A BIDDING GAME IN A CONTINUUM LIMIT ORDER BOOK

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Abstract. The paper is concerned with a continuum model of the limit order book, viewed as a noncooperative game for n players. An external buyer asks for a random amount $X > 0$ of a given asset. This amount will be bought at the lowest available price, as long as the price does not exceed a given upper bound $\overline{P}$. One or more sellers offer various quantities of the asset at different prices, competing to fulfill the incoming order, whose size is not known a priori.

The first part of the paper deals with solutions to the measure-valued optimal pricing problem for a single player, proving an existence result and deriving necessary and sufficient conditions for optimality. The second part is devoted to Nash equilibria. For a general class of random variables X and an arbitrary number of players, the existence and uniqueness of the corresponding Nash equilibrium is proved, explicitly determining the pricing strategy of each player. For a different class of random variables, it is shown that no Nash equilibrium can exist.

The paper also describes the asymptotic limit as the total number of players approaches infinity, and provides formulas for the price impact produced by an incoming order.

Key words. measure-valued optimization, optimality conditions, Nash equilibrium, bidding game, optimal pricing strategy, limit order book, price impact

AMS subject classifications. 49K21, 49J21, 91A06, 91A13, 91A60.

1. Introduction. This paper is concerned with a continuum model of the limit order book in a stock market, viewed as a noncooperative game for n players. Our main goal is to study the existence and uniqueness of a Nash equilibrium, determining the optimal bidding strategies of the various agents who submit limit orders.

We consider a one-sided limit order book. In our basic setting, we assume that an external buyer asks for a random amount of $X > 0$ of shares of a certain asset. This external agent will buy the amount X at the lowest available price, as long as this price does not exceed a given upper bound $\overline{P}$. One or more sellers offer various quantities of this asset at different prices, competing to fulfill the incoming order, whose size is not known a priori.

Having observed the prices asked by his competitors, each seller must determine an optimal strategy, maximizing his expected payoff. Of course, when other sellers are present, asking a higher price for a stock reduces the probability of selling it.

In our model we assume that the i-th player owns an amount κ_i of stock. He can put all of it on sale at a given price, or offer different portions at different prices. In general, his strategy will thus be described by a measure μ_i on $I\!R_+$, where $\mu_i([0,p])$ denotes the total amount of shares put on sale by the i-th player at a price $\leq p$.

In practice, it is clear that prices can take only a discrete set of values. However, by studying a continuum model where strategies are described by Radon measures one obtains clear-cut results on existence or non-existence of Nash equilibria, and clean, explicit solution formulas. In general, it turns out that the Nash equilibrium consists of measures which are absolutely continuous w.r.t. Lebesgue measure.

Several recent papers ([9], [12], [5]) deal with the modeling of the limit order book from the point of view of the agents who submit the limit orders. These models are intrinsically discrete in the price variable: limit orders can be submitted at prices $\{p_1, \ldots, p_N\}$ and to each price there corresponds a queue of limit orders, which are

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to be executed according to a first-in-first-out schedule. The shape of the limit order book is determined by the prices at which the various agents decide to submit their limit orders.

On the other hand, in [?], [8], [10], [1] the prices are continuous and the shape of the limit order book is described by a *density*. An important achievement of these models is that, when the shape of the limit order book is given, this determines a corresponding price impact function. The *price impact function* describes how the execution of a market order affects the underlying asset prices, i.e. it describes how the bid and ask prices change after the execution of a market order. Clearly, this is a quantity of key importance in the modeling of financial markets and an understanding of the price impact function allows us to gain insight in the mechanism of price formation. In [8] the limit order book has a *block shape* and this gives rise to a linear price impact. In [10] and [1] the limit order book density has a general shape which is described by a measure. These papers, given the order book shape as input, mainly consider the problem of optimal execution of trades by means of market orders.

In our model, prices are allowed to vary in a continuum of values but the shape of the limit order book is not given *a priori*. Indeed, we prove that this shape can be endogenously determined as the unique Nash equilibrium, resulting from the optimal pricing strategies implemented by the selling agents.

The paper is organized as follows. In Section 2 we consider the optimization problem for a single agent, who observes the limit orders submitted by his competitors and wishes to optimally price the sale of his own assets. We also introduce a fundamental distinction between two classes of random variables: Type A and Type B. These two types yield completely different results when Nash equilibria are studied.

Under general assumptions, the existence of an optimal pricing strategy is proved in Section 3. Necessary conditions for optimality are derived in Sections 4 and 5. For random variables of Type B, these imply that the optimal strategy always consists in putting all the assets for sale at the same price. In Section 6 we prove some sufficient conditions for optimality.

Sections 7 and 8 are devoted to the study of Nash equilibria. We consider n players, putting on sale quantities $\kappa_1, \ldots, \kappa_n$ of the same asset. We say that an n-tuple of pricing strategies $(\mu_1, \ldots, \mu_n)$ provides a Nash equilibrium if each μ_i provides an optimal strategy for the i-th agent, in reply to the bidding strategies of all the other agents. When the random buying order X is a random variable of type A, we prove that this noncooperative game admits a unique Nash equilibrium, which is explicitly determined. On the other hand, if the random variable X is of type B, we show that no Nash equilibrium can exist.

In Section 9 we consider an asymptotic limit, where the total number of sellers approaches infinity, while the amount of asset put on sale by each agent approaches zero. In this case, the limit order book approaches a well defined shape, determined by the probability distribution of the random variable X. From this model, one can deduce the price impact of an incoming buying order of size X. Some explicit examples are provided in Section 10.

In addition to the classical paper [7], for an introduction to non-cooperative games and Nash equilibria we refer to [3, 6, 13, 14].

2. The optimization problem for a single player. A general optimization problem for one agent can be formulated as follows. Let X be a non-negative random variable, with distribution function

$$Prob.\{X \leq s\} \;=\; 1 - \psi(s)\,. \tag{2.1}$$

Throughout the following we shall assume

(A1) *The map $s \mapsto \psi(s)$ is continuously differentiable and satisfies*

$$\psi(0) \;=\; 1\,, \qquad \psi(+\infty) \;=\; 0, \qquad \psi'(s) \;<\; 0 \qquad \textit{for all } s>0\,.$$

We shall consider two main classes of random variables, depending on the decay properties of the function ψ.

Definition 1. *We say that a probability distribution is*

$$\textit{of} \quad \textbf{type A} \quad \textit{if} \quad (\ln\psi(s))'' \;\geq\; 0 \quad \textit{for all } s>0\,. \tag{2.2}$$

$$\textit{of} \quad \textbf{type B} \quad \textit{if} \quad (\ln\psi(s))'' \;<\; 0 \quad \textit{for all } s>0\,. \tag{2.3}$$

For example, the probability distributions determined by $\psi_1(s) = e^{-\lambda s}$, $\lambda > 0$, $\psi_2(s) = \frac{1}{(1+s)^\alpha}$, $\alpha > 0$, are of type A, while $\psi_3(s) = e^{-s^2}$ yields a probability distribution of type B. Roughly speaking, a probability distribution is of type A if its tail decays not faster than a negative exponential. Of course, one can consider more general probability distributions, where $(\ln\psi)''$ changes sign. For such random variables, the analysis will likely be more difficult.

Let $\Phi_0 : [0,\overline{P}] \mapsto I\!R_+$ be a non-negative, nondecreasing function. For every p, we think of $\Phi_0(p)$ as the total amount of stock offered for sale at a price $\leq p$ by the other agents.
Consider an additional seller entering the market, owning an amount κ of stock.

Definition 2. *A* **pricing strategy** *for the new player is a nondecreasing map* $\phi : [0,\kappa] \mapsto [p_0,\overline{P}]$.

Using the Lagrangian variable $\beta \in [0,\kappa]$ to label a particular share in possession of the new agent, by $\phi(\beta)$ we thus denote the price at which this particular share is put on sale. The total amount of shares that the new agent offers for sale at price $\leq p$ is thus computed by

$$\mu_1([0,p]) \;=\; meas\Big(\{\beta\in[0,\kappa]\,;\quad \phi(\beta)\leq p\}\Big). \tag{2.4}$$

This is the push-forward of the Lebesgue measure on $[0,\kappa]$ w.r.t. the map ϕ.

Next, assume that the incoming order has size X. The total amount of stock sold by the new agent is

$$\beta(X) \;=\; \sup\;\Big\{\beta\in[0,\kappa]\,;\quad \beta+\Phi_0(\phi(\beta)) \;\leq\; X\Big\}\,, \tag{2.5}$$

yielding the payoff

$$\int_0^{\beta(X)} (\phi(\beta)-p_0)\,d\beta\,.$$

Here $p_0>0$ is the value that the new player attaches to a unit amount of stock. For example, it could be the mean bid-ask price.

The optimization problem for the new seller can thus be formulated as

$$\text{Maximize:} \quad J(\phi) \doteq E\Big[\int_0^{\beta(X)} (\phi(\beta) - p_0)\, d\beta\Big] \tag{2.6}$$

among all pricing strategies $\phi : [0, \kappa] \mapsto [0, \overline{P}]$. Here $E[\cdot]$ denotes the expectation w.r.t. the probability distribution of the random variable X.
Observe that, by (2.1) and (2.9), we have the equivalent representation

$$J(\phi) = \int_0^{\kappa} (\phi(\beta) - p_0)\, \psi\Big(\beta + \Phi_0(\phi(\beta))\Big)\, d\beta\,. \tag{2.7}$$

Remark 1. If Φ_0 has a jump at a point ξ, this means that a positive amount of stock is offered for sale by the other agents at the price ξ. Two main cases can arise.

CASE 1: Φ_0 is left continuous, i.e. $\Phi_0(\xi) = \Phi_0(\xi-)$. This means that the new agent has selling priority. If he also puts on sale a positive amount of stock at the same price ξ, his stock will be the first to be sold.

CASE 2: Φ_0 is right continuous, i.e. $\Phi_0(\xi) = \Phi_0(\xi+)$. This means that the new agent does not have selling priority. If he also puts on sale a positive amount of stock at the same price ξ, his stock will be the last to be sold.

Notice that in Case 1 the function Φ_0 is lower semicontinuous. This property will play a key role in the proof of existence of an optimal strategy.

3. Existence of an optimal strategy. Our first result shows the existence of an optimal strategy for the new agent, assuming that he has selling priority.

THEOREM 3.1 (existence). *Let X be a random variable satisfying the assumptions (A1). Let $\Phi_0 : [0, \overline{P}] \mapsto I\!R_+$ be a left-continuous, nondecreasing function, and let $\kappa > 0$. Then there exists an optimal pricing strategy $\phi^* : [0, \kappa] \mapsto [p_0, \overline{P}]$ for the new agent, maximizing the expected payoff (2.10).*

Proof. Let $(\phi_\nu)_{\nu \geq 1}$ be a maximizing sequence of pricing strategies. Since all functions ϕ_ν are non-decreasing, using Helly's compactness theorem (see for example [11], p. 372), by extracting a subsequence and relabeling we can achieve the pointwise convergence

$$\phi_\nu(\beta) \to \phi^*(\beta) \qquad \text{for all } \beta \in [0, \kappa]\,.$$

We claim that the strategy ϕ^* is optimal.

