Optimal Shapes for Tree Roots

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April 30, 2022

Abstract

The paper studies a class of variational problems, modeling optimal shapes for tree roots. Given a measure μ describing the distribution of root hair cells, we seek to maximize a harvest functional $\mathcal{H}$, computing the total amount of water and nutrients gathered by the roots, subject to a cost for transporting these nutrients from the roots to the trunk. Earlier papers had established the existence of an optimal measure, and a priori bounds. Here we derive necessary conditions for optimality. Moreover, in space dimension $d = 2$, we prove that the support of an optimal measure is nowhere dense.

1 Introduction

Variational problems related to the optimal shape of tree roots were recently considered in [12, 14]. Here one seeks an optimal measure μ, describing the distribution of root hair cells. The goal is to maximize a payoff, measuring the amount of water and nutrient absorbed by the roots, minus a cost for transporting these nutrients to the base of the trunk. As in [11], given an open set $\Omega \subset I\!R^d$, the density of nutrients is modeled by the solution to an elliptic equation with measure coefficients.

$$\Delta u + f(u) - u\mu \;=\; 0, \qquad\qquad x \in \Omega,$$

with Neumann boundary conditions. Here

$$\int u\, d\mu$$

yields the total harvest. In addition, a ramified transportation cost is present. For a given $0 < \alpha < 1$, this is described by the α-irrigation cost of the measure μ from the origin [3, 24, 29].

The existence of an optimal measure was first proved in [14] under a constraint on the total mass of the measure μ, and then in [12] in a more general setting.

In the present paper we initiate a study of the properties of these optimal measures. Our first result provides necessary conditions for optimality. These take the form

$$\Phi(x) \;=\; cZ(x) \qquad\qquad \text{for all } x\in \mathrm{Supp}(\mu), \tag{1.1}$$

for a suitable constant $c>0$. Here $\Phi(x)$ measures the rate of increase of the total harvest, if the measure μ is locally increased at the point $x\in\Omega$. On the other hand, $Z(x)$ is the landscape function [3, 28]. This is proportional to the rate of increase of the irrigation cost, if the measure μ is locally increased at the point x.

In the second part of the paper we perform a detailed study of the equation (1.1), in dimension $d=2$. Our main result, Theorem 4.1, shows that the support of an optimal measure μ is nowhere dense. To appreciate the physical meaning of this fact, one may observe that water and nutrients can be moved around either by diffusion, or by ramified transport. Diffusion comes for free, but it is only effective at short distances. A network of roots is thus needed to transport water and nutrients over longer distances, while at small scales one can rely on diffusion alone.

The proof of Theorem 4.1 exploits the fact that the two functions Φ and Z have very different regularity properties, hence the set where they coincide must be small. To help the reader, we outline here the main ideas.

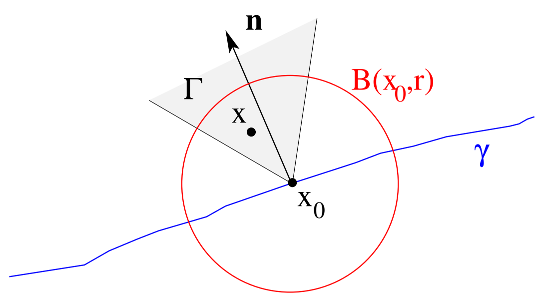

Figure 1: Proving that $\Phi(x)<cZ(x)$, at several points x near x_0.

Consider any particle path $s\mapsto\gamma(s)$, $s\in[0,\overline{s}]$, in an optimal irrigation plan for the measure μ. Given a point $x_0=\gamma(s_0)$, let $\mathbf{n}$ be the unit vector perpendicular to γ at x_0, and consider the shaded region in Fig. 1

$$\Gamma \;\doteq\; \left\{ x\in I\!R^2\,;\; \left\langle \mathbf{n}\,,\,\frac{x-x_0}{|x-x_0|}\right\rangle \;>\; \frac{1}{2}\right\}.$$

By an argument based on Riesz' sunrise lemma [23] we show that, for a.e. $s_0\in[0,\overline{s}]$, the landscape function Z satisfies a lower Hölder estimate on Γ, namely

$$Z(x)-Z(x_0)\;\geq\;\delta_0\cdot|x-x_0|^\alpha \qquad\qquad \text{for all } x\in\Gamma, \tag{1.2}$$

for some constant $\delta_0>0$ depending on x_0.

On the other hand, the function Φ can be bounded above by an auxiliary function Φ^+, which satisfies

$$\left\{\begin{array}{rcll} \Delta\Phi^+ &=& f \qquad & \text{on }\;\Omega,\\[2mm] \Phi^+ &\leq& cZ \qquad & \text{on }\;\gamma, \end{array}\right. \tag{1.3}$$

for some source function f. Since we only know that $f \in \mathbf{L}^1(\Omega)$, from (1.3) we do not obtain any pointwise upper bound on Φ^+. However, one can look at the average value of Φ^+ over balls centered at x_0 with small radius $r > 0$. Relying on Vitali's covering theorem, together with an estimate on the Green's function for the Laplacian on a suitable domain, we eventually obtain the averaged integral estimate

$$\fint_{\Gamma \cap B(x_0,r)} \Phi^+(x)\, dx - \Phi^+(x_0) \;\leq\; C_0\, r^{1-\varepsilon}, \tag{1.4}$$

where $\varepsilon > 0$ can be chosen arbitrarily small.

Choosing $\varepsilon < 1-\alpha$ and letting $r \to 0$, from (1.2)–(1.4) we conclude that there exists a sequence of points $x_n \to x_0$ such that

$$\Phi(x_n) \;\leq\; \Phi^+(x_n) \;<\; cZ(x_n) \qquad\qquad \text{for all } n \geq 1.$$

Since Φ is upper semicontinuous while Z is lower semicontinuous, this implies that the strict inequality $\Phi(x) < cZ(x)$ holds on an open set containing all the points x_n. Observing that the same conclusion can be reached for almost every point x_0 along every irrigation path γ, this achieves the proof.

The remainder of the paper is organized as follows. Section 2 reviews the definition of the harvest functional [11, 12, 14], and some basic properties of ramified transport and the landscape function [3, 24, 28, 29]. In Section 3 we formulate the optimization problem for tree roots, and derive a set of necessary conditions for optimality, stated in Theorems 3.2 and 3.3. The proofs are an adaptation of the arguments in [13], where similar necessary conditions were established for a fishery model. The key ideas are taken from [15], where some shape optimization problems involving measures were first studied.

The main new result of the paper, on the support of the optimal measure μ, is stated in Theorem 4.1. The proof is worked out in Section 4, relying on two key lemmas. Lemma 4.2, providing a lower Hölder estimate on the landscape function, is then proved in Section 5. Finally, Lemma 4.3, establishing an upper bound on suitable averages of the function Φ^+, is proved in Section 6.

An alternative approach to the Hölder continuity of the landscape function can be found in [7, 8]. A different variational problem involving a ramified transportation cost has been recently studied in [18, 27]. Additional properties of optimal transportation plans were studied in [4, 9, 10, 17, 25, 26]. For a survey, see also [30].

2 Review of the basic functionals

2.1 Harvest functionals

We consider a utility functional associated with plant roots. Here the main goal is to collect moisture and nutrients from the ground. To model the efficiency of a root, we let $u(x)$ be the density of water+nutrients at the point x, and consider a positive Radon measure μ describing the distribution of root hair cells.

To fix ideas, let $\Omega \subset \mathbb{R}^d$, be an open set of dimension $d \geq 2$, with $\mathcal{C}^2$ boundary. We assume that μ is a positive, bounded Radon measure, supported on the closure $\overline{\Omega}$, and absolutely continuous w.r.t. capacity. This property is expressed by the implication

$$\mathrm{cap}_2(V) \;=\; 0 \qquad \Longrightarrow \qquad \mu(V) \;=\; 0. \tag{2.1}$$

For the definition and basic properties of capacity we refer to [2, 22]. Based on physical considerations, following [11, 12, 14] we consider the solution to the elliptic problem with measure source

$$\Delta u + f(u) - u\,\mu \;=\; 0 \qquad\qquad \text{on } \Omega, \tag{2.2}$$

and Neumann boundary conditions

$$\partial_{\mathbf{n}(x)} u \;=\; 0 \qquad \text{on } \partial\Omega\,. \tag{2.3}$$

Here $\mathbf{n}(x)$ denotes the unit outer normal vector at the boundary point $x \in \partial\Omega$, while $\partial_{\mathbf{n}} u$ is the derivative of u in the normal direction. Suitable assumptions on the source function f will be given at (2.6).

Elliptic problems with measure data have been studied in various papers [5, 6, 16], and are now well understood. A key fact is that, roughly speaking, the Laplace operator "does not see" sets with zero capacity. Therefore, a measure concentrated on a set with zero capacity does not affect the solution of (2.2). Following [5, 6], we denote by $\mathcal{M}_0$ the family of all bounded Radon measures which vanish on sets with zero capacity.

Definition 2.1 *Let μ be a positive, bounded Radon measure on $\overline{\Omega}$, which is absolutely continuous w.r.t. capacity. A function $u \in \mathbf{L}^\infty(\Omega) \cap H^1(\Omega)$ is a solution to the elliptic problem (2.2)-(2.3) if*

$$-\int_\Omega \nabla u \cdot \nabla \varphi\, dx + \int_\Omega f(u)\varphi\, dx - \int_{\overline{\Omega}} u\varphi\, d\mu \;=\; 0 \tag{2.4}$$

for every test function $\varphi \in \mathcal{C}_c^\infty(I\!\!R^d)$.

Definition 2.2 *In connection with a solution u of (2.2)-(2.3), the* **total harvest** *is defined as*

$$\mathcal{H}(u,\mu) \;\doteq\; \int_{\overline{\Omega}} u\, d\mu\,. \tag{2.5}$$

For reader's convenience we collect the main assumptions used throughout the paper.

(A1) *$\Omega \subset I\!\!R^d$ is a bounded, connected open set with $\mathcal{C}^2$ boundary. Moreover, $0 \notin \overline{\Omega}$.*

(A2) *$f : I\!\!R \mapsto I\!\!R$ is a $\mathcal{C}^2$ function such that, for some constants $u_{max}, K > 0$,*

$$f(u_{max}) = 0, \qquad\quad 0 \le f(u) \le K, \qquad f''(u) < 0 \quad \text{for all } u \in [0, u_{max}]. \tag{2.6}$$

(A3) *The space dimension is $d \ge 2$. The exponent α in the irrigation cost satisfies*

$$\alpha \;>\; 1 - \frac{1}{d-1}\,. \tag{2.7}$$

Remark 2.3 If μ is a general measure and u is a discontinuous function, the integral (2.5) may not be well defined. To resolve this issue, calling

$$\fint_V u\,dx \;=\; \frac{1}{\mathrm{meas}(V)}\int_V u\,dx$$

the average value of u on a set V, for each $x \in \overline{\Omega}$ we consider the limit

$$u(x) \;=\; \lim_{r\downarrow 0}\ \fint_{\Omega\cap B(x,r)} u(y)\,dy. \tag{2.8}$$

As proved in [20], if $u \in H^1(\Omega)$ then the above limit exists at all points $x \in \overline{\Omega}$ with the possible exception of a set whose capacity is zero. If $\mu \in \mathcal{M}_0$, then the integral (2.5) is well defined. Our present setting is actually even better, because in (2.2) u and μ are positive while f is bounded. Therefore, if the constant C is chosen large enough, the function $u + C|x|^2$ is subharmonic [2]. As a consequence, the limit (2.8) is well defined at every point $x \in \overline{\Omega}$.

