there exists at least one admissible family of departure rates $\{u^*_{k,p}\}$ which yields a Nash equilibrium solution.*

*(ii) In every Nash equilibrium solution, all departure rates are uniformly bounded and have compact support.*

Before proving the theorem, we establish a "modulus of continuity" for the exit time. Namely, for drivers traveling along a given path $\Gamma_p$, the arrival time $\tau_p(t)$ is a uniformly continuous function of the departure time $t$. In the following, we first consider a single arc $\gamma_{ij}$ with flux function $F_{ij}(\cdot)$ satisfying (A1). As in figure 1, $g_{ij}$ denotes the inverse function while $g^*_{ij}$ is the Legendre transform. For a driver who enters the arc $\gamma_{ij}$ at time $t$ (possibly joining a queue), we denote by $\tau_{ij}(t)$ his exit time.

**Lemma 4.1.** *Given constants $G, M > 0$, there exists a continuous function $\phi_{ij} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ depending on the flux function $F_{ij}$ in (2.1) and on $M, G$, such that $\phi_{ij}(0) = 0$ and moreover the following holds. Let $U^-_{ij}(\cdot)$ be a Lipschitz continuous departure distribution, such that $0 \le u_{ij}(t) = \frac{d}{dt}U^-_{ij}(t) \le M$ for a.e. $t$ and $\int u_{ij}(t)\,dt \le G$. Then the exit time $\tau_{ij}(\cdot)$ satisfies*

$$\tau_{ij}(t_2) - \tau_{ij}(t_1) \;\le\; \phi_{ij}(t_2 - t_1) \qquad \textit{whenever} \quad t_1 \le t_2 . \tag{4.1}$$

**Proof. 1.** Let $L_{ij}$ be the length of the arc $\gamma_{ij}$. Following [2], consider the function

$$h_{ij}(s) \;\doteq\; - L_{ij}\, g^*_{ij}\left(\frac{-s}{L_{ij}}\right) . \tag{4.2}$$

Since the Legendre transform $g^*_{ij}$ is convex, the function $h_{ij}$ is concave. Moreover, calling $\mu_{ij} \doteq L_{ij}/v_{ij}(0)$ the minimum time needed to drive across the arc $\gamma_{ij}$ (= length of the arc divided by the maximum speed), one has

$$\begin{array}{rcl} h_{ij}(s) &=& 0 \quad \text{if} \quad s \ge -\mu_{ij} , \\ h_{ij}(s) &<& 0 \quad \text{if} \quad s < -\mu_{ij} . \end{array} \tag{4.3}$$

Using the solution formula (2.24), the amount of drivers that exit from the arc $\gamma_{ij}$ before time $t$ is computed by (see Fig. 3)

$$U^+_{ij}(\tau) \;=\; \min_s \;\Big\{ U^-_{ij}(s) - h_{ij}(s - \tau)\Big\} . \tag{4.4}$$

In other words, $U^+_{ij}(t)$ is the amount by which one can shift upward the graph of $h(\cdot - \tau)$, before hitting the graph of $U^-_{ij}(\cdot)$. In turn, the exit time of a driver who enters the arc $\gamma_{ij}$ at time $t$ is given by

$$\tau_{ij}(t) \;=\; (t + \mu_{ij}) \;\vee\; \inf \Big\{ \tau \, ; \;\; U^-_{ij}(t) + h_{ij}(s - \tau) \le U^-_{ij}(s) \quad \text{for all } s < t \Big\} , \tag{4.5}$$

where we used the notation $a \vee b \doteq \max\{a, b\}$.

**2.** Let $t_1 < t_2$ be given. Two cases can be considered.

CASE 1 (Fig. 4, left). The minimum

$$U^+_{ij}(\tau_{ij}(t_2)) \;=\; \min_s \;\Big\{ U^-_{ij}(s) - h_{ij}(s - \tau_{ij}(t_2))\Big\} . \tag{4.6}$$

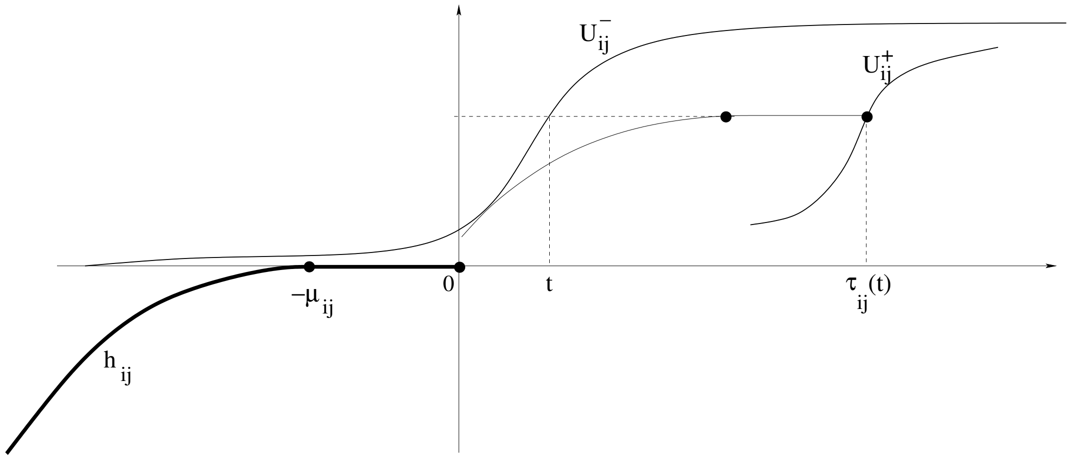


Figure 3: A geometric construction of the exit distribution $t \mapsto U^+_{ij}(t)$ from the entrance distribution $U^-_{ij}$, using the formula (4.4).

