Optimal Open-loop Strategies in a Debt Management Problem

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Abstract

The paper studies optimal strategies for a borrower who needs to repay his debt, in an infinite time horizon. An instantaneous bankruptcy risk is present, which increases with the size of the debt. This induces a pool of risk-neutral lenders to charge a higher interest rate, to compensate for the possible loss of part of their investment.

Solutions are interpreted as Stackelberg equilibria, where the borrower announces his repayment strategy $u(t)$ at all future times, and lenders adjust the interest rate accordingly. This yields a highly nonstandard problem of optimal control, where the instantaneous dynamics depends on the entire future evolution of the system. Our analysis shows the existence of optimal open-loop controls, deriving necessary conditions for optimality and characterizing possible asymptotic limits as $t \to +\infty$.

Key Words: optimal control, open-loop strategies, necessary conditions, infinite time horizon, debt management, bankruptcy risk.

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1 Introduction

We consider an optimization problem for a borrower, who needs to repay his debt, in an infinite time horizon. The main feature of our model is the presence of a bankruptcy risk. As a consequence, the interest rate payed on the loan is not a priori given, but must be determined as part of the solution.

The debt is financed by selling bonds, which promise a stream of payments to the investors. When bankruptcy happens, the borrower incurs in a very large cost B, while the lenders recover only a fraction $\theta \in [0, 1]$ of their outstanding capital. For this reason, bonds are sold at a fraction $p \in [0,1]$ of their nominal value, to compensate investors for the possible loss of part of their capital.

A related model was recently considered in [9], formulated in terms of two state variables: the total amount of debt X and the average interest A payed on outstanding loans, at any given time t. In the present paper we introduce a simpler model, involving one single state variable x, describing the nominal value of the debt. This new model still captures the heart of the matter. While the analysis in [9] was mainly focused on existence of an optimal solution, here we can also derive necessary conditions for optimality, and determine the asymptotic limit of the solution as $t \to +\infty$.

Our solutions should be interpreted in a Stackelberg sense. The borrower announces a repayment strategy $u = u(t)$ for all future times. In turn, the discounted bond price $p(t)$ is determined by the competition among a pool of risk-neutral lenders, based on the bankruptcy risk at all future times. Since this risk grows with the total size of the debt, we obtain a highly non-standard optimal control problem for the borrower, where the instantaneous dynamics depends on the entire future trajectory.

The remainder of the paper is organized as follows. In Section 2, we introduce the model and collect all the assumptions of the parameters and functions. Also, we formulate the optimal control problem for the borrower in open-loop form, given the initial value of the debt.

Section 3 contains a careful analysis of the dynamics of the system. Given an initial data $x(0) = x_0$ and a control function $u = u(t)$, we show that our evolution equations always admit at least one solution. Differently from usual control systems, here the solution may not be unique. Indeed, the competition among lenders may yield multiple Nash equilibria. In practical terms, this can be explained as follows.

If lenders regard their investment as safe, they will buy bonds at almost full price $p \approx 1$. This lowers the burden of servicing the debt, thus reducing the chance of bankruptcy.
On the other hand, given the same initial debt size, if lenders regard their investment as risky, they will buy bonds only at a deeply discounted price $p << 1$. In turn, this forces the borrower to sell a larger amount of bonds to raise the same amount of cash, pushing up the total debt and hence the risk of bankruptcy. In this respect, our model captures what is commonly called a "self-fulfilling prophecy".

In Section 4 we prove that, for any given initial amount of debt, there exists at least one optimal open-loop control $u^*(\cdot)$ and a corresponding solution $x^*(\cdot)$, minimizing the expected total cost to the borrower.

Section 5 is concerned with necessary conditions for optimality. Two main cases are considered: either (i) bankruptcy occurs with probability one within a bounded interval of time, or (ii) as $t \to +\infty$, the debt approaches an asymptotic equilibrium state x_∞. In both cases, optimality conditions are obtained, relying on the Pontryagin Maximum Principle. Finally, Section 6 provides an example where the optimal strategies are analytically described and numerically computed.

For the basic theory of optimal control and the Pontryagin Maximum Principle we refer to [10, 12]. Necessary conditions for control problems with infinite time horizon were derived in [2, 3, 14, 16]. An introduction to non-cooperative differential games can be found in [6, 7]. In the economics literature, some related models of debt and bankruptcy can be found in

[1, 4, 11, 13]. Very recently, in [15] Nuño and Thomas proposed a model where the yearly income of the debtor is a stochastic process. Bankruptcy occurs at the random time T_b when the debt-to-income ratio reaches a given threshold x^*. Optimal solutions in feedback form, and the dependence of the total expected cost on the choice of x^*, are studied in the forthcoming paper [8].

2 The infinite horizon optimal control problem

The model includes the following variables:

- t = time, measured in years,
- $x(t)$ = total debt, measured as a fraction of the yearly income of the borrower,
- $u(t) \in [0, 1]$ = payment rate, as a fraction of the borrower's income,
- $p(t) \in [0, 1]$ = discounted bond price,
- T_b = random time when bankruptcy occurs.

In addition, we consider the functions

- $L(u)$ = cost to the borrower by implementing the control u,
- $\rho(x)$ = instantaneous risk of bankruptcy,
- $\theta(x(T_b)) \in [0, 1]$ = salvage rate, as a fraction of the outstanding capital which can be recovered by the lenders in case of bankruptcy,

and the constants

- r = discount rate,
- λ = rate at which the principal is payed back,
- B = bankruptcy cost to the borrower,
- M = maximum size of the debt, beyond which bankruptcy immediately occurs.

We regard $x(\cdot)$ as the one-dimensional state variable, while $u(\cdot)$ is the control variable for the borrower.

When an investor (i.e., a lender) purchases a coupon of unit nominal value at time $t = 0$, he receives the promise of a stream of payments for all future times. The primary capital is payed back at rate λ, so that the outstanding value of the loan at time $t > 0$ is $e^{-\lambda t}$. In addition, the borrower pays an interest r. The repayment rate is thus

$$\psi(t) \;=\; (\lambda + r)e^{-\lambda t}.$$

The total payoff for the lender, exponentially discounted in time, is computed by

$$\Psi \;=\; \int_0^{+\infty} e^{-rt}(\lambda+r)e^{-\lambda t}\,dt \;=\; 1\,. \tag{2.1}$$

However, if bankruptcy occurs at a random time T_b, payments will stop at time $t = T_b$, and the total payoff will be

$$\Psi \;=\; \int_0^{T_b} e^{-rt}(\lambda+r)e^{-\lambda t}\,dt + e^{-rT_b}\theta(x(T_b))e^{-\lambda T_b}\,. \tag{2.2}$$

Notice that the last term in (2.2) accounts for

[exponential discount] $\times$ [salvage rate] $\times$ [outstanding capital at the time of bankruptcy].

If $\theta = 1$, so that the outstanding capital is recovered in full, then again $\Psi = 1$. In general, however, $\theta < 1$. Assuming that the market price results from the perfect competition of a pool of risk-neutral lenders, the coupon will be bought at the discounted price

$$p \;=\; E\left[\int_0^{T_b} e^{-rt}(\lambda+r)e^{-\lambda t}\,dt + e^{-rT_b}\theta(x(T_b))e^{-\lambda T_b}\right]. \tag{2.3}$$

Here E denotes the expected value of the given quantity, depending on the random variable T_b.

The distribution of random time T_b at which bankruptcy occurs is determined as follows. If at time τ the borrower is not yet bankrupt and the total debt is $x(\tau) = y$, then the probability that bankruptcy will occur shortly after time τ is measured by

$$\text{Prob.}\Big\{T_b \in [\tau,\,\tau+\varepsilon]\;\Big|\;T_b > \tau\,,\; x(\tau) = y\Big\} \;=\; \rho(y)\cdot\varepsilon + o(\varepsilon). \tag{2.4}$$

Here $\rho(y)$ measures an "instantaneous bankruptcy risk", while $o(\varepsilon)$ denotes a higher order infinitesimal as $\varepsilon \to 0$.