2.2 Optimal irrigation plans

Given $\alpha \in [0,1]$ and a positive measure μ on $I\!R^d$, the minimum cost for irrigating the measure μ from the origin will be denoted by $\mathcal{I}^\alpha(\mu)$. Following [24], this cost can be defined as follows. Let $M = \mu(I\!R^d)$ be the total mass to be transported and let $\Theta = [0, M]$. We think of each $\theta \in \Theta$ as a "water particle".

Definition 2.4 *A measurable map*

$$\chi : \Theta \times I\!R_+ \;\mapsto\; I\!R^d \tag{2.9}$$

is called an **admissible irrigation plan** *for the measure μ if*

(i) For every $\theta \in \Theta$, the map $t \mapsto \chi(\theta, t)$ is 1-Lipschitz. More precisely, for each θ there exists a stopping time $T(\theta)$ such that, calling

$$\dot\chi(\theta,t) \;=\; \frac{\partial}{\partial t}\,\chi(\theta,t)$$

the partial derivative w.r.t. time, one has

$$\big|\dot\chi(\theta,t)\big| \;=\; \begin{cases} 1 & \text{for a.e. } t \in [0, T(\theta)],\\[2mm] 0 & \text{for } t \geq T(\theta). \end{cases} \tag{2.10}$$

(ii) At time $t = 0$ all particles are at the origin:

$$\chi(\theta, 0) \;=\; \mathbf{0} \;\in\; I\!R^d \qquad \text{for all} \quad \theta \in \Theta.$$

(iii) The push-forward of the Lebesgue measure on $\Theta = [0, M]$ through the map $\theta \mapsto \chi(\theta, T(\theta))$ coincides with the measure μ. In other words, for every open set $A \subset I\!R^n$ there holds

$$\mu(A) \;=\; meas\Big(\big\{\theta \in \Theta\,;\;\; \chi(\theta, T(\theta)) \in A\big\}\Big), \tag{2.11}$$

where $meas(\cdot)$ denotes the Lebesgue measure on Θ.

In order to define the corresponding transportation cost, we first consider the amount of paths which go through a point $x \in I\!\!R^d$:

$$|x|_\chi \;=\; \mathrm{meas}\Big(\big\{\theta \in \Theta\,;\;\; \chi(\theta,t) = x \quad \text{for some} \;\; t \geq 0\big\}\Big). \qquad (2.12) \quad \boxed{\texttt{chi}}$$

We think of $|x|_\chi$ as the **total flux through the point** x.

Definition 2.5 *For a given $\alpha \in [0,1]$, the total cost of the irrigation plan χ is*

$$\mathcal{E}^\alpha(\chi) \;\doteq\; \int_\Theta \left(\int_{I\!\!R_+} \big|\chi(\theta,t)\big|_\chi^{\alpha-1} \cdot |\dot\chi(\theta,t)|\,dt\right) d\theta. \qquad (2.13) \quad \boxed{\texttt{TCg}}$$

*The α-**irrigation cost** of a measure μ is defined as*

$$\mathcal{I}^\alpha(\mu) \;\doteq\; \inf_\chi \mathcal{E}^\alpha(\chi), \qquad (2.14) \quad \boxed{\texttt{Idef}}$$

where the infimum is taken over all admissible irrigation plans.

We say that μ is α-irrigable if $\mathcal{I}^\alpha(\mu) < +\infty$.

A lower bound on the transportation cost is provided by

Lemma 2.6 *For any positive Radon measure μ on $I\!\!R^d$ and any $\alpha \in [0,1]$, one has*

$$\mathcal{I}^\alpha(\mu) \;\geq\; \int_0^{+\infty} \Big[\mu\big(\{x \in I\!\!R^d;\;\; |x| \geq r\}\big)\Big]^\alpha dr\,. \qquad (2.15) \quad \boxed{\texttt{Ilb}}$$

In particular, for every $r > 0$ one has

$$\mathcal{I}^\alpha(\mu) \;\geq\; r \cdot \Big[\mu\big(\{x \in I\!\!R^d;\;\; |x| \geq r\}\big)\Big]^\alpha. \qquad (2.16) \quad \boxed{\texttt{Irb}}$$

We recall that optimal irrigation plans satisfy

Single Path Property: *If $\chi(\theta,\tau) = \chi(\theta',\tau')$ for some $\theta, \theta' \in \Theta$ and $0 < \tau \leq \tau'$, then*

$$\chi(\theta,t) \;=\; \chi(\theta',t) \qquad \textit{for all } t \in [0,\tau]. \qquad (2.17) \quad \boxed{\texttt{SPP}}$$

Remark 2.7 In the case $\alpha = 1$, the expression ($\overset{\texttt{TCg}}{2.13}$) reduces to

$$\mathcal{E}^\alpha(\chi) \;\doteq\; \int_\Theta \left(\int_{I\!\!R_+} |\dot\chi_t(\theta,t)|\,dt\right) d\theta \;=\; \int_\Theta [\text{total length of the path } \chi(\theta,\cdot)]\,d\theta\,.$$

Of course, this length is minimal if every path $\chi(\cdot,\theta)$ is a straight line, joining the origin with $\chi(\theta, T(\theta))$. Hence

$$\mathcal{I}^\alpha(\mu) \;\doteq\; \inf_\chi \mathcal{E}^\alpha(\chi) \;=\; \int_\Theta |\chi(\theta, T(\theta))|\,d\theta \;=\; \int |x|\,d\mu\,.$$

On the other hand, when $\alpha < 1$, moving along a path which is traveled by few other particles comes at a high cost. Indeed, in this case the factor $\big|\chi(\theta,t)\big|_\chi^{\alpha-1}$ becomes large. To reduce the total cost, is thus convenient that particles travel along the same path as far as possible.

2.3 The landscape function

Let $\chi : \Theta \times I\!\!R_+ \mapsto I\!\!R^d$ be a (possibly not optimal) irrigation plan which satisfies the single path property. The **landscape function** $Z : I\!\!R^d \mapsto I\!\!R_+$ is defined as follows [3, 28].

(i) As a first step, consider the set

$$\Gamma \doteq \{x \in I\!\!R^d\,;\ |x|_\chi > 0\}. \tag{2.18}$$

Gamdef

We think of Γ as the union of all irrigation paths. For $x \in \Gamma$, choose any particle $\theta \in \Theta$ that reaches x, so that $\chi(\theta, \tau) = x$ at some time $\tau \geq 0$. We then define

$$Z_\chi(x) \doteq \int_0^\tau \big|\chi(\theta, t)\big|_\chi^{\alpha-1}\, dt. \tag{2.19}$$

Zdef

By the single path property (2.17), the above integral is independent of the choice of θ.

(ii) The landscape function Z is now defined as the lower semicontinuous envelope of Z_χ. Namely,

$$Z(x) \doteq \liminf_{y \in \Gamma, y \to x} Z_\chi(y), \tag{2.20}$$

Zd2

with the understanding that $Z(x) = +\infty$ if $x \notin \overline{\Gamma}$.

The following results were originally proved for a probability measure. However, by a rescaling, it is clear that they remain valid for any bounded positive measure.

Lemma 2.8 *Let μ be a α-irrigable measure on $I\!\!R^d$, and let χ be an optimal irrigation plan for μ. Calling Z the landscape function of χ, one has*

$$\mathcal{I}^\alpha(\mu) = \int_{I\!\!R^d} Z(x)\, d\mu\,. \tag{2.21}$$

4x0

Proof. See Corollary 4.5 in [28].

Theorem 2.9 *Let μ be an α-irrigable measure on $I\!\!R^d$ and let g be a measurable function such that $\|g\|_{L^\infty(\mu)} \leq 1$. Consider the measure $\nu \doteq (1+g)\mu$. Then*

$$\mathcal{I}^\alpha(\nu) \leq \mathcal{I}^\alpha(\mu) + \alpha \int_{I\!\!R^d} Z(x) g(x)\, d\mu\,. \tag{2.22}$$

4x1

Proof. See Theorem 4.7 in [28]. Notice that the condition $\|g\|_{L^\infty(\mu)} \leq 1$ guarantees that ν is a positive measure.

1:LH **Lemma 2.10** *Let χ be an optimal α-irrigation plan for the measure μ, and let $Z(\cdot)$ be the corresponding landscape function. Then, for any two point $x \in \Gamma, y \in \Gamma$, one has*

$$Z(x) - Z(y) \leq \frac{1}{\alpha} |x|_\chi^{\alpha-1} |x - y|\,. \tag{2.23}$$

0xa0

Proof. See Corollary 3.10 in [7].

We conclude this section by observing that, for any branch in Γ, the arc-length can be bounded below in terms of the Euclidean distance.

l:112 **Lemma 2.11** *Let χ be an optimal irrigation plan for the measure μ. Consider any path $t \mapsto \gamma(t) = \chi(\theta, t)$ for some particle $\theta \in \Theta$. Assume that its multiplicity is bounded from below:*

$$|\gamma(s)|_\chi \;\geq\; \delta_0 \;>\; 0 \qquad\qquad \text{for all } s \in [0, \bar{s}]. \tag{2.24}$$ mbb

Then there exists a constant C such that

$$|t-s| \;\leq\; C\,|\gamma(t)-\gamma(s)| \qquad\qquad \text{for all } s,t \in [0, \bar{s}]. \tag{2.25}$$ distlb

Proof. To fix ideas, assume that $s < t$, $x = \gamma(s)$, $y = \gamma(t)$. By optimality, the multiplicity function $\tau \mapsto |\gamma(\tau)|_\chi$ is non-increasing along γ. Therefore

$$Z(y) - Z(x) \;=\; \int_s^t |\gamma(\tau)|_\chi^{\alpha-1}\,d\tau \;\geq\; |\gamma(s)|_\chi^{\alpha-1}(t-s)\,. \tag{2.26}$$ 0xd5

On the other hand, according to Lemma 2.10 we have

$$Z(y) - Z(x) \;\leq\; \frac{1}{\alpha}|\gamma(t)|_\chi^{\alpha-1}|x-y|\,. \tag{2.27}$$ 0xd6

Calling M the total mass of the irrigated measure, combining (2.26) with (2.27) one obtains

$$t-s \;\leq\; \frac{1}{\alpha}\left(\frac{|\gamma(t)|_\chi}{|\gamma(s)|_\chi}\right)^{\alpha-1}|y-x| \;\leq\; \frac{1}{\alpha}\left(\frac{\delta_0}{M}\right)^{\alpha-1}\big|\gamma(t)-\gamma(s)\big|. \tag{2.28}$$ 0xd7

□

3 Necessary conditions for optimal tree roots

s:3

Following [12, 14], the optimization problem for tree roots can be stated as

(OPR) *Maximize the functional*

$$\mathcal{H}(u,\mu) - c\mathcal{I}^\alpha(\mu), \tag{3.1}$$ mH

among all couples (u,μ), where μ is a positive measure on $\overline{\Omega}$, and u is a solution to (2.2)-(2.3).

Existence of solutions was proved in [12].

t:20 **Theorem 3.1** *Let the assumptions* **(A1)–(A3)** *hold. Then the problem* **(OPR)** *has at least one optimal solution (u^*,μ), satisfying*

$$\begin{cases} \Delta u^* + f(u^*) - u^*\mu \;=\; 0 & x \in \Omega\,, \\ \mathbf{n}\cdot\nabla u^* \;=\; 0 & x \in \partial\Omega\,. \end{cases} \tag{3.2}$$ harveq

The measure μ on $\overline{\Omega}$ has bounded total mass.