is attained at a point $\bar s \in [t_1, t_2]$. In this case we have

$$\tau_{ij}(t_1) \;\geq\; t_1 + \mu_{ij}\,,$$

$$\tau_{ij}(t_2) \;\leq\; t_2 + \inf\Big\{\tau\,;\quad Ms \geq h_{ij}(s-\tau) \quad \text{for all } s \in [-M(t_2-t_1)\,,\ 0]\Big\}.$$

Hence

$$\tau_{ij}(t_2) - \tau_{ij}(t_1) \;\leq\; \phi^\sharp_{ij}(t_2 - t_1) \tag{4.7}$$

with

$$\phi^\sharp_{ij}(\xi) \;\doteq\; \xi + \inf\Big\{\tau\,;\quad Ms \geq h_{ij}(s-\tau) \quad \text{for all } s \in [-M\xi\,,\ 0]\Big\} - \mu_{ij}\,. \tag{4.8}$$

CASE 2 (Fig. 4, right). The minimum in (4.6) is attained at a point $\bar s < t_1$. In this case we have

$$U^-_{ij}(t_1) + h_{ij}(\bar s - \tau_{ij}(t_1)) \;\leq\; U^-_{ij}(\bar s) \;\;=\;\; U^-_{ij}(t_2) + h_{ij}(\bar s - \tau_{ij}(t_2))\,.$$

Since $h_{ij}$ is concave and $\tau_{ij}(t_1) - t_1 \geq \mu_{ij}, \quad \tau_{ij}(t_2) - t_2 \geq \mu_{ij}$, one has

$$\begin{aligned} 0 \;\;&\geq\;\; h_{ij}\left(\tau_{ij}(t_1) - \tau_{ij}(t_2) - \mu_{ij}\right) \;\;\geq\;\; h_{ij}(\bar s - \tau_{ij}(t_2)) - h_{ij}(\bar s - \tau_{ij}(t_1)) \\ &\geq\;\; U^-_{ij}(t_1) - U^-_{ij}(t_2) \;\;\geq\;\; M(t_1 - t_2). \end{aligned}$$

In this case, we conclude

$$\tau_{ij}(t_2) - \tau_{ij}(t_1) \;\leq\; \phi^\flat_{ij}(t_2 - t_1), \tag{4.9}$$

where the continuous function $\phi^\flat_{ij}$ is implicitly defined by

$$h_{ij}(-\mu_{ij} - \phi^\flat_{ij}(\xi)) \;\;=\;\; -\,M\xi\,. \tag{4.10}$$

**3.** The two functions $\phi^\sharp_{ij}$, $\phi^\flat_{ij}$ defined at (4.8), (4.10) are both continuous and vanish at the origin. Defining

$$\phi_{ij}(\xi) \;\doteq\; \max\,\{\phi^\sharp_{ij}(\xi)\,,\ \phi^\flat_{ij}(\xi)\}\,, \tag{4.11}$$

the conclusion of the lemma is satisfied. □

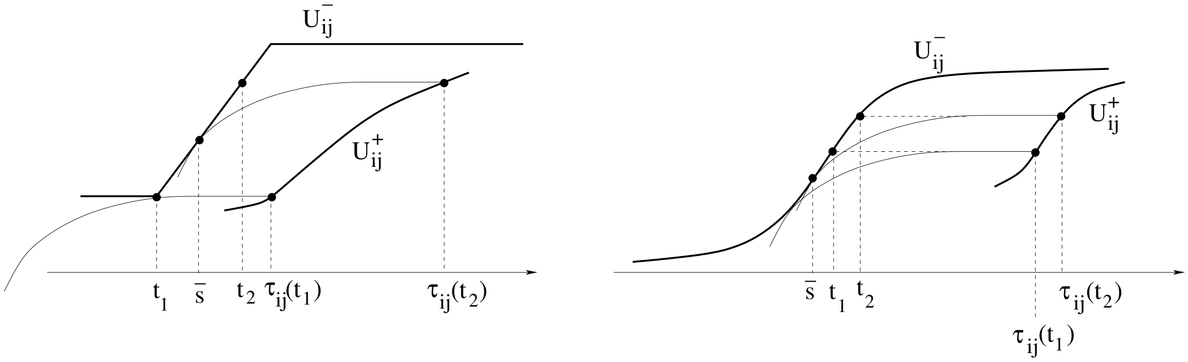


Figure 4: The two cases considered in the proof of Lemma 4.1.

The next lemma extends the result of Lemma 4.1 to a general path $\Gamma_p$. Here $\tau_p(t)$ denotes the arrival time of a driver starting at time $t$ and traveling along $\Gamma_p$.