Indeed, since Φ_0 is lower semicontinuous and ψ is strictly decreasing, the composite map $s \mapsto \psi(\Phi_0(s))$ is upper semicontinuous and for every $\beta \in [0, \kappa]$ we have

$$\limsup_{\nu \to \infty}\; \psi\Big(\Phi_0(\phi_\nu(\beta))\Big) \leq \psi\Big(\Phi_0(\phi^*(\beta))\Big).$$

In turn, this yields

$$\begin{aligned}
\sup_\phi J(\phi) &= \lim_{\nu \to \infty} J(\phi_\nu) = \lim_{\nu \to \infty} \int_0^{\kappa} (\phi_\nu(\beta) - p_0)\, \psi\Big(\beta + \Phi_0(\phi_\nu(\beta))\Big)\, d\beta \\
&\leq \int_0^{\kappa} \limsup_{\nu \to \infty} \Big\{ (\phi_\nu(\beta) - p_0)\, \psi\Big(\beta + \Phi_0(\phi_\nu(\beta))\Big) \Big\}\, d\beta \\
&\leq \int_0^{\kappa} (\phi^*(\beta) - p_0)\, \psi\Big(\beta + \Phi_0(\phi^*(\beta))\Big)\, d\beta = J(\phi^*)\,.
\end{aligned}$$

□

Example 1. If the new player does not have priority, an optimal strategy may fail to exist. For example, assume that the other sellers offer a total amount of stock κ_0, all at the same price $\overline{P}$. This situation is described by the right continuous function

$$\Phi_0(p) \;=\; \begin{cases} 0 & \text{if} \quad p < \overline{P}\,, \\ \kappa_0 & \text{if} \quad p = \overline{P}\,. \end{cases} \tag{3.1}$$

Assume that the new player has an amount κ of stock to put on sale. For each $\nu \geq 1$, consider the pricing strategy $\phi_\nu(\beta) \equiv \overline{P} - \nu^{-1}$. Then $(\phi_\nu)_{\nu \geq 1}$ is a maximizing sequence. Writing $a \wedge b \doteq \min\{a, b\}$, $a_+ \doteq \max\{a, 0\}$, the expected payoffs are

$$J(\phi_\nu) \;=\; (\overline{P} - \nu^{-1} - p_0) \cdot E[X \wedge \kappa]\,.$$

However, the expected payoff $(\overline{P} - p_0) \cdot E[X \wedge \kappa]$ could be achieved only if the new agent puts all his stock for sale at the maximum price $\overline{P}$ and has selling priority over the other agents (that would correspond to Φ_0 being left continuous). However, if Φ_0 is the function in (3.2), the new agent does not have priority. With the strategy $\phi^*(\beta) \equiv \overline{P}$ he only achieves

$$J(\phi^*) \;=\; (\overline{P} - p_0) \cdot E\Big[(X - \kappa_0)_+ \wedge \kappa\Big].$$

4. Necessary conditions. In this section we seek necessary conditions for the optimality of a pricing strategy ϕ for the new agent. For this purpose given a nonnegative, nondecreasing function $\Phi_0 : [0, \overline{P}] \mapsto I\!R_+$ as in (2.9), we introduce the functions

$$G^\beta(p) \;\doteq\; -\,\psi\Big(\beta + \Phi_0(p)\Big) \cdot \Big[(p - p_0)\psi'\Big(\beta + \Phi_0(p)\Big)\Big]^{-1}. \tag{4.1}$$

For $0 \leq a < b \leq \kappa$ we shall also consider the integrated function

$$G^{[a,b]}(p) \;\doteq\; -\int_a^b \psi\Big(\beta + \Phi_0(p)\Big)d\beta \cdot \left[(p - p_0)\int_a^b \psi'\Big(\beta + \Phi_0(p)\Big)d\beta\right]^{-1}.$$

Remark 2. If the random variable X is of type A, then for every p the map $\beta \mapsto G^\beta(p)$ is non-decreasing. On the other hand, if X is of type B, then the maps $\beta \mapsto G^\beta(p)$ are strictly decreasing.

In this section we do not make any assumption on the left or right continuity of Φ_0. It will thus be convenient to define the left continuous function

$$\Phi_0^\flat(p) \;\doteq\; \Phi_0(p-)\,.$$

In other words, $\Phi_0^\flat$ is the unique left continuous function that coincides with Φ_0 everywhere with the possible exception of countably many points of jump. Call $J^\flat(\phi)$ the expected payoff achieved by a pricing strategy $\phi : [0, \kappa] \mapsto [0, \overline{P}]$ when Φ_0 is replaced by $\Phi_0^\flat$.

LEMMA 4.1. *In the above setting, for every $\Phi_0 : [0, \overline{P}] \mapsto I\!R_+$ and $\kappa > 0$ one has*

$$\sup_\phi J(\phi) \;=\; \max_\phi J^\flat(\phi)\,. \tag{4.2}$$

Proof. By Theorem 3.1, the maximum expected payoff on the right hand side of (4.4) is attained. Namely, there exists a pricing strategy ϕ^* such that

$$J^\flat(\phi^*) \;=\; \max_\phi J^\flat(\phi).$$

Consider the strategies $\phi_n(\beta) \;=\; \phi^*(\beta) - \frac{1}{n}$. The corresponding payoffs satisfy

$$\begin{aligned} J(\phi_n) \;&=\; \int_0^\kappa \left(\phi^*(\beta) - \frac{1}{n} - p_0\right)\psi\left(\beta + \Phi_0\left(\phi^*(\beta) - \frac{1}{n}\right)\right)\,d\beta \\ &\geq\; \int_0^\kappa (\phi^*(\beta) - p_0)\,\psi\left(\beta + \Phi_0\left(\phi^*(\beta) - \frac{1}{n}\right)\right)\,d\beta - \frac{\kappa}{n} \\ &\geq\; \int_0^\kappa (\phi^*(\beta) - p_0)\,\psi\left(\beta + \Phi_0^\flat\,(\phi^*(\beta))\right)\,d\beta - \frac{\kappa}{n} \;=\; J^\flat(\phi^*) - \frac{\kappa}{n}. \end{aligned}$$

Therefore

$$\sup_\phi J(\phi) \;\geq\; \sup_n J(\phi_n) \;\geq\; \sup_n \left\{J^\flat(\phi^*) - \frac{\kappa}{n}\right\} \;=\; J^\flat(\phi^*) \;=\; \sup_\phi J^\flat(\phi).$$

The converse inequality is clear. Indeed, $\Phi_0^\flat(p) \leq \Phi_0(p)$ for every p. Hence $J^\flat(\phi) \geq J(\phi)$ for every admissible strategy $\phi : [0,\kappa] \mapsto [0,\overline{P}]$. □

Given a nondecreasing map $\phi : [0,\kappa] \mapsto [0,\overline{P}]$ one can isolate countably many disjoint intervals $S_j \doteq [a_j, b_j] \subseteq [0,\kappa]$ such that ϕ is constant on each S_j and strictly increasing elsewhere. Namely, defining

$$S \;\doteq\; \bigcup_j S_j\,,$$

one has

$$\beta_1 \;\notin\; S\,, \qquad \beta_1 \;<\; \beta_2 \qquad \Longrightarrow \qquad \phi(\beta_1) \;<\; \phi(\beta_2)\,.$$

In connection with the measure μ_1 introduced at (2.8), we observe that the atomic part of μ_1 is the measure μ_1^a concentrated on the points $\phi(a_j) \;=\; \phi(b_j)$. Indeed, $\mu_1^a\Big(\{\phi(a_j)\}\Big) = b_j - a_j > 0$.

THEOREM 4.2 (necessary conditions for optimality). *Let the random variable X satisfy the assumptions (A1) and let $\Phi_0 : [0,\overline{P}] \mapsto I\!R_+$ be a nondecreasing map. If $\phi : [0,\kappa] \mapsto [p_0,\overline{P}]$ is an optimal pricing strategy, then the following holds.*

(i) For almost every $\beta \in [0,\kappa] \setminus S$, setting $x \doteq \phi(\beta)$ one has

$$\limsup_{\varepsilon\to 0-} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon} \;\leq\; G^\beta(x) \;\leq\; \liminf_{\varepsilon\to 0+} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon}\,. \tag{4.3}$$

(ii) If $\beta \in [a_i, b_i] \subset S$, with $\phi(\beta') = x < \overline{P}$ for all $\beta' \in [a_i, b_i]$, then

$$\begin{aligned} G^{a_i}(x) \;&\geq\; \limsup_{\varepsilon\to 0-} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon}\,, \\ G^{b_i}(x) \;&\leq\; \liminf_{\varepsilon\to 0+} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon}\,, \end{aligned} \tag{4.4}$$

$$\limsup_{\varepsilon\to 0-} \frac{\Phi_0(x+\varepsilon)-\Phi_0(x)}{\varepsilon} \;\leq\; G^{[a_i,b_i]}(x) \;\leq\; \liminf_{\varepsilon\to 0+} \frac{\Phi_0(x+\varepsilon)-\Phi_0(x)}{\varepsilon}\,. \tag{4.5}$$

Proof. **1.** Assume that the second inequality in (4.9) does not hold at some $\beta_0 \in [0,\kappa]\setminus S$. Setting $x_0 \doteq \phi(\beta_0)$, this clearly implies

$$L \;\doteq\; \liminf_{\varepsilon\to 0+} \frac{\Phi_0(x_0+\varepsilon)-\Phi_0(x_0)}{\varepsilon} \;<\; \infty\,.$$

Hence the nondecreasing function Φ_0 is right continuous at the point $x_0 = \phi(\beta_0)$. By continuity we can thus find λ and $\delta > 0$ such that

$$\begin{array}{ll} L \;<\; \lambda \;<\; G^{\beta}(p), & \beta\in[\beta_0,\beta_0+\delta],\;\; p\in[x_0,\;x_0+\delta]\,,\\ \psi(\zeta)+(p-p_0)\psi'(\zeta)\lambda>0, & p\in[x_0,\;x_0+\delta]\,,\;\zeta\in\beta_0+\Phi_0(x_0)+[0,\delta\lambda]\,. \end{array} \tag{4.6}$$

2. We claim that there exists $\varepsilon\in\,]0,\delta]$ such that the following conditions hold.

$$\begin{array}{ll} \Phi_0(p) \;\geq\; \Phi_0(x_0+\varepsilon)+\lambda\,(p-x_0-\varepsilon) & \text{for all } p\in[x_0,\,x_0+\varepsilon],\\ \beta_1 \;\doteq\; \sup\{\beta\,;\;\; \phi(\beta)<x_0+\varepsilon\} \;<\; \beta_0+\delta\,. & \end{array} \tag{4.7}$$

Indeed, by definition of liminf there exists $\varepsilon_2\in\,]0,\delta]$ such that

$$\Phi_0(x_0+\varepsilon_2) \;<\; \Phi_0(x_0)+\lambda\varepsilon_2\,.$$

Consider the modified function

$$\Phi_0^\flat(p) \;\doteq\; \left\{\begin{array}{lll} \Phi_0(p) & \text{if} & p\notin\,]x_0\,,\;x_0+\varepsilon_2]\,,\\ \Phi_0(p-) & \text{if} & p\in\,]x_0\,,\;x_0+\varepsilon_2]\,. \end{array}\right.$$

By lower semicontinuity, the function $\eta\;\mapsto\;\Phi_0^\flat(x_0+\eta)-\lambda\eta$ attains a strictly negative minimum on the interval $[0,\varepsilon_2]$. If

$$\varepsilon\in\operatorname*{argmin}_{\eta\in[0,\varepsilon_2]}\Big\{\Phi_0^\flat(x_0+\eta)-\lambda\eta\Big\}$$

is a point where this minimum is attained, then (4.15) holds.