Indeed, the result in [12] established the existence of an optimal pair (u^*, μ), in the more general case where the set $\Omega \subseteq I\!\!R^d$ may be unbounded, possibly also with $0 \in \overline{\Omega}$. The analysis in [12] also shows that the irrigation cost for the optimal measure is bounded: $\mathcal{I}^\alpha(\mu) < +\infty$. By itself, this does not guarantee that the total mass of the measure μ is bounded, because μ may concentrate an infinite amount of mass near the origin, where the transportation cost is almost zero. In the present setting however, thanks to the additional assumption $0 \notin \overline{\Omega}$ in **(A1)**, by (2.16) we conclude that the total mass of μ is bounded by

$$\mu(\overline{\Omega}) \;\leq\; \left(\frac{\mathcal{I}^\alpha(\mu)}{r_0}\right)^{1/\alpha}, \qquad\qquad r_0 \doteq d(0, \overline{\Omega}) \;=\; \min\bigl\{|x|\,;\; x \in \overline{\Omega}\bigr\}.$$

The main goal of this section is to derive necessary conditions for optimality.

Theorem 3.2 *Let the assumptions* **(A1)–(A3)** *hold. Let (u^*, μ) be an optimal solution to the problem* **(OPR)**, *satisfying (3.2). Let χ be an optimal irrigation plan for the measure μ, and let Z be the corresponding landscape function.*

Then there exists a bounded solution $\psi \geq 0$ to the adjoint equation

$$\left\{\begin{array}{rcll} \Delta\psi + f'(u^*)\psi - \psi\mu &=& -\,\mu & \qquad x \in \Omega\,, \\ \mathbf{n}\cdot\nabla\psi &=& 0 & \qquad x \in \partial\Omega\,, \end{array}\right. \tag{3.3}$$

such that, μ-almost everywhere, one has

$$(1-\psi)u^* \;=\; c\,\alpha Z. \tag{3.4}$$

Proof. We follow the same steps as in the proof of Theorem 2.1 in [13], with minor modifications.

1. We begin by proving that the solution u^* of (3.2) is uniformly positive on $\overline{\Omega}$. Indeed, since $0 \notin \overline{\Omega}$, recalling (2.26) we can choose a constant $0 < \delta_0 < u_{max}$ such that the landscape function satisfies

$$c\alpha\, Z(x) \;\geq\; \delta_0 \;>\; 0 \qquad\qquad \text{for all } x \in \overline{\Omega}. \tag{3.5}$$

We now claim that the optimal measure μ vanishes on the set where $u^* < c\alpha Z$, namely

$$\mu\Bigl(\bigl\{x \in \overline{\Omega}\,;\;\; u^*(x) \;<\; c\alpha Z(x)\bigr\}\Bigr) \;=\; 0. \tag{3.6}$$

Indeed, if (3.6) fails, we can consider the reduced measure $\mu_0 \doteq g\,\mu$, where

$$g(x) \;=\; \left\{\begin{array}{ll} 1 & \quad\text{if } \; u^*(x) \geq c\alpha Z(x), \\ 0 & \quad\text{if } \; u^*(x) < c\alpha Z(x). \end{array}\right.$$

Let u_0 be the solution to

$$\left\{\begin{array}{rcll} \Delta u + f(u) - u\mu_0 &=& 0 & \qquad x \in \Omega\,, \\ \mathbf{n}\cdot\nabla u &=& 0 & \qquad x \in \partial\Omega\,. \end{array}\right. \tag{3.7}$$

In view of the assumptions (2.6) on the source function f, the existence of such a solution follows from the analysis in [11]. Moreover, a comparison argument yields

$$0 \;\leq\; u_0(x) \;\leq\; u_{max} \qquad\qquad \text{for all } x \in \Omega.$$

Since u^* provides a subsolution to (3.7), we have $u^* \leq u_0$. Hence, by (2.22),

$$\begin{aligned}\mathcal{H}(u_0,\mu_0) - \mathcal{H}(u^*,\mu) - c\Big[\mathcal{I}^\alpha(\mu_0) - \mathcal{I}^\alpha(\mu)\Big] \;&=\; \int_\Omega u_0\, d\mu_0 - \int_\Omega u^*\, d\mu - c\Big[\mathcal{I}^\alpha(\mu_0) - \mathcal{I}^\alpha(\mu)\Big]\\ &\geq\; \int_\Omega (g-1)u^*\, d\mu - c\alpha\int_\Omega (g-1)Z\, d\mu \;>\; 0,\end{aligned}$$

against the optimality of (u^*,μ). Therefore, $\mu_0 = \mu$ and (3.6) holds.

Next, in view of (3.6), the function

$$\widetilde{u}(x) \;\doteq\; \max\big\{\delta_0\,,\; u^*(x)\big\}$$

is a subsolution of (3.7). Indeed, on the set where $\widetilde{u}(x) = \delta_0$ we have

$$\Delta\widetilde{u} + f(\widetilde{u}) - \widetilde{u}\,\mu \;=\; f(\widetilde{u}) \;=\; f(\delta_0) \;>\; 0.$$

On the other hand, by (2.6) the constant function $u(x) = u_{max}$ is trivially a supersolution. We thus conclude that

$$u_{max} \;\geq\; u^*(x) \;\geq\; \widetilde{u}(x) \;\geq\; \delta_0 \;>\; 0 \qquad\qquad \text{for all } x \in \overline{\Omega}\,. \tag{3.8}$$

2. Consider a family of perturbed measures, of the form

$$\mu_\varepsilon \;=\; \mu + \varepsilon\nu, \tag{3.9}$$

where

$$\nu \;=\; g\,\mu, \qquad \text{with} \quad \|g\|_{\mathbf{L}^\infty} \leq 1. \tag{3.10}$$

Let u_ε be the corresponding solution of

$$\Delta u_\varepsilon + f(u_\varepsilon) - u_\varepsilon\,\mu_\varepsilon \;=\; 0 \qquad\qquad \text{on } \;\overline{\Omega}, \tag{3.11}$$

with Neumann boundary conditions.

When the measure μ is replaced by μ_ε, by (2.22) the irrigation cost satisfies

$$\mathcal{I}^\alpha(\mu_\varepsilon) \;\leq\; \mathcal{I}^\alpha(\mu) + \alpha\varepsilon\int_\Omega Z(x)g(x)\, d\mu(x). \tag{3.12}$$

In the next steps we shall derive a formula computing the corresponding change in the harvest functional $\mathcal{H}(u_\varepsilon,\mu_\varepsilon)$.

3. Following [13, 15], consider the space $X_\mu \doteq \mathbf{H}^1(\Omega)\cap\mathbf{L}^2(\mu)$, and its dual space X'_μ. As shown in [13], assuming that μ is not the zero measure, the space X_μ is a Hilbert space with the equivalent inner product

$$\langle u,v\rangle_{X_\mu} \doteq \int_\Omega Du\cdot Dv\; dx + \int_\Omega uv\; d\mu\,. \tag{3.13}$$

Given $F \in X'_\mu$, consider the problem of finding $u \in X_\mu$ which satisfies

$$\Delta u \;-\; u\mu \;=\; F, \tag{3.14}$$

with Neumann boundary conditions (2.3). We define the resolvent operator $R_\mu : X'_\mu(\Omega) \to X_\mu(\Omega)$ by setting $R_\mu(F) = u$, where u is the unique solution of (3.14). By Riesz' theorem, R_μ is a bounded linear operator from $X'_\mu(\Omega)$ onto $X_\mu(\Omega)$, and thus continuously differentiable.

4. Now let μ_ε, u_ε be as in (3.9), (3.11). Notice that (3.11) is equivalent to

$$\Delta u_\varepsilon \;-\; u_\varepsilon \mu \;=\; -f(u_\varepsilon) + \varepsilon u_\varepsilon \nu. \tag{3.15}$$

Using the resolvent operator, (3.15) can be written as

$$u_\varepsilon \;=\; R_\mu\big(-f(u_\varepsilon) + \varepsilon u_\varepsilon \nu\big)\,. \tag{3.16}$$

To prove that the map $\varepsilon \mapsto u_\varepsilon$ is differentiable, consider the function $\Psi : I\!R \times X_\mu \to X_\mu$ defined as

$$\Psi(\varepsilon, w) \;\doteq\; w - R_\mu\Big(-f(w) + \varepsilon w\,\nu\Big). \tag{3.17}$$

Being the composition of the linear operator R_μ and a smooth map, it is clear that Ψ is continuously differentiable. When $\varepsilon = 0$ we already know that

$$\Psi(0, u^*) \;=\; 0 \;\in\; X_\mu\,. \tag{3.18}$$

We claim that, for ε in a neighborhood of zero, the equation

$$\Psi(\varepsilon, w) \;=\; 0 \tag{3.19}$$

implicitly defines a function $w(\varepsilon) = u_\varepsilon$, providing the solution to (3.15).

As shown in step **6** of the proof of Theorem 2.1 in [13], the linear operator

$$w \;\mapsto\; w + R_\mu\big(f'(u^*)w\big) \tag{3.20}$$

has a bounded inverse on X_μ. By the implicit function theorem, it follows that the map $\varepsilon \mapsto u_\varepsilon$ is well defined, and differentiable in a neighborhood of the origin.

Having established the differentiability of the map $\varepsilon \mapsto u_\varepsilon$, its derivative at $\varepsilon = 0$ can be computed by differentiating (3.16). This yields

$$v \;\doteq\; \frac{du_\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} \;=\; R_\mu\big(-f'(u^*)v + u^*\nu\big).$$

Therefore v satisfies the linear, non-homogeneous equation

$$\begin{cases} \Delta v + f'(u^*)v - v\mu \;=\; u^*\nu & x \in \Omega\,, \\ \mathbf{n}\cdot\nabla v \;=\; 0 & x \in \partial\Omega\,. \end{cases} \tag{3.21}$$

Notice that (3.21) could be formally obtained by inserting the expansion

$$u_\varepsilon \;=\; u^* + \varepsilon v + o(\varepsilon)$$

in (3.15), and retaining terms of order $\mathcal{O}(\varepsilon)$.

5. In this step we show that the adjoint problem (3.3) has a uniformly bounded solution $\psi \in X_\mu$. Toward this goal, we first choose $\lambda > 0$ large enough so that

$$f'(u)(\lambda u + 1) \ < \ \lambda f(u) \qquad \text{for all } u \in [\delta_0, u_{max}]. \tag{3.22}$$

Notice that such a constant exist, thanks to (3.8) and the assumptions (2.6).

We now claim that, the function

$$\psi^+ \ = \ \lambda u^* + 1 \tag{3.23}$$

is a supersolution to (3.3). Indeed, inserting (3.23) in (3.3) and using (3.2), we obtain

$$\begin{aligned} \Delta\psi^+ + f'(u^*)\psi^+ + (1-\psi^+)\mu \ &= \ \lambda\Delta u^* + f'(u^*)(\lambda u^* + 1) - \lambda u^* \mu \\ &= \ f'(u^*)(\lambda u^* + 1) - \lambda f(u^*) \ \leq \ 0. \end{aligned}$$

This holds for every $x \in \overline{\Omega}$, because of (3.8) and (3.22). We thus conclude that ψ satisfies the uniform bounds

$$0 \ \leq \ \psi(x) \ \leq \ \lambda\, u^*(x) + 1 \ \leq \ \lambda\, u_{max} + 1. \tag{3.24}$$

6. Next, let ψ be the solution to the adjoint problem (3.3). Using v as test function and integrating by parts one obtains

$$\int v\, d\mu \ = \ \int \nabla\psi \cdot \nabla v\, dx - \int f'(u^*)\psi v\, dx + \int \psi v\, d\mu \ = \ - \int \psi u^* d\nu. \tag{3.25}$$

Notice that the last identlity follows from the fact that v is a weak solution to (3.21), using ψ as test function.