Assume that, as long as bankruptcy does not occur, the size of the debt $x(t)$ is determined at all future times. Define

$$T^M \;\doteq\; \inf\big\{t > 0\,;\;\; x(t) = M\big\} \;\in\; I\!R \cup \{+\infty\} \tag{2.5}$$

the first time when the debt reaches the maximum possible value M. The probability that the borrower is not yet bankrupt at a time $t > 0$ is then computed as

$$P(t) \;\doteq\; Prob.\{T_b > t\} \;=\; \begin{cases} \exp\left\{-\displaystyle\int_0^t \rho(x(\tau))\,d\tau\right\} & \text{if} \quad t < T^M, \\ 0 & \text{if} \quad t \geq T^M. \end{cases} \tag{2.6}$$

Notice that this depends on the function $\tau \mapsto x(\tau)$. If debt is maintained at a higher level, then the probability of bankruptcy increases. Using (2.6) to compute the expectation in (2.3), for any $t \geq 0$ one obtains

$$\begin{aligned} p(t) \;&=\; 1 - \int_t^{T^M} \big[1-\theta(x(s))\big]\,\rho(x(s))\exp\left\{-\int_t^s \big[r+\lambda+\rho(x(\tau))\big]\,d\tau\right\} ds \\ &\qquad -\big[1-\theta(M)\big]\exp\left\{-\int_t^{T^M} [r+\lambda+\rho(x(\tau))]d\tau\right\} \\ &\doteq\; 1 - \mathcal{L}_0(t) - \mathcal{L}_M(t)\,. \end{aligned} \tag{2.7}$$

The first integral $\mathcal{L}_0(t)$ on the right hand side of (2.7) accounts for the lost value if bankruptcy occurs at any time $s \in]t, T^M[$. The second integral $\mathcal{L}_M(t)$ accounts for the lost value if bankruptcy occurs exactly at time T^M. Clearly, if $T^M = +\infty$, then $\mathcal{L}_M(t) \equiv 0$.

Denoting by an upper dot the derivative w.r.t. time, from (2.7) we obtain

$$\begin{aligned} \dot{p}(t) &= \big[1-\theta(x(t))\big]\,\rho(x(t)) + \big[r+\lambda+\rho(x(t))\big](p(t)-1) \\ &= \big[p(t)-\theta(x(t))\big]\,\rho(x(t)) + (r+\lambda)(p(t)-1)\,. \end{aligned} \tag{2.8}$$

Next, we write an evolution equation for the nominal value $x(t)$ of the debt. Calling $u(t)$ the rate of payments made by the borrower, one has

$$\dot{x}(t) = \begin{cases} -\lambda x(t) + \dfrac{(\lambda+r)x(t)-u(t)}{p(t)} & \text{if } 0 < x(t) < M, \\ 0 & \text{if } x(t) \in \{0, M\}\,. \end{cases} \tag{2.9}$$

Indeed, (2.9) is motivated by the following observations.

- The nominal value of outstanding loans decreases, since they are payed back at the fixed rate λ.
- To service the current debt, the borrower should make a stream of payments with rate $(\lambda+r)x(t)$. If his instantaneous payment rate $u(t)$ is smaller than this value, new loans must be initiated. Accounting for the discounted price $p(t)$, the nominal value of these new loans is given by $\dfrac{(\lambda+r)x(t)-u(t)}{p(t)}$. It may also happen that $u(t) > (\lambda+r)x(t)$. In this case, the borrower is simply buying back some of his debt from the market.
- The second alternative in (2.9) guarantees that the state constraint $x(t) \in [0, M]$ is satisfied, for all $t \geq 0$. Clearly, if $x(\tau) = 0$, then at time τ the debt is completely extinguished and equals zero at all future times $t > \tau$. On the other hand, if $x(\tau) = M$, then again the evolution stops, because of immediate bankruptcy.

The equation (2.9) is supplemented by the initial datum

$$x(0) = x_0\,, \tag{2.10}$$

specifying the initial value of the debt.

Remark 1. In a standard control problem, the initial data (2.10) and the control function $u(\cdot)$ completely determine the evolution of the state $x(\cdot)$. This is not the case here. Indeed, the evolution equation (2.9) also involves the discounted price $p(t)$, which by (2.7) is determined by all future values of $x(\cdot)$. The existence of a solution is not an obvious fact, and will be proved in Section 3. Uniqueness does not hold, in general.

Given a control $u(\cdot)$ and a trajectory $x(\cdot)$, to compute the total expected cost to the borrower, exponentially discounted in time, we first introduce the function

$$\gamma(t) \doteq e^{-rt} \exp\left\{-\int_0^t \rho(x(s))\,ds\right\}. \tag{2.11}$$

Notice that this is the product of the exponential discount, times the probability of not being bankrupt at time t. We now compute

$$\begin{aligned}
J(u,x) &= E\left[\int_0^{T_b} e^{-rt} L(u(t))\,dt + B\,e^{-rT_b}\right] \\
&= \int_0^{T^M} \gamma(t)\Big\{\rho(x(t))\,B + L(u(t))\Big\}\,dt \;+\; \gamma(T^M)\,B \\
&= B\int_0^{T^M} e^{-rt}\exp\left\{-\int_0^t \rho(x(s))\,ds\right\}\rho(x(t))\,dt + \int_0^{T^M}\gamma(t)L(u(t))\,dt + \gamma(T^M)B \\
&= B - \gamma(T^M)B - rB\int_0^{T^M}\gamma(t)\,dt + \int_0^{T^M}\gamma(t)L(u(t))\,dt + \gamma(T^M)B \\
&= B + \int_0^{T^M}\gamma(t)[L(u(t)) - rB]\,dt
\end{aligned} \tag{2.12}$$

For a given initial value (2.10) of the debt we can now formulate the optimal control problem for the borrower, in open-loop form.

(DMP) Debt Management Problem with bankruptcy risk. *Given an initial size $x(0) = x_0$ of the debt, find a control $t \mapsto u(t) \in [0,1]$ and a corresponding map $t \mapsto (x(t), p(t))$, which minimize the expected cost*

$$J(u,x) \doteq B + \int_0^{T^M}\gamma(t)[L(u(t)) - rB]\,dt\,, \tag{2.13}$$

subject to the dynamics (2.9) and the constraint (2.7).

In the remainder of the paper we will prove that this problem has at least one solution, and derive necessary conditions for optimality in the form of a maximum principle.

Concerning the functions ρ, L, θ, and the constants r, λ, B, M, we shall assume

(A1) *All constants r, λ, B, M are strictly positive. Moreover, $rM > 1$.*

(A2) *The function ρ is continuously differentiable. There exists $R_0 \geq 0$ such that*

$$\rho(x) = 0 \quad \textit{for} \quad x \in [0, R_0], \qquad \rho'(x) > 0 \quad \textit{for} \quad x \in]R_0, M[, \qquad \lim_{x\to M-} \rho(x) \;=\; +\infty. \tag{2.14}$$

(A3) *The map $\theta : [0, M] \mapsto [0,1]$ is Lipschitz continuous, nonincreasing, and strictly positive.*

(A4) *The cost function L is twice continuously differentiable for $u \in [0,1[$ and satisfies*

$$L(0) = 0, \qquad L' > 0, \quad L'' > 0, \qquad L(1) \;=\; \lim_{u\to 1-} L(u) \;=\; +\infty. \tag{2.15}$$

To motivate **(A3)**, assume that the borrower owns an amount R_0 of collateral (real estate, gold reserves, etc...) to back up his debt. In case of bankruptcy, this will be divided among lenders. In this case the function θ will have the form

$$\theta(x) \;=\; \min\left\{1,\; \frac{R_0}{x}\right\}. \tag{2.16}$$

3 Construction of solutions

Given a control $u(\cdot)$ and an initial datum x_0, the following analysis shows the existence of a solution to our system of evolution equations. Since here we are not solving a Cauchy problem, one should be aware that solutions may not be unique.

Theorem 1. *Let the assumptions* **(A1)-(A3)** *hold. Let a measurable function* $u : [0,\infty[\mapsto [0,1]$ *and an initial state* $x_0 \in [0,M]$ *be given. Then the equations (2.7), (2.9), (2.10) admit at least one solution* $t \mapsto (x(t), p(t))$*, defined for all* $t \geq 0$.

Proof. Let a measurable function u and an initial state x_0 be given.

1. Choose a constant $0 < \mu < r$ and let X be the Banach space of all continuous functions $f : [0,\infty[\mapsto I\!R$ such that

$$\|f\| \doteq \sup_{t\geq 0} e^{-\mu t}|f(t)| < +\infty. \tag{3.1}$$

Within the space X, consider the closed, convex subset

$$Y \doteq \Big\{p \in X\,;\ \ p(t) \in \big[\theta(M),\, 1\big] \quad \text{for all } t \geq 0\Big\}. \tag{3.2}$$

We recall that, by **(A2)**, one has $\theta(M) > 0$.

2. For any $p \in Y$, let $x(\cdot) = \Lambda_1(p)$ be the solution to (2.9)-(2.10). We observe that this solution is unique. Indeed, for every $t \geq 0$, the map $x \mapsto -\lambda x + \frac{(\lambda+r)x-u(t)}{p(t)}$ is Lipschitz continuous. Hence the Cauchy problem has a unique solution defined up to the first time T where either $x(T) = 0$ or $x(T) = M$. By (2.9), the solution remains constant for all $t \geq T$.