3. Let $\phi^{\varepsilon+}$ be the perturbed strategy defined by

$$\phi^{\varepsilon+}(\beta) \;\doteq\; \left\{\begin{array}{lll} \phi(\beta) & \text{if} & \beta\notin[\beta_0,\beta_1]\,,\\ x_0+\varepsilon & \text{if} & \beta\in[\beta_0,\beta_1]\,. \end{array}\right.$$

Since $\psi'<0$, using (4.15) and then (4.13), one obtains

$$\begin{aligned} &J^\flat(\phi^{\varepsilon+})-J(\phi) = \\ &= \int_{\beta_0}^{\beta_1}\Big[(x_0+\varepsilon-p_0)\psi\Big(\beta+\Phi_0^\flat(x_0+\varepsilon)\Big)-(\phi(\beta)-p_0)\psi\Big(\beta+\Phi_0(\phi(\beta))\Big)\Big]d\beta\\ &\geq\;\int_{\beta_0}^{\beta_1}\int_{\phi(\beta)}^{x_0+\varepsilon}\frac{d}{dp}\Big[(p-p_0)\psi\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)\Big]\;dp\;d\beta\\ &=\;\int_{\beta_0}^{\beta_1}\int_{\phi(\beta)}^{x_0+\varepsilon}\Big[\psi\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)\\ &\qquad+(p-p_0)\psi'\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)\lambda\Big]\,dp\,d\beta\geq\;\delta_0\;>\;0\,, \end{aligned}$$

for some positive constant δ_0. Using Lemma 4.1 we conclude

$$J(\phi) \;=\; \sup_{\varphi} J(\varphi) \;=\; \sup_{\varphi} J^\flat(\varphi) \;\geq\; J^\flat(\phi^{\varepsilon+}) \;\geq\; J(\phi)+\delta\,,$$

reaching a contradiction. The first inequality in (4.9) can be proved by an entirely similar argument.

4. The two statements (4.10)-(4.11) will be deduced as consequences of the more general necessary conditions

$$\begin{aligned} G^{[\xi,b_i]}(x) &\leq \liminf_{\varepsilon\to 0+} \frac{\Phi_0(x+\varepsilon)-\Phi_0(x)}{\varepsilon}, \text{ for all } \xi\in[a_i,b_i],\\ G^{[a_i,\xi]}(x) &\geq \limsup_{\varepsilon\to 0-} \frac{\Phi_0(x+\varepsilon)-\Phi_0(x)}{\varepsilon}, \text{ for all } \xi\in[a_i,b_i]. \end{aligned} \tag{4.8}$$

Indeed, the two inequalities in (4.10) are obtained by observing that

$$\lim_{\xi\to b_i-} G^{[\xi,b_i]}(x) = G^{b_i}(x), \quad \lim_{\xi\to a_i+} G^{[a_i,\xi]}(x) = G^{a_i}(x).$$

Moreover, (4.11) follows from the two inequalities in (4.22), choosing $\xi = b_i$ and $\xi = a_i$, respectively.

5. It now remains to prove (4.22). Assume that the first inequality in (4.22) fails at $\beta_0\in[a_i,b_i]$, and call $x_0=\phi(\beta_0)$. Then by continuity we can find λ and $\delta>0$ such that

$$\liminf_{\varepsilon\to 0+} \frac{\Phi_0(x_0+\varepsilon)-\Phi_0(x_0)}{\varepsilon} \;<\; \lambda \;<\; G^{[\xi,b_i]}(p) \qquad \text{for all } p\in[x,\ x+\delta],$$

which implies that there exists $c_0>0$ such that

$$\int_\xi^{b_i} \psi(\sigma)\,d\sigma + \lambda(p-p_0)\int_\xi^{b_i}\psi'(\sigma)\,d\sigma \geq c_0>0, \qquad \text{for all } p\in[x_0,\ x_0+\delta]. \tag{4.9}$$

Choose $\varepsilon\in\,]0,\delta]$ such that the following conditions hold.

$$\begin{aligned} \Phi_0(p) &\geq \Phi_0(x+\varepsilon)+\lambda\cdot(p-x-\varepsilon) \qquad \text{for all } p\in[x_0,\,x_0+\varepsilon],\\ \beta_1 &\doteq \sup\{\beta\,;\ \ \phi(\beta)<x+\varepsilon\} \;<\; b_i+\delta(\varepsilon)\,. \end{aligned} \tag{4.10}$$

where $\delta(\varepsilon)\downarrow 0$, as $\varepsilon\to 0$. Let $\phi^{\xi,\varepsilon+}$ be the perturbed strategy defined by

$$\phi^{\xi,\varepsilon+}(\beta) \;\doteq\; \begin{cases} x_0+\varepsilon & \text{if } \ \beta\in\phi^{-1}([x_0,x_0+\varepsilon])\cap[\xi,\infty)\,,\\ \phi(\beta) & \text{otherwise.}\end{cases}$$

One obtains

$$\begin{aligned} &J^\flat(\phi^{\xi,\varepsilon+})-J(\phi)\\ &\quad=\int_\xi^{\beta_1}\Big[(x_0+\varepsilon-p_0)\psi\Big(\beta+\Phi_0^\flat(x_0+\varepsilon)\Big)-(\phi(\beta)-p_0)\psi\Big(\beta+\Phi_0(\phi(\beta))\Big)\Big]\,d\beta\\ &\quad\geq\int_{x_0}^{x_0+\varepsilon}\left(\int_\xi^{b_i}+\int_{b_i}^{\phi^{-1}(p)}\right)\Big[\psi\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)+\\ &\qquad\qquad +(p-p_0)\psi'\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)\lambda\Big]\,dp\,d\beta\\ &\quad\geq c_0\varepsilon+\varepsilon\delta(\varepsilon)=c_0\varepsilon+o(\varepsilon)>0 \end{aligned}$$

for $\varepsilon > 0$ sufficiently small. Notice that the last inequality follows from (4.23) and (4.24). Using Lemma 4.1 we reach a contradiction. □

COROLLARY 4.3. *Assume that $\Phi_0(\cdot)$ is piecewise $\mathcal{C}^1$, and let $\phi(\cdot)$ be an optimal strategy. Then for almost every $\beta \in [0,\kappa] \setminus S$ one has*

$$\frac{d}{dp}\Phi_0(\phi(\beta)) \;=\; G^\beta(\phi(\beta)). \tag{4.11}$$

Indeed, for a.e. $\beta \in [0,\kappa] \setminus S$ one has

$$\limsup_{\varepsilon\to 0-} \frac{\Phi_0(\phi(\beta)+\varepsilon) - \Phi_0(\phi(\beta))}{\varepsilon} = \liminf_{\varepsilon\to 0+} \frac{\Phi_0(\phi(\beta)+\varepsilon) - \Phi_0(\phi(\beta))}{\varepsilon} = \frac{d}{dp}\Phi_0(\phi(\beta)).$$

Hence (4.28) follows from (4.9).

Example 2. Assume that the random variable X has exponential distribution, so that

$$\psi(s) \;=\; Prob.\{X < s\} \;=\; e^{-\lambda s}\,. \tag{4.12}$$

Let Φ_0 be continuous, piecewise $\mathcal{C}^1$. If $\phi : [0,\kappa] \mapsto [0,\overline{P}]$ is an optimal pricing strategy, then the necessary conditions imply that the range of ϕ should by contained in the set

$$\left\{ p \in \,]p_0, \overline{P}[\,;\quad \Phi_0'(p) = \frac{1}{\lambda(p-p_0)} \right\} \cup \{\overline{P}\}\,.$$

5. Atomic optimal strategies. Our next goal is to prove that, if the random variable X is of type B, then any optimal pricing strategy for the new agent must be constant. Namely, all stock should be offered for sale at the same price. A preliminary lemma will be needed.

LEMMA 5.1. *Let X be a random variable of type B. Assume that $\phi : [0,\kappa] \mapsto [0,\overline{P}]$ is a pricing strategy taking exactly two values, say p_1 and p_2. Then one of the two constant strategies $\phi_1(\beta) \equiv p_1$ or $\phi_2(\beta) \equiv p_2$ yields an expected payoff strictly larger than ϕ.*

Proof. **1.** Fix $p_1 < p_2 \in [p_0, \overline{P}]$. For $\theta \in [0,\kappa]$ consider the pricing strategy

$$\phi^\theta(\beta) \;\doteq\; \begin{cases} p_1 & \text{if } \beta \in [0,\theta], \\ p_2 & \text{if } \beta \in \,]\theta,\kappa]. \end{cases} \tag{5.1}$$

The corresponding payoff is

$$J(\phi^\theta) \;=\; (p_1 - p_0)\int_0^\theta \psi(\beta + \Phi_0(p_1))\,d\beta + (p_2 - p_0)\int_\theta^\kappa \psi(\beta + \Phi_0(p_2))\,d\beta\,.$$

We claim that the maximum of $J(\phi^\theta)$ can be attained only if $\theta = 0$ or $\theta = \kappa$.

2. Assume, on the contrary, that

$$0 \;<\; \theta^* \;<\; \kappa\,, \qquad\qquad J(\phi^{\theta^*}) \;=\; \max_{\theta\in[0,\kappa]} J(\phi^\theta).$$

The optimality conditions yield

$$\frac{d}{d\theta}J(\phi^\theta)\bigg|_{\theta=\theta^*} \;=\; 0\,, \qquad\qquad \frac{d^2}{d\theta^2}J(\phi^\theta)\bigg|_{\theta=\theta^*} \;\leq\; 0\,.$$

In turn, these imply

$$\begin{cases} (p_1-p_0)\psi(\theta^*+\Phi_0(p_1)) & = & (p_2-p_0)\psi(\theta^*+\Phi_0(p_2)), \\ (p_1-p_0)\psi'(\theta^*+\Phi_0(p_1)) & \leq & (p_2-p_0)\psi'(\theta^*+\Phi_0(p_2)). \end{cases} \tag{5.2}$$

We now recall that X is of type B, hence $\left(\frac{\psi'}{\psi}\right)' < 0$. Therefore

$$s_1 < s_2 \quad\Longrightarrow\quad \frac{\psi'(s_1)}{\psi(s_1)} > \frac{\psi'(s_2)}{\psi(s_2)}. \tag{5.3}$$

From (5.5) we obtain

$$\frac{\psi'(\theta^*+\Phi_0(p_1))}{\psi(\theta^*+\Phi_0(p_1))} \leq \frac{\psi'(\theta^*+\Phi_0(p_2))}{\psi(\theta^*+\Phi_0(p_2))}\,. \tag{5.4}$$

Since $p_1 < p_2$, the first equality in (5.5) implies that $\psi(\theta^*+\Phi_0(p_1)) > \psi(\theta^*+\Phi_0(p_2))$, hence

$$s_1 \doteq \theta^*+\Phi_0(p_1) < \theta^*+\Phi_0(p_2) \doteq s_2\,.$$

The inequality (5.7) is thus in contradiction with (5.6). This achieves the proof. □
The same argument used in the proof of Lemma 5.1 yields

COROLLARY 5.2. *Let $\varphi : [0,\kappa] \mapsto [0,\overline{P}]$ be a pricing strategy taking finitely many values $p_1 < p_2 < \ldots < p_m$. For a given $k \in \{1,\ldots,m-1\}$, consider the two strategies*

$$\varphi^{k-}(\beta) = \begin{cases} \varphi(\beta) & \text{if } \varphi(\beta) \notin \{p_k, p_{k+1}\}\,, \\ p_k & \text{if } \varphi(\beta) \in \{p_k, p_{k+1}\}\,, \end{cases}$$

$$\varphi^{k+}(\beta) = \begin{cases} \varphi(\beta) & \text{if } \varphi(\beta) \notin \{p_k, p_{k+1}\}\,, \\ p_{k+1} & \text{if } \varphi(\beta) \in \{p_k, p_{k+1}\}\,. \end{cases}$$

Then

$$J(\varphi) \leq \max\Big\{J(\varphi^{k-})\,,\ J(\varphi^{k+})\Big\}\,.$$

In other words, we can always replace a strategy taking m distinct values with a new strategy taking $m-1$ values and achieving at least the same payoff.