Differentiating the harvest functional w.r.t. ε and using (3.25) one obtains

$$\frac{d}{d\varepsilon}\mathcal{H}(u_\varepsilon, \mu_\varepsilon)\Big|_{\varepsilon=0} \ = \ \int_\Omega u^*\, d\nu + \int_\Omega v\, d\mu \ = \ \int_\Omega (1-\psi)u^*\, d\nu. \tag{3.26}$$

7. Since (u^*, μ) yield an optimal solution, in view of (3.26), (3.10), and (3.12), we obtain

$$\begin{aligned} 0 \ &\geq \ \limsup_{\varepsilon\to 0+} \left[\frac{\mathcal{H}(u_\varepsilon, \mu_\varepsilon) - \mathcal{H}(u^*, \mu)}{\varepsilon} - c\, \frac{\mathcal{I}^\alpha(\mu_\varepsilon) - \mathcal{I}^\alpha(\mu)}{\varepsilon} \right] \\ &\geq \ \int_\Omega \Big((1-\psi)u^* - c\alpha\, Z \Big) g\, d\mu\,. \end{aligned} \tag{3.27}$$

Since the function $g \in \mathbf{L}^\infty$ can be chosen arbitrarily, we conclude that the identity (3.4) must hold almost everywhere w.r.t. the measure μ. $\square$

Outside the support of μ, the identity (3.4) may fail. Yet, we expect that it can be replaced by an inequality. A result in this direction, valid in dimension $d = 2$, is now proved.

Theorem 3.3 *Assume $d = 2$. In the same setting as Theorem 3.2, let (u^*, μ) be an optimal solution to the problem* **(OPR)**, *and let χ be an optimal irrigation plan for the measure μ.*

Consider any particle path $s\mapsto\gamma(s)\doteq\chi(\bar\theta,s)$, $s\in[0,\bar s]$, where the multiplicity remains strictly positive. Then,

$$(1-\psi)u^* ~\leq~ c\,\alpha Z. \tag{3.28}$$

at almost every point $x=\gamma(s)$, $s\in[0,\bar s]$, such that $\gamma(s)\in\Omega$.

Proof. 1. Assume that the conclusion does not hold. Then the set

$$S~\doteq~\Big\{s\in[0,\bar s]\,;~~\gamma(s)\in\Omega,~~\big(1-\psi(\gamma(s))u^*\big(\gamma(s)\big)~>~c\,\alpha Z\big(\gamma(s)\big)\Big\},$$

where the inequality (3.28) fails, has positive Lebesgue measure.

Let ν be the measure supported along the 1-dimensional curve γ, obtained as the push-forward of Lebesgue measure on S, via the map $s\mapsto\gamma(s)$.

For $\varepsilon>0$, consider the measures

$$\mu_\varepsilon~\doteq~\mu+\varepsilon\nu,$$

and let u_ε be the corresponding solutions to (2.2)-(2.3). Since the dimension is $d=2$, in view of Lemma 2.11 it follows that the measure ν is absolutely continuous w.r.t. capacity. By a similar argument as in the proof of the previous theorem, the derivative of the harvest functional is computed by

$$\frac{d}{d\varepsilon}\mathcal{H}(u_\varepsilon,\mu_\varepsilon)\bigg|_{\varepsilon=0}~=~\int_\Omega(1-\psi)u^*\,d\nu\,. \tag{3.29}$$

2. It remains to estimate the change in the irrigation cost. The measure μ_ε has total mass

$$\mu_\varepsilon(\overline\Omega)~=~\mu(\overline\Omega)+\varepsilon\nu(\Omega)~=~M+\varepsilon\,\mathrm{meas}(S),$$

where meas$(\cdot)$ always denotes Lebesgue measure. Since the measure ν is supported along the curve γ, it is natural to consider an irrigation plan

$$\chi_\varepsilon:[0,M+\varepsilon\,\mathrm{meas}(S)]\times I\!\!R_+~\mapsto~I\!\!R^2$$

which coincides with χ for $\theta\in[0,M]$, while all the additional particles $\theta\in\big]M,M+\varepsilon\,\mathrm{meas}(S)\big]$ are transported to destination along the same path γ. The change in multiplicity at points $x=\gamma(s)$ is thus

$$\big|\gamma(s)\big|_{\chi_\varepsilon}~=~\big|\gamma(s)\big|_\chi+\varepsilon\,\mathrm{meas}\big(S\cap[s,\bar s]\big).$$

In turn, the increase in the irrigation cost is computed as

$$\begin{aligned}\mathcal{E}^\alpha(\chi_\varepsilon)-\mathcal{E}^\alpha(\chi)~&=~\int_0^{\bar s}\Big(\big|\gamma(s)\big|_{\chi_\varepsilon}^\alpha-\big|\gamma(s)\big|_\chi^\alpha\Big)\,ds\\&=~\varepsilon\,\alpha\int_0^{\bar s}\big|\gamma(s)\big|_\chi^{\alpha-1}\mathrm{meas}(S\cap[s,\bar s]\big)ds+o(\varepsilon)\\&=~\varepsilon\,\alpha\int_S Z(\gamma(s))\,ds+o(\varepsilon).\end{aligned} \tag{3.30}$$

Here the last identity follows from

$$\begin{aligned}\int_0^{\bar s} \big|\gamma(s)\big|_\chi^{\alpha-1} \mathrm{meas}(S\cap[s,\bar s])\,ds &= \int_0^{\bar s}\Big(\frac{d}{ds}\int_0^s \big|\gamma(t)\big|_\chi^{\alpha-1}\,dt\Big)\Big(\int_s^{\bar s}\mathbf{1}_S(t)\,dt\Big)\,ds \\ &= \Big[Z(s)\int_s^{\bar s}\mathbf{1}_S(t)\,dt\Big]_0^{\bar s} - \int_0^{\bar s} Z(s)(-\mathbf{1}_S(s))\,ds,\end{aligned}$$

where we have used the fact that $Z(0)=0$.

Since χ is an optimal irrigation plan for the measure μ, we conclude

$$\liminf_{\varepsilon\to 0+}\frac{\mathcal{I}^\alpha(\mu_\varepsilon)-\mathcal{I}^\alpha(\mu)}{\varepsilon} \leq \lim_{\varepsilon\to 0+}\frac{\mathcal{E}^\alpha(\chi_\varepsilon)-\mathcal{E}^\alpha(\chi)}{\varepsilon} = \int \alpha Z\,d\nu. \tag{3.31}$$ dIe

Combining (3.29) with (3.31) we obtain a contradiction to the optimality of the solution (u^*,μ). □

Remark 3.4 The above argument would fail in higher space dimensions because, when $d\geq 3$, a measure ν supported on a 1-dimensional arc is not absolutely continuous w.r.t. capacity.

4 The support of the optimal measure

s:4

Let (u^*,μ) be an optimal solution to the problem **(OPR)**, describing optimal shapes for tree roots. According to the Theorem 3.2, the support of the optimal measure μ is contained in the set where the two functions $(1-\psi)u^*$ and $c\alpha Z$ coincide. In the remainder of this paper, by showing that these functions have very different regularity properties, we will prove that the coincidence set is indeed very small, at least in the case of dimension $d=2$. We recall that the support of a positive measure μ is defined as

$$\mathrm{Supp}(\mu) \doteq \Big\{x\in I\!\!R^d\,;\ \mu\big(B(x,r)\big)>0 \quad \text{for all } r>0\Big\}.$$

t:41 **Theorem 4.1** *Let the assumptions* **(A1)–(A3)** *hold, and assume $d=2$, $0<\alpha<1$. Let (u^*,μ) be an optimal solution to* **(OPR)**. *Then the support of the measure μ is nowhere dense.*

Toward a proof, we begin with a few remarks.

(i) The two functions u^*,ψ in (3.2)-(3.3) are non-negative and bounded above. In particular, there exists constants $K,K'>0$ such that

$$0\leq f(u^*(x))\leq K, \qquad \big|f'(u^*(x))\psi(x)\big|\leq K' \qquad \text{for all } x\in\overline{\Omega}. \tag{4.1}$$ Kdef

(ii) By (3.2) we have

$$\Delta u^* = u^*\mu - f(u^*) \geq -K. \tag{4.2}$$ usub

Moreover, by (3.4) and (3.8) it follows

$$\mu\Big(\big\{x\in\overline{\Omega}\,;\ \ 1-\psi(x)<0\big\}\Big) = 0. \tag{4.3}$$ muoo

Hence the measure $(1-\psi)\mu$ is non-negative. By (3.3) we thus have

$$\Delta\psi \;=\; -(1-\psi)\mu - f'(u^*)\psi(x) \;\leq\; K'. \tag{4.4}$$

As a consequence of (4.2)-(4.4), the function $u^*+K|x|^2$ is sub-harmonic, while $\psi-K'|x|^2$ is super-harmonic. In particular (see [2] for details), u^* is upper semicontinuous and ψ is lower semicontinuous. Both u^* and ψ are Borel measurable. Their values are well defined at every point $x\in\overline{\Omega}$.

(iii) In addition, since the measure μ is absolutely continuous w.r.t. capacity, we have the regularity estimates

$$u^*\;\in\; H^1(\Omega),\qquad \psi\;\in\; H^1(\Omega),\qquad \Phi\;\doteq\;(1-\psi)\,u^*\;\in\; H^1(\Omega)\cap\mathbf{L}^\infty(\Omega)\,. \tag{4.5}$$

Indeed, from (3.8), (3.24), and the fact that $\psi\geq 0$, we obtain

$$-(\lambda\, u_{max})\cdot u_{max}\;\leq\;\Phi(x)\;\leq\; u^*(x)\;\leq\; u_{max} \tag{4.6}$$

for all $x\in\Omega$.

Combining the previous estimates, we further study the regularity of the product function Φ in (4.5). By (3.2)-(3.3) it follows

$$\begin{aligned}
\Delta\big((1-\psi)u^*\big) \;&=\; -\,u^*\,\Delta\psi+(1-\psi)\,\Delta u^* - 2\nabla\psi\cdot\nabla u^*\\
&=\;\big[f'(u^*)\psi+(1-\psi)\mu\big]u^* - \big[f(u^*)-u^*\mu\big](1-\psi)-2\,\nabla\psi\cdot\nabla u^* \qquad (4.7)\\
&=\; f'(u^*)u^*\psi - f(u^*)(1-\psi)+2(1-\psi)u^*\,\mu - 2\,\nabla\psi\cdot\nabla u^*.
\end{aligned}$$

Therefore, the function $\Phi\doteq(1-\psi)u^*$ provides a bounded solution to the linear elliptic equation with measure-valued coefficients:

$$\Delta\Phi \;=\; 2\Phi\,\mu+\phi, \tag{4.8}$$

where

$$\phi\;\doteq\;\big[f'(u^*)u^*+f(u^*)\big]\psi - f(u^*) - 2\,\nabla\psi\cdot\nabla u^*. \tag{4.9}$$

Notice that the product $\Phi\,\mu=(1-\psi)u^*\mu$ is always a positive measure, because of (4.3). However, Φ can attain both positive and negative values. We also observe that $\phi\in\mathbf{L}^1(\Omega)$, because $\psi,u^*\in H^1(\Omega)$. In addition, at boundary points $x\in\partial\Omega$, the Neumann boundary condition holds:

$$\nabla\Phi\cdot\mathbf{n}\;=\;\nabla\big((1-\psi)u^*\big)\cdot\mathbf{n}\;=\;u^*\left(\nabla(1-\psi)\cdot\mathbf{n}\right)+(1-\psi)\left(\nabla u^*\cdot\mathbf{n}\right)\;=\;0. \tag{4.10}$$

The proof of Theorem 4.1 will rely on two complementary lemmas.