Next, given a solution $x = \Lambda_1(p)$ of (2.9)-(2.10), let T^M be as in (2.5) and define the function $p^\sharp \doteq \Lambda_2(x)$ by setting

$$\begin{aligned} p^\sharp(t) \ &\doteq\ 1 - \int_t^{T^M} \big[1-\theta(x(s))\big]\,\rho(x(s)) \exp\left\{-\int_t^s \big[r+\lambda+\rho(x(\tau))\big]\,d\tau\right\} ds \\ &\qquad -\big[1-\theta(M)\big] \exp\left\{-\int_t^{T^M} [r+\lambda+\rho(x(\tau))]d\tau\right\} \\ &=\ 1 - \mathcal{L}_0(t) - \mathcal{L}_M(t), \end{aligned} \tag{3.3}$$

for $t \in [0, T^M[$, while

$$p^\sharp(t) \ =\ \theta(M) \qquad\qquad \text{for} \quad t \geq T^M. \tag{3.4}$$

In the next steps we shall prove that the composition

$$p \ \mapsto\ p^\sharp \ \doteq\ \Lambda_2(\Lambda_1(p)) \tag{3.5}$$

is a continuous, compact operator from Y into itself, with the distance induced by the norm (3.1).

3. We begin by proving that the map $p \mapsto x = \Lambda_1(p)$ is continuous. More precisely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $p, \tilde{p} \in Y$ and $\|p - \tilde{p}\| \leq \delta$, then the corresponding solutions of the Cauchy problem (2.9), (2.10) satisfy

$$e^{-\mu t}|x(t) - \tilde{x}(t)| \;\leq\; \varepsilon \qquad \text{for all } t \geq 0. \tag{3.6}$$

Observing that $x(t), \tilde{x}(t) \in [0, M]$, the inequality in (3.6) will certainly hold for all $t \geq \tau_\varepsilon \doteq \frac{1}{\mu}\log(\frac{M}{\varepsilon})$.

For $t \in [0, \tau_\varepsilon]$, recalling that $p, \tilde{p}$ take values inside the interval $[\theta(M)\,,\,1]$, from (2.9) we deduce

$$\begin{aligned}
\frac{d}{dt}|x(t) - \tilde{x}(t)| \;&\leq\; \left|\frac{(\lambda + r)x(t) - u(t)}{p(t)} - \frac{(\lambda + r)\tilde{x}(t) - u(t)}{\tilde{p}(t)}\right| \\
&\leq\; \frac{r + \lambda}{\theta(M)}\,|x(t) - \tilde{x}(t)| + [(\lambda + r)M + 1]\left|\frac{1}{p(t)} - \frac{1}{\tilde{p}(t)}\right| \\
&\leq\; \frac{r + \lambda}{\theta(M)}\,|x(t) - \tilde{x}(t)| + \frac{(\lambda + r)M + 1}{\theta^2(M)}\,|p(t) - \tilde{p}(t)|.
\end{aligned}$$

Since $x(0) = \tilde{x}(0) = x_0$, if

$$\sup_{t \geq 0} e^{-\mu t}|p(t) - \tilde{p}(t)| \;\leq\; \delta$$

for some $\delta > 0$ suitably small, an application of Gronwall's estimate on the interval $[0, \tau_\varepsilon]$ now yields (3.6).

4. In this step we prove that the set of functions $Z \doteq \{\Lambda_2(\Lambda_1(p))\,;\;\; p \in Y\}$ is equicontinuous and $Z \subset Y$.

Let $\varepsilon > 0$ be given, and let $p^\sharp = \Lambda_2(x)$, for some $x = \Lambda_1(p)$ with $p \in Y$. From (2.9) we deduce

$$-\frac{1}{\theta(M)} \;\leq\; \frac{rx(t) - 1}{p(t)} \;\leq\; \dot{x}(t) \;\leq\; \frac{\lambda + r}{\theta(M)}\,M. \tag{3.7}$$

In particular, $x(\cdot)$ is uniformly Lipschitz continuous. In turn, by (2.16) the function $t \mapsto \theta(x(t))$ is Lipschitz continuous as well.

Since we assume $rM > 1$, by the last inequality in (3.7) there exist constants $M_0 < M$ and $c_0 > 0$ such that

$$x(t) \;\geq\; M_0 \qquad \Longrightarrow \qquad \dot{x}(t) \;\geq\; c_0\,. \tag{3.8}$$

Introducing the time

$$t_0 \;\doteq\; \inf\big\{t \geq 0\,;\;\; x(t) \geq M_0\big\}, \tag{3.9}$$

by (3.8) we have

$$T^M \;\leq\; t_0 + \frac{M - M_0}{c_0}\,. \tag{3.10}$$

To understand the properties of $p^\sharp(\cdot)$, we introduce the integral function

$$\varphi(x) \;\doteq\; \int_0^x \rho(y)\,dy\,. \tag{3.11}$$

Various cases will be considered.

Case 1: $T^M = +\infty$. In this case, by (3.10) we must have $x(t) < M_0$ for all $t \geq 0$. Differentiating (3.3), as in (2.8) we obtain

$$\dot{p}^\sharp(t) \;=\; \big[p^\sharp(t) - \theta(x(t))\big]\rho(x(t)) + (r+\lambda)(p^\sharp(t) - 1), \tag{3.12}$$

$$|\dot{p}^\sharp(t)| \;\leq\; \rho(M_0) + (r+\lambda). \tag{3.13}$$

Hence $p^\sharp = \Lambda(p)$ is uniformly Lipschitz continuous.

Case 2: $T^M < +\infty$ and $\varphi(M) < +\infty$. In this case there is a positive probability that bankruptcy occurs exactly at time $t = T^M$. Moreover,

$$\lim_{t\to T^M-} \mathcal{L}_0(t) \;=\; 0, \qquad\qquad \lim_{t\to T^M-} \mathcal{L}_M(t) \;=\; 1 - \theta(M),$$

hence $p^\sharp$ is continuous at $t = T^M$.

Let t_0 be as in (3.9). By (3.13) the function $p^\sharp$ is uniformly Lipschitz continuous on the interval $[0, t_0]$. On the other hand, for $t_0 \leq t < t' \leq T^M$, by (3.3) we have

$$\begin{aligned}
|p^\sharp(t') - p^\sharp(t)| \;&\leq\; \int_t^{t'} \rho(x(s))\,ds + \int_t^{T^M} \rho(x(s)) \left(\int_t^{t'} r + \lambda + \rho(x(\tau))\,d\tau\right) ds \\
&\qquad + \int_t^{t'} r + \lambda + \rho(x(\tau))\,d\tau \\
&\leq\; \left(\frac{\varphi(M)}{c_0} + 2\right) \left[(r+\lambda)(t'-t) + \int_t^{t'} \rho(x(\tau))\,d\tau\right] \\
&\leq\; \left(\frac{\varphi(M)}{c_0} + 2\right) \left[(r+\lambda)(t'-t) + \int_{x(t)}^{x(t')} \left(\rho(x)\cdot\frac{d\tau}{dx}\right) dx\right] \\
&\leq\; \left(\frac{\varphi(M)}{c_0} + 2\right) \left[(r+\lambda)(t'-t) + \frac{\varphi(x(t')) - \varphi(x(t))}{c_0}\right].
\end{aligned} \tag{3.14}$$

By (3.7) it follows

$$|x(t') - x(t)| \;\leq\; \frac{\lambda + r}{\theta(M)} M \cdot |t' - t|\,.$$

In turn, this yields a bound on the difference $\varphi(x(t')) - \varphi(x(t))$ in terms of $|t' - t|$ and the modulus of continuity of the function φ. By (3.14), every function $p^\sharp$ in (3.3) satisfies a uniform modulus of continuity.

Case 3: $T^M < +\infty$ and $\varphi(M) = +\infty$. Then, with probability one, bankruptcy occurs before time $t = T^M$. Moreover,

$$\lim_{t\to T^M-} \mathcal{L}_0(t) \;=\; 1 - \theta(M), \qquad\qquad \mathcal{L}_M(t) \;\equiv\; 0.$$

Therefore, we again conclude that $p^\sharp$ is continuous at $t = T^M$.

The expression (3.3) can now be rewritten as

$$p^\sharp(t) \;=\; 1 + \int_t^{T^M} [1 - \theta(x(s))] e^{-(r+\lambda)(s-t)} \cdot \frac{d}{ds} \exp\left\{-\int_t^s \rho(x(\tau))\,d\tau\right\} ds\,. \tag{3.15}$$

Let t_0 be as in (3.9). By (3.13) the function $p^\sharp$ is Lipschitz continuous on $[0, t_0]$, and it is of course constant for $t \geq T^M$. We claim that $p^\sharp$ is uniformly Lipschitz continuous also on the time interval $[t_0, T^M]$. Namely,

$$|p^\sharp(\tau) - p^\sharp(t)| \;\leq\; (C_1 + C_2)\,|\tau - t|, \tag{3.16}$$

where C_1, C_2 are constants such that

$$\begin{aligned} \left| \frac{d}{ds}\Big([1 - \theta(x(s))] e^{-(r+\lambda)(s-t)} \Big) \right| \;&\leq\; C_1 \qquad \text{for all} \quad 0 < t < s, \\ \frac{e^{(r+\lambda)s} - 1}{s} \;&\leq\; C_2 \qquad \text{for all } 0 < s < \tfrac{M - M_0}{c_0}. \end{aligned} \tag{3.17}$$

In view of (3.7) such constants exist, and are independent of $x(\cdot)$.