**Lemma 4.2.** *Let all departure rates $\overline{u}_{k,p}(t)$ be uniformly bounded, so that*

$$\overline{u}_{k,p}(t) \;\leq\; M_0 \qquad \textit{for all} \quad k,p,t\,. \tag{4.12}$$

*Then, for any viable path $\Gamma_p$, there exists continuous function $\phi_p : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that*

$$\phi_p(0) \;=\; 0\,, \qquad\qquad \big(\tau_p(t_2) - \tau_p(t_1)\big) \;\leq\; \phi_p(t_2 - t_1) \qquad\qquad \textit{whenever} \quad t_1 \leq t_2\,. \tag{4.13}$$

**Proof.** The assumption (4.12), together with the fact that the flux through each arc $\gamma_{ij}$ cannot be greater than the maximum flux $F_{ij}^{max}$, implies that the maximum incoming flux through each arc is bounded by some constant $M$. For each arc $\gamma_{ij}$, let $\phi_{ij}$ be the modulus of continuous dependence constructed in Lemma 4.1. If $\Gamma$ is the path in (1.2), obtained as the concatenation of the arcs $\gamma_{i(\ell-1),i(\ell)}$, $\ell = 1,\ldots,\nu$, it now suffices to define the function $\phi_p$ as the composition of the corresponding functions $\phi_{i(\ell-1),i(\ell)}$, namely

$$\phi_p \;\doteq\; \phi_{i(\nu-1),i(\nu)} \circ \cdots \circ \phi_{i(1),i(2)} \circ \phi_{i(0),i(1)}\,.$$

□

Our final lemma shows that the arrival times, and hence the cost functions, depend continuously on the departure rates.

**Lemma 4.3.** *Consider a sequence of departure rates $u^\nu = (u^\nu_{k,p})$ which are uniformly bounded and supported inside a common interval $I = [-T,T]$, namely*

$$0 \;\leq\; u^\nu_{k,p}(t) \;\leq\; M_0 \qquad \textit{for all} \quad t \in I\,, \qquad\qquad u^\nu_{k,p}(t) \;=\; 0 \qquad \textit{if} \;\; t \notin I\,, \tag{4.14}$$

*for every $\nu \geq 1$. Assume that for all $k,p$ one has the weak convergence*

$$u^\nu_{k,p} \;\rightharpoonup\; u^*_{k,p}\,. \tag{4.15}$$

*For each viable path $\Gamma_q$, call $\tau^\nu_q(t)$, $\tau^*_q(t)$ the corresponding arrival times of a driver who departs at time $t$ and travels along $\Gamma_q$. Then, as $\nu \to \infty$, one has the uniform convergence*

$$\|\tau^\nu_q - \tau^*_q\|_{\mathcal{C}([-T,T])} \;\to\; 0\,. \tag{4.16}$$

**Proof. 1.** We first consider the case of a single arc $\gamma = \gamma_{ij}$. Assume that the departure rates $u^\nu$ satisfy

$$\begin{array}{rclcl} u^\nu(t) & \in & [0, M_0] & \qquad \text{if} & t \in [-T, T]\,, \\ u^\nu(t) & = & 0 & \qquad \text{if} & t \notin [-T, T]\,, \end{array}$$

and converge weakly: $u^\nu \rightharpoonup u$. Define the integrated functions

$$U^\nu(t) \doteq \int_{-\infty}^t u^\nu(s)\, ds, \qquad\qquad U^*(t) \doteq \int_{-\infty}^t \bar u(s)\, ds\,.$$

Our assumptions imply the convergence $U^\nu(t) \to U(t)$, uniformly for $t \in \mathbb{R}$. In turn, this implies that the exit distributions

$$U^\nu_+(t) \doteq \min_\tau \left\{ U^\nu(\tau) + L_{ij}\, g^*_{ij}\Big(\frac{t-\tau}{L_{ij}}\Big)\right\}, \qquad\qquad U_+(t) \doteq \min_\tau \left\{ U(\tau) + L_{ij}\, g^*_{ij}\Big(\frac{t-\tau}{L_{ij}}\Big)\right\}, \tag{4.17}$$

satisfy the uniform convergence $U^\nu_+ \to U_+$. Indeed, by (4.17) we have

$$\|U^\nu_+ - U_+\|_{\mathbf{L}^\infty} \leq \|U^\nu - U\|_{\mathbf{L}^\infty} \to 0 \qquad\qquad \text{as} \quad \nu \to \infty\,. \tag{4.18}$$

**2.** Given $\varepsilon > 0$ choose $\hat\nu$ large enough so that

$$|U^\nu(t) - U(t)| \leq \varepsilon \qquad\qquad \text{for all } \nu \geq \hat\nu, \,, \quad t \in \mathbb{R}\,. \tag{4.19}$$

Fix a time $t \in [-T, T]$. To estimate the difference $|\tau^\nu(t) - \tau(t)|$ between the corresponding arrival times, consider the modified departure distribution

$$U^\sharp(s) \doteq \left\{ \begin{array}{cl} U(s) & \quad \text{if} \quad s \leq t\,, \\ U(t) + (1 + M_0)(s - t) & \quad \text{if} \quad s \in [t, T]\,, \\ U(t) + (1 + M_0)(T - t) & \quad \text{if} \quad s \geq T\,. \end{array}\right. \tag{4.20}$$

Call $\tau^\sharp(t)$ the arrival time of a driver departing at time $t$, relative to the distribution $U^\sharp$. By Lemma 4.1, the function $t \mapsto \tau^\sharp(t)$ satisfies a uniform modulus of continuity say

$$\tau^\sharp(t_2) - \tau^\sharp(t_1) \leq \phi(t_2 - t_1) \qquad\qquad \text{for all } t_1 < t_2\,,$$

for some continuous function $\phi$ with $\phi(0) = 0$. In particular,

$$\tau^\sharp(t + \varepsilon) - \tau^\sharp(t) \leq \phi(\varepsilon)\,. \tag{4.21}$$