Given an optimal pair (u^*,μ), let $\chi:\Theta\times I\!R_+\mapsto I\!R^2$ be an optimal irrigation plan for the measure μ. Moreover, consider any particle trajectory

$$s\mapsto\gamma(s)\doteq\chi(\bar\theta,s), \tag{4.11}$$

for some $\bar{\theta} \in \Theta$. By (2.10), γ is 1-Lipschitz and hence a.e. differentiable. We denote by $\mathbf{t}(s) = \dot{\gamma}(s)$ the tangent vector. We also assume that, on some initial interval, the multiplicity remains uniformly positive:

$$m(s) \;\doteq\; |\gamma(s)|\chi \;\geq\; \delta \;>\; 0 \qquad \text{for all } s \in [0, \overline{s}]. \tag{4.12}$$ mulg

The first lemma establishes a lower Hölder estimate on the landscape function Z.

l:42 **Lemma 4.2** *In the above setting, for a.e. $s_0 \in [0, \overline{s}]$ there exist constants $r_0, c_0 > 0$ such that the following holds. Calling $x_0 = \gamma(s_0)$, the landscape function Z satisfies*

$$Z(x) - Z(x_0) \;\geq\; c_0 |x - x_0|^\alpha \quad \textit{whenever} \quad |x - x_0| < r_0, \qquad \left| \left\langle \mathbf{t}(s_0)\,,\, \frac{x - x_0}{|x - x_0|} \right\rangle \right| \;\leq\; \frac{1}{3}\,. \tag{4.13}$$ hola

Next, we claim that a converse inequality holds for the function $\Phi = (1 - \psi)u^*$.

l:43 **Lemma 4.3** *In the same setting as Lemma 4.2, let $\beta \doteq (1 + \alpha)/2$, Then for a.e. $s_0 \in [0, \overline{s}]$ there exists a constant $c_1 > 0$ and an infinite sequence of points $x_k \to x_0 = \gamma(s_0)$ such that*

$$\Phi(x_k) - \Phi(x_0) \;\leq\; c_1 |x_k - x_0|^\beta, \qquad \left| \left\langle \mathbf{t}(s_0)\,,\, \frac{x_k - x_0}{|x_k - x_0|} \right\rangle \right| \;\leq\; \frac{1}{3}\,, \qquad \textit{for all } k \geq 1. \tag{4.14}$$ holb

A proof of Lemma 4.2 will be given in Section 5, while Lemma 4.3 will be proved in Section 6.

Relying on the two above lemmas, we can now give a proof of Theorem 4.1. Let $y \in I\!R^2$ be a point inside the support of the measure μ. Then, for any given $\varepsilon > 0$, there exists a particle $\bar{\theta} \in \Theta$ and a path (4.11) which satisfies (4.12) for some $\delta > 0$ and such that

$$\big|\gamma(\overline{s}) - y\big| \;<\; \varepsilon.$$

By Lemmas 4.2 and 4.3, for a.e. $s_0 \in [0, \overline{s}]$, at the point $x_0 = \gamma(s_0)$ both (4.13) and (4.14) are satisfied. In particular, we can choose s_0 close enough to $\overline{s}$ so that

$$|x_0 - y| \;\leq\; \big|\gamma(s_0) - \gamma(\overline{s})\big| + \big|\gamma(\overline{s}) - y\big| \;<\; \varepsilon.$$

By (4.13) and (4.14), since $\alpha < \beta < 1$, there exists a point x_k sufficiently close to x_0 such that

$$\Phi(x_k) - \Phi(x_0) \;\leq\; c_1 |x_k - x_0|^\beta \;<\; c\alpha \cdot c_0\, |x_k - x_0|^\alpha, \qquad\qquad |x_k - y| \;<\; \varepsilon.$$

We now observe that, restricted to the set

$$\Omega^+ \;\doteq\; \big\{x \in \overline{\Omega}\,;\;\; 1 - \psi(x) \;>\; 0\big\} \;=\; \big\{x \in \overline{\Omega}\,;\;\; \Phi(x) \;>\; 0\big\},$$

the function $\Phi = (1 - \psi)u^*$ is the product of two positive, upper semicontinuous functions. Therefore it is upper semicontinuous. We can thus find an open neighborhood V_k of x_k such that

$$\Phi(x) - \Phi(x_0) \;<\; c\alpha \cdot c_0\, |x - x_0|^\alpha \qquad \text{for all } x \in V_k\,. \tag{4.15}$$ ph5

By Theorem 3.3 it follows that $\Phi(x_0) \leq c\alpha Z(x_0)$. Together with (4.15) and (4.13), this yields

$$\Phi(x) \ < \ c\alpha\, Z(x_0) + c\alpha \cdot c_0\, |x - x_0|^\alpha \ \leq \ c\alpha Z(x) \qquad \text{for all } x \in V_k\,. \tag{4.16}$$

Hence by Theorem 3.2, the open set V_k does not intersect the support of μ.

Since $\varepsilon > 0$ was arbitrary, we conclude that every point $y \in \text{Supp}(\mu)$ lies in the closure of an open set V which does not intersect $\text{Supp}(\mu)$. This shows that the closed set $\text{Supp}(\mu)$ has empty interior, completing the proof. □

5 A lower Hölder estimate for the landscape function

Aim of this section is to give a proof of Lemma 4.2. Actually, the result remains valid more generally for any positive, bounded Radon measure μ on $I\!R^d$. For a given $0 < \alpha < 1$, let χ be an optimal irrigation plan for a measure μ, and let Z be the corresponding landscape function.

Let $s \mapsto \gamma(s)$ be a particle trajectory with uniformly positive multiplicity, as in (4.11)-(4.12). For a given constant $\kappa > 0$, consider the set

$$J_\kappa \ \doteq \ \Big\{ s_0 \in [0, \overline{s}]\,;\ \big|m(s) - m(s_0)\big| \ \leq \ \kappa\, |s - s_0| \quad \text{for all } s \in [0, \overline{s}] \Big\}. \tag{5.1}$$

Since the multiplicity m is bounded and nonincreasing, by Riesz' sunrise lemma (see for example [23], p.319) it follows

$$\text{meas}(J_\kappa) \ \geq \ \overline{s} - \frac{2m(0)}{\kappa}\,. \tag{5.2}$$

Therefore

$$\lim_{\kappa \to +\infty} \text{meas}(J_\kappa) \ = \ \overline{s}. \tag{5.3}$$

In the following, we say that a point $x_0 = \gamma(s_0)$ with $0 < s_0 < \overline{s}$ is a *good point*, and write $x \in \mathcal{G}$, provided that

(i) s_0 is a Lebesgue point of the map $s \mapsto \mathbf{t}(s)$,

(ii) $s_0 \in J_\kappa$ for some κ large enough.

By (5.3) and the fact that γ is 1-Lipschitz, it follows that the set of good points has full measure. Namely, $\gamma(s_0) \in \mathcal{G}$ for a.e. $s_0 \in [0, \overline{s}]$.

We claim that, at every good point x_0, the property (4.13) holds.

As shown in Fig. 2, consider a square Q_ε whose side has length $\varepsilon > 0$, centered at x_0, with two sides parallel to $\mathbf{t}(s_0)$. Let $Q_{2\varepsilon}$ be the concentric square with side of length 2ε, and let N be a constant large enough so that

$$\frac{N^{1-\alpha}}{2} - \frac{\sqrt{2}}{\alpha} \ \geq \ 1. \tag{5.4}$$

By choosing $\varepsilon > 0$ small enough, we can assume that γ is the only path of multiplicity $> \delta_0/N$ that intersects $Q_{2\varepsilon}$.

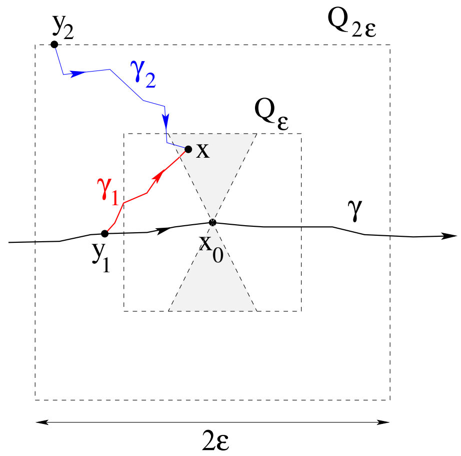

Figure 2: As shown in the proof of Lemma 4.2, in the shaded region inside Q_ε the landscape function grows at least at a Hölder rate. f:ir154

Let x be a point such that

$$x \in Q_\varepsilon\,, \qquad\qquad \left|\left\langle \mathbf{t}(s_0)\,,\, \frac{x-x_0}{|x-x_0|}\right\rangle\right| ~\leq~ \frac{1}{3}\,. \tag{5.5}$$ xqe

To establish a lower bound on $Z(x)$, two cases will be considered.

CASE 1: Assume that x is reached by a "short branch" γ_1, that bifurcates from γ at some point $y_1 = \gamma(s_1) \in Q_{2\varepsilon}$.

In this case, at every point along this short branch γ_1 the multiplicity m_1 is bounded by the downward jump in the multiplicity along γ. Since $s_0 \in \mathcal{J}_\kappa$, this implies

$$m_1 ~\leq~ m(s_1-) - m(s_1+) ~\leq~ \kappa \cdot |s_0 - s_1|.$$

Therefore

$$Z(x) ~\geq~ Z(y_1) + \Big(\kappa |s_0 - s_1|\Big)^{\alpha-1} |x - y_1|. \tag{5.6}$$ Zx1

Since along γ the multiplicity is bounded below by (4.12), we have

$$Z(y_1) ~\geq~ Z(x_0) - \delta_0^{\alpha-1} |s_0 - s_1|. \tag{5.7}$$ Zy1

The assumption that γ is differentiable at $x_0 = \gamma(s_0)$ implies that, by choosing $\varepsilon > 0$ small enough in view of (5.5) we can assume

$$|s_0 - s_1| ~\leq~ 2|x_0 - y_1|, \qquad\qquad \left|\left\langle \frac{x_0 - y_1}{|x_0 - y_1|}\,,\, \frac{x - x_0}{|x - x_0|}\right\rangle\right| ~\leq~ \frac{1}{2}\,. \tag{5.8}$$ ss01

In particular, the angle between these two vectors is larger than $\pi/3$. By elementary trigonometry, this implies

$$|x - y_1| ~\geq~ \frac{|x - x_0|}{2} + \frac{|x_0 - y_1|}{2}\,. \tag{5.9}$$ xxy

Calling

$$\sigma = |x - x_0|, \qquad\quad r = |x_0 - y_1|,$$

from (5.6)–(5.8) we now obtain

$$
\begin{aligned}
Z(x) - Z(x_0) &= \big[Z(x) - Z(y_1)\big] + \big[Z(y_1) - Z(x_0)\big] \\
&\geq \Big(2\kappa|x_0 - y_1|\Big)^{\alpha-1} |x - y_1| - 2\delta_0^{\alpha-1}|x_0 - y_1| \\
&\geq (2\kappa)^{\alpha-1} r^{\alpha-1} \frac{\sigma + r}{2} - 2\delta_0^{\alpha-1} r \\
&\geq 2^{\alpha-2}\kappa^{\alpha-1} r^{\alpha-1}\sigma + 2^{\alpha-3}\kappa^{\alpha-1} r^{\alpha}.
\end{aligned}
\tag{5.10}
$$

Notice that the last inequality follows from the fact that $r \leq \sqrt{2}\,\varepsilon$, with $\varepsilon > 0$ small. Since the minimum of the right hand side of (5.10) is achieved when

$$
r = \frac{2(1-\alpha)}{\alpha}\sigma,
$$

we conclude

$$
Z(x) - Z(x_0) \geq c_0\,\sigma^{\alpha}, \tag{5.11}
$$

for a suitable constant c_0.