To prove our claim, observe that the function

$$s \;\mapsto\; \phi^t(s) \;\doteq\; \exp\left\{ - \int_t^s \rho(x(\zeta))\, d\zeta \right\} \tag{3.18}$$

is monotone decreasing and satisfies

$$\phi^t(t) \;=\; 1\,, \qquad\qquad \phi^t(T^M) \;=\; 0$$

For each $0 < y < 1$, call $s^t(y)$ the time such that

$$\phi^t(s^t(y)) \;=\; y.$$

Notice that, since the function $t \mapsto \rho(x(t))$ is increasing, for $t_0 < t < \tau < T^M$ we have

$$s^\tau(y) - s^t(y) \;\leq\; \tau - t\,.$$

Using (3.15) we thus obtain the estimate

$$\begin{aligned} &|p^\sharp(\tau) - p^\sharp(t)| \\ &= \left| \int_0^1 \Big\{ \big[1 - \theta(x(s^\tau(y)))\big] e^{-(r+\lambda)(s^\tau(y) - \tau)} - \big[1 - \theta(x(s^t(y)))\big] e^{-(r+\lambda)(s^t(y) - t)} \Big\}\, dy \right| \\ &\leq\; C_1 \int_0^1 |s^\tau(y) - s^t(y)|\, dy + \left| \int_0^1 \Big\{ \big[1 - \theta(x(s^t(y)))\big] e^{-(r+\lambda)(s^t(y) - t)} \big(e^{(r+\lambda)(\tau - t)} - 1\big) \Big\}\, dy \right| \\ &\leq\; C_1\,|\tau - t| + |e^{(r+\lambda)(\tau - t)} - 1| \\ &\leq\; (C_1 + C_2)\,|\tau - t|, \end{aligned}$$

proving (3.16).

To check $Z \subset Y$, we need to show that

$$\theta(M) \;\leq\; p^\sharp(t) \;\leq\; 1 \qquad\qquad \text{for all} \quad t \geq 0. \tag{3.19}$$

Since $\theta(M) \leq \theta(x(t)) \leq 1$, for any $t \leq T^M$ we obtain

$$
\begin{aligned}
1 \ \geq \quad p^\sharp(t) \ &\geq \ 1 - \int_t^{T^M} \big[1-\theta(M)\big]\,\rho(x(s)) \exp\left\{-\int_t^s \big[r+\lambda+\rho(x(\tau))\big]\,d\tau\right\} ds \\
&\qquad - \big[1-\theta(M)\big] \exp\left\{-\int_t^{T^M} [r+\lambda+\rho(x(\tau))]d\tau\right\} \\
&\geq \ 1-(1-\theta(M)) \ = \ \theta(M).
\end{aligned}
$$

Since by definition $p^\sharp(t) = \theta(M)$ for all $t \geq T^M$, we conclude that $Z \subset Y$.

5. In this step we prove that Λ_2 is continuous on the range of Λ_1. Let $x \in \Lambda_1(Y)$ and $\varepsilon > 0$ be given. We claim that there exists $\delta > 0$ such that, if $\tilde{x} \in \Lambda_1(Y)$ and $\|x - \tilde{x}\| \leq \delta$, then the corresponding functions $p = \Lambda_2(x)$ and $\tilde{p} = \Lambda_2(\tilde{x})$ satisfy

$$
e^{-\mu t}|p(t) - \tilde{p}(t)| \ \leq \ \varepsilon \qquad \text{for all } t \geq 0. \tag{3.20}
$$

To prove the claim, we first observe that, since $p(t), \tilde{p}(t) \in [0,1]$, the inequality (3.20) certainly holds for all $t \geq T_\varepsilon \doteq \frac{1}{\mu} \ln \frac{1}{\varepsilon}$.

Call T^M and $\widetilde{T}^M \in [0, +\infty]$ the first times when $x(t) = M$ and $\tilde{x}(t) = M$, respectively.

As long as $t < T^M$, the price $p(t)$ satisfies the linear ODE (2.8). Therefore, choosing any $T < T^M$, we have the representation

$$
p(t) \ = \ p(T) e^{-\int_t^T [r+\lambda+\rho(x(\tau))]\,d\tau} + \int_t^T e^{-\int_t^s [r+\lambda+\rho(x(\tau))]\,d\tau} [r+\lambda+\theta(x(s))\rho(x(s))]\,ds.
$$

Using a similar representation for $\tilde{p}$, for $t \leq T < \min\{T^M, \widetilde{T}^M\}$ the difference $p(t) - \tilde{p}(t)$ can thus be expressed as

$$
p(t) - \tilde{p}(t) \ = \ I_1(t) + I_2(t)\,, \tag{3.21}
$$

where

$$
I_1(t) \ \doteq \ p(T) \exp\left\{-\int_t^T [r+\lambda+\rho(x(\tau))]\,d\tau\right\} - \tilde{p}(T) \exp\left\{-\int_t^T [r+\lambda+\rho(\tilde{x}(\tau))]\,d\tau\right\} \tag{3.22}
$$

and

$$
\begin{aligned}
I_2(t) \ \doteq \ & \int_t^T \exp\left\{-\int_t^s [r+\lambda+\rho(x(\tau))]\,d\tau\right\} [r+\lambda+\theta(x(s))\rho(x(s))]\,ds \\
& - \int_t^T \exp\left\{-\int_t^s [r+\lambda+\rho(\tilde{x}(\tau))]\,d\tau\right\} [r+\lambda+\theta(\tilde{x}(s))\rho(\tilde{x}(s))]\,ds.
\end{aligned} \tag{3.23}
$$

Two cases will be considered.

Case 1: $T^M = +\infty$. We then choose $T > T_\varepsilon \doteq \frac{1}{\mu} \ln \frac{1}{\varepsilon}$ large enough so that

$$
\exp\left\{-\int_{T_\varepsilon}^T (r+\lambda)\,d\tau\right\} \ < \ \frac{\varepsilon}{3}\,.
$$

For all $t \in [0,T]$, this choice implies $|I_1(t)| \leq \frac{\varepsilon}{2}$. Then we choose $\delta > 0$ so small that $\|x-\tilde{x}\| \leq \delta$ implies $|I_2(t)| \leq \frac{\varepsilon}{2}$ for all $t \in [0,T]$. This achieves (3.20).

Case 2: $T^M < +\infty$.

By step **4**, all functions $p^\sharp \in \Lambda_2(\Lambda_1(Y))$ are equicontinuous. Hence there exists $\delta_1 > 0$ such that $M - \delta_1 \geq M_0$ and

$$|t - t'| \ \leq\ \delta_1 \qquad \Longrightarrow \qquad |p^\sharp(t) - p^\sharp(t')| \ \leq\ \frac{\varepsilon}{3}. \tag{3.24}$$

Recalling (3.8), define

$$M_\varepsilon \ \doteq\ M - \frac{c_0\delta_1}{2}$$

and call T_ε the unique time such that $x(T_\varepsilon) = M_\varepsilon$.

By choosing $\delta > 0$ sufficiently small, the inequality $\|x - \tilde{x}\| \leq \delta$ implies

$$|T^M - \widetilde{T}^M| \ <\ \frac{\delta_1}{2}, \qquad\qquad |x(T_\varepsilon) - \tilde{x}(T_\varepsilon)| \ \leq\ \frac{c_0\delta_1}{2}.$$

Since $p(T^M) = \tilde{p}(\widetilde{T}^M) = \theta(M)$, we have the estimate

$$|p(T_\varepsilon) - \tilde{p}(T_\varepsilon)| \ \leq\ |p(T^M) - p(T_\varepsilon)| + |\tilde{p}(\widetilde{T}^M) - \tilde{p}(T_\varepsilon)| \ \leq\ \frac{\varepsilon}{3} + \frac{\varepsilon}{3}. \tag{3.25}$$

Indeed, the last inequality follows from (3.24) together with

$$|T^M - T_\varepsilon| \ \leq\ \frac{1}{c_0}(M - x(T_\varepsilon)) \ =\ \frac{1}{c_0}\frac{c_0\delta_1}{2},$$

$$|\widetilde{T}^M - T_\varepsilon| \ \leq\ \frac{1}{c_0}(M - \tilde{x}(T)) \ \leq\ \frac{|M - x(T_\varepsilon)|}{c_0} + \frac{|x(T_\varepsilon) - \tilde{x}(T_\varepsilon)|}{c_0} \ \leq\ \frac{\delta_1}{2} + \frac{\delta_1}{2}.$$

For every $t \geq T$, the same argument used in (3.25) yields

$$|p(t) - \tilde{p}(t)| \ \leq\ \frac{2\varepsilon}{3}.$$

On the other hand, to estimate the difference (3.21) for $t < T$, we set

$$E(t) \ \doteq \exp\left\{-\int_t^T [r + \lambda + \rho(x(\tau))]\, d\tau\right\}, \qquad\qquad \widetilde{E}(t) \ \doteq \exp\left\{-\int_t^T [r + \lambda + \rho(\tilde{x}(\tau))]\, d\tau\right\}.$$

By possibly shrinking the value of δ, for every $t \in [0,T]$ we can achieve

$$|I_2(t)| \ \leq\ \frac{\varepsilon}{6}, \qquad\qquad |E(t) - \widetilde{E}(t)| \ \leq\ \frac{\varepsilon}{6}. \tag{3.26}$$

This implies

$$|I_1(t)| \ =\ \left|p(T)E(T) - \tilde{p}(T)\widetilde{E}(T)\right| \ \leq\ |p(T) - \tilde{p}(T)| \cdot E(T) + \tilde{p}(T) \cdot |E(T) - \widetilde{E}(T)| \leq\ \frac{2\varepsilon}{3} + \frac{\varepsilon}{6}. \tag{3.27}$$

Using (3.26)-(3.27), we conclude that the right hand side of (3.21) is $\leq \varepsilon$, for all $t \in [0,T]$.