Observing that

$$U^\sharp(t + \varepsilon) - U^\sharp(s) \geq U^\nu(t + \varepsilon) - U^\nu(s) \qquad\qquad \text{for all } \nu \geq \hat\nu\,, \quad s \leq t + \varepsilon$$

and using the representation formula (4.5) with $h_{ij}$ given by (4.2), we obtain

$$\begin{array}{l} \tau^\sharp(t+\varepsilon) \\[2mm] \quad = (t + \varepsilon + \mu_{ij}) \vee \inf\left\{\tau;\ U^\sharp(t+\varepsilon) - U^\sharp(s) \leq L_{ij}\, g^*_{ij}\Big(\frac{\tau - s}{L_{ij}}\Big) \qquad \text{for all} \quad s \leq t + \varepsilon\right\} \\[2mm] \quad \geq (t + \varepsilon + \mu_{ij}) \vee \inf\left\{\tau;\ U^\nu(t+\varepsilon) - U^\nu(s) \leq L_{ij}\, g^*_{ij}\Big(\frac{\tau - s}{L_{ij}}\Big) \qquad \text{for all} \quad s \leq t + \varepsilon\right\} \\[2mm] \quad = \tau^\nu(t + \varepsilon)\,. \end{array}$$

Therefore

$$\tau^\nu(t) \;\le\; \tau^\nu(t+\varepsilon) \;\le\; \tau^\sharp(t+\varepsilon) \;\le\; \tau^\sharp(t)+\phi(\varepsilon) \;=\; \tau(t)+\phi(\varepsilon) \qquad\qquad \text{for all } \nu\ge\hat\nu\,. \tag{4.22}$$

Switching the roles of $U, U^\nu$ and hence defining

$$U^\sharp(s) \;\doteq\; \begin{cases} U^\nu(s) & \text{if } \ s\le t\,,\\ U^\nu(t)+(1+M_0)(s-t) & \text{if } \ s\in[t,T]\,,\\ U^\nu(t)+(1+M_0)(T-t) & \text{if } \ s\ge T\,,\end{cases}$$

we obtain the converse inequality

$$\tau(t) \;\le\; \tau^\nu(t)+\phi(\varepsilon) \qquad\qquad \text{for all } \nu\ge\hat\nu\,. \tag{4.23}$$

**3.** For each arc $\gamma_{ij}$, the inequalities (4.22)-(4.23) show that the functions $t\mapsto\tau^\nu_{ij}(t)$ converge uniformly to the corresponding functions $t\mapsto\tau_{ij}(t)$. Given any path $\Gamma_q$, the uniform convergence $\tau^\nu_q\to\tau^*_q$ is now obtained by a straightforward argument, using induction on the number of arcs contained in $\Gamma_q$. □

We are now ready to prove the main result of this paper.

**Proof of Theorem 2.**

**1.** We claim that there exists a time interval $I=[-T_0,\,T_0]$ so large that, in any Nash equilibrium, no driver will depart or arrive at a time $t\notin I$. Indeed, given the $n$-tuple $(G_1,\ldots,G_n)$, the travel time along any viable path $\Gamma_p=\big(\gamma_{i(0),i(1)},\ldots,\gamma_{i(\nu-1),i(\nu)}\big)$ is a priori bounded by

$$T^{max}_p \;\doteq\; \sum_{\ell=1}^{\nu}\left\{\frac{G}{F^{max}_{i(\ell-1),i(\ell)}}+\frac{L_{i(\ell-1),i(\ell)}}{v_{i(\ell-1),i(\ell)}\big(\rho^*_{i(\ell-1),i(\ell)}\big)}\right\}. \tag{4.24}$$

Here and in the sequel, we call

$$G \;\doteq\; G_1+\cdots+G_n \tag{4.25}$$

the total number of drivers. Notice that, in each summand on the right hand side of (4.24), the first term is an upper bound for the time spent waiting in the queue (total number of drivers divided by the maximum flux) while the second term is an upper bound on the actual travel time (length divided by the minimum speed). Let

$$T^{max} \;\doteq\; \max_p\; T^{max}_p$$

be an upper bound on the travel time along all viable paths. In view of assumption (A2), there exists $T_0$ large enough such that

$$\min_k\big\{\varphi_k(t)+\psi_k(t)\big\} \;>\; \max_k\big\{\varphi_k(0)+\psi_k(T^{max})\big\} \qquad \text{for all } t\notin I\doteq[-T_0,\,T_0]\,. \tag{4.26}$$

Therefore, in a Nash equilibrium no driver will depart or arrive outside $[-T_0,\,T_0]$. Otherwise, he would achieve a strictly lower cost by departing at time $t=0$.

