On the Optimal Shape of Tree Roots and Branches

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Abstract

This paper introduces two classes of variational problems, determining optimal shapes for tree roots and branches. Given a measure μ, describing the distribution of leaves, we introduce a sunlight functional $\mathcal{S}(\mu)$ computing the total amount of light captured by the leaves. On the other hand, given a measure μ describing the distribution of root hair cells, we consider a harvest functional $\mathcal{H}(\mu)$ computing the total amount of water and nutrients gathered by the roots. In both cases, we seek to maximize these functionals subject to a ramified transportation cost, for transporting nutrients from the roots to the trunk and from the trunk to the leaves. The main results establish various properties of these functionals, and the existence of optimal distributions. In particular, we prove the upper semicontinuity of $\mathcal{S}$ and $\mathcal{H}$, together with a priori estimates on the support of optimal distributions.

1 Introduction

Living organisms come in an immense variety of shapes, such as roots, branches, leaves, and flowers in plants, or bones in animals. In many cases, it is expected that through natural selection, these organisms have evolved into a "best possible" shape. From a mathematical perspective, it is thus of interest to study functionals whose minimizers may determine some of the many shapes found in the biological world.

As a step in this direction, in this paper we consider two functionals, defined on a space of positive measures on $\mathbb{R}^d$, and show how they can be used to describe the optimal configurations of roots and branches in a tree.

The first one, which we call the "sunlight functional", models the total amount of sunlight captured by the leaves of a tree. Here we think of a measure μ as the density of leaves. To achieve a realistic model, our functional $\mathcal{S}(\mu)$ will take different forms in the case of a free-standing tree in the middle of a prairie, or a tree in a forest, whose lower branches are partially shielded by the surrounding vegetation. The model also accounts for the fact that light rays come from different directions at different times of the day.

The second one, which we call the "harvest functional", models the total amount of water and nutrients collected by the roots. In this case, we think of a measure μ as the density of root hair cells in the soil. A similar harvest functional $\mathcal{H}(\mu)$ was introduced in [13], in connection with a problem of optimal harvesting of marine resources. In the present paper, both Dirichlet and Neumann boundary conditions will be considered.

The above functionals will be combined with a "ramified transportation cost", for transporting nutrients from the roots to the base of the trunk, or from the base of the trunk to the leaves. For a given measure μ on $I\!R^d$, this is modeled by the minimum α-irrigation cost $\mathcal{I}^\alpha(\mu)$ from the origin, introduced in [24, 31]. The lower semicontinuity of this cost plays an essential role toward the existence of optimal solutions. For a comprehensive introduction to irrigation problems we refer to [7]. Further results on the structure of optimal irrigation patterns can be found in [11, 16].

The optimal shape of branches is now determined by the variational problem

$$\text{maximize:} \quad \mathcal{S}(\mu) - c\mathcal{I}^\alpha(\mu) \tag{1.1}$$

for some constants $0 < \alpha < 1$ and $c > 0$. We study this maximization problem among all positive measures with a given total mass:

$$\mu(I\!R^d) = \kappa_0 \,. \tag{1.2}$$

Notice that, to maximize the gathered sunlight, the leaves should be spread out as wide as possible. On the other hand, this makes it more costly to transport nutrients from the root to all the leaves.

Similarly, the optimal structure of a root system can be related to the problem

$$\text{maximize:} \quad \mathcal{H}(\mu) - c\mathcal{I}^\alpha(\mu). \tag{1.3}$$

The remainder of the paper is organized as follows. In Section 2 we introduce a sunlight functional and prove some of its properties. These include the upper semicontinuity and various estimates. Section 3 is concerned with the harvest functional, recalling the main definitions and extending some of the results in [13] to different boundary conditions. In Section 4 we briefly review the theory of optimal ramified transport, proving some estimates on the minimum α-irrigation cost for a measure μ, for later use. The optimization problems for the shape of tree branches and tree roots are studied in Sections 5 and 6, respectively. Using the semicontinuity of the various functionals, together with a priori bounds on the supports of a sequence of optimizing measures, in both cases we establish the existence of an optimal solution. Some concluding remarks are given in the last section.

2 The sunlight functional

Throughout the following, $B(x_0, r)$ denotes an open ball centered at x_0 with radius r, while $S^{d-1} = \{x \in I\!R^d\,;\ |x| = 1\}$ denotes the unit sphere in $I\!R^d$. We write $\overline{\Omega}$ for the closure of a set Ω, and $\mathcal{L}^d$ for the d-dimensional Lebesgue measure.