6. By the previous analysis, the map $p \mapsto p^\sharp \doteq \Lambda_2(\Lambda_1(p))$ is continuous from the closed convex set Y into itself. We now claim that this map is compact. Indeed, let $(p_n)_{n\geq 1}$ be a sequence of functions in Y. By the equicontinuity of all functions $p^\sharp$, proved in step **4**, for every $\varepsilon > 0$ we can find a subsequence p_{n_j} such that $p^\sharp_{n_j} \doteq \Lambda_2(\Lambda_1(p_{n_j}))$ converges to some limit function $\bar{p}$ uniformly on bounded intervals. Since all p_n take values inside $[0,1]$, this implies the convergence in norm: $\|p^\sharp_{n_j} - \bar{p}\| \to 0$.

7. An application of Schauder's theorem yields the existence of a fixed point $\bar{p} = \Lambda_2(\Lambda_1(\bar{p}))$. We then consider the function $\bar{x} = \Lambda_1(\bar{p})$. By the definitions of Λ_1 and Λ_2 it follows that the map $t \mapsto (\bar{x}(t), \bar{p}(t))$ provides a solution to the equations (2.7), (2.9), (2.10). □

4 Existence of optimal solutions

In this section, given an initial size x_0 of the debt, we prove the existence of an optimal strategy that minimizes the expected cost to the borrower.

Theorem 2. *Under the assumptions* **(A1)–(A4)***, the optimization problem* **(DMP)** *admits an optimal solution (u, x, p), which minimizes the expected cost (2.13).*

Proof. 1. For any initial data x_0, the trivial control $u(t) \equiv 0$ yields a total cost $J(u) \leq B$. Indeed, this cannot be worse than the cost of immediate bankruptcy. It thus suffices to prove the theorem assuming that

$$0 \;\leq\; m \;\doteq\; \inf_{(u,x,p)\in\mathcal{S}} J(u,x) \;<\; B\,. \tag{4.1}$$

Here the infimum is taken over the set $\mathcal{S}$ of all measurable controls $u : [0,\infty[\mapsto [0,1]$ and all solutions (x,p) of the system (2.7), (2.9), (2.10).

Consider a minimizing sequence $(u_n, x_n, p_n) \in \mathcal{S}$, so that

$$J(u_n, x_n) \;\to\; m \qquad\qquad \text{as} \qquad n \to \infty. \tag{4.2}$$

Define the corresponding functions γ_n as in (2.11), with x replaced by x_n. By the estimates derived in the proof of Theorem 1, the functions x_n, p_n are uniformly equicontinuous. By possibly taking a subsequence, we can thus assume the weak convergence $u_n \rightharpoonup u$ together with the convergence $x_n \to x$, $p_n \to p$ uniformly on bounded intervals $[0,T]$. Using the convexity of the cost function L and the fact that u enters linearly in the equations (2.9), we will prove that the triple of limit functions (u, x, p) is optimal.

2. We first consider the case where

$$\liminf_{n\to\infty} \; T_n^M \;=\; \infty. \tag{4.3}$$

By possibly taking a subsequence, we can assume that

$$(x_n, p_n, \gamma_n)(t) \;\to\; (x, p, \gamma)(t) \qquad\qquad \text{as} \;\; n \to \infty, \tag{4.4}$$

uniformly on every bounded interval $[0,\, T]$.

Since u enters linearly in the equations (2.9), by (4.4) and the weak convergence $u_n \rightharpoonup u$ it is clear that (2.7) and (2.9) are satisfied, together with the initial condition (2.10).

If $J(u,x) > m$, recalling (2.12) then there exists a bounded interval $[0,T]$ such that

$$B + \int_0^T \gamma(t)\Big\{L(u(t)) - rB\Big\}\, dt \;>\; m + e^{-rT}B\,. \tag{4.5}$$

Recalling that the cost function L is non-negative and convex, and observing that $\gamma(t) \le e^{-rt}$, we obtain

$$\begin{aligned} m \;&<\; B + \int_0^T \gamma(t)\Big\{L(u(t)) - rB\Big\}\, dt - e^{-rT}B \\ &\le\; B + \liminf_{n\to\infty} \int_0^T \gamma_n(t)\Big\{L(u_n(t)) - rB\Big\}\, dt - e^{-rT}B \\ &\le\; B + \liminf_{n\to\infty} \int_0^{T_n} \gamma_n(t)\Big\{L(u_n(t)) - rB\Big\}\, dt \;\le\; m\,. \end{aligned}$$

This contradiction shows that $J(u,x) \le m$, proving the optimality of the solution (u,x,p).

3. Next, consider the case where

$$T^M \;\doteq\; \liminf_{n\to\infty} T_n^M \;<\; \infty. \tag{4.6}$$

By possibly taking a subsequence, we can assume that $T_n^M \to T^M$, as well as the convergence (4.4) uniformly on every subinterval of the form $[0,\, T^M - \delta]$, with $\delta > 0$. As before, one checks that the limit functions (u,x,p) satisfy the conditions (2.7), (2.9), and (2.10).

If $J(u,x) \;>\; m$, then we can choose $\varepsilon, \delta > 0$ small enough so that

$$B + \int_0^{T^M-\delta} \gamma(t)\Big\{L(u(t)) - rB\Big\}\, dt \;>\; m + \varepsilon\,, \qquad\qquad \delta rB \;<\; \varepsilon\,. \tag{4.7}$$

Using again the convexity of the cost function L and recalling that $\gamma_n(t) \le 1$, by the two inequalities in (4.7) we obtain

$$\begin{aligned} m \;&=\; B + \liminf_{n\to\infty} \int_0^{T_n^M} \gamma_n(t)\Big(L(u_n(t)) - rB\Big)\, dt \\ &\ge\; B + \int_0^{T^M-\delta} \gamma(t)\Big\{L(u(t)) - rB\Big\}\, dt - \delta rB \;>\; m\,. \end{aligned} \tag{4.8}$$

This contradiction shows that $J(u,x) \le m$, completing the proof. □

5 Necessary conditions for optimality

Given an initial value of the debt $x_0 \in\,]0, M[$, let $t \mapsto u^*(t)$ be an optimal control, and let $t \mapsto (x^*(t), p^*(t))$ be a corresponding optimal solution of **(DMP)**. We seek necessary conditions for optimality. One easy case is covered by the next lemma. We recall that R_0 is the constant in (2.14).

Lemma 1. *Assume that $x^*(t) \leq R_0$ for all $t \geq \tau$. Then the optimal trajectory is constant:*

$$x^*(t) \;=\; x^*(\tau)\,, \qquad p^*(t) \;=\; 1, \qquad u^*(t) \;=\; rx^*(\tau) \qquad\qquad \textit{for all} \quad t \geq \tau. \tag{5.1}$$

Proof. The assumption yields $\rho(x^*(t)) = 0$ and hence $p^*(t) = 1$ for all $t \geq \tau$. The control u^* thus provides the minimum cost for the optimal control problem

$$\text{minimize: } \int_\tau^\infty e^{-rt} L(u(t))\,dt,$$

with dynamics and state constraint

$$\dot{x} \;=\; rx - u, \qquad x(\tau) \;=\; \bar{x}, \qquad\qquad x(t) \in [0, R_0] \quad \text{for all} \;\; t \geq \tau.$$

Here $\bar{x} = u^*(\tau)$. One readily checks that the value function for this problem is

$$V(\bar{x}) \;=\; \frac{1}{r} L(r\bar{x}),$$

and the unique optimal control is $u^*(t) \equiv r\bar{x}$, keeping the debt constant in time. This achieves the proof. □

Corollary 1. *If $x^*(t) = x_0 < R_0$ for some all $t \geq \tau$, then $x^*(t) = x_0$ for all $t \geq 0$.*

In the remainder of this section, we shall study optimal trajectories taking values in the interval $[R_0, M]$, where the bankruptcy risk is positive. Consider the function

$$\gamma^*(t) \;=\; e^{-rt} \cdot \exp\left\{ -\int_0^t \rho(x^*(s))\,ds \right\}, \tag{5.2}$$

and the time

$$T^M \;\doteq\; \inf\Big\{ t > 0\,; \;\; x^*(t) = M \Big\} \;\in\; I\!R \cup \{+\infty\}\,. \tag{5.3}$$