**2.** Let $F^{max}=\max\limits_{i,j}F^{max}_{ij}$ be an upper bound for the flux over all arcs. Call

$$\varphi'_{max} \;\doteq\; \max_{1\le k\le n}\,\max_{t\in I}\;|\varphi'_k(t)|\,, \qquad\qquad \psi'_{min} \;\doteq\; \min_{1\le k\le n}\,\min_{t\in I}\;\psi'_k(t).$$

Observe that $\psi'_{min} > 0$, because of the assumption (A2). We claim that, in a Nash equilibrium, all departure rates $u_{k,p}$ must satisfy the priori bound

$$u_{k,p}(t) \;\leq\; \kappa \;\doteq\; \frac{\varphi'_{max} \cdot F_{max}}{\psi'_{min}} \qquad\qquad \text{for a.e. } t\,. \tag{4.27}$$

Indeed, consider drivers of the $k$-th family traveling along the path $\Gamma_p$. Let $t_1 < t_2$ be any two departure times, and call $\tau_1 < \tau_2$ the corresponding arrival times. The total costs for these two drivers must be the same, hence

$$\varphi(t_1) + \psi(\tau_1) \;=\; \varphi(t_2) + \psi(\tau_2)\,.$$

On the other hand, the upper bound on the flux implies

$$\tau_2 - \tau_1 \;\geq\; \frac{1}{F_{max}} \int_{t_1}^{t_2} u_{k,p}(t)\,dt\,.$$

Therefore

$$(t_2 - t_1)\,\varphi'_{max} \;\geq\; \varphi(t_1) - \varphi(t_2) \;=\; \psi(\tau_2) - \psi(\tau_1) \;\geq\; (\tau_2 - \tau_1)\,\psi'_{min} \;\geq\; \frac{\psi'_{min}}{F_{max}} \int_{t_1}^{t_2} u_{k,p}(t)\,dt\,.$$

We thus conclude

$$\frac{\varphi'_{max} \cdot F_{max}}{\psi'_{min}} \;\geq\; \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} u_{k,p}(t)\,dt\,.$$

Since this bound is valid for every interval $[t_1, t_2] \subseteq I$, the pointwise bound (4.27) must hold. Moreover, for $t \notin I$ we already know that $u_{k,p}(t) = 0$. The last statement of the Theorem is thus proved.

**3.** Choose the time

$$T \;\doteq\; T_0 + \frac{G}{\kappa}\,, \tag{4.28}$$

where $G \doteq \sum_k G_k$.

Consider the family of admissible departure rates

$$\mathcal{U} \;\doteq\; \Bigg\{ (u_{k,p})_{1\leq k\leq n,\,1\leq p\leq N}\;; \qquad u_{k,p} : \mathbb{R} \mapsto [0, 4\kappa]\,, \qquad u_{k,p}(t) = 0 \;\text{ for }\; t \notin [-T, T]\,,$$
$$\sum_p \int u_{k,p}(t)\,dt \;=\; G_k \quad \text{for every } k \Bigg\}. \tag{4.29}$$

It is understood that $u_{k,p} \equiv 0$ if the path $\Gamma_p$ does not connect $A_{d(k)}$ with $A_{a(k)}$. Notice that $\mathcal{U}$ is a closed convex subset of $\mathbf{L}^1(\mathbb{R}; \mathbb{R}^{n\times N})$.

For each fixed $\nu \geq 1$, we consider a finite dimensional subset $\mathcal{U}_\nu \subset \mathcal{U}$ consisting of all $u = (u_{k,p})$ which are piecewise constant on time intervals of length $T/\nu$. Introducing the points

$$t^\nu_\ell \;\doteq\; \frac{\ell}{\nu} T\,, \qquad\qquad -\nu \leq \ell \leq \nu\,,$$

we thus define

$$\mathcal{U}_\nu \doteq \Big\{ u = (u_{k,p}) \in \mathcal{U}\,; \qquad \text{every function } u_{k,p} \text{ is constant on each subinterval } I_\ell^\nu \doteq \,]t_{\ell-1}^\nu,\, t_\ell^\nu]\Big\}. \tag{4.30}$$

Observe that every $u \in \mathcal{U}_\nu$ has the form

$$u = (u_{k,p})\,, \qquad u_{k,p}(t) = u_{k,p,\ell} \qquad \text{for all} \quad t \in ]t_{\ell-1}^\nu,\, t_\ell^\nu]\,, \tag{4.31}$$

for some constants $u_{k,p,\ell} \in [0,\, 4\kappa]$.

**4.** Given $u = (u_{k,p}) \in \mathcal{U}$, let $\tau_q(t)$ be the arrival time of a driver starting at time $t$ and traveling along the path $\Gamma_q$. Clearly, this arrival time depends on the overall traffic conditions, hence on all functions $u_{k,p}$. If this driver belongs to the $j$-th family, his total cost is

$$\Phi_{j,q}^{(u)}(t) = \varphi_j(t) + \psi_j(\tau_q(t))\,.$$

We now observe that, for each $\nu \geq 1$, the domain $\mathcal{U}_\nu$ is a finite dimensional, compact, convex subset of $\mathbf{L}^2([-T,T];\, \mathbb{R}^{n\times N})$. Moreover, by Lemma 4.3 the maps $u \mapsto \Phi_{k,p}^{(u)}(\cdot)$ are continuous from $\mathcal{U}_\nu$ into $\mathbf{L}^2$. Hence, by the theory of variational inequalities [12], there exists a function $\bar u^\nu = (\bar u_{j,q}^\nu) \in \mathcal{U}_\nu$ which satisfies

$$\sum_{j,q} \int_{-T}^{T} \Phi_{j,q}^{(\bar u^\nu)}(t) \cdot \Big( v_{j,q}(t) - \bar u_{j,q}^\nu(t)\Big)\, dt \;\geq\; 0 \qquad \text{for all } v \in \mathcal{U}_\nu\,. \tag{4.32}$$

