Entropy Admissibility of the Limit Solution for a Nonlocal Model of Traffic Flow

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Abstract

We consider a conservation law model of traffic flow, where the velocity of each car depends on a weighted average of the traffic density ρ ahead. The averaging kernel is of exponential type: $w_\varepsilon(s) = \varepsilon^{-1}e^{-s/\varepsilon}$. For any decreasing velocity function v, we prove that, as $\varepsilon \to 0$, the limit of solutions to the nonlocal equation coincides with the unique entropy-admissible solution to the scalar conservation law $\rho_t + (\rho v(\rho))_x = 0$.

1 Introduction

We consider a nonlocal PDE model for traffic flow, where the traffic density $\rho = \rho(t,x)$ satisfies a scalar conservation law with nonlocal flux

$$\rho_t + (\rho v(q))_x \;=\; 0. \tag{1.1}$$

Here $\rho \mapsto v(\rho)$ is a decreasing function, modeling the velocity of cars depending on the traffic density, while the integral

$$q(x) \;=\; \int_x^{+\infty} \varepsilon^{-1} e^{(x-y)/\varepsilon}\, \rho(y)\, ds \tag{1.2}$$

computes a weighted average of the density of cars ahead. As in [2], we shall assume

(A1) *The velocity function $v : [0, \rho_{jam}] \mapsto \mathbb{R}_+$ is $\mathcal{C}^2$, and satisfies*

$$v(\rho_{jam}) \;=\; 0, \qquad\qquad v'(\rho) \;\leq\; -\,\delta_* \;<\; 0, \qquad \text{for all } \rho \in [0, \rho_{jam}]. \tag{1.3}$$

One can think of ρ_{jam} as the maximum possible density of cars along the road, when all cars are packed bumper-to-bumper and nobody moves. The conservation equation (1.1) will be solved with initial data

$$\rho(0,x) \;=\; \bar\rho(x) \;\in\; [0, \rho_{jam}]\,. \tag{1.4}$$

As $\varepsilon \to 0+$, the weight function $w_\varepsilon(s) = \varepsilon^{-1}e^{-s/\varepsilon}$ converges to a Dirac mass at the origin, and the nonlocal equation (1.1)-(1.2) formally converges to the scalar conservation law

$$\rho_t + \big(\rho v(\rho)\big)_x \;=\; 0. \tag{1.5}$$

Assuming that the initial datum $\bar\rho$ has bounded total variation and takes uniformly positive values, the recent analysis in [2] has established:

(i) For every $\varepsilon > 0$, the Cauchy problem with non-local flux (1.1), (1.2), (1.4), has a unique solution $\rho = \rho_\varepsilon(t, x)$. Its total variation satisfies a uniform bound

$$\text{Tot.Var.}\{\rho_\varepsilon(t,\cdot)\} \;\leq\; M \tag{1.6}$$

where the constant M is independent of t, ε.

(ii) As $\varepsilon \to 0$, by possibly taking a subsequence, one obtains the convergence $\rho_\varepsilon \to \rho$ in $\mathbf{L}^1_{loc}$. The limit function $\rho = \rho(t, x)$ provides a weak solution to the Cauchy problem (1.4)-(1.5).

A major issue, which was not fully resolved in [2], is the entropy admissibility of the limit solution ρ. Aim of the present note is to resolve this question in the affirmative. Namely, we prove:

Theorem. *Let v satisfy the assumptions* **(A1)**, *and let ρ_ε be a sequence of solutions to the nonlocal Cauchy problem (1.1), (1.2) and (1.4), satisfying the uniform BV bounds (1.6). Assume that, as $\varepsilon \to 0$, we have the convergence $\rho_\varepsilon \to \rho$ in $\mathbf{L}^1_{loc}$.*
Then ρ is the unique entropy admissible solution to the Cauchy problem (1.4)-(1.5).

The above result was proved in [2] in the special case where the velocity is affine: $v(\rho) = a - b\rho$. The earlier proof was based on the Hardy-Littlewood inequality. In the next section we give a simpler proof, valid for a general class of velocity functions v.

For a more general class of averaging kernels, assuming that the initial datum $\bar\rho$ satisfies a one-sided Lipschitz condition, the convergence to the unique entropy admissible solution was recently proved in [3]. Our result requires an exponential kernel, but it applies to any BV initial data. In particular, $\bar\rho$ can be piecewise constant.

For the general theory of conservation laws we refer to [1, 5, 6]. A brief review of literature on hyperbolic conservation laws with nonlocal flux can be found in [2].