Remark 3. Consider the continuous function

$$F(p_1,p_2,\theta,q_1,q_2) \doteq \max\left\{(p_1-p_0)\int_0^\kappa \psi(\beta+q_1)\,d\beta\,,\ (p_2-p_0)\int_0^\kappa \psi(\beta+q_2)\,d\beta\right\}$$
$$-(p_1-p_0)\int_0^\theta \psi(\beta+q_1)\,d\beta - (p_2-p_0)\int_\theta^\kappa \psi(\beta+q_2)\,d\beta\,.$$

Let $\kappa_0 \doteq \Phi_0(\overline{P})$. The proof of Lemma 5.1 shows that $F > 0$ on the set

$$\Omega \doteq \Big\{(p_1,p_2,\theta,q_1,q_2)\,;\ \ 0 \leq p_1 < p_2 \leq \overline{P}\,,\ \ 0 < \theta < \kappa\,,\ \ 0 \leq q_1 \leq q_2 \leq \kappa_0\Big\}\,.$$

Given any $\varepsilon > 0$, consider the compact subset

$$\Omega_\varepsilon \doteq \Big\{(p_1,p_2,\theta,q_1,q_2)\,;\, 0 \leq p_1 \leq p_2-\varepsilon \leq p_2 \leq \overline{P}\,,\, \varepsilon < \theta < \kappa-\varepsilon\,,\, 0 \leq q_1 \leq q_2 \leq \kappa_0\Big\}\,.$$

Since F is strictly positive on the compact set Ω_ε, it attains a strictly positive minimum $\delta(\varepsilon) > 0$ on Ω_ε. In particular, this shows that given a positive ε, we can find $\delta(\varepsilon) > 0$ such that the following holds. Assume that $0 \le p_1 \le p_2 - \varepsilon < p_2 \le \overline{P}$ and $\theta \in [\varepsilon, \kappa - \varepsilon]$. Then the pricing strategy ϕ^θ in (5.1) satisfies

$$J(\phi^\theta) \ \le \ \max_{\alpha\in[0,\kappa]} J(\phi^\alpha) - \delta(\varepsilon). \tag{5.5}$$

THEOREM 5.3. *Assume that the random variable X is of type B and satisfies the assumption (A1). Then, given any nondecreasing map Φ_0, any optimal solution ϕ of the problem (2.10) must be constant.*

Proof. Let ϕ be an optimal solution. Assuming that ϕ is not constant, we shall derive a contradiction.

1. Choose $\varepsilon > 0$ and points $0 < a < a + 2\varepsilon < b < \overline{P}$ so that

$$meas\Big(\{\beta \in [0,\kappa]\,;\ \ \phi(\beta) < a\}\Big) \ > \ \varepsilon, \qquad meas\Big(\{\beta \in [0,\kappa]\,;\ \ \phi(\beta) > b + \varepsilon\}\Big) \ > \ \varepsilon\,.$$

Let $\delta(\varepsilon) > 0$ be the corresponding constant in (5.9), and choose an integer n large enough so that

$$\frac{\kappa}{n} \ < \ \min\{\varepsilon\,,\ \delta(\varepsilon)\}\,.$$

Introduce the points $p_j \ \doteq \ j/n$ and consider the approximate, piecewise constant strategy

$$\phi_n(\beta) \ = \ p_j \qquad \text{if} \quad p_j \le \phi(\beta) < p_{j+1}\,.$$

This definition yields

$$J(\phi_n) \ \ge \ J(\phi) - \frac{\kappa}{n} \ > \ J(\phi) - \delta(\varepsilon)\,.$$

2. By construction, ϕ_n takes only finitely many values $p_0, \ldots, p_N$. Since the random variable X is of type B, by repeatedly using Corollary 5.2 we can replace the strategy ϕ_n with a strategy $\phi^\flat$ taking only three distinct values, P_1, P_2, P_3. More precisely, we can find three prices $P_1, P_2, P_3 \in \{p_1, \ldots, p_N\}$, with

$$0 < \ P_1 \ \le \ a \ < \ P_2 \ < \ b - \varepsilon \ \le \ P_3 \ \le \ \overline{P},$$

such that the following holds. Defining the piecewise constant strategy

$$\phi^\flat(\beta) \ = \ \begin{cases} P_1 & \text{if} \quad \phi_n(\beta) \le a\,,\\ P_2 & \text{if} \quad a < \phi_n(\beta) < b\,,\\ P_3 & \text{if} \quad \phi_n(\beta) \ge b\,,\end{cases}$$

one has $J(\phi^\flat) \ \ge \ J(\phi_n)$.

3. If now $P_2 - P_1 \le P_3 - P_2$, we apply once again Corollary 5.2 and obtain a strategy $\phi^\sharp$ of the form

$$\phi^\sharp(\beta) \ = \ \begin{cases} Q_1 & \text{if} \quad \phi^\flat(\beta) \in \{P_1, P_2\}\,,\\ Q_2 & \text{if} \quad \phi^\flat(\beta) = P_3\,.\end{cases}$$

with

$$Q_1 \in \{P_1, P_2\}\,, \quad Q_2 = P_3\,, \qquad\qquad J(\phi^\sharp) \;\geq\; J(\phi^\flat)\,.$$

On the other hand, if now $P_2 - P_1 > P_3 - P_2$, we use Corollary 5.2 to obtain a strategy $\phi^\sharp$ of the form

$$\phi^\sharp(\beta) \;=\; \begin{cases} Q_1 & \text{if} \quad \phi^\flat(\beta) = P_1\,, \\ Q_2 & \text{if} \quad \phi^\flat(\beta) \in \{P_2, P_3\}\,. \end{cases}$$

with $Q_1 = P_1\,, Q_2 \in \{P_2, P_3\}\,, J(\phi^\sharp) \;\geq\; J(\phi^\flat)$. In both cases we obtain a strategy $\phi^\sharp$ taking exactly two values Q_1, Q_2, with $Q_2 - Q_1 \geq \varepsilon$. Moreover

$$meas\Big(\{\beta \in [0,\kappa]\,;\;\; \phi^\sharp(\beta) = Q_1\}\Big) \;\geq\; \varepsilon\,, \qquad meas\Big(\{\beta \in [0,\kappa]\,;\;\; \phi^\sharp(\beta) = Q_2\}\Big) \;\geq\; \varepsilon\,. \tag{5.6}$$

4. Finally, consider the two constant strategies

$$\phi_1^*(\beta) = Q_1\,, \qquad\qquad \phi_2^*(\beta) = Q_2\,.$$

By (5.16) and (5.9), we conclude

$$\max\Big\{J(\phi_1^*)\,,\; J(\phi_2^*)\Big\} \;\geq\; J(\phi^\sharp) + \delta(\varepsilon) \;\geq\; J(\phi_n) + \delta(\varepsilon) \;\geq\; J(\phi) - \frac{\kappa}{n} + \delta(\varepsilon) \;>\; J(\phi).$$

This contradicts the optimality of ϕ, proving the theorem. □

6. Sufficient conditions. We now consider a case where all strategies $\phi : [0,\beta] \mapsto [p_0, \overline{P}]$ which satisfy the necessary conditions stated in Theorem 4.2 are in fact optimal. We make the following assumption on the regularity of Φ_0.

(A2) *The map $s \mapsto \Phi_0(s)$ is continuous on the half-open interval $[0, \overline{P}[$. Moreover, its derivative $\Phi_0'(p)$ is piecewise continuous.*

THEOREM 6.1 (sufficient conditions). *Let the assumptions (A1)-(A2) hold, and let X be a random variable of type A, so that (2.3) holds. Moreover, assume that one has*

$$\begin{aligned} G^\beta(p) \;&\geq\; \Phi_0'(p) \quad \textit{for all} \;\; p \in [p_0, \phi(\beta)], \\ G^\beta(p) \;&\leq\; \Phi_0'(p) \quad \textit{for all} \;\; p \in [\phi(\beta), \overline{P}]. \end{aligned} \tag{6.1}$$

Then ϕ is optimal.

Proof. Assuming that the new agent has priority, by Theorem 3.1 an optimal strategy ϕ^* exists.

Let now ϕ be any admissible strategy which satisfies the conditions (6.1). Consider the interpolated strategy

$$\phi^\theta(\beta) \;\doteq\; \theta\phi(\beta) + (1-\theta)\phi^*(\beta).$$

Since ϕ^* is optimal, to prove that ϕ is also optimal it thus suffices to show that

$$\frac{d}{d\theta} J\Big(\theta\phi + (1-\theta)\phi^*(\beta)\Big) \;\geq\; 0\,. \tag{6.2}$$

We have

$$\frac{d}{d\theta}J(\phi^\theta) = \int_0^\kappa (\phi(\beta) - \phi^*(\beta))(\phi^\theta(\beta) - p_0)\psi'\big(\beta + \Phi_0(\phi^\theta(\beta))\big) \cdot \\ \cdot \big[\Phi_0'(\phi^\theta(\beta)) - G^\beta(\phi^\theta(\beta))\big]\, d\beta,$$

and (6.3) follows from the fact that $\psi'(s) < 0$ for every s, and

$$\begin{cases} \phi(\beta) \leq \phi^*(\beta) \implies \phi^\theta(\beta) \geq \phi(\beta) \implies \Phi_0'(\phi^\theta(\beta)) \geq G^\beta(\phi^\theta(\beta)), \\ \phi(\beta) \geq \phi^*(\beta) \implies \phi^\theta(\beta) \leq \phi(\beta) \implies \Phi_0'(\phi^\theta(\beta)) \leq G^\beta(\phi^\theta(\beta)). \end{cases}$$

Hence the integrand is nonnegative for every β. □

Example 3. Assume that X is exponentially distributed, as in (4.29), and that there exists a subinterval $[x_1, x_2] \subset [p_0, \overline{P}]$ such that

$$\Phi_0'(x) - \frac{1}{\lambda(x - p_0)} \qquad \begin{cases} < 0 & \text{if } x < x_1\,, \\ = 0 & \text{if } x \in [x_1, x_2]\,, \\ > 0 & \text{if } x > x_2\,. \end{cases}$$

Then a pricing strategy $\phi : [0, \kappa] \mapsto [0, \overline{P}]$ is optimal if and only if it takes values inside the interval $[x_1, x_2]$.
Indeed, in this particular case the function

$$G^\beta(p) = \frac{1}{\lambda(p - p_0)} \qquad \text{for all } p \in [x_1, x_2], \quad \beta \in [0, \kappa]$$

does not depend on β. The result follows directly from Theorem 6.1 and Example 2.

7. Nash Equilibria. We now assume that n traders compete, selling different amounts of the same asset. For $i = 1, \ldots, n$, let κ_i be the amount of stock put on sale by the i-th agent and let $\phi_i : [0, \kappa_i] \mapsto I\!R_+$ be his pricing strategy. We wish to study Nash non-cooperative equilibria, where the strategy of each player is an optimal reply to the strategies adopted by all the other players. In the following, we assume that all traders have the same payoff function, and they all assign the same probability distribution to the random size X of the incoming order.

Definition 3. *Let $\phi_i^* : [0, \kappa_i] \mapsto [0, \overline{P}]$ be the pricing strategy of the i-th player. Define the right continuous, non-decreasing functions*

$$\Phi_i(p) \doteq \sum_{j \neq i} meas\Big(\{\beta \in [0, \kappa_j]\,;\ \ \phi_j(\beta) \leq p\}\Big), \qquad i = 1, \ldots, n\,. \tag{7.1}$$

Then the n-tuple of strategies $(\phi_1^*, \ldots, \phi_n^*)$ is a **Nash equilibrium solution** *to the bidding game if each ϕ_i^* provides an optimal pricing strategy for the problem*

$$\text{Maximize:} \qquad J_i(\phi) \doteq \int_0^{\kappa_i} (\phi(\beta) - p_0)\, \psi\Big(\beta + \Phi_i(\phi(\beta))\Big)\, d\beta\,. \tag{7.2}$$

Remark 4. The above definition does not mention the possible priority of one seller over another. Indeed, priority is irrelevant, because in any Nash equilibrium it is not possible that two sellers offer positive amounts of asset at the same price p^*. If this happens, the agent that does not have priority could offer his amount at price $p^* - \varepsilon$ with $\varepsilon > 0$ sufficiently small, and achieve a strictly larger expected payoff. This motivates our choice (7.1) of right-continuous functions Φ_i.