**5.** We now let $\nu \to \infty$. By the previous steps, there exists a sequence of piecewise constant functions $\bar u^\nu = (\bar u_{k,p}^\nu) \in \mathcal{U}_\nu$ such that (4.32) holds for every $\nu \geq 1$. Since all functions $\bar u_{k,p}^\nu$ are uniformly bounded and supported inside the interval $I = [-T,T]$, by taking a subsequence we can assume the weak convergence

$$(\bar u_{k,p}^\nu) \rightharpoonup (u_{k,p}^*) \tag{4.33}$$

for some function $u^* = (u_{k,p}^*) \in \mathcal{U}$. We claim that the departure rates $u_{k,p}^*$ yield a Nash equilibrium solution. More precisely:

(NE) Given any $k,p$, any $t_1 \in Supp(u_{k,p}^*)$, $t_2 \in \mathbb{R}$ and any path $\Gamma_q$ with the same initial and final nodes as $\Gamma_p$, one has

$$\Phi_{k,p}^*(t_1) \;\leq\; \Phi_{k,q}^*(t_2). \tag{4.34}$$

Indeed, (4.34) implies that no $k$-driver can lower his own cost by switching to the time $t_2$ or choosing the alternative path $\Gamma_q$ to reach destination. We recall that $t$ is in the support of a function $f \in \mathbf{L}^1$ if and only if $\int_{t-\varepsilon}^{t+\varepsilon} |f(s)|\, ds \neq 0$ for every $\varepsilon > 0$.

**6.** By Lemma 4.2, all maps $t \mapsto \tau_{k,p}^\nu(t)$ have the same modulus of continuity. Since these arrival times are uniformly bounded, we can apply Ascoli's compactness theorem. Choosing a subsequence and relabeling, as $\nu \to \infty$ we thus achieve the convergence

$$\tau_{k,p}^\nu(t) \;\to\; \tau_{k,p}^*(t) \qquad \text{for all } k,p, \text{ uniformly for } t \in [-T,T]. \tag{4.35}$$

By the assumption **(A2)**, the cost functions $\varphi_k(\cdot)$, $\psi_k(\cdot)$ are continuous. Therefore, the functions $\Phi^\nu_{k,p}(\cdot) = \varphi_k(\cdot) + \psi_k\big(\tau^\nu_{k,p}(\cdot)\big)$ converge to $\Phi^*_{k,p}(\cdot) = \varphi_k(\cdot) + \psi_k\big(\tau^*_{k,p}(\cdot)\big)$, uniformly for $t \in [-T, T]$. By Lemma 4.3, $\tau^*_{k,p}(t)$ is indeed the arrival time of a $k$-driver departing at time $t$ and following the path $\Gamma_p$, in the case where the departure rates of all drivers are given by $u^* = (u^*_{j,q})$.