CASE 2: Assume that x is reached by a "long branch" γ_2, that enters $Q_{2\varepsilon}$ at a point y_2 which does not lie on the curve γ. By (2.23) we have

$$
Z(x_0) - Z(y_2) \leq \frac{1}{\alpha}\delta_0^{\alpha-1}|x_0 - y_2|. \tag{5.12}
$$

By construction, this long branch has length $\geq \varepsilon/2$. Moreover, inside $Q_{2\varepsilon}$, all of its points have multiplicity $\leq \delta_0/N$. In this case, we would have

$$
Z(x) \geq Z(y_2) + \left(\frac{\delta_0}{N}\right)^{\alpha-1}\frac{\varepsilon}{2}. \tag{5.13}
$$

Together, (5.12) and (5.13) yield

$$
\begin{aligned}
Z(x) - Z(x_0) &\geq -\frac{1}{\alpha}\delta_0^{\alpha-1}\sqrt{2}\,\varepsilon + \left(\frac{\delta_0}{N}\right)^{\alpha-1}\frac{\varepsilon}{2} \\
&\geq \delta_0^{\alpha-1}\varepsilon\left[\frac{N^{1-\alpha}}{2} - \frac{\sqrt{2}}{\alpha}\right] \geq \delta_0^{\alpha-1}\varepsilon\,,
\end{aligned}
\tag{5.14}
$$

because of our choice of the constant N at (5.4). Notice that the right hand side of (5.14) remains uniformly positive for all $x \in Q_\varepsilon$.

Combining (5.11) with (5.14), by choosing $r_0 < \varepsilon/2$ small enough, we achieve (4.13). This achieves the proof of Lemma 4.2.

6 Proof of Lemma 4.3

The proof will be worked out in several steps.

1. Let $0 < \beta < 1$ be given. To prove the inequality in (4.14), we need to establish some upper bound on the function Φ, relying on the fact that it provides a solution to (4.8), with $\phi \in \mathbf{L}^1$. Since μ is a positive measure, this can be achieved by constructing a supersolution to

$$\Delta\Phi \;=\; -|\phi|, \tag{6.1}$$ EM2

with suitable boundary conditions. We recall that, by Theorem 3.3, along the curve γ the upper bound (3.28) holds. Moreover, at every point $x \in \overline{\Omega}$, by (3.14) we have

$$\Phi(x) \;\leq\; u^*(x) \;\leq\; u_{max}\,. \tag{6.2}$$ EM3

2. As a preliminary, for any integer $\kappa \geq 1$, consider the sets

$$S_\kappa \;=\; S'_\kappa \cap S''_\kappa\,, \tag{6.3}$$ Skap

where

$$S'_\kappa \;\doteq\; \Big\{ s_0 \in [0,\overline{s}]\,;\quad Z(\gamma(s)) - Z(\gamma(s_0)) \leq \kappa|s-s_0| \quad \text{for all } s \in [0,\overline{s}]\Big\},$$

$$S''_\kappa \;\doteq\; \left\{ s_0 \in [0,\overline{s}]\,;\quad \limsup_{r\to 0+} \frac{1}{r}\int_{B(\gamma(s_0),r)} \big|\phi(x)\big|\,dx \;\leq\; \kappa\right\}.$$

We claim that the union of the sets S_κ has full measure in $[0,\overline{s}]$. Indeed, since the map $s \mapsto Z(\gamma(s))$ is monotone increasing, by Riesz' sunrise lemma (see for example [23]) we have

$$\mathrm{meas}\Big([0,\overline{s}]\setminus S'_\kappa\Big) \;\leq\; 2\,\frac{Z\big(\gamma(\overline{s})\big) - Z\big(\gamma(0)\big)}{\kappa}\,.$$

This already shows that the union of the sets S'_κ has full measure in $[0,\overline{s}]$. Next, consider the set

$$\widehat{S} \;\doteq\; [0,\overline{s}]\setminus \bigcup_{\kappa\geq 1} S''_\kappa.$$

If $\mathrm{meas}(\widehat{S}) > 0$, a contradiction is obtained as follows. For a given $\kappa > 0$, and every $s \in \widehat{S}$, we can find a sequence of radii $r_i \downarrow 0$ such that

$$\int_{B(\gamma(s),r_i)} \big|\phi(x)\big|\,dx \;\geq\; \kappa r_i\,.$$

Let C be the constant in (2.25). As s varies in $\widehat{S}$, the corresponding intervals $[s - Cr_i, s + Cr_i]$ trivially cover $\widehat{S}$. By Vitali's covering theorem (see [22]), we can extract a countable family of disjoint intervals $I_j = [s_j - Cr_j, s_j + Cr_j]$, $j \in \mathcal{J}$, so that the collection of intervals $[s_j - 5Cr_j, s_j + 5Cr_j]$ covers $\widehat{S}$. In particular, this implies

$$\sum_j r_j \;\geq\; \frac{1}{10C}\mathrm{meas}(\widehat{S}).$$

By (2.25), the balls $B(\gamma(s_j), r_j)$ are mutually disjoint. Hence

$$\sum_{j\in\mathcal{J}} \int_{B(\gamma(s_j),r_j)} |\phi|\,dx \;\leq\; \|\phi\|_{\mathbf{L}^1}\,. \tag{6.4}$$ 70

On the other hand,

$$\sum_{j\in\mathcal{J}}\int_{B(\gamma(s_j),r_j)}|\phi|\,dx \;\geq\; \sum_j \kappa r_j \;\geq\; \kappa\cdot\frac{1}{10C}\mathrm{meas}(\widehat{S}). \tag{6.5}$$

Since κ can be arbitrarily large, if $\mathrm{meas}(\widehat{S})>0$, from (6.5) we obtain a contradiction with (6.4).

We now define a point $x=\gamma(s)$ to be a *good point* if the tangent vector $\mathbf{t}(s)=\dot\gamma(s)$ is well defined, and if $s\in S_\kappa$ for some $\kappa\geq 1$. In the remainder of the proof, we will show that (4.14) holds for every good point x_0.

3. Toward a future comparison, we study two elliptic problems on a domain $\mathcal{D}_\delta\subset I\!\!R^2$ which, in polar coordinates, has the form

$$\mathcal{D}_\delta \;\doteq\; \left\{(r,\theta)\,;\quad r\in[0,1],\quad \theta\in\left[-\frac{\pi}{2}-\delta\,,\ \frac{\pi}{2}+\delta\right]\right\}, \tag{6.6}$$

for some $\delta>0$ small. On this domain, let Φ_1 be the solution to

$$\left\{\begin{array}{rcll}\Delta\Phi_1(x) &=& 0 & \text{if}\quad x\in\mathcal{D}_\delta\,,\\[2mm] \Phi_1(x) &=& |x| & \text{if}\quad x\in\partial\mathcal{D}_\delta\,.\end{array}\right. \tag{6.7}$$

In addition, we consider the solution Φ_2 to the Poisson problem

$$\left\{\begin{array}{rcll}\Delta\Phi_2 &=& -\,\phi & \text{on}\ \ \mathcal{D}_\delta,\\[2mm] \Phi_2 &=& 0 & \text{on}\ \ \partial\mathcal{D}_\delta\,,\end{array}\right. \tag{6.8}$$

assuming that $\phi\geq 0$ and, for some $\kappa\geq 1$,

$$\int_{\mathcal{D}_\delta\cap B(0,r)}\phi\,dx \;\leq\; \kappa r \qquad\qquad \text{for all } r\in[0,1]. \tag{6.9}$$

Both problems (6.7)-(6.8) are more conveniently studied by constructing a conformal map, transforming the domain $\mathcal{D}_\delta$ into the half disc

$$\mathcal{D}' \;=\; \left\{(r,\theta)\,;\quad r\in[0,1],\quad \theta\in\left[-\frac{\pi}{2}\,,\ \frac{\pi}{2}\right]\right\}.$$

Choosing $p=1+\frac{2\delta}{\pi}$, the transformation $z\mapsto\Lambda(z)=z^p$ in the complex plane is then a conformal map from D' onto $\mathcal{D}_\delta$, as shown in Fig. 3. In polar coordinates, this takes the form

$$(\tilde r,\tilde\theta) \;=\; \Lambda(r,\theta) \;=\; (r^p, p\theta). \tag{6.10}$$

4. To construct a solution to (6.7), we now solve the problem on the half disc

$$\left\{\begin{array}{rcll}\Delta v &=& 0 & x\in\mathcal{D}',\\[2mm] v &=& r^p & x\in\partial\mathcal{D}'.\end{array}\right. \tag{6.11}$$

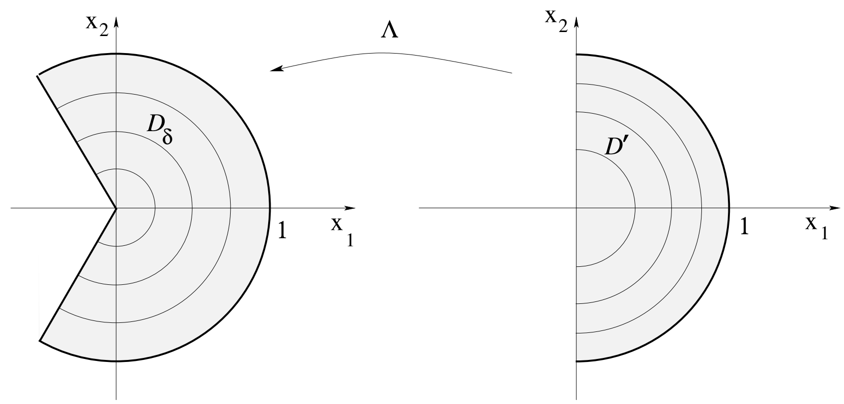

Figure 3: Taking $p = 1 + \frac{2\delta}{\pi}$, the conformal map $z \mapsto z^p$ transforms the half disc D' into the domain D_δ. f:ir148

The function

$$\Phi_1(z) \;\doteq\; v(z^{1/p}) \tag{6.12}$$ Udef

will then provide the desired solution to (Ph1 6.7).

In the following, we shall not need an explicit expression for Φ_1, but only an estimate on its asymptotic behavior near the origin. To construct v, we can use the Green's function on the half space, then add a correction (smooth in a neighborhood of the origin) to take into account the effect of the boundary at $|z| = 1$. Restricted to the subdomain

$$\mathcal{D}'' \;\doteq\; \left\{ x = (x_1, x_2)\,;\;\; |x| < \frac{1}{2}\,,\;\; x_1 > 0 \right\},$$

this leads to

$$v(x_1, x_2) \;=\; \frac{x_1}{\pi} \int_{-1}^{1} \frac{|y_2|^p}{x_1^2 + (y_2 - x_2)^2} dy_2 + e(x_1, x_2), \tag{6.13}$$ vrep

for some smooth correction term e, with $e(0,0) = 0$.