For any $T < T^M$, a standard argument (see for example [14]) shows that the control u^* and the functions (x^*, p^*, γ^*) provide a solution to an optimization problem on the subinterval $[0, T]$, namely:

(OT) *Minimize the cost functional on the interval* $[0, T]$:

$$J(\gamma, u) \;\doteq\; \int_0^T f_0\big(u(t), \gamma(t)\big)\,dt\,, \qquad\qquad f_0(u, \gamma) \;\doteq\; [L(u) - rB]\gamma\,, \tag{5.4}$$

subject to

$$\begin{cases} \dot{x} \;=\; f_1(u, x, p) \;\doteq\; -\lambda x + \dfrac{(\lambda + r)x - u}{p}\,, \\[2mm] \dot{p} \;=\; f_2(x, p) \;\doteq\; \big[r + \lambda + \rho(x)\big]p - \big[r + \lambda + \theta(x)\rho(x)\big]\,, \\[2mm] \dot{\gamma} \;\doteq\; f_3(x, \gamma) \;\doteq\; -\,(r + \rho(x))\gamma\,, \end{cases} \tag{5.5}$$

with constraints and boundary data

$$u(t) \in [0,1]\,, \tag{5.6}$$

$$\begin{cases} x(0) &= x_0, \\ \gamma(0) &= 1, \end{cases} \tag{5.7}$$

$$\begin{cases} x(T) &= x^*(T), \\ p(T) &= p^*(T), \\ \gamma(T) &= \gamma^*(T). \end{cases} \tag{5.8}$$

Introducing the adjoint variables $(x^\dagger, p^\dagger, \gamma^\dagger)$ and a constant $\mu \geq 0$, we consider the Hamiltonian function

$$H(u, x, p, \gamma, x^\dagger, p^\dagger, \gamma^\dagger, \mu) \doteq \mu\, f_0(u, \gamma) + x^\dagger f_1(u, x, p) + p^\dagger f_2(x, p) + \gamma^\dagger f_3(x, \gamma). \tag{5.9}$$

An application of the Pontryagin Maximum Principle (PMP) to the above problem yields the existence of a constant $\mu \in \{0,1\}$ and functions $(x^\dagger(t), p^\dagger(t), \gamma^\dagger(t)) : [0,T] \to \mathbb{R}^3$ which satisfy the adjoint linear system

$$\begin{aligned} \begin{pmatrix} \dot{x}^\dagger \\ \dot{p}^\dagger \\ \dot{\gamma}^\dagger \end{pmatrix} &= - \begin{pmatrix} \partial H/\partial x \\ \partial H/\partial p \\ \partial H/\partial \gamma \end{pmatrix} \\ &= \begin{pmatrix} \lambda - \frac{\lambda+r}{p} & \theta'(x)\rho(x) - \rho'(x)(p-\theta(x)) & \rho'(x)\gamma \\ \frac{(\lambda+r)x-u}{(p)^2} & -(\lambda+r+\rho(x)) & 0 \\ 0 & 0 & r+\rho(x) \end{pmatrix} \begin{pmatrix} x^\dagger \\ p^\dagger \\ \gamma^\dagger \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ L(u) - rB \end{pmatrix}, \end{aligned} \tag{5.10}$$

together with the initial condition

$$p^\dagger(0) = 0, \tag{5.11}$$

and the non-degeneracy condition

$$(x^\dagger(0), p^\dagger(0), \gamma^\dagger(0), \mu) \neq (0,0,0,0). \tag{5.12}$$

Moreover, for a.e. $t \in [0,T]$, one has

$$u^*(t) = \arg\min_{\omega \in [0,1]} \left\{ -\frac{x^\dagger(t)}{p(t)}\omega + \mu\, \gamma(t) L(\omega) \right\}. \tag{5.13}$$

From the pointwise minimality condition (5.13) one can recover the optimal control u as a function of $x^\dagger$, p, and γ. More precisely, define the function $U^\sharp(x^\dagger, p, \gamma)$ implicitly by the relations

$$\begin{cases} L'\big(U^\sharp(x^\dagger, p, \gamma)\big) = \dfrac{x^\dagger}{p\,\gamma} & \text{if } \dfrac{x^\dagger}{p\,\gamma} > L'(0)\,, \\ U^\sharp(x^\dagger, p, \gamma) = 0 & \text{if } \dfrac{x^\dagger}{p\,\gamma} \leq L'(0). \end{cases} \tag{5.14}$$

Inserting $u = U^\sharp(x^\dagger, p, \gamma)$ in (5.5) and (5.10), one obtains a system of 6 scalar ODEs for the variables $x, p, \gamma, x^\dagger, p^\dagger, \gamma^\dagger$, with the three boundary conditions in (5.7) and (5.11).

The goal of the following analysis is two-fold: (i) show that the optimization problem is "normal", hence in the (PMP) one can take $\mu = 1$, and (ii) determine three additional asymptotic conditions as $t \to +\infty$, in order to compute the optimal solution for all $t \geq 0$.

Lemma 2. *For every solution of (5.5)–(5.7) on an interval* $[0, T]$, *with* $x(t) \in [0, M]$ *and terminal data* $p(T) \in [\theta(M),\ 1]$, *one has*

$$p(t) \in [\theta(M), 1], \qquad\qquad \gamma(t) \in [0, 1]\,. \tag{5.15}$$

Proof. The first inclusion follows from the implications

$$\begin{array}{ccrcl} p = \theta(M) & \Longrightarrow & \dot{p} & = & (r+\lambda)(\theta(M)-1) + \rho(x)(\theta(M)-\theta(x)) \ \leq\ 0, \\ p = 1 & \Longrightarrow & \dot{p} & = & \rho(x)(1-\theta(x)) \ \geq\ 0\,. \end{array}$$

This shows that the interval $[\theta(M), 1]$ is backward invariant for the second ODE in (5.5).

Since $r + \rho(x) > 0$, the second inclusion in (5.15) is obvious. □

We will study the equations determined by the PMP in two main cases.

CASE 1: $T^M \doteq \inf\Big\{t > 0\,;\ \ x^*(t) = M\Big\} < +\infty\,.$ In this case $(u^*, x^*, p^*, \gamma^*)$ is optimal for the problem (5.4)–(5.7), with terminal constraints

$$\left\{ \begin{array}{rcl} x(T) & = & M, \\ p(T) & = & \theta(M), \end{array} \right. \tag{5.16}$$

while T is regarded as a free terminal time. The PMP (see for example [10, 12]) now yields the existence of a constant $\mu \in \{0, 1\}$ and an adjoint vector $(x^\dagger, p^\dagger, \gamma^\dagger)$ satisfying (5.10), so that the six boundary conditions in (5.7), (5.16), and

$$\left\{ \begin{array}{rcl} p^\dagger(0) & = & 0, \\ \gamma^\dagger(T) & = & 0, \end{array} \right. \tag{5.17}$$

hold, together with the optimality condition (5.13). To show that $\mu = 1$, assume on the contrary that $\mu = 0$. Then we must have $x^\dagger(t) \equiv 0$. In turn, (5.10) and the boundary conditions (5.17) imply

$$p^\dagger(t) \ = \ \exp\left\{-\int_0^t \big(\lambda + r + \rho(x^*(t))\big)\,dt\right\} p^\dagger(0) \ = \ 0, \tag{5.18}$$

$$\gamma^\dagger(t) \ = \ \exp\left\{-\int_t^T \big(r + \rho(x^*(t))\big)\,dt\right\} \gamma^\dagger(T) \ = \ 0, \tag{5.19}$$

contradicting the non-degeneracy assumption.

Since the time T is free, we need an additional boundary condition. This can be obtained by the vanishing of the Hamiltonian function:

$$\begin{aligned} 0 &= H\big(u^*(t), x^*(t), p^*(t), \gamma^*(t), x^\dagger(t), p^\dagger(t), \gamma^\dagger(t)\big) \\ &= [L(u^*) - rB]\gamma^* + x^\dagger \left[-\lambda x^* + \frac{(\lambda+r)x^* - u^*}{p^*} \right] \\ &\qquad + p^\dagger \Big[(r+\lambda+\rho(x^*))p^* - (r+\lambda+\theta(x^*)\rho(x^*)) \Big] - \gamma^\dagger (r+\rho(x^*))\gamma^*. \end{aligned} \tag{5.20}$$

At time $t = 0$ we obtain

$$[L(u^*(0)) - rB] + x^\dagger(0) \left[-\lambda x_0 + \frac{(\lambda+r)x_0 - u^*(0)}{p^*(0)} \right] - r\gamma^\dagger(0) \;=\; 0. \tag{5.21}$$

By (5.13), this yields

$$\min_{\omega\in[0,1]} \left\{ -\frac{x^\dagger(0)}{p^*(0)}\omega + L(\omega) \right\} \;=\; rB + r\gamma^\dagger(0) + x^\dagger(0)\lambda x_0 - \frac{(\lambda+r)x_0}{p^*(0)}\,. \tag{5.22}$$

Notice that in this case the quantity in (5.22) can be negative, without leading to any contradiction.