If the random variable X is of type A, in this section we shall prove that a Nash equilibrium solution always exists, and explicitly determine the strategies of the various players. On the other hand, if X is of type B, we prove that no Nash equilibrium solution can exist.

As a preliminary example, given a random variable X of type A we construct the Nash equilibrium in the special case when all players have the exact same amount of shares to offer for sale.

LEMMA 7.1 (Nash equilibrium for identical players). *Assume that X is of type A and satisfies the assumptions (A1). Consider n players, each one putting on sale the same amount $\kappa = \kappa_1 = \cdots = \kappa_n$ of asset. Then the pricing strategies*

$$\phi_1^*(\beta) \;=\; \phi_2^*(\beta) \;=\; \cdots \;=\; \phi_n^*(\beta) = \phi(\beta), \tag{7.3}$$

with

$$\phi(\beta) \;\doteq\; p_0 + [\overline{P} - p_0] \left(\frac{\psi(n\beta)}{\psi(n\kappa)}\right)^{\frac{1-n}{n}}, \qquad\qquad \beta \in [0, \kappa], \tag{7.4}$$

provide a Nash equilibrium solution to the bidding game (7.2).

Proof. **1.** Since $\psi' < 0$, the pricing strategies in (7.3)-(7.4) are strictly increasing. Moreover, for $i = 1, \ldots, n$, the functions $\Phi_1 = \cdots = \Phi_n = \Phi$ in (7.1) are all equal and satisfy

$$\Phi_i(\phi(\beta)) \;=\; \Phi(\phi(\beta)) \;=\; (n-1)\beta, \qquad\qquad \Phi'(\phi(\beta)) \;=\; \frac{n-1}{\phi'(\beta)}. \tag{7.5}$$

By (7.4)-(7.5), a direct computation shows that

$$\Phi(\overline{P}) \;=\; (n-1)\kappa\,, \qquad \Phi(p) \;=\; 0 \quad \text{for} \quad p \;\leq\; p_A \;\doteq\; p_0 + [\overline{P} - p_0]\,(\psi(n\kappa))^{\frac{n-1}{n}}\,,$$

$$\Phi(p) \;>\; 0, \qquad \Phi'(p) \;=\; \frac{-\psi\left(\frac{n}{n-1}\Phi(p)\right)}{(p-p_0)\psi'\left(\frac{n}{n-1}\Phi(p)\right)} \qquad\qquad \text{for} \;\; p_A < p < \overline{P}\,. \tag{7.6}$$

The ask price p_A is the minimum price at which some of the asset is offered for sale.

2. In order to check the necessary condition (4.28), we compute

$$G^\beta(p) \;=\; -\,\frac{\psi\left(\beta + \Phi(p)\right))}{(\phi(\beta) - p_0)\psi'\left(\beta + \Phi(p)\right))}\,.$$

By (7.5) and (7.7), this yields

$$G^\beta(\phi(\beta)) \;=\; -\,\frac{\psi\left(n\beta\right)}{(\phi(\beta) - p_0)\psi'\left(n\beta\right)} \;=\; \Phi'(\phi(\beta)),$$

showing that (4.28) holds.

3. To prove that the n-tuple of pricing strategies in (7.3)-(7.4) provides a Nash equilibrium, we need to show that each strategy satisfies the sufficient conditions for optimality (6.1).

Fix any value $\beta^* \in [0,\kappa]$ and call $p^* \doteq \phi(\beta^*)$.

Consider any two prices $p_1, p_2 \in [p_0, \overline{P}]$, with $p_1 < p^* < p_2$. As observed in Remark 2, since the random variable X is of type A, the map $\beta \mapsto G^\beta(p)$ is nondecreasing. Hence

$$G^{\beta^*}(p_2) \;\leq\; G^{\beta_2}(p_2) \;=\; \Phi_0'(p_2)\,, \tag{7.7}$$

where $\beta_2 > \beta^*$ is such that $p_2 = \phi(\beta_2)$.

Next, if $p_1 > \phi(0)$, there exists $\beta_1 < \beta^*$ such that $\phi(\beta_1) = p_1$ and

$$G^{\beta^*}(p_1) \;\geq\; G^{\beta_1}(p_1) \;=\; \Phi_0'(p_1)\,. \tag{7.8}$$

On the other hand, if $p_1 \leq \phi(0)$, then $\Phi_0'(p_1) = 0$ and clearly

$$G^{\beta^*}(p_1) \;>\; \Phi_0'(p_1)\,. \tag{7.9}$$

The three inequalities (7.10), (7.11), (7.12) show that the sufficient conditions (6.1) are satisfied, and therefore $(\phi_1^*, \ldots, \phi_n^*)$ provides a Nash equilibrium. □

Remark 5. In this Nash equilibrium the expected payoff of each agent is

$$J(\phi) \;=\; \int_0^\kappa (\phi(\beta) - p_0)\cdot \psi(n\beta)\,d\beta \;=\; \frac{1}{n}\,(\psi(n\kappa))^{\frac{n-1}{n}} \cdot (\overline{P} - p_0)\cdot \int_0^{n\kappa} \psi(s)^{\frac{1}{n}}\,ds.$$

We now extend the previous result to an arbitrary number of players, putting on sale different amounts of the asset.

THEOREM 7.2 (existence of a Nash equilibrium). *Let X be a random variable of type A, satisfying the assumptions (A1). Given $n \geq 2$ players, putting on sale the amounts $\kappa_1, \ldots, \kappa_n > 0$ of the same asset, the bidding game (7.2) has a Nash equilibrium.*

Proof. **1.** Without loss of generality, we can assume that

$$0 \;<\; \kappa_1 \;\leq\; \kappa_2 \;\leq\; \cdots \;\leq \kappa_n\,.$$

Define

$$\begin{cases} p_n & \doteq \;\; \overline{P}, \\ p_j & \doteq \;\; p_0 + \Big(\psi((n-j+1)h_j)\Big)^{\frac{n-j}{n-j+1}} \cdot [p_{j+1} - p_0] \;\; \text{if} \;\; j = 1, \ldots, n-1. \end{cases}$$

We claim that a Nash equilibrium solution is provided by the following pricing strategies:

$$
\begin{aligned}
\phi_1(\beta) &\doteq p_0 + [p_2 - p_0]\left(\frac{\psi(n\kappa_1)}{\psi(n\beta)}\right)^{\frac{n-1}{n}} \qquad \beta\in[0,\kappa_1],\\
&\vdots\\
\phi_j(\beta) &\doteq \begin{cases} \phi_{j-1}(\beta) & \beta\in[0,\kappa_{j-1}],\\[2mm] p_0+[p_{j+1}-p_0]\left(\dfrac{\psi((n-j+1)(\kappa_j-\kappa_{j-1}))}{\psi((n-j+1)(\beta-\kappa_{j-1}))}\right)^{\frac{n-j}{n-j+1}} & \beta\in[\kappa_{j-1},\kappa_j],\end{cases}\\
&\vdots\\
\phi_n(\beta) &\doteq \begin{cases} \phi_{n-1}(\beta) & \beta\in[0,\kappa_{j-1}],\\[2mm] \overline{P} & \beta\in[\kappa_{n-1},\kappa_n].\end{cases}
\end{aligned}
\tag{7.10}
$$

2. Starting from the explicit formulas (7.15), a direct computation shows that the corresponding functions Φ_i in (7.1) satisfy

$$
0 \;\leq\; \Phi_n(p) \;\leq\; \Phi_{n-1}(p) \;\leq\; \cdots \;\leq\; \Phi_1(p), \qquad \text{for all } p\in[p_0,\overline{P}[\,. \tag{7.11}
$$

Moreover, for every $j=1,\ldots,n$ one has (see Fig. 7.1)

$$
\Phi_j(p) \;=\; \begin{cases} \Phi_n(p) & \text{for all } p\in[p_0,p_{j+1}[\,,\\[2mm] \dfrac{n+1-\ell}{n-\ell}\,\Phi_n(p) & \text{for all } p\in[p_\ell,p_{\ell+1}[\,, \quad \ell>j\,.\end{cases} \tag{7.12}
$$

To determine all functions Φ_j, it thus suffices to compute Φ_n. This is a continuous, nondecreasing, piecewise $\mathcal{C}^1$ function on $[0,\overline{P}[\,$, which satisfies

$$
\begin{aligned}
\Phi_n(p) &\;=\; 0 && \text{if} \quad p\leq p_1\,,\\
\Phi_n'(p) &\;=\; \frac{-\psi\left(\frac{n+1-j}{n-j}\Phi_n(p)\right)}{(p-p_0)\psi'\left(\frac{n+1-j}{n-j}\Phi_n(p)\right)} && \text{if} \quad p_j<p\leq p_{j+1}\,.
\end{aligned}
\tag{7.13}
$$

By (7.17) it follows

$$
\Phi_j'(p) = \begin{cases} \frac{n+1-\ell}{n-\ell}\,\Phi_n(p) & p\in[p_\ell,p_{\ell+1}], \quad \ell>j\,,\\[2mm] \Phi_n'(p) & p<p_{j+1}\,.\end{cases} \tag{7.14}
$$

In particular, by (7.18)-(7.19) it follows that the necessary conditions $\Phi_i'(\phi_i(\beta)) = G^\beta(i(\phi_i(\beta))$, stated in Corollary 4.3, are satisfied.

FIG. 7.1. *A Nash equilibrium with four players. Here $\psi(s)=e^{-s}$, $p_0=0$, $\overline{P}=8$. The amounts κ_i are chosen so that $p_A=2$, $p_2=4$, $p_3=6$.*

3. In order to apply the sufficient condition for optimality stated in Theorem 6.1, given any $p^* = \phi_i(\beta^*)$, we need to check that

$$\begin{cases} \Phi_i'(p) \ \leq \ G_i^{\beta^*}(p) & \text{if} \quad p < p^* , \\ \\ \Phi_i'(p) \ \geq \ G_i^{\beta^*}(p) & \text{if} \quad p > p^* , \end{cases}$$

where $G_i^\beta(p)$ is defined as in (4.1), with Φ_0 replaced by Φ_i:

$$G_i^\beta(p) \ \doteq \ -\psi\Big(\beta + \Phi_i(p)\Big) \cdot \Big[(p-p_0)\psi'\Big(\beta+\Phi_i(p)\Big)\Big]^{-1}.$$

We observe that, since the random variable X is of type A, from (7.16) it follows

$$G_n^\beta(p) \ \leq \ G_{n-1}^\beta(p) \ \leq \ \cdots \ \leq \ G_1^\beta(p).$$

To fix the ideas, assume $p^* \in [p_i, p_{i+1}]$. As in the proof of Lemma 7.1, we shall consider various cases.

CASE 1: $p < p_1$. In this case $\Phi_i'(p) = 0$ and the inequality $G_i^\beta(p) > \Phi_i'(p)$ is trivial.