2 Proof of the theorem

1. According to [4, 7], to prove uniqueness it suffices to prove that the limit solution dissipates one single strictly convex entropy. We thus consider the entropy and entropy flux pair

$$\eta(\rho) \;=\; \frac{\rho^2}{2}\,, \qquad\qquad \psi(\rho) \;=\; \int_0^\rho \big[s v(s) + s^2 v'(s)\big]\,ds. \tag{2.1}$$

For future use, we observe that (1.2) implies

$$\rho \;=\; q - \varepsilon q_x. \tag{2.2}$$

Moreover, we introduce the function

$$W(\rho) \;\doteq\; \int_0^\rho s^2 v'(s)\,ds\,. \tag{2.3}$$

The equation (1.1) can now be written as

$$\rho_t + (\rho v(\rho))_x \;=\; \Big(\rho(v(\rho)-v(q))\Big)_x.$$

Multiplying both sides by $\eta'(\rho)=\rho$, we obtain

$$\eta(\rho)_t + \psi(\rho)_x \;=\; \rho\Big(\rho(v(\rho)-v(q))\Big)_x. \tag{2.4}$$

2. Given a test function $\varphi\in\mathcal{C}^1_c(\mathbb{R})$, $\varphi\geq 0$, using (2.2) we estimate the quantity

$$\begin{aligned} J \;&\doteq\; 2\int \rho\Big(\rho(v(\rho)-v(q))\Big)_x \varphi\,dx \\ &=\; \int (\rho^2)_x\big(v(\rho)-v(q)\big)\,\varphi\,dx + \int 2\rho^2\big(v(\rho)-v(q)\big)_x\varphi\,dx \\ &=\; -\int \rho^2\big(v(\rho)-v(q)\big)\varphi_x\,ds + \int \rho^2\big(v(\rho)-v(q)\big)_x\,\varphi\,dx \\ &\doteq\; J_1+J_2\,. \end{aligned} \tag{2.5}$$

Concerning the second integral, using (2.2) we obtain

$$\begin{aligned} J_2 \;&=\; \int \rho^2 v'(\rho)\,\rho_x\,\varphi\,dx - \int \rho q\,v'(q)q_x\,\varphi\,dx + \int \rho\varepsilon\,(q_x)^2\,v'(q)\,\varphi\,dx \\ &\doteq\; J_{21}+J_{22}+J_{23}\,. \end{aligned} \tag{2.6}$$

Using (2.2) once again, we now compute

$$\begin{aligned} J_{21}+J_{22} \;&=\; \int \rho^2 v'(\rho)\,\rho_x\,\varphi\,dx - \int q^2\,v'(q)q_x\,\varphi\,dx + \int q\varepsilon\,(q_x)^2 v'(q)\,\varphi\,dx \\ &\doteq\; J_3+J_4+J_5\,. \end{aligned} \tag{2.7}$$

Since $\rho, q,\varphi\geq 0$ while $v'\leq 0$, from (2.6) and (2.7) we immediately see that

$$J_{23}\;\leq\; 0, \qquad\qquad J_5\;\leq\; 0. \tag{2.8}$$

On the other hand, integrating by parts and recalling (2.3), we obtain

$$\begin{aligned} J_3+J_4 \;&=\; \int \big[W(\rho)\big]_x\,\varphi\,dx - \int \big[W(q)\big]_x\varphi\,dx \\ &=\; -\int \big[W(\rho)-W(q)\big]\,\varphi_x\,dx\,. \end{aligned} \tag{2.9}$$

3. To conclude, consider a sequence of solutions ρ_ε to (1.1)-(1.2), (1.4). Assume that, as $\varepsilon \to 0$, we have the convergence $\rho_\varepsilon \to \rho$ in $\mathbf{L}^1_{loc}$. Notice that this implies $q_\varepsilon \to \rho$ in $\mathbf{L}^1_{loc}$ as well. Hence, the integrals J_1 and $J_3 + J_4$ both approach zero. By the previous analysis,

$$
\begin{aligned}
&2\iint \big\{\eta(\rho_\varepsilon)\varphi_t + \psi(\rho_\varepsilon)\varphi_x\big\}\,dxdt \\
&\qquad \geq \ \iint \rho_\varepsilon^2\big(v(\rho_\varepsilon) - v(q_\varepsilon)\big)\varphi_x\,dxdt + \iint \big[W(\rho_\varepsilon) - W(q_\varepsilon)\big]\,\varphi_x\,dxdt\,.
\end{aligned}
$$

Letting $\varepsilon \to 0$, since the right hand side converges to zero, we obtain

$$
\iint \big\{\eta(\rho)\varphi_t + \psi(\rho)\varphi_x\big\}\,dxdt \ \geq \ 0.
$$

This proves that the limit solution ρ is entropy admissible. In particular, by [4, 7], ρ is the unique entropy weak solution to the Cauchy problem (1.4)-(1.5). □

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