CASE 2: The optimal control u^* and a corresponding optimal trajectory (x^*, p^*) are defined for all $t \geq 0$. Moreover, we assume that, for some constants κ, τ,

$$0 \;<\; u^*(t) \;\leq\; \kappa \;<\; 1, \qquad \text{for all } t \geq \tau, \tag{5.23}$$

and that there exists the limit

$$x_\infty \;\doteq\; \lim_{t\to+\infty} x^*(t) \;>\; R_0. \tag{5.24}$$

According to the assumption **(A2)**, by (2.14) this implies $\rho(x_\infty) > 0$.

We claim that, in the necessary conditions (5.10)–(5.13), one can always take $\mu = 1$. Indeed, if $\mu = 0$, the minimum (5.13) can be attained only if $x^\dagger \equiv 0$. This identity, together with $p^\dagger(0) = 0$, implies $p^\dagger(t) \equiv 0$. Therefore, the non-degeneracy of the adjoint vector $(x^\dagger, p^\dagger, \gamma^\dagger)$ implies $\gamma^\dagger(t) \neq 0$ for all $t \geq 0$. However, since $\gamma(t) \neq 0$, the first equation in (5.10) yields $\rho'(x^*(t)) = 0$, and hence $x^*(t) \in [0, R_0]$ for all $t \geq 0$. This remaining case has already been covered in Lemma 1, showing that the optimal trajectory $x^*(\cdot)$ is constant. Our claim is thus proved.

Since the optimality condition determines the pointwise control value

$$u \;=\; U^\sharp(x^\dagger, p, \gamma) \;=\; (L')^{-1}(x^\dagger/p\gamma), \tag{5.25}$$

it is convenient to replace $x^\dagger, p^\dagger$ by the ratios $\tilde{x}^\dagger \doteq \frac{x^\dagger}{\gamma}$ and $\tilde{p}^\dagger \doteq \frac{p^\dagger}{\gamma}$. Using these rescaled variables, from the PMP we obtain a system of five ODEs:

$$\begin{cases} \dot{x} &=& -\lambda x + \dfrac{(\lambda+r)x - (L')^{-1}(\tilde{x}^\dagger/p)}{p}\,, \\ \dot{p} &=& \big[p-\theta(x)\big]\,\rho(x) + (r+\lambda)(p-1)\,, \\ \dot{\tilde{x}}^\dagger &=& \left(\lambda + r + \rho(x) - \dfrac{\lambda+r}{p}\right)\tilde{x}^\dagger + \Big[\theta'(x)\rho(x) - \rho'(x)(p-\theta(x))\Big]\tilde{p}^\dagger + \rho'(x)\gamma^\dagger, \\ \dot{\tilde{p}}^\dagger &=& \dfrac{(\lambda+r)x - (L')^{-1}(\tilde{x}^\dagger/p)}{p^2}\tilde{x}^\dagger - \lambda\tilde{p}^\dagger, \\ \dot{\gamma}^\dagger &=& (r+\rho(x))\gamma^\dagger + L\Big((L')^{-1}(\tilde{x}^\dagger/p)\Big) - rB\,. \end{cases} \tag{5.26}$$

The initial data are

$$\begin{cases} x(0) &=& x_0\,, \\ \tilde{p}^\dagger(0) &=& 0. \end{cases} \tag{5.27}$$

Notice that in (5.26) the evolution equation for γ has been omitted, because the variable γ is not present in any of the equations for the remaining five variables.

By (5.24) and (2.8) it follows

$$\begin{aligned} \lim_{t\to+\infty} p(t) \;=\; p_\infty \;&\doteq\; 1 - [1-\theta(x_\infty)]\rho(x_\infty)\int_0^\infty e^{-s[r+\lambda+\rho(x_\infty)]}\,ds \\ &=\; \frac{r+\lambda+\theta(x_\infty)\rho(x_\infty)}{r+\lambda+\rho(x_\infty)}\,. \end{aligned} \tag{5.28}$$

Moreover, by (5.23) and (5.25), the variable $\tilde{x}^\dagger$ remains uniformly bounded for $t \in [\tau, +\infty[$. In turn, the fifth equation in (5.26) implies that $\tilde{p}^\dagger$ remains bounded.

Finally, we claim that also the variable $\gamma^\dagger$ is uniformly bounded on $[\tau. +\infty[$. Otherwise, the last equation in (5.26) would imply that $\gamma^\dagger$ grows at an exponential rate, say

$$\big|\gamma^\dagger(t)\big| \;>\; \gamma_0\, e^{rt/2}. \tag{5.29}$$

Using (5.29) in the fourth equation, since $\rho'(x_\infty) > 0$, one obtains $|\tilde{x}^\dagger(t)| \to +\infty$, reaching a contradiction.

The above arguments show that all variables $\tilde{x}^\dagger, \tilde{p}^\dagger, \gamma^\dagger$ remain uniformly bounded on $[\tau, +\infty[$. Since they satisfy a system of ODEs with Lipschitz continuous right hand side, we conclude that they are uniformly bounded on the entire domain $[0, +\infty[$.

In particular, the fourth equation in (5.26) implies that $\tilde{x}^\dagger$ is uniformly Lipschitz continuous. Hence the optimal control $u^*(t) = (L')^{-1}\big(\tilde{x}^\dagger(t)/p(t)\big)$ is Lipschitz continuous as well.

By the first equation in (5.26), the limits (5.24) and (5.28) and the Lipschitz continuity of the control function $u^*(\cdot)$ imply

$$\lim_{t\to+\infty} u^*(t) \;=\; u_\infty \;\doteq\; (\lambda + r - \lambda p_\infty)x_\infty\,. \tag{5.30}$$

From the optimality condition (5.13), if $u^*(t) > 0$ it follows

$$\tilde{x}^\dagger(t) \;=\; p(t)L'(u(t)). \tag{5.31}$$

Letting $t \to +\infty$, this yields

$$\lim_{t\to+\infty} \tilde{x}^\dagger(t) \;=\; p_\infty L'(u_\infty). \tag{5.32}$$

The last two equations in (5.26) imply that the dual variables $\tilde{p}^\dagger(t)$ and $\gamma^\dagger$ also have limits as $t \to +\infty$. Indeed

$$\lim_{t\to+\infty} \tilde{p}^\dagger(t) \;=\; \tilde{p}^\dagger_\infty \;\doteq\; \frac{(\lambda+r)x_\infty - u_\infty}{\lambda p_\infty^2} L'(u_\infty)p_\infty \;=\; x_\infty L'(u_\infty). \tag{5.33}$$

$$\lim_{t\to+\infty} \gamma^\dagger(t) \;=\; \gamma^\dagger_\infty \;=\; \frac{rB - L(u_\infty)}{r + \rho(x_\infty)}. \tag{5.34}$$

By the previous analysis, from the asymptotic size of the debt x_∞ one can determine the limit values p_∞, u_∞ of the discounted bond price and of the control. We now show that the limit x_∞ cannot be arbitrary.

Observing that the right hand side of the fourth equation in (5.26) is uniformly Lipschitz continuous, and moreover $\tilde{x}(t) \to \tilde{x}_\infty$, we conclude that $\dot{\tilde{x}}^\dagger(t) \to 0$. This yields the identity

$$\left(\lambda + r + \rho(x_\infty) - \frac{\lambda+r}{p_\infty}\right)\tilde{x}^\dagger_\infty + \Big[\theta'(x_\infty)\rho(x_\infty) - \rho'(x_\infty)(p_\infty - \theta(x_\infty))\Big]\tilde{p}^\dagger_\infty + \rho'(x_\infty)\gamma^\dagger_\infty\,, \tag{5.35}$$

Using the identities (5.32)–(5.34), we eventually obtain the additional equation

$$L'(u_\infty)\Big[\rho(x_\infty)\theta(x_\infty) + x_\infty\big[\theta'(x_\infty)\rho(x_\infty) - \rho'(x_\infty)(p_\infty - \theta(x_\infty))\big]\Big] + \rho'(x_\infty)\frac{rB - L(u_\infty)}{r+\rho(x_\infty)} \;=\; 0.$$

Since $x_\infty\theta'(x_\infty) = -R_0/x_\infty = -\theta(x_\infty)$, the above equation can be written in the simpler form

$$-x_\infty L'(u_\infty)(p_\infty - \theta(x_\infty)) + \frac{rB - L(u_\infty)}{r+\rho(x_\infty)} \;=\; 0. \tag{5.36}$$

Summarizing the above analysis we obtain

Theorem 3. *Under the assumptions* **(A1)–(A4)**, *let (u^*, x^*, p^*) be an optimal solution to the debt management problem* **(DMP)**, *defined for all $t \geq 0$. Assume that, for some constants κ, τ, the conditions (5.23)-(5.24) hold. Then there exists adjoint variables $\tilde{x}^\dagger, \tilde{p}^\dagger, \gamma^\dagger$ satisfying the system (5.26), the initial conditions (5.27), and the asymptotic limits*

$$\Big(x^*, p^*, \tilde{x}^\dagger, \tilde{p}^\dagger, \gamma^\dagger\Big)(t) \;\to\; \left(x_\infty\,,\; p_\infty\,,\; L'(u_\infty)p_\infty\,,\; L'(u_\infty)x_\infty\,,\; \frac{rB - L(u_\infty)}{r+\rho(x_\infty)}\right), \tag{5.37}$$

as $t \to +\infty$, where

$$u_\infty = \lim_{t\to+\infty} u^*(t) = (\lambda + r - \lambda p_\infty)x_\infty\,, \qquad p_\infty = \lim_{t\to+\infty} p^*(t) = \frac{r + \lambda + \theta(x_\infty)\rho(x_\infty)}{r + \lambda + \rho(x_\infty)}\,. \tag{5.38}$$

Moreover, the limit value x_∞ satisfies the identity (5.36).