To estimate $v(x)$ we shall rely on the following identity, valid for $x_1 > 0$.

$$\frac{x_1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x_1^2 + (t - x_2)^2}\, dt \;=\; 1. \tag{6.14}$$ DPint

The integral in (vrep 6.13) can be bounded as

$$\begin{aligned} &\frac{x_1}{\pi} \int_{-1}^{1} \frac{|y_2|^p}{x_1^2 + (y_2 - x_2)^2}\, dy_2 \\ &\quad = \frac{x_1}{\pi} \int_{-2|x|}^{2|x|} \frac{|y_2|^p}{x_1^2 + (y_2 - x_2)^2}\, dy_2 + \frac{x_1}{\pi} \int_{\{2|x| < |y_2| \le 1\}} \frac{|y_2|^p}{x_1^2 + (y_2 - x_2)^2}\, dy_2 \\ &\quad \doteq\; I_1 + I_2\,. \end{aligned} \tag{6.15}$$ I12

By (DPint 6.14) we now have

$$I_1 \;\le\; 2^p |x|^p. \tag{6.16}$$ I1

To estimate the second integral, we observe that $|y_2| > 2|x|$ implies

$$|y_2| \;\le\; \big|(0, y_2) - (x_1, x_2)\big| + |x| < \sqrt{x_1^2 + (y_2 - x_2)^2} + \frac{1}{2}|y_2|.$$

Therefore $x_1^2 + (y_2 - x_2)^2 > \frac{1}{4}|y_2|^2$, and hence

$$\frac{x_1}{\pi}\int_{\{2|x|<|y_2|\le 1\}} \frac{|y_2|^p}{x_1^2+(y_2-x_2)^2}\,dy_2 \;<\; \frac{4x_1}{\pi}\int_{\{2|x|<|y_2|\le 1\}} |y_2|^{p-2}\,dy_2.$$

We thus conclude that, if $|x| < 1/2$, the second integral can be estimated as

$$I_2 \;\le\; \frac{4x_1}{\pi}\int_{\{2|x|<|y_2|\le 1\}} |y_2|^{p-2}\,dy_2 \;=\; \frac{8x_1}{\pi}\int_{2|x|}^{1} t^{p-2}\,dt \;=\; \frac{8x_1}{(p-1)\pi}\big(1-(2|x|)^{p-1}\big). \qquad (6.17) \qquad \boxed{\texttt{I2}}$$

Furthermore, since the correction term e is smooth, it can be bounded above by some linear function: $e(r,\theta) \le Cr$.

Here and throughout the following, for notational convenience we denote by $C > 0$ a positive constant, whose value can change at each step.

Combining (6.16)-(6.17), we obtain the estimate

$$\big|v(x)\big| \;\le\; C\,|x| \qquad\qquad \text{for all } x \in \mathcal{D}', \qquad (6.18) \qquad \boxed{\texttt{vlin}}$$

for a suitable constant C. In turn, by (6.12), this implies

$$\big|\Phi_1(x)\big| \;\le\; C|x|^{1/p}. \qquad (6.19) \qquad \boxed{\texttt{phi1}}$$

5. In addition to the upper bound (6.19), we observe that the solution Φ_1 of (6.7) satisfies the lower bound

$$\Phi_1(x) \;\ge\; |x| \qquad\qquad x \in \mathcal{D}_\delta\,. \qquad (6.20) \qquad \boxed{\texttt{P1up}}$$

Indeed, one immediately checks that the function $\varphi(x) = |x|$ is a subsolution to (6.7).

6. We now consider the solution to (6.8). Using polar coordinates, if $\Phi_2 = \Phi_2(\tilde r, \tilde\theta)$ is a solution to (6.8) on $\mathcal{D}_\delta$, then the function $u(r,\theta) \;=\; \Phi_2(r^p, p\theta)$ satisfies

$$\begin{aligned}
\Delta u(r,\theta) \;&=\; u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\
&=\; p^2 r^{2p-2}\Phi_{2,rr} + p^2 r^{p-2}\Phi_{2,r} + p^2 r^{-2}\Phi_{2,\theta\theta} \\
&=\; p^2 r^{2p-2}\Delta\Phi_2(r^p, p\theta).
\end{aligned}$$

We thus set

$$f(r,\theta) \;\doteq\; p^2 r^{2p-2}\phi(r^p, p\theta),$$

and consider the Poisson problem on the half disc

$$\begin{cases} \Delta u \;=\; -f & \text{on } \mathcal{D}', \\ \;\;u \;=\; 0 & \text{on } \partial\mathcal{D}'. \end{cases} \qquad (6.21) \qquad \boxed{\texttt{PPu}}$$

Notice that

$$\|f\|_{\mathbf{L}^1(\mathcal{D}')} \;=\; \int_{-\pi/2}^{\pi/2}\int_0^1 p^2 r^{2p-1}\phi(r^p, p\theta)\,dr\,d\theta \;=\; \int_{\mathcal{D}_\delta} p^2 r^{2p-1}\phi(\tilde r, \tilde\theta)\frac{d\tilde r}{p r^{p-1}}\,\frac{d\tilde\theta}{p} \;=\; \|\phi\|_{\mathbf{L}^1(\mathcal{D}_\delta)}\,.$$

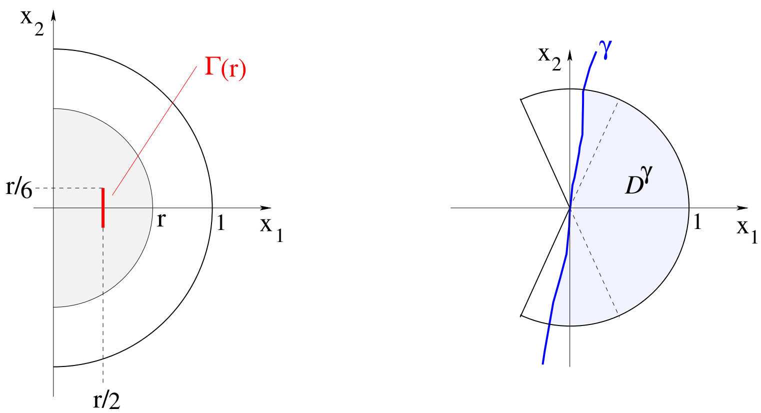

Figure 4: Left: the interval $\Gamma(r)$, where the average value for the solution u of (6.21) can be estimated. Right: the domain $\mathcal{D}^\gamma$ considered at (6.39). f:ir155

We observe that, since the function f is only in $\mathbf{L}^1$, pointwise bounds on u cannot be deduced from a Sobolev embedding theorem. However, we can establish a bound on the average value of u on an interval $\Gamma(r)$, as shown in Fig. 4, left.

Fix a radius $0 < r \leq 1$ and consider the interval

$$\Gamma(r) \doteq \Big\{(x_1,x_2)\,;\quad x_1 = \frac{r}{2},\quad |x_2| < \frac{r}{6}\Big\}, \tag{6.22}$$

shown in Fig. 4, left. Setting

$$y = (y_1,y_2), \qquad y' = (-y_1,y_2),$$

an upper bound for the solution u of (6.21) will be obtained by using the Green's formula for the half space $I\!R^2_+ \doteq \{(x_1,x_2)\,;\ x_1 \geq 0\}$. For a given radius $r > 0$, it will be convenient to split the function f as

$$f(x) = f^\flat + f^\sharp \;=\; f\cdot \mathbf{1}_{\{|x|\leq r\}} + f\cdot \mathbf{1}_{\{|x|>r\}}\,.$$

This leads to

$$\begin{aligned} u(x) \;&\leq\; u^+(x) \doteq \frac{1}{2\pi}\int_{I\!R^2_+} \big(\ln|x-y'| - \ln|x-y|\big) f(y)\,dy \\ &=\; \frac{1}{2\pi}\int_{I\!R^2_+} \big(\ln|x-y'| - \ln|x-y|\big)\big(f^\flat(y) + f^\sharp(y)\big)\,dy \;\doteq\; u^\flat(x) + u^\sharp(x)\,. \end{aligned} \tag{6.23}$$

In the next steps, we shall prove an integral bound on $u^\flat$ and a pointwise bound on $u^\sharp$.

7. Setting

$$z_2 \;=\; x_2 - y_2\,, \qquad F^\flat(y_1) \;=\; \int_{I\!R} f^\flat(y_1,y_2)\,dy_2\,,$$

$$\widetilde G(x_1,y_1) \;\doteq\; \frac{1}{4\pi}\int_{I\!R} \Big(\ln\big((x_1+y_1)^2 + z_2^2\big) - \ln\big((x_1-y_1)^2 + z_2^2\big)\Big)\,dz_2\,, \tag{6.24}$$

we now compute

$$
\begin{aligned}
U^\flat(x_1) &\doteq \frac{1}{2\pi}\int_{I\!R}\int_{I\!R^2_+}\big(\ln|x-y'|-\ln|x-y|\big)\,f^\flat(y)\,dy\,dx_2\\
&= \frac{1}{4\pi}\int_{I\!R}\int_{I\!R}\int_0^\infty\Big(\ln\big((x_1+y_1)^2+z_2^2\big)-\ln\big((x_1-y_1)^2+z_2^2\big)\Big)f^\flat(y_1,y_2)dy_1dy_2\,dz_2\\
&= \int_0^\infty \widetilde G(x_1,y_1)\,F^\flat(y_1)dy_1\,.
\end{aligned}
\tag{6.25}
$$

ax1

From the representation (6.24), since $x_1\geq 0$, we have

$$
\widetilde G(x_1,y_1)\;\geq\;0\,,\qquad \widetilde G(x_1,-y_1)\;=\;-\,\widetilde G(x_1,y_1)\,,\qquad\text{for all } y_1\;\geq\;0\,.
\tag{6.26}
$$

ax00

We now claim that, when $r>0$ is sufficiently small, one has

$$
U^\flat(x_1)\;\leq\;Cx_1|\ln x_1|\cdot\|f^\flat\|_{\mathbf{L}^1}.
\tag{6.27}
$$

ax2

Indeed, by (6.25) we can write

$$
\begin{aligned}
U^\flat(x_1) &= \int_{0\leq y_1\leq 2x_1}\widetilde G(x_1,y_1)F^\flat(y_1)\,dy_1+\int_{y_1>2x_1}\widetilde G(x_1,y_1)F^\flat(y_1)\,dy_1\\
&= \int_{0\leq y_1\leq 2x_1}\widetilde G(x_1,y_1)F^\flat(y_1)\,dy_1,
\end{aligned}
\tag{6.28}
$$

ax3

because $2x_1=r$ and $F^\flat(y_1)$ vanishes for $|y_1|>r$. By the definition (6.24) it follows

$$
\begin{aligned}
\widetilde G(x_1,y_1) &= \frac{1}{4\pi}\int_{I\!R}\Big(\ln(z_2^2+(x_1+y_1)^2)-\ln(z_2^2+(x_1-y_1)^2)\Big)\,dz_2\\
&= \frac{1}{2\pi}\int_0^{|y_1|}\Big(\ln(z_2^2+(x_1+|y_1|)^2)-\ln(z_2^2+(x_1-|y_1|)^2)\Big)\,dz_2\\
&\quad+\frac{1}{2\pi}\int_{|y_1|}^\infty\ln\left(1+\frac{4x_1|y_1|}{z_2^2+(x_1-|y_1|)^2}\right)\,dz_2\\
&\doteq A+B.
\end{aligned}
\tag{6.29}
$$