Let μ be a positive, bounded Radon measure on $I\!R^d$. Thinking of μ as the distribution of leaves on a tree, we seek a functional $\mathcal{S}(\mu)$ describing the total amount of sunlight captured by the leaves.

To begin with a simple setting, fix a unit vector $\mathbf{n} \in \mathbb{R}^d$ and assume that all light rays come parallel to $\mathbf{n}$. Moreover, assume that the measure μ is absolutely continuous with density f w.r.t. Lebesgue measure on $\mathbb{R}^d$. Call $E_{\mathbf{n}}^{\perp}$ the $(d-1)$-dimensional subspace perpendicular to $\mathbf{n}$ and let $\pi_{\mathbf{n}} : \mathbb{R}^d \mapsto \mathbf{n}^{\perp}$ be the perpendicular projection. As shown in Fig. 1, each point $x \in \mathbb{R}^d$ can be expressed uniquely as

$$x \;=\; y + s\mathbf{n} \tag{2.1}$$

with $y \in E_{\mathbf{n}}^{\perp}$ and $s \in \mathbb{R}$.

Our basic modeling assumption is that the rate at which sunlight is absorbed is proportional to the local density of leaves. For each fixed $y \in E_{\mathbf{n}}^{\perp}$, calling $s \mapsto \phi(y,s)$ the amount of sunlight reaching the point $x = y + s\mathbf{n}$, we thus assume

$$\frac{\partial}{\partial s}\phi(y,s) \;=\; f(y+s\mathbf{n})\phi(y,s)$$

$$\lim_{s\to+\infty} \phi(y,s) \;=\; 1.$$

For simplicity, we here assign unit values to the absorption rate, and to the amount of light arriving from infinity per unit $(d-1)$-dimensional volume in $E_{\mathbf{n}}^{\perp}$. This implies

$$\phi(y,s) \;=\; \exp\left\{-\int_s^{+\infty} f(y+t\mathbf{n})\,dt\right\}. \tag{2.2}$$

Integrating over the perpendicular plane $E_{\mathbf{n}}^{\perp}$, the total amount of light which is absorbed by the leaves is thus

$$\mathcal{S}^{\mathbf{n}}(\mu) \;=\; \int_{E_{\mathbf{n}}^{\perp}} 1 \;-\; \exp\left\{-\int_{-\infty}^{+\infty} f(y+t\mathbf{n})\,dt\right\} dy. \tag{2.3}$$

We now observe that the formula (2.3) can be easily extended to the case of a general measure μ, not necessarily absolutely continuous w.r.t. Lebesgue measure.

On the perpendicular subspace $E_{\mathbf{n}}^{\perp}$ consider the projected measure $\mu^{\mathbf{n}}$ defined by setting

$$\mu^{\mathbf{n}}(A) \;=\; \mu\Big(\{x \in \mathbb{R}^d\,;\;\; \pi_{\mathbf{n}}(x) \in A\}\Big) \tag{2.4}$$

for every open set $A \subseteq E_{\mathbf{n}}^{\perp}$. Call $\Phi^{\mathbf{n}}$ the density of the absolutely continuous part of $\mu^{\mathbf{n}}$ w.r.t. the $(d-1)$-dimensional Lebesgue measure on $E_{\mathbf{n}}^{\perp}$.

Definition 2.1 *The total amount of sunlight from the direction* $\mathbf{n}$ *absorbed by a measure* μ *on* $\mathbb{R}^d$ *is defined as*

$$\mathcal{S}^{\mathbf{n}}(\mu) \;\doteq\; \int_{E_{\mathbf{n}}^{\perp}} \Big(1 - \exp\{-\Phi^{\mathbf{n}}(y)\}\Big)\,dy\,. \tag{2.5}$$

Next, we model the fact that sunlight does not always come from the same direction. Instead, there exists a density function $\eta : S^{d-1} \mapsto \mathbb{R}_+$ which describes the total amount of light coming from the direction $\mathbf{n}$ during the course of a day.