Proof. Consider a sequence of times $T_n \to +\infty$. For every $n \geq 1$, taking $T = T_n$, the functions u^*, x^*, p^* provide a solution to the optimization problem (5.4)–(5.8) on the time interval $[0, T_n]$. By the PMP, there exist an adjoint vector $(\tilde{x}_n^\dagger, \tilde{p}_n^\dagger, \gamma_n^\dagger)$ satisfying the system (5.26) with initial data (5.27). Thanks to the assumption (5.23), by the previous analysis this implies that on the domain $[\tau, T_n]$ all variables $\tilde{x}_n^\dagger, \tilde{p}_n^\dagger, \gamma_n^\dagger$ satisfy a uniform bound, independent of n. In turn, the equations (5.26) imply that all these functions are uniformly bounded and uniformly Lipschitz continuous on the entire half line $[0, +\infty[$. By Ascoli's theorem, we can thus extract a subsequence which converges uniformly on compact sets:

$$(\tilde{x}_n^\dagger, \tilde{p}_n^\dagger, \gamma_n^\dagger)(t) \;\to\; (\tilde{x}^\dagger, \tilde{p}^\dagger, \gamma^\dagger)(t).$$

It is now clear that the limit functions satisfy the last three equations in (5.26), and

$$u^*(t) \;=\; (L')^{-1}\left(\frac{\tilde{x}_n^\dagger(t)}{\tilde{p}_n(t)}\right) \;=\; (L')^{-1}\left(\frac{\tilde{x}^\dagger(t)}{\tilde{p}(t)}\right),$$

for all $n \geq 1$, $t \geq 0$.

By the previous analysis, as $t \to +\infty$ the limits (5.37)-(5.38) hold, together with the identity (5.36). □

Remark 2. The necessary conditions given in Theorem 2 consist of a system of ODEs for the five variables $(x, p, \tilde{x}^\dagger, \tilde{p}^\dagger, \gamma^\dagger)$, together with the two initial data for $x, \tilde{p}$ in (5.27) and the five equations in (5.36)–(5.38) for the asymptotic limits of these variables as $t \to +\infty$. At first sight, the problem may appear to be overdetermined (7 boundary values for 5 variables). However, this is the correct number of boundary conditions provided that the linearization of the system (5.26) around the asymptotic limit

$$\left(x_\infty\,,\; p_\infty\,,\; L'(u_\infty)p_\infty\,,\; L'(u_\infty)x_\infty\,,\; \frac{rB - L(u_\infty)}{r + \rho(x_\infty)}\right) \tag{5.39}$$

has a 3-dimensional unstable subspace.

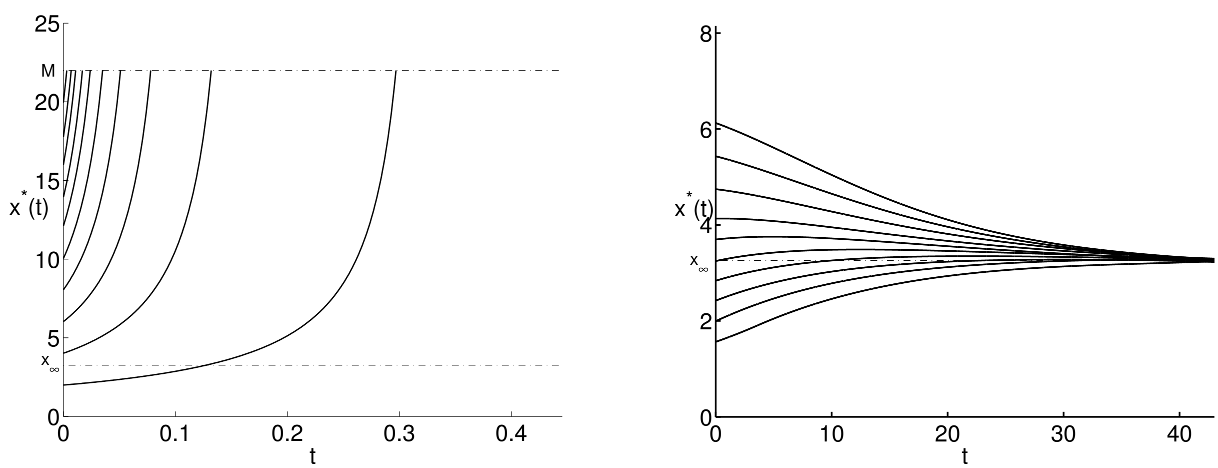

Figure 1: Left: solutions to (5.26) which reach the bankruptcy level M in finite time. Right: solutions which approach the steady state x_∞ as $t \to +\infty$.

6 An example

Consider the debt management problem **(DMP)**, taking

$$r = 0.05, \quad \lambda = 0.1, \quad R_0 = 0.1, \quad M = 22, \quad B = 10,$$

$$L(u) = \frac{u}{1-u}, \qquad \theta(x) = \min\left\{1, \frac{R_0}{x}\right\},$$

$$\rho(x) = \begin{cases} \ln \dfrac{M-R_0}{M-x} + \dfrac{R_0 - x}{M - R_0} & \text{if} \quad x \in [R_0, M[\,, \\ 0 & \text{if} \quad x \in [0, R_0]. \end{cases}$$

In this case, a numerical simulation yields the asymptotic values

$$x_\infty \approx 3.2598, \qquad p_\infty \approx 0.9308, \qquad u_\infty \approx 0.1855,$$

$$\tilde{x}^\dagger_\infty \approx 1.4032, \qquad \tilde{p}^\dagger_\infty \approx 4.9142, \qquad \gamma^\dagger_\infty \approx 4.4234.$$

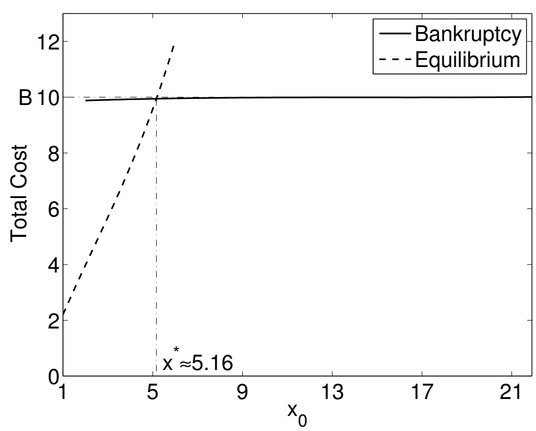

Figure 2: A comparison of the expected total cost of solutions to (5.26), depending on the initial value x_0 of the debt. The dashed line yields the cost of trajectories approaching a steady state x_∞ as $t \to +\infty$. The solid line yields the cost of trajectories reaching bankruptcy in finite time. Notice that this cost is just slightly smaller than the bankruptcy cost $B = 10$.

Linearizing the system (5.26) around these values, we obtain a 5×5 matrix whose eigenvalues are approximately computed as

$$\eta_1 \approx 0.1187, \qquad \eta_2 \approx 0.1150, \qquad \eta_3 \approx 0.1040, \qquad \eta_4, \eta_5 \approx -0.0765 \pm 0.0361\sqrt{-1}.$$

As expected, exactly three of these eigenvalues have positive real part.

Figure 1 shows some numerical results. The plots refer to the x-component of solutions to the system of equations (5.26), satisfied by optimal trajectories. Solutions where the debt $x(t)$ reaches the bankruptcy level M in finite time are shown on the left. Solutions where the debt $x(t)$ approaches asymptotically the steady state x_∞ are shown on the right. The total expected costs of these solutions are plotted in Fig. 2. When the initial size x_0 of the debt is small, i.e. $x_0 \leq x^* \approx 5.16$, strategies approaching the steady state x_∞ yield a lower cost. On the other hand, when the initial debt is large, i.e. $x_0 \geq x^*$, it is convenient to reach bankruptcy in finite time.

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