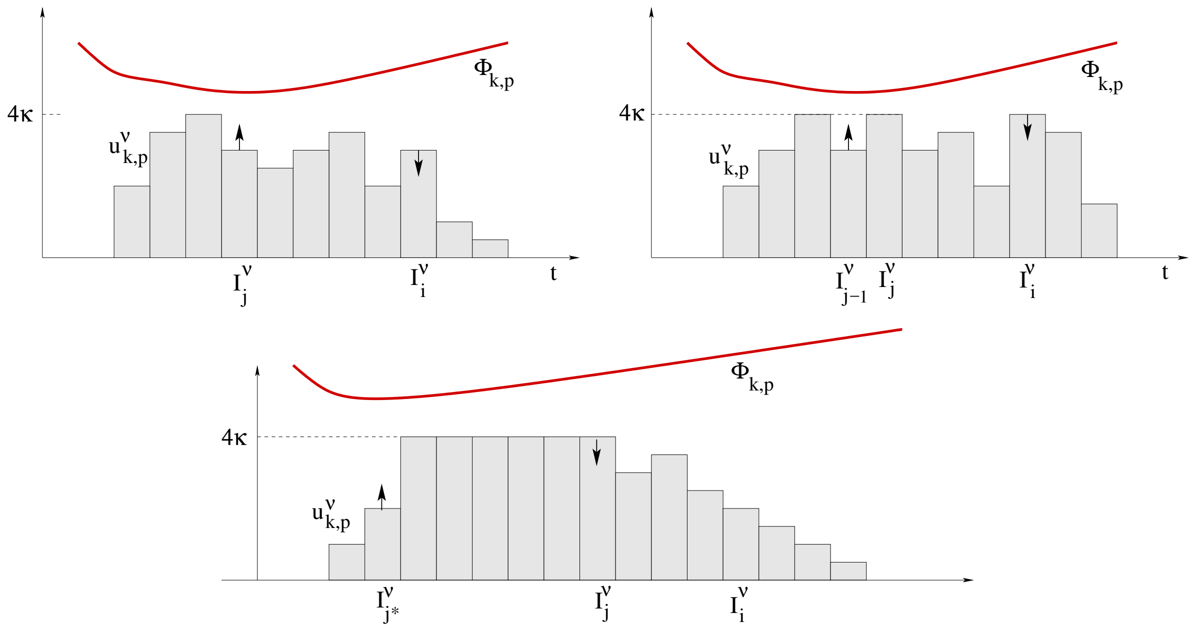


Figure 5: The three cases considered in the proof of Theorem 2. In Case 1 (top left) the average of the cost $\Phi_{k,p}$ on the interval $I^\nu_i$ is higher than on the interval $I^\nu_j$. To obtain a contradiction with (4.32) we simply move some of the mass from $I^\nu_i$ to $I^\nu_j$. In Case 2a (top right) one cannot increase the value of $u^\nu_{k,p}$ on the interval $I^\nu_j$ because of the constraint $u \le 4\kappa$. However, some mass can be moved from $I^\nu_i$ to the previous interval $I^\nu_{j-1}$. In Case 2b (bottom) there are several adjacent intervals where $u^\nu_{k,p} \equiv 4\kappa$. In this case, if $I^\nu_{j^*}$ is the first interval to the left of $I^\nu_j$ where $u^\nu_{k,p} < 4\kappa$, we argue that (i) $t^\nu_{j^*} > -T$, and (ii) the average of the cost $\Phi_{k,p}$ on $I^\nu_{j^*}$ is strictly less than on $I^\nu_j$. In this last case, to obtain a contradiction with (4.32) we move some mass from $I^\nu_{j^*}$ to $I^\nu_j$.

**7.** If (4.34) fails, then by continuity there exists $\delta > 0$ such that

$$\Phi^*_{k,p}(t) \;>\; \Phi^*_{k,q}(t') + 2\delta \qquad\qquad \text{whenever } |t - t_1| \le 2\delta, \quad |t' - t_2| \le 2\delta\,. \tag{4.36}$$

By uniform convergence, for all $\nu$ large enough we have

$$\Phi^\nu_{k,p}(t) \;>\; \Phi^\nu_{k,q}(t') + \delta \qquad\qquad \text{whenever } |t - t_1| \le 2\delta, \quad |t' - t_2| \le 2\delta\,. \tag{4.37}$$

Observe that it is not restrictive to assume that $t_2 \in [-T_0, T_0]$. Indeed, if (4.34) fails for some $t_2 \notin [-T_0, T_0]$, then (4.26) implies

$$\Phi^*_{k,p}(t_1) \;>\; \Phi^*_{k,q}(t_2) \;>\; \Phi^*_{k,q}(0),$$

and we can simply replace $t_2$ with zero.

The weak convergence (4.33), together with the assumption on the support of the function $u^*_{k,p}$, now implies

$$\lim_{\nu\to\infty} \int_{t_1-\delta}^{t_1+\delta} \bar u^\nu_{k,p}(t)\,dt \;=\; \int_{t_1-\delta}^{t_1+\delta} u^*_{k,p}(t)\,dt \;>\; 0\,.$$

Therefore, for every $\nu$ sufficiently large we can find two intervals

$$I_i^\nu \;=\; ]t_{i-1}^\nu,\, t_i^\nu] \;\subset\; [t_1-\delta\,,\; t_1+\delta], \qquad\qquad I_j^\nu \;=\; ]t_{j-1}^\nu,\, t_j^\nu] \;\subset\; [t_2-\delta\,,\; t_2+\delta] \tag{4.38}$$

with $t_j^\nu > t_2$ and $\bar u_{k,p}^\nu(t) > 0$ for $t\in I_i^\nu$.

**8.** We now derive a contradiction, showing that, for $\nu$ sufficiently large, the departure rates $\bar u_{k,p}^\nu$ do not satisfy the variational inequality (4.32). Two possibilities can arise (see fig.5).

CASE 1: $\bar u_{k,q,j}^\nu \;<\; 4\kappa$. In this case we define a new set of departure rates $v^\varepsilon = (v_{k,p}^\varepsilon)\in\mathcal{U}_\nu$ by setting

$$\begin{aligned} v_{k,p}^\varepsilon(t) &\;=\; u_{k,p}^\nu(t)-\varepsilon \qquad \text{if}\quad t\in I_i^\nu\,,\\ v_{k,q}^\varepsilon(t) &\;=\; u_{k,q}^\nu(t)+\varepsilon \qquad \text{if}\quad t\in I_j^\nu\,,\end{aligned}$$

and setting $v_{h,r}^\varepsilon(t) = u_{h,r}^\nu(t)$ in all other cases. Notice that, if $\varepsilon = \min\{u_{k,p,i}^\nu,\; 4\kappa - u_{k,q,j}^\nu\}$ then these new departure rates are still admissible. By (4.37) and (4.38), this construction yields

$$\sum_{h,r}\int_{-T}^T \Phi_{h,r}^{(\bar u^\nu)}(t)\cdot\Big(v_{h,r}^\varepsilon(t)-\bar u_{h,r}^\nu(t)\Big)\,dt \;=\; \varepsilon\int_{I_j^\nu}\Phi_{k,q}^{(\bar u^\nu)}(t)\,dt-\varepsilon\int_{I_i^\nu}\Phi_{k,p}^{(\bar u^\nu)}(t)\,dt \;\leq\; -2\varepsilon\delta\,, \tag{4.39}$$

providing a contradiction with (4.32).

CASE 2: $\bar u_{k,q,j}^\nu \;=\; 4\kappa$. If this equality holds, consider the index

$$j^* \;\doteq\; \max\;\{i<j\,;\;\; \bar u_{k,q,i}^\nu \;<\; 4\kappa\}\,.$$

Notice that $t_{j^*}^\nu > -T$. Indeed, by construction $t_2 > -T_0$. If $t_{j^*}^\nu \leq -T$, by (4.28) this would imply

$$\bar u_{k,q}^\nu(t) =\; 4\kappa \qquad\qquad \text{for all}\;\; t\in[t_{j^*}^\nu\,, t_j^\nu] \;\supseteq\; [-T,\, -T_0]\,,$$

$$\int \bar u_{k,q}^\nu(t)\,dt \;\geq\; 4\kappa(t_j^\nu - t_{j^*}^\nu) \;\;\geq\; 4\kappa(T-T_0) >\; G\,,$$

reaching a contradiction. We consider two subcases.