Definition 2.2 *If light comes from different directions with variable intensity* $\eta = \eta(\mathbf{n})$*, the total amount of sunlight captured by a measure* μ *on* $\mathbb{R}^d$ *is defined as*

$$\mathcal{S}^{\eta}(\mu) \;\doteq\; \int_{S^{d-1}} \mathcal{S}^{\mathbf{n}}(\mu)\,\eta(\mathbf{n})\,d\mathbf{n}\,. \tag{2.6}$$

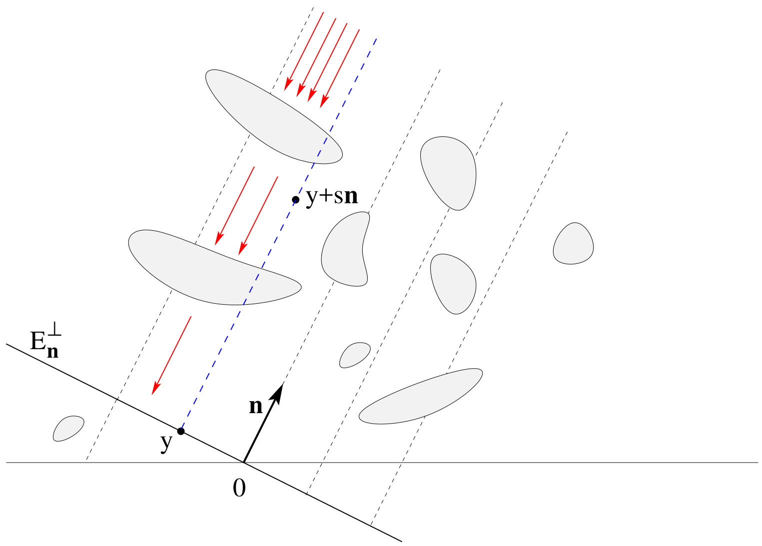

Figure 1: Sunlight arrives from the direction parallel to $\mathbf{n}$. Part of it is absorbed by the measure μ, supported on the grey regions.

Remark 2.1 The above definitions apply to a general Radon measure μ on $I\!R^d$. However, measures which are singular w.r.t. the $(d-1)$-dimensional Hausdorff measure are irrelevant. More precisely, if $\mu = \mu_1 + \mu_2$ and μ_2 is concentrated on a set whose $(d-1)$-dimensional measure is zero, then for every unit vector $\mathbf{n} \in I\!R^d$ we have $\mathcal{S}^{\mathbf{n}}(\mu) = \mathcal{S}^{\mathbf{n}}(\mu_1)$, while $\mathcal{S}^{\mathbf{n}}(\mu_2) = 0$.

Remark 2.2 A case of particular interest is when light comes uniformly from all directions of the positive half sphere.

$$S_+^{d-1} \doteq \Big\{\mathbf{n} = (n_1, \ldots, n_d)\,; \quad |\mathbf{n}| = 1\,, \quad n_d > 0\Big\},$$

We shall model this situation by taking

$$\eta(\mathbf{n}) = \begin{cases} \sigma_d/2 & \text{if} \quad \mathbf{n} \in S_+^{d-1}\,, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Here σ_d denotes the $(d-1)$-dimensional measure of the surface of the unit ball in $I\!R^d$.

The next lemma collects some elementary properties of the functional $\mathcal{S}^{\mathbf{n}}$. In the following, we denote by μ^λ the measure such that

$$\mu^\lambda(A) = \mu(\lambda^{-1}A) \tag{2.8}$$

for every open set $A \subset I\!R^d$, so that

$$Supp(\mu^\lambda) = \lambda \cdot Supp(\mu) = \{\lambda x\,; \quad x \in Supp(\mu)\}. \tag{2.9}$$

Moreover, ω_d denotes the volume of the unit ball in $I\!R^d$.

Lemma 2.1 *Let $\mu, \tilde{\mu}$ be positive Radon measures on $I\!R^d$. For any unit vector $\mathbf{n} \in S^{d-1}$, the following holds.*

(i) $\mathcal{S}^{\mathbf{n}}(\mu) \ \leq \ \mu(I\!R^d)$,

(ii) If the measure μ is supported inside a ball of radius r, then $\mathcal{S}^{\mathbf{n}}(\mu) \leq \omega_{d-1}\, r^{d-1}$.

(iii) $\mathcal{S}^{\mathbf{n}}(\mu) \ \leq \ \mathcal{S}^{\mathbf{n}}(\mu+\tilde{\mu}) \ \leq \ \mathcal{S}^{\mathbf{n}}(\mu) + \mathcal{S}^{\mathbf{n}}(\tilde{\mu})$.

(iv) $\mathcal{S}^{\mathbf{n}}(\lambda\mu) \ \leq \ \lambda S^{\mathbf{n}}(\mu)$, *for every* $\lambda \geq 1$.