CASE 2: $p_1 < p < p^*$. We can then find $\beta \in [0, \beta^*]$ such that $\phi_i(\beta) = p$. Since the random variable X is of type A, by Remark 2 this implies

$$G_i^{\beta^*}(p) \ \geq \ G_i^\beta(p) \ = \ \Phi_i'(p).$$

CASE 3: $p^* < p < p_{i+1}$. Choose $\beta \in]\beta^*, \kappa_i]$ such that $\phi_i(\beta) = p$. Again by Remark 2, this implies

$$G_i^{\beta^*}(p) \ \leq \ G_i^\beta(p) \ = \ \Phi_i'(p).$$

CASE 4: $p > p_{i+1}$. In this case, we can choose $\beta \in]\kappa_i, \kappa_n]$ such that $\phi_n(\beta) = p$. This yields

$$G_i^{\beta^*}(p) \ \leq \ G_i^{\kappa_i}(p) \ = \ G_n^\beta(p) \ = \ \Phi_n'(p) \ \leq \ \Phi_i'(p).$$

□

THEOREM 7.3 (nonexistence of a Nash equilibrium). *Let X be a random variable of type B, satisfying the assumptions (A1). Then, for any number $n \geq 2$ of players offering amounts $\kappa_1, \ldots, \kappa_n > 0$ of the same asset for sale, a Nash equilibrium cannot exist (regardless of the selling priorities established among the players).*

Proof. **1.** Assume, on the contrary, that a Nash equilibrium $(\phi_1^*, \ldots, \phi_n^*)$ exists. By Theorem 5.3, each pricing strategy ϕ_i^* must be constant, say

$$\phi_i^*(\beta) \ \equiv \ p_i, \qquad\qquad i = 1, \ldots, n.$$

We claim that

$$i \neq j \quad \Longrightarrow \quad p_i \neq p_j .$$

Otherwise, since one of the two players does not have the priority over the other, he could increase his expected payoff by pricing all his asset at $p_i - \varepsilon$, for some ε small enough.

2. Let $\varepsilon \ \doteq \ \min_{i \neq j} |p_i - p_j|$. Choose $k \in \{1, \ldots, n\}$ such that $p_k < \overline{P}$. Then the k-th player can unilaterally increase his payoff by using the strategy

$$\tilde\phi_k^*(\beta) \ = \ p_k + \frac{\varepsilon}{2}.$$

This contradiction shows that no Nash equilibrium can exist. □

8. Uniqueness of the Nash equilibrium. In this section we prove that, if the random variable X is of type A, then the Nash equilibrium constructed in Theorem 7.2 is unique.

In the following, given an n-tuple of pricing strategies $\phi : [0, \kappa_i] \mapsto [0, \overline{P}]$, we denote by

$$F_i(p) \doteq \sup \{\beta \in [0, \kappa_i] ; \quad \phi_i(\beta) \le p\}$$

the amount of asset put on sale at price $\le p$ by the i-th player. Moreover, we define

$$F(p) = \sum_{i=1}^{n} F_i(p).$$

Observe that, with these definitions, the functions Φ_i in (7.1) are expressed by

$$\Phi_i(p) = \sum_{j \ne i} F_j(p) = F(p) - F_i(p) .$$

Lemma 8.1. *Let the n-tuple $(\phi_1, \ldots, \phi_n)$ be a Nash equilibrium. Then the following holds.*

(i) There exists a Lipschitz constant C such that

$$F(p_2) - F(p_1) \le C(p_2 - p_1) \qquad \text{for all } p_0 < p_1 < p_2 < \overline{P} .$$

(ii) At most one of the functions F_i can have an upward jump at $p = \overline{P}$, while all the others are Lipschitz continuous on the entire interval $[0, \overline{P}]$.

(iii) There exists a minimum ask price p_A and a constant $\delta_0 > 0$ such that

$$F(p) = 0 \quad \text{for all } p \le p_A , \qquad F'(p) \ge \delta_0 \quad \text{for a.e. } p \in [p_A, \overline{P}].$$

Proof. **1.** Let

$$p_* = p_0 + \psi(K)(\overline{P} - p_0) > p_0,$$

and

$$C \doteq \max \Big\{ G_i^{\beta}(p) \ ; \ \beta \in [0, \kappa], \ p \in [p_*, \overline{P}], \ i \in \{1, 2, \ldots, n\} \Big\} + 1. \tag{8.1}$$

We claim that for every $i \in \{1, \ldots, n\}$, the set

$$S_i \doteq \Big\{ p \in [p_*, \overline{P}] ; \quad F_i(p) > F_i(q) + C(p - q) \quad \text{for some} \quad q < p \Big\} \tag{8.2}$$

is empty.

Indeed, if $S_i \ne \emptyset$, we can write S_i as a union of intervals, say

$$S_i = \bigcup_k \,]a_k, b_k] .$$

Consider any other player, say the j-th player, with $j \ne i$. Then

$$meas\Big(\{\beta \in [0, \kappa_j] ; \quad \phi_j(\beta) \in]a_k, b_k[\ \} \Big) = 0. \tag{8.3}$$

Otherwise, the j-th player could get a strictly higher expected payoff by using the strategy

$$\tilde{\phi}_j(\beta) \doteq \begin{cases} a_k & \text{if } \phi_j(\beta) \in]a_k, b_k[\,, \\ \phi_j(\beta) & \text{otherwise,} \end{cases}$$

as the following computation shows:

$$\begin{aligned} J(\tilde{\phi}_j) - J(\phi_j) &= \int_{\{\beta\ ;\ \phi_j(\beta)\in[a_k,b_k]\}} (a_k - p_0)\cdot\psi(\beta + \Phi_j(a_k)) - (\phi_j(\beta) - p_0)\cdot\psi(\beta + \Phi_j(\phi_j(\beta)) \\ &\geq -\int_{\{\beta\ ;\ \phi_j(\beta)\in[a_k,b_k]\}} \int_{a_k}^{\phi_j(\beta)} (p-p_0)\psi'\left(\beta + \Phi_j(a_k) - C(a_k - p)\right)\cdot \\ &\qquad \cdot\left(C - \frac{\psi\Big(\beta + \Phi_j(a_k) - C(a_k-p)\Big)}{(p-p_0)\psi'\Big(\beta+\Phi_j(a_k) - C(a_k-p)\Big)}\right) dp\, d\beta \\ &\geq -\int_{\{\beta\ ;\ \phi_j(\beta)\in[a_k,b_k]\}} \int_{a_k}^{\phi_j(\beta)} (p-p_0)\psi'\left(\beta + \Phi_j(a_k) - C(a_k - p)\right)\cdot \\ &\qquad \cdot\left(C - G_j^{\beta}(a_k)\right) dp\, d\beta \ > \ 0. \end{aligned}$$

The first inequality follows from (8.6) and the Fundamental Theorem of Calculus, the third inequality follows from the fact that X is of Type A, and the strict inequality follows from the definition (8.5).

However, if (8.7) holds for every $j \neq i$, then the strategy ϕ_i for the i-th player is not optimal. Indeed, he could achieve a strictly higher payoff by setting

$$\tilde{\phi}_i(\beta) \doteq \begin{cases} b_k - \varepsilon & \text{if } \phi_i(\beta) \in [a_k, b_k[\,, \\ \phi_i(\beta) & \text{otherwise,} \end{cases}$$

for some $\varepsilon > 0$ sufficiently small. This proves that Φ_j is Lipschitz on the interval $[p_*, \overline{P}]$ for every $j \in \{1, \ldots, n\}$. Since

$$F = \frac{1}{n-1}\sum_{j=1}^{n} \Phi_j,$$

we conclude that F is Lipschitz continuous on $[p_*, \overline{P}[$.

Let p be a point such that $F'(p) > 0$. Then at least one agent is putting some shares on sale at the price p. From the necessary conditions (4.28) on the best reply of any of the n players, if $F' > 0$, then it satisfies the inequality

$$F'(p) \ \geq\ \Phi_j'(p) \ =\ \frac{-\psi(F)}{(p-p_0)\psi'(F)}\,, \qquad F(\overline{P}) \ \leq\ K \doteq \sum_{i=1}^{n} \kappa_i \quad p \in [p_*, \overline{P}[.$$

Denote by $Y(p)$ the solution to the terminal value problem

$$Y' \ =\ \frac{-\psi(Y)}{(p-p_0)\psi'(Y)}\,, \qquad Y(\overline{P}) \ =\ K\,.$$

By direct computation we see that

$$\psi(Y(p)) \;=\; \frac{\overline{P}-p_0}{p-p_0}\psi(K),$$

which implies that $Y(p_*)=0$. By comparison, we see that $Y(p)\geq F(p)$ and therefore

$$p_A \;\doteq\; \inf\{p : F(p)>0\}\geq p_*>p_0.$$

This proves the first assertion of the Lemma.

2. The second assertion is clear: if two players put a positive amount of asset for sale at the same price $\overline{P}$, the one that does not have priority can improve his expected payoff by selling the asset at price $\overline{P}-\varepsilon$.

3. Toward a proof of (iii), we show that there exists $\delta_0>0$ small enough so that, for any $p^*<\overline{P}$, the following implication holds:

$$F'(p^*)\leq\delta_0 \qquad\Longrightarrow\qquad F(p)=0\quad\text{for all}\quad p\in[0,p^*]\,.$$

Indeed, let

$$\delta_0 \;\doteq\; \frac{1}{2}\min\Big\{G_i^\beta(p)\;;\;\beta\in[0,\kappa],\;p\in[p_0,\overline{P}],\;i\in\{1,2,\ldots,n\}\Big\},$$

and observe that $\delta_0>0$. By (i) it follows that F is differentiable at a.e. point $p\in[0,\overline{P}]$. Assume $F'(p^*)\leq\delta_0$ and consider the non-empty set

$$S^* \;\doteq\; \big\{p<p^*\,;\quad F(p)>F(p^*)-2\delta_0(p^*-p)\big\}\,.$$

If $F(p)=F(p^*)$ for all $p\in S^*$, recalling that F is Lipschitz continuous we conclude that $F(p)=F(p^*)=0$ for all $p\leq p^*$, as claimed.

In the opposite case, there exist $p'<p^*$ such that

$$F(p')\;<\;F(p^*)\,,\qquad F(p)\;\geq\;F(p^*)-2\delta_0(p^*-p)\qquad\text{for all } p\in[p',p^*]. \tag{8.4}$$

Clearly, at least one the the players is putting some assets for sale within the price interval $[p',p^*]$, say, the i-th player. This leads to a contradiction, because by (8.9)

$$\Phi_i(p)\;\geq\;\Phi_i(p^*)-2\delta_0(p^*-p)\,.$$

Hence the strategy

$$\tilde\phi_i(\beta)\;=\;\begin{cases} p^* & \text{if}\quad \phi_i(\beta)\in[p',p^*]\,,\\ \phi_i(\beta) & \text{otherwise}\,,\end{cases}$$

yields a strictly higher expected payoff:

$$J(\tilde\phi_i)-J(\phi_i)\;=\;\int_{\{\beta;\,\phi_i(\beta)\in[p',p^*]\}}\int_{\phi_i(\beta)}^{p^*}(p-p_0)\psi'\left(\beta+\Phi_i(p)\right)\cdot\Big(\Phi_i'(p)-G_i^\beta(p)\Big)\,dp\,d\beta$$

$$\geq\;\int_{\{\beta;\,\phi_i(\beta)\in[p',p^*]\}}\int_{\phi_i(\beta)}^{p^*}(p-p_0)\psi'\left(\beta+\Phi_i(p)\right)\cdot\Big(2\delta_0-G_i^\beta(p)\Big)\,dp\,d\beta>0.$$

□

THEOREM 8.2. *In the same setting of Theorem 7.2, the Nash equilibrium is unique.*

Proof. **1.** Let $(\phi_1, \ldots, \phi_n)$ be a Nash equilibrium. By Lemma 8.1, the corresponding functions F_i are Lipschitz continuous on $[0, \overline{P}[$, and all except at most one of them are Lipschitz continuous on the closed interval $[0, \overline{P}]$. Moreover, there exists a minimum ask price p_A such that (iii) in Lemma 8.1 holds.

2. By Rademacher's theorem, every function F_i is differentiable a.e. on $[0, \overline{P}[$. For each p, consider the set of indices

$$\mathcal{I}(p) \doteq \{i\,;\quad F'(p) > 0\}$$

and call $N(p) \doteq \#\mathcal{I}(p)$ the cardinality of this set. By Lemma 8.1 the function $N(\cdot)$ is well defined and Lebesgue measurable. Moreover, $N(p) \geq 2$ for a.e. $p \in [p_A, \overline{P}]$.