CASE 2a: $j^* \,=\, j-1$. In this case, since it is not restrictive to assume $\frac{T}{\nu} < \frac{\delta}{4}$, we have $I_{j-1}^\nu = [t_{j-2}^\nu, t_{j-1}^\nu] \subset [t_2-\delta,\; t_2+\delta]$. We can thus derive a contradiction as in CASE 1, simply replacing $j$ by $j-1$.

CASE 2b: $j^*\leq j-2$. Observe that, for all $s_1<s_2$,

$$\tau_{k,q}^\nu(s_2)-\tau_{k,q}^\nu(s_1) \;\geq\; \frac{1}{F_{max}}\int_{s_1}^{s_2}\bar u_{k,q}^\nu(\xi)\,d\xi\,. \tag{4.40}$$

In particular, for any $s_1\in I_{j^*}^\nu$ and $s_2\in I_j^\nu$ we have

$$\tau_{k,q}^\nu(s_2)-\tau_{k,q}^\nu(s_1) \;\geq\; \frac{1}{F_{max}}\int_{s_1}^{s_2}\bar u_{k,q}^\nu(\xi)\,d\xi \;\geq\; \frac{1}{F_{max}}\,4\kappa\,[t_{j-1}-t_{j^*}] \;\geq\; \frac{4\kappa(s_2-s_1)}{3F_{max}}\,. \tag{4.41}$$

This yields the estimate

$$\psi_k\big(\tau_{k,q}^\nu(s_2)\big)-\psi_k\big(\tau_{k,q}^\nu(s_1)\big) \;\geq\; \psi'_{min}\big(\tau_{k,q}^\nu(s_2)-\tau_{k,q}^\nu(s_1)\big) \;\geq\; \psi'_{min}\cdot\frac{4\kappa(s_2-s_1)}{3F_{max}}\,. \tag{4.42}$$

On the other hand, we have

$$\varphi_k(s_2) - \varphi_k(s_1) \;\geq\; -\varphi'_{max}(s_2 - s_1)\,. \tag{4.43}$$

Recalling the definition of the constant $\kappa$ in (4.27), from (4.42)-(4.43) we obtain

$$\Phi^\nu_{k,q}(s_2) - \Phi^\nu_{k,q}(s_1) \;\geq\; \left(\frac{4\kappa\psi'_{min}}{3F_{max}} - \varphi'_{max}\right)(s_2 - s_1) \;=\; \frac{1}{3}\varphi'_{max}\cdot(s_2 - s_1) \tag{4.44}$$

for all $s_1 \in I^\nu_{j^*}$ and $s_2 \in I^\nu_j$ .

We now choose the departure rates $v^\varepsilon = (v^\varepsilon_{k,p}) \in \mathcal{U}_\nu$ by setting

$$\begin{array}{rcll} v^\varepsilon_{k,p}(t) & = & u^\nu_{k,q}(t) - \varepsilon & \text{if} \quad t \in I^\nu_j\,, \\ v^\varepsilon_{k,q}(t) & = & u^\nu_{k,q}(t) + \varepsilon & \text{if} \quad t \in I^\nu_{j^*}\,, \end{array} \tag{4.45}$$

and setting $v^\varepsilon_{h,r}(t) = u^\nu_{h,r}(t)$ in all other cases. Notice that, if $\varepsilon = \min\{u^\nu_{k,q,j},\; 4\kappa - u^\nu_{k,q,j^*}\}$ then these new departure rates are still admissible.

Using (4.44) with $s_1 = t$, $s_2 = t + t^\nu_j - t^\nu_{j^*}$ we compute

$$\begin{array}{l} \displaystyle\sum_{h,r}\int_{-T}^{T}\Phi^{(\bar u^\nu)}_{h,r}(t)\cdot\Big(v^\varepsilon_{h,r}(t) - \bar u^\nu_{h,r}(t)\Big)\,dt \;=\; \varepsilon\left(\int_{I^\nu_{j^*}}\Phi^{(\bar u^\nu)}_{k,q}(t)\,dt - \int_{I^\nu_j}\Phi^{(\bar u^\nu)}_{k,q}(t)\,dt\right) \\ \displaystyle =\; \varepsilon\int_{I^\nu_{j^*}}\Big(\Phi^{(\bar u^\nu)}_{k,q}(t) - \Phi^{(\bar u^\nu)}_{k,q}(t + t^\nu_j - t^\nu_{j^*})\Big)\,dt \;\leq\; -\frac{\varepsilon}{3}\varphi'_{max}\cdot(t^\nu_j - t^\nu_{j^*}) \;<\; 0\,. \end{array} \tag{4.46}$$

Once again we reached a contradiction with (4.32), completing the proof. □

**Remark 4.4.** The above proof is based on a topological method, and does not yield any information about the uniqueness of the Nash equilibrium. Another important question is the dynamic stability of this equilibrium solution. This issue was investigated numerically in [4]. In the case of a single group of drivers traveling on a single road, the uniqueness of the Nash equilibrium solution was proved in [2], but the stability issue remains unresolved.

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