(v) For every $\lambda > 0$ one has

$$\mathcal{S}^{\mathbf{n}}(\lambda^{d-1}\mu^{\lambda}) \ = \ \lambda^{d-1}\mathcal{S}^{\mathbf{n}}(\mu). \tag{2.10}$$

(vi) If μ is absolutely continuous w.r.t. Lebesgue measure, then

$$\lim_{\lambda\to 0+} \frac{\mathcal{S}^{\mathbf{n}}(\lambda\mu)}{\lambda} \ = \ \mu(I\!R^d) \ = \ \lim_{\lambda\to+\infty} \mathcal{S}^{\mathbf{n}}(\mu^{\lambda}). \tag{2.11}$$

Proof. 1. To prove (i), consider any unit vector $\mathbf{n}$ and call $\Phi^{\mathbf{n}}$ the density of the absolutely continuous part of of the projected measure $\mu^{\mathbf{n}}$ w.r.t. the $(d-1)$-dimensional Lebesgue measure on $E_{\mathbf{n}}^{\perp}$. Then

$$\mathcal{S}^{\mathbf{n}}(\mu) \ \doteq \ \int_{E_{\mathbf{n}}^{\perp}} \Big(1-\exp\{-\Phi^{\mathbf{n}}(y)\}\Big)\, dy \ \leq \ \int_{E_{\mathbf{n}}^{\perp}} \Phi^{\mathbf{n}}(y)\, dy \ \leq \ \mu^{\mathbf{n}}(E_{\mathbf{n}}^{\perp}) \ = \ \mu(I\!R^n). \tag{2.12}$$

2. To prove (ii), let μ be supported inside the ball $B(x_0, r)$, centered at x_0 with radius r. Call y_0 the perpendicular projection of x_0 on the space $E_{\mathbf{n}}^{\perp}$. Then $\Phi^{\mathbf{n}}(y) = 0$ whenever $|y-y_0| > r$. Hence

$$\mathcal{S}^{\mathbf{n}}(\mu) \ \doteq \ \int_{E_{\mathbf{n}}^{\perp}} \Big(1-\exp\{-\Phi^{\mathbf{n}}(y)\}\Big)\, dy \ \leq \ \int_{E_{\mathbf{n}}^{\perp}\cap B(y_0,r)} 1\, dy \ = \ \omega_{d-1} r^{d-1}.$$

3. Concerning (iii), the first inequality is an immediate consequence of the monotonicity of the function $1-e^{-x}$. To prove the second inequality, denote by $\Phi^{\mathbf{n}}, \widetilde{\Phi}^{\mathbf{n}}$ the density functions of the projected measures $\mu^{\mathbf{n}}, \tilde{\mu}^{\mathbf{n}}$ on the perpendicular space $E_{\mathbf{n}}^{\perp}$. Observing that $\Phi^{\mathbf{n}} + \widetilde{\Phi}^{\mathbf{n}}$ is the density function of $(\mu+\tilde{\mu})^{\mathbf{n}}$, one obtains

$$\begin{aligned} &\mathcal{S}^{\mathbf{n}}(\mu+\tilde{\mu}) - \mathcal{S}^{\mathbf{n}}(\mu) - \mathcal{S}^{\mathbf{n}}(\tilde{\mu}) \\ &\quad = \ \int_{E_{\mathbf{n}}^{\perp}} \Big(1-\exp\{-\Phi^{\mathbf{n}}(y)-\widetilde{\Phi}^{\mathbf{n}}(y)\}\Big) - \Big(1-\exp\{-\Phi^{\mathbf{n}}(y)\}\Big) - \Big(1-\exp\{-\widetilde{\Phi}^{\mathbf{n}}(y)\}\Big)\, dy \\ &\quad = \ \int_{E_{\mathbf{n}}^{\perp}} \Big[\exp\{-\Phi^{\mathbf{n}}(y)\} + \exp\{-\widetilde{\Phi}^{\mathbf{n}}(y)\} - \exp\{-\Phi^{\mathbf{n}}(y)-\widetilde{\Phi}^{\mathbf{n}}(y)\} - 1\Big]\, dy \ \leq \ 0. \end{aligned} \tag{2.13}$$

Indeed, the last inequality is obtained by checking that

$$h(x_1, x_2) \ \doteq \ e^{-x_1} + e^{-x_2} - e^{-x_1-x_2} - 1 \ \leq \ 0$$

for every $x_1, x_2 \geq 0$.

4. To prove (iv), consider the function

$$h(x) \doteq 1 - e^{-\lambda x} - \lambda + \lambda e^{-