For $p \in [p_A, \overline{P}[$, $i \in \mathcal{I}(p)$, let $\beta_i \in [0, \kappa_i]$ be such that $\phi_i(\beta_i) = p$. Recalling (4.1), from the necessary conditions (4.9) we deduce

$$\Phi_i'(p) \;=\; G_i^{\beta_i}(p) \;=\; \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))} \qquad\qquad i \in \mathcal{I}(p)\,.$$

Observing that

$$\Phi_i'(p) \;=\; \sum_{j\neq i} F_j'(p),$$

one obtains

$$\begin{cases} \Phi_i'(p) \;=\; \dfrac{N(p)-1}{N(p)} F'(p)\,, \qquad & F_i'(p) \;=\; \dfrac{F'(p)}{N(p)} \qquad & \text{for} \quad i \in \mathcal{I}(p)\,, \\[3mm] \Phi_i'(p) \;=\; F'(p), & F_i'(p) \;=\; 0 & \text{for} \quad i \notin \mathcal{I}(p)\,. \end{cases}$$

The Lipschitz function F thus satisfies the ODE

$$F'(p) \;=\; \frac{N(p)}{N(p)-1} \cdot \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))}$$

at a.e. point $p \in [p_A, \overline{P}]$.

3. We claim that, for each $i \in \{1, \ldots, n\}$, the set of prices where the i-th player offers assets for sale is an interval $[p_A, p_{i+1}]$. Assume, on the contrary, that this is not the case. To derive a contradiction, call $S_i \doteq \{p \in [p_0, \overline{P}]\,;\ \ F_i' > 0\}$ and $\mathcal{L}_i \doteq \{p \in [p_0, \overline{P}]\,;\ \ p \text{ is a Lebesgue point of } F_i'\}$. Let $q \in \big([p_A, \overline{P}] \cap \mathcal{L}_i\big) \setminus S_i$ and assume that

$$[q, \overline{P}] \cap \mathcal{L}_i \cap S_i \;\neq\; \emptyset.$$

Let $q^* \doteq \inf [q, \overline{P}] \cap \mathcal{L}_i \cap S_i$. Then, for any $\delta_1 > 0$, the following two sets are non-empty:

$$A \;\doteq\; ([q^* - \delta_1, q^*] \cap \mathcal{L}_i) \setminus S_i \;\neq\; \emptyset, \qquad\qquad B \;\doteq\; [q^*, q^* + \delta_1] \cap \mathcal{L}_i \cap S_i \;\neq\; \emptyset.$$

Indeed, B is nonempty, by the definition of infimum. Moreover, if $q^* = q$ then $q^* \in A \neq \emptyset$, otherwise A is nonempty by the definition of infimum.

From the necessary conditions (4.28) we deduce

$$\Phi_i'(p) \;=\; F'(p) \;=\; \frac{N(p)}{N(p)-1}\cdot \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))} \;\geq\; \frac{n}{n-1}\cdot\frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))}$$

for $p \notin S_i$, while

$$\Phi_i'(p) \;=\; \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))}$$

for $p \in S_i$. Choose the intermediate slope

$$\lambda \;\doteq\; \left(\frac{2n-1}{2n-2}\right)\frac{-\psi(F(q^*))}{(q^*-p_0)\psi'(F(q^*))}\,.$$

By continuity we can choose $\delta_0 < \delta_1$ small enough so that

$$\lambda - \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))} \;>\; 0, \quad \text{for all } p \in [q^*-\delta_0, q^*+\delta_0].$$

Finally, let $p_1 \in A$ and $p_2 \in B$ be Lebesgue points of F_i' and consider the two lines

$$\gamma_1(p) \;=\; \Phi_i(p_1) + \lambda(p-p_1)\,, \qquad\qquad \gamma_2(p) \;=\; \Phi_i(p_2) + \lambda(p-p_2)\,.$$

We split the analysis into two cases.

CASE 1: $\gamma_1 \geq \gamma_2$. We then consider the intermediate point

$$p' \;\doteq\; \min\big\{p > p_1\,;\;\; \Phi_i(p) = \gamma_1(p)\big\}\,.$$

Observe that $p' > p_1$, because $p_1 \in \mathcal{L}_i \setminus \mathcal{S}_i$ and $\Phi_i'(p_1) > \lambda$.
Then the new pricing strategy

$$\tilde\phi_i(\beta) \;=\; \begin{cases} p_1 & \text{if} \quad \phi_i(\beta) \in [p_1, p']\,, \\ \phi_i(\beta) & \text{otherwise,} \end{cases}$$

yields a strictly better expected payoff:

$$\begin{aligned} J(\tilde\phi_i) - J(\phi_i) \;&=\; \int_{\{\beta\ ;\ \phi_i(\beta)\in[p_1,p']\}} (p_1-p_0)\cdot\psi(\beta+\Phi_i(p_1)) - (\phi_i(\beta)-p_0)\cdot\psi(\beta+\Phi_i(\phi_i(\beta))) \\ &\geq\; -\int_{\{\beta\ ;\ \phi_i(\beta)\in[p_1,p']\}}\int_{p_1}^{\phi_i(\beta)} (p-p_0)\psi'\left(\beta+\Phi_i(p)\right)\cdot \\ &\qquad\qquad \cdot\left(\lambda - \frac{\psi\Big(\beta+\gamma_1(p)\Big)}{(p-p_0)\psi'\Big(\beta+\gamma_1(p)\Big)}\right) dp\, d\beta \;>\; 0. \end{aligned}$$

CASE 2: $\gamma_1 < \gamma_2$. We then consider the intermediate point

$$p' \;\doteq\; \max\{p < p_2\,;\;\; \Phi_i(p) = \gamma_2(p)\}\,.$$

An entirely similar argument now shows that the new pricing strategy

$$\tilde{\phi}_i(\beta) \;=\; \begin{cases} p' & \text{if} \quad \phi_i(\beta) \in [p', p_2]\,, \\ \phi_i(\beta) & \text{otherwise,} \end{cases}$$

yields a strictly better expected payoff. In both cases we showed that ϕ_i is not optimal, thus reaching a contradiction.

4. From the previous step it follows

$$p_1 \;\leq\; p_2 \;\leq\; \cdots \;\leq\; p_n \;\leq\; p_{n+1} \;\leq\; \overline{P}.$$

We claim that $p_n = p_{n+1} = \overline{P}$. Indeed, if $p_n < p_{n+1}$, this means that the n-th player is the only seller in the interval $[p_n, p_{n+1}]$. He could achieve a better expected payoff by taking all his assets originally on sale at a price $p \in [p_n, p_{n+1}]$ and offering them at the price p_{n+1} instead. This shows that $p_n = p_{n+1}$.

Finally we show that $p_n = \overline{P}$. Indeed, if this were not the case, we would have

$$F'(p) = 0, \quad \text{for all } p \in\,]p_n, \overline{P}]\,,$$

contradicting the third statement in Lemma 8.1. □

9. A large number of small agents. In this section we study the limiting case where the number of sellers approaches infinity, but the total amount of asset offered for sale remains bounded.

Example 3. Consider the simple case of n players, each one selling the same amount K/n of asset. By (7.7) in the proof of Lemma 7.1, the total amount $Z_n(p) = \frac{n}{n-1}\Phi(p)$ of asset put on sale at price $\leq p$ is found by solving the ODE

$$\frac{n-1}{n} Z_n' \;=\; \frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n)}\,, \qquad\qquad Z_n(\overline{P}) = K\,.$$

As $n \to \infty$, the limit distribution $Z(p) = \lim_{n\to\infty} Z_n(p)$ is clearly obtained by solving

$$Z' \;=\; \frac{-\psi(Z)}{(p-p_0)\psi'(Z)}\,, \qquad\qquad Z(\overline{P}) \;=\; K\,. \tag{9.1}$$

We wish to show that the same limit holds, without assuming that all players put on sale exactly the same amount of asset. Consider a sequence of bidding games, satisfying:

- **(G1)** The n-th game involves n distinct players, selling the amounts $\kappa_{n,1}, \ldots, \kappa_{n,n}$ of the same asset.
- **(G2)** The total amount of asset put on sale in the n-th game is $K_n \doteq \sum_{i=1}^n \kappa_{n,i}$, with $\lim_{n\to\infty} K_n = K$.
- **(G3)** The largest amount of asset put on sale by any player in the n-th game approaches zero: $\lim_{n\to\infty} \left(\sup_{1\leq i\leq n} \kappa_{n,i}\right) = 0$.

The next result shows that, with the above assumptions, as $n \to \infty$ the limit order book approaches a well defined shape. In the following, we call $Z_n(p)$ the amount of asset offered for sale at price $< p$, in the Nash equilibrium solution (7.15) for the n-th game. Moreover, we let $Z(p)$ to be the solution to the Cauchy problem (9.1).

Observe that the right hand side of the ODE in (9.1) is well defined and uniformly positive as long as $Z \in [0, K]$. Indeed,

$$Z'(p) \ \geq \ \frac{C_0}{p - p_0}$$

for some constant $C_0 > 0$. By a comparison argument we conclude that there exists a value $p_A > p_0$ such that the solution of (9.1) satisfies

$$Z(p_A) \ = \ 0\,, \qquad\qquad Z(p) \ > \ 0 \quad \text{for} \quad p_A < p < \overline{P}\,. \tag{9.2}$$

We then extend the function Z to the entire interval $[0, \overline{P}]$ by setting

$$Z(p) \ \doteq \ 0 \qquad\qquad \text{for} \qquad p \in [0, p_A]\,. \tag{9.3}$$

THEOREM 9.1. *Let X be a random variable of type A, satisfying the assumptions (A1). Consider a sequence of games for n players, satisfying (G1)–(G3).*

Then, for any $\varepsilon > 0$, the following holds.

$$\lim_{n\to\infty} Z_n(p) \ = \ Z(p) \qquad \textit{uniformly for all } p \in [0, \overline{P}]\,, \tag{9.4}$$

$$\lim_{n\to\infty} Z_n'(p) \ = \ Z'(p) \qquad \textit{uniformly for all } p \in [0, p_A - \varepsilon] \cup [p_A + \varepsilon\,,\ \overline{P} - \varepsilon]\,, \tag{9.5}$$

where Z is defined by (9.1), (9.3), and p_A is determined by (9.2).