ax4

A direct computation yields

$$
|A|\;\leq\;\frac{1}{2\pi}\int_0^{|y_1|}|\ln(z_2^2+(x_1+|y_1|)^2)|\,dz_2+\frac{1}{2\pi}\int_0^{|y_1|}|\ln(z_2^2+(x_1-|y_1|)^2)|\,dz_2\,.
\tag{6.30}
$$

ax5

Choosing r small enough, since we have $|y_1|\leq 2x_1=r$, in (6.30) we can assume

$$
z_2^2+(x_1+|y_1|)^2\;\leq\;1\,,\qquad z_2^2+(x_1-|y_1|)^2\;\leq\;1.
$$

This yields the estimate

$$
|A|\;\leq\;\frac{1}{\pi}\int_0^{|y_1|}|\ln(z_2^2)|\,dz_2\;\leq\;\frac{2}{\pi}\big(|y_1|\ln(|y_1|)+|y_1|\big)\;\leq\;Cx_1|\ln(x_1)|\,.
\tag{6.31}
$$

ax6

To estimate B in (6.29), using the inequality $\ln(1+s)\leq s$ for $s\geq 0$, we obtain

$$
|B|\;\leq\;\frac{1}{2\pi}\int_{|y_1|}^\infty\frac{4x_1|y_1|}{z_2^2}\,dz_2\;=\;\frac{4x_1}{2\pi}\;\leq\;Cx_1|\ln(x_1)|\,.
\tag{6.32}
$$

ax7

Using (6.29) together with the bounds (6.31) and (6.32), from (6.28) we conclude that, for $r > 0$ sufficiently small,

$$U^\flat(x_1) \;\leq\; \int_{0\leq y_1\leq 2x_1} (|A|+|B|)F^\flat(y_1)\,dy_1 \;\leq\; Cx_1|\ln(x_1)|\int_{0\leq y_1\leq 2x_1} F^\flat(y_1)\,dy_1\,. \tag{6.33}$$

8. It now remains to estimate

$$u^\sharp(x) \;\doteq\; \frac{1}{2\pi}\int_{I\!\!R^2_+}\big(\ln|x-y'|-\ln|x-y|\big)f^\sharp(y)\,dy \tag{6.34}$$

for x on the segment $\Gamma(r)$ at (6.22). Setting

$$F^\sharp(s) \;\doteq\; \int_{r<|y|<s} f^\sharp(y)\,dy \;\leq\; \kappa s,$$

we compute

$$\begin{aligned}\int_{r<|y|<1}\frac{f^\sharp(y)}{|y|}\,dy \;&=\; \int_r^1 \frac{1}{s}\cdot\left(\frac{d}{ds}F^\sharp(s)\right)ds \;=\; \left[\frac{F^\sharp(s)}{s}\right]_r^1 + \int_r^1\frac{1}{s^2}\cdot F^\sharp(s)\,ds\\ &\leq\; \kappa+\int_r^1\frac{\kappa}{s}\,ds \;\leq\; \kappa\big(1+|\ln r|\big).\end{aligned} \tag{6.35}$$

Next, we observe that $x\in\Gamma(r)$ implies $|x|\leq\frac{\sqrt{10}}{6}r$. This leads to the bound

$$\ln|x-y'|-\ln|x-y| \;=\; \frac{1}{2}\ln\left(1+\frac{4x_1y_1}{|x-y|^2}\right)\leq\frac{2x_1y_1}{(|y|-|x|)^2} \;<\; \frac{2|x||y|}{(|y|/3)^2}.$$

Using this bound in (6.34), for $x\in\Gamma(r)$ we obtain

$$u^\sharp(x) \;\leq\; C\,|x|\int_{|y|>r}\frac{f^\sharp(y)}{|y|}\,dy \;\leq\; Cr\,\kappa\big(1+|\ln r|\big), \tag{6.36}$$

for some constant C.

9. Since $u^+ = u^\flat + u^\sharp$, combining the integral bound (6.27) with the pointwise bound (6.36) we obtain

$$\int_{\Gamma(r)} u^+(x)\,dx \;=\; \int_{-r/6}^{r/6} u^+\left(\frac{r}{2},\,x_2\right)dx_2 \;\leq\; Cr|\ln r|\cdot\|f^\flat\|_{\mathbf{L}^1}+\frac{r}{2}\cdot Cr\kappa\big(1+|\ln r|\big). \tag{6.37}$$

We now recall that, by the assumption (6.9),

$$\|f^\flat\|_{\mathbf{L}^1} \;=\; \int_{|x|<r} f(x)\,dx \;=\; \int_{|x|<r^p}\phi(x)\,dx \;\leq\; Cr^p,$$

for some constant C and $p>1$. Using this inequality in (6.37) and (6.23), we conclude that the average value of u over $\Gamma(r)$ satisfies the bound

$$⨍_{\Gamma(r)} u\,dx \;\leq\; ⨍_{\Gamma(r)} u^+\,dx \;\leq\; Cr\big(1+|\ln r|\big), \tag{6.38}$$

for a suitable constant C and all $r > 0$ sufficiently small.

10. Next, consider the more general problem

$$\begin{cases} \Delta\Phi(x) \;=\; -\,\phi(x), \qquad x \in \mathcal{D}^\gamma, \\[2mm] \Phi(x) \;=\; |x|, \qquad x \in \partial\mathcal{D}^\gamma\,. \end{cases} \tag{6.39}$$

As shown in Fig. 4, right, the domain $\mathcal{D}^\gamma$ is the portion of the unit disc to the right of a Lipschitz curve γ, with

$$\gamma \;\subset\; \widehat{\mathcal{D}}_\delta \;\doteq\; \left\{(r,\theta)\,;\; r\in[0,1],\quad \theta\in\left[-\frac{\pi}{2}-\delta\,,\; -\frac{\pi}{2}+\delta\right]\cup\left[\frac{\pi}{2}-\delta\,,\;\frac{\pi}{2}+\delta\right]\right\}. \tag{6.40}$$

As before, we assume that (6.9) holds, for some $\kappa \geq 1$. Moreover, since we are seeking an upper bound on the solution Φ of (6.39), w.l.o.g. we can assume that $\phi \geq 0$. To cover the general case it suffices to replace ϕ by its positive part $\phi^+(x) \doteq \max\{\phi(x), 0\}$.

In view of (6.20) and the fact that $\Phi_2 \geq 0$, we have the comparison

$$\Phi(x) \;\leq\; \Phi_1(x) + \Phi_2(x) \qquad\qquad \text{for all } x\in\mathcal{D}^\gamma, \tag{6.41}$$

where Φ_1 and Φ_2 are the solutions to (6.7) and (6.8), respectively.

11. Thanks to (6.38), we can now construct a sequence of points $P_k \in \Gamma(r_k)$, with $r_k \to 0$, such that

$$u(P_k) \;\leq\; u^+(P_k) \;\leq\; C|P_k|\Big(1 + \big|\ln|P_k|\big|\Big).$$

Setting $Q_k = \Lambda(P_k) = (P_k)^p$ we now obtain

$$\Phi_2(Q_k) \;\doteq\; u(P_k) \;\leq\; C|P_k|\Big(1+\big|\ln|P_k|\big|\Big) \;\leq\; C|Q_k|^{1/p}\Big(1+\big|\ln|Q_k|\big|\Big). \tag{6.42}$$

We recall that C always denotes a positive constant, whose precise value may change at each occurrence.

Given $0 < \beta < 1$, we can now choose $\delta > 0$ small enough so that

$$\frac{1}{p} \;=\; \left[1 + \frac{2\delta}{\pi}\right]^{-1} \;>\; \beta.$$

Combining (6.19) with (6.42), we thus obtain

$$\Phi(Q_k) \;\leq\; \Phi_1(Q_k) + \Phi_2(Q_k) \;\leq\; C|Q_k|^{1/p} + C|Q_k|^{1/p}\Big(1+\big|\ln|Q_k|\big|\Big) \;<\; C|Q_k|^\beta. \tag{6.43}$$

By (6.22), the assumption $P_k = (P_{k1}, P_{k2}) \in \Gamma(r_k)$ implies

$$|P_{k2}| \;\leq\; \frac{P_{k1}}{3}\,.$$

Hence, calling $\mathbf{e}_2 \doteq (0,1)$, we have

$$\left|\left\langle \mathbf{e}_2\,,\,\frac{P_k}{|P_k|}\right\rangle\right| \;=\; \frac{|P_{k2}|}{\sqrt{P_{k1}^2 + P_{k2}^2}} \;\leq\; \frac{1}{\sqrt{10}} \;<\; \frac{1}{3}\,.$$

In turn, if $\delta > 0$ was chosen sufficiently small, then $Q_k = (Q_{k1}, Q_{k2}) = \Lambda(P_k)$ still satisfies

$$\left|\left\langle \mathbf{e}_2\,,\ \frac{Q_k}{|Q_k|}\right\rangle\right| \;\leq\; \frac{1}{3}\,. \tag{6.44}$$

12. At last, we can now complete the proof of Lemma 4.3. Let $x_0 = \gamma(x_0)$ be a *good point*, as defined at the end of step **2.** By a possible rotation of coordinates, we can assume that the tangent vector is $\mathbf{t}(s_0) = \mathbf{e}_2 = (0,1)$. We then choose a small radius $\rho > 0$ and consider an affine transformation

$$x \;\mapsto\; y \;=\; \mathcal{T}x$$

mapping the disc $B(x_0, \rho)$ centered at x_0 with radius ρ onto the disc $B(0,1)$ centered at the origin with unit radius.

Restricted to the disc $B(x_0, \rho)$, the function Φ satisfies the elliptic equation (4.8) together with the lower bound

$$\Phi(x) \;\leq\; c\alpha Z(x) \qquad\qquad \text{for}\ \ x \in \gamma.$$

Since x_0 is a *good point*, we can choose $\lambda, \rho > 0$ small enough so that the corresponding function

$$\widetilde{\Phi}(y) \;=\; \lambda\big[\Phi(\mathcal{T}^{-1}y) - \Phi(x_0)\big] \qquad\qquad \text{for}\ \ |y| \leq 1$$

satisfies a system of the form

$$\begin{cases} \Delta\widetilde{\Phi}(y) \;\geq\; -\,\phi(y), & y \in \mathcal{D}^\gamma, \\[2mm] \ \ \widetilde{\Phi}(y) \;\leq\; |y|, & y \in \partial\mathcal{D}^\gamma\,. \end{cases} \tag{6.45}$$

In other words, $\widetilde{\Phi}$ provides a subsolution to (6.39). By the previous analysis, there exists an infinite sequence of points $Q_k \to 0$ such that

$$\widetilde{\Phi}(Q_k) \;\leq\; C\,|Q_k|^\beta.$$

Going back to the original coordinate $x \in B(x_0, \rho)$, this yields a sequence of points $q_k = \mathcal{T}^{-1}Q_k$ such that

$$q_k \to x_0, \qquad\qquad \Phi(q_k) - \Phi(x_0) \;\leq\; C\,|q_k - x_0|^\beta.$$

Notice that the inequality

$$\left|\left\langle \mathbf{t}(s_0)\,,\ \frac{q_k - x_0}{|q_k - x_0|}\right\rangle\right| \;\leq\; \frac{1}{3} \tag{6.46}$$

is an immediate consequence of (6.44). This completes the proof. □

Acknowledgments. The research of A. Bressan and Q. Sun was partially supported by NSF with grant DMS-1714237, "Models of controlled biological growth". The research of S. T. Galtung was partially supported by the grant "Wave Phenomena and Stability – a Shocking Combination (WaPheS)" (project no. 286822) from the Research Council of Norway.

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