Proof. **1.** For a given $n \geq 1$, it is not restrictive to assume $\kappa_{n,1} \leq \kappa_{n,2} \leq \cdots \leq \kappa_{n,n}$. For $1 < i \leq n$ call $h_{n,i} \doteq \kappa_{n,i} - \kappa_{n,i-1}$. Moreover, set $h_{n,1} \doteq \kappa_{n,1}$. In the Nash equilibrium solution for the n-th game, the total amount $Z_n(p)$ put on sale at price $< p$ is characterized by the equations

$$Z_n(\overline{P}) \ = \ K_n - h_{n,n}\,, \qquad\qquad Z_n(p) \ = \ 0 \qquad \text{for} \ \ p \in [0, p_{n,1}]\,, \tag{9.6}$$

$$Z_n'(p) \ = \ \frac{n-i+1}{n-i} \cdot \frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))} \qquad \text{for} \ \ p_{n,i} < p < p_{n,i+1}\,, \quad 1 \leq i < n\,. \tag{9.7}$$

Here the prices $p_{n,i}$ are determined by the inductive rule

$$p_{n,n} \ = \ \overline{P}, \qquad\qquad \int_{p_{n,i}}^{p_{n,i+1}} \frac{Z_n'(p)}{n-i+1}\, dp \ = \ h_{n,i-1} \qquad \text{for} \ \ i \geq 1\,. \tag{9.8}$$

Recalling that $\psi > 0$, $\psi' < 0$, from (9.7) we deduce

$$\frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))} \ \leq \ Z_n'(p) \ \leq \ \frac{-2\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))}\,, \qquad p_{n,1} < p < \overline{P}, \tag{9.9}$$

$$\frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))} \ \leq \ Z_n'(p) \ \leq \ \frac{-(m+1)}{m} \frac{\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))} \qquad \text{for} \ p_{n,n-m} < p < \overline{P}\,. \tag{9.10}$$

2. For any fixed $m \geq 1$, we claim that

$$p_{n,n-m} \ \to\ \overline{P}\,, \qquad Z_n(p_{n,n-m}) \ \to\ K \qquad \text{as} \quad n \to \infty. \tag{9.11}$$

Indeed, by (9.9) it follows that all maps $Z_n(\cdot)$ are increasing and uniformly Lipschitz continuous, say

$$Z_n(\overline{P}) - C(\overline{P} - p) \ \leq\ Z_n(p) \ \leq\ Z_n(\overline{P}) \qquad \text{for all} \ \ p \in \left[\frac{p_A + \overline{P}}{2}\,,\ \overline{P}\right],$$

for some Lipschitz constant C. Since $Z_n(\overline{P}) = K_n - h_n \to K$ as $n \to \infty$, we can find $\delta > 0$ such that

$$\frac{K}{2} \ \leq\ Z_n(p) \ \leq\ 2K \qquad \text{for all} \ \ p \in [\overline{P} - \delta,\, \overline{P}] \tag{9.12}$$

and all n sufficiently large. By (9.8) one has

$$\begin{aligned}\int_{p_{n,n-m}}^{\overline{P}} Z_n'(p)\,dp \quad &\leq (m+1) \sum_{i=n-m}^{n-1} \int_{p_{n,i}}^{p_{n,i+1}} \frac{Z_n'(p)}{n-i+1}\,dp \ =\ (m+1) \sum_{i=n-m}^{n} h_{n,i-1} \\ &\leq\ (m+1)\,(\kappa_{n,n-1} - \kappa_{n-m-1}) \ \leq\ (m+1)\,\kappa_{n,n} \ \to\ 0\end{aligned} \tag{9.13}$$

as $n \to \infty$. Together, (9.13) and (9.14) imply (9.11). Indeed, using (9.10), (9.13) and the assumption (A1), it follows that, if $p_{n,n-m} < \overline{P} - \delta$, then

$$\int_{p_{n,n-m}}^{\overline{P}} Z_n'(p)\,dp \ \geq\ \int_{p_{n,n-m}}^{\overline{P}} \frac{-\psi(K/2)}{(\overline{P} - p_0)\,\psi'(Z_n(p))}\,dp \ \geq\ m_0 \frac{\psi(K/2)}{(\overline{P} - p_0)}\delta\,,$$

where

$$m_0 \ \doteq\ \min_{s \in [K/2,\, K]} \frac{-1}{\psi'(s)} \ >\ 0.$$

By (9.14) we thus have $p_{n,n-m} \geq \overline{P} - \delta$ for all n sufficiently large. Therefore

$$\frac{K}{2} \cdot (\overline{P} - p_{n,n-m}) \ \leq\ \int_{p_{n,n-m}}^{\overline{P}} Z_n'(p)\,dp \ \to\ 0,$$

showing that $p_{n,n-m} \to \overline{P}$ as $n \to \infty$. In turn, this implies

$$\begin{aligned}|Z_n(p_{n,n-m}) - K| \quad &\leq\ |Z_n(p_{n,n-m}) - Z_n(\overline{P})| + |Z_n(\overline{P}) - K| \\ &\leq\ C(\overline{P} - p_{n,n-m}) + |K - K_n| + \kappa_{n,n} \ \to\ 0.\end{aligned}$$

3. By the previous step, the function Z_n satisfies the differential inequalities

$$-\frac{m+1}{m} \frac{\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))} \ \leq\ Z_n'(p) \ \leq\ \frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))}\,, \quad p_{n,1} < p < p_{n,n-m}, \tag{9.14}$$

with terminal conditions at $p = p_{n,n-m}$ satisfying (9.11). We now compare (9.15) and (9.11) with (9.1). By standard results on the continuous dependence of solutions to a Cauchy problem, for any $\varepsilon > 0$ we have the convergence

$$Z_n(p) \ \to\ Z(p), \qquad Z_n'(p) \ \to\ Z'(p), \tag{9.15}$$

uniformly on the interval $[p_A+\varepsilon,\, \overline{P}-\varepsilon]$.

By (9.9), on the region where $Z_n>0$ the derivative satisfies $Z_n'(p)\geq c_0$ for some constant $c_0>0$ and all $p>0$, $n\geq 2$. Since in (9.16) we can choose $\varepsilon>0$ arbitrarily small, we conclude that the value p_n in (9.6) satisfy

$$\lim_{n\to\infty} p_{n,1} \;=\; p_A \tag{9.16}$$

Observing that

$$\begin{array}{rclll} Z_n(p) & = & 0 & \text{for} & p\in[0,p_{n,1}]\,,\\ Z(p) & = & 0 & \text{for} & p\in[0,p_A]\,, \end{array}$$

and that all functions Z, Z_n are uniformly Lipschitz continuous, from (9.16) and (9.17) we deduce the convergence (9.4)-(9.5). □

10. Examples. In this section we consider in more detail some particular cases, when the probability distribution of size of incoming market order is given.

Example 4. Assume that the size of the incoming market order is exponentially distributed, with mean λ^{-1}. Two competing agents put on sale the amounts $\kappa_1<\kappa_2$ of shares. The Nash equilibrium (7.15) is given by

$$\phi_2^*(\beta) \;=\; \begin{cases} p_0+e^{-\lambda\kappa_1+\lambda\beta}\cdot[\overline{P}-p_0], & \beta\in[0,\kappa_1]\,,\\ \overline{P} & \beta\in[\kappa_1,\kappa_2]\,, \end{cases}$$

$$\phi_1^*(\beta) \;=\; p_0+e^{-\lambda\kappa_1+\lambda\beta}\cdot[\overline{P}-p_0], \qquad \beta\in[0,\kappa_1]\,.$$

The *cumulative* limit order book is thus given by

$$F(p)=\begin{cases} 0 & p\in[p_0,p_A[,\\ \dfrac{2}{\lambda}\ln\dfrac{(p-p_0)}{\overline{P}-p_0} & p\in[p_A,\overline{P}[,\\ \kappa_1+\kappa_2 & p=\overline{P}. \end{cases}$$

This corresponds to a limit order book *density*

$$F'(p) \;=\; \frac{2}{\lambda(p-p_0)}\chi_{[p_A,\overline{P}]}(p)+(\kappa_2-\kappa_1)\cdot\delta_{\overline{P}},$$

where $\delta_{\overline{P}}$ denotes a unit Dirac mass located at $p=\overline{P}$, and the ask price p_A is given by

$$p_A \;=\; p_0+(\overline{P}-p_0)\cdot e^{-\lambda\kappa_1}.$$

The expected payoffs of the two agents in the Nash equilibrium configuration are given by

$$J_1 \;=\; \int_0^{\kappa_1}(\phi_1^*(\beta)-p_0)\cdot e^{-2\lambda\beta}\,d\beta \;=\; \frac{e^{-\lambda\kappa_1}(1-e^{-\lambda\kappa_1})(\overline{P}-p_0)}{\lambda}$$

$$J_2 \;=\; J_1+E[(X-\kappa_1)_+\wedge\kappa_2] \;=\; \frac{e^{-\lambda\kappa_1}(1-e^{-\lambda(\kappa_2+2\kappa_1)})(\overline{P}-p_0)}{\lambda}\,.$$

We observe that an increase in the total amount put on sale by the smaller player (hence by both players) lowers the ask price, and also decreases the expected payoff of both competitors. On the other hand, the larger player can increase his expected payoff by increasing the total amount of shares he puts on sale:

$$J_2 \nearrow \frac{e^{-\lambda\kappa_1}(\overline{P}-p_0)}{\lambda}, \quad \text{as } \kappa_2 \to \infty, \quad \kappa_1 \text{ fixed.}$$

Finally, using the explicit expression of the limit order book resulting from the Nash equilibrium, we can also derive an expression for the *price impact function* $\rho(X)$, which represents the increase in the ask price in response to a market order of size X. Indeed, $\rho(X)$ is defined by the following implicit equation

$$Z(p_A + \rho(X)) \;=\; X\,.$$

This yields

$$\rho(X) = \begin{cases} (e^{\frac{\lambda X}{2}}-1)(\overline{P}-p_0)e^{-\lambda\kappa_1} & \text{if } \; X \le 2\kappa_1\,, \\ \overline{P}-p_A & \text{if } \; X > 2\kappa_1\,. \end{cases}$$

Example 5. Consider the asymptotic limit of a large number of small agents, putting on sale a total amount of K shares. Assume that the size of the incoming market order is exponentially distributed with mean λ^{-1}. In this case, the Cauchy problem (9.1) simplifies to

$$Z'(p) \;=\; \frac{1}{\lambda(p-p_0)}\,, \qquad\qquad Z(\overline{P}) \;=\; K\,.$$

The expected payoff per unit amount of asset put on sale by any agent is given by

$$J_u \;=\; (\overline{P}-p_0)e^{-\lambda K}$$

The ask price is $p_A = p_0 + (\overline{P}-p_0)\cdot e^{-\lambda K}$, while the price impact function is given by

$$\rho(X) = \begin{cases} (e^{\lambda X}-1)(\overline{P}-p_0)e^{-\lambda K} & \text{if } \; X \le K\,, \\[2ex] \overline{P}-p_A & \text{if } \; X > K\,. \end{cases}$$

Example 6. Assume that the random size X of the incoming buying order is distributed according to the power law distribution $\psi(s) = (1+s)^{-\alpha}$. Consider n players, each one putting on sale the same amount κ of shares, for a total amount of $K = n\kappa$. The Nash equilibrium is thus given by (7.4):

$$\phi_1^*(\beta) \;=\; \cdots \;=\; \phi_n^*(\beta) \;=\; \phi(\beta) \;\doteq\; p_0 + [\overline{P}-p_0]\cdot\left(\frac{1+n\beta}{1+n\kappa}\right)^{\frac{n-1}{n}\alpha},$$

and the corresponding ask price is $p_A^n \;=\; \phi(0) \;=\; p_0 + [\overline{P}-p_0]\cdot(1+n\kappa)^{\frac{1-n}{n}\alpha}$. The cumulative limit order book is thus given by

$$Z_n(p) \;=\; (1+n\kappa)\left(\overline{P}-p_0\right)^{-\frac{1}{\alpha}\frac{n}{n-1}}\cdot(p-p_0)^{\frac{1}{\alpha}\frac{n}{n-1}} - 1, \qquad p\in[p_A^n,\overline{P}].$$

The corresponding order book density is then

$$Z_n'(p) \;=\; \frac{n}{\alpha(n-1)}\cdot(1+n\kappa)\left(\overline{P}-p_0\right)^{-\frac{1}{\alpha}\frac{n}{n-1}}\cdot(p-p_0)^{\frac{n(1-\alpha)+\alpha}{n\alpha-\alpha}}, \qquad p\in[p_A^n,\overline{P}].$$

From the above expressions we can easily compute the asymptotic limit as the number of players goes to infinity, for $K=n\kappa$ fixed. The ask price is $p_A=p_0+[\overline{P}-p_0]\cdot(1+K)^{-\alpha}$ and the limit order book is given by

$$Z'(p) \;=\; \frac{1}{\alpha}\cdot(1+K)\left(\overline{P}-p_0\right)^{-\frac{1}{\alpha}}\cdot(p-p_0)^{\frac{1-\alpha}{\alpha}}, \qquad p\in[p_A,\overline{P}].$$

In this case, the price impact function is given by

$$\rho(X)=\frac{\overline{P}-p_0}{(1+K)^\alpha}\cdot[(1+X)^\alpha-1], \quad X\leq K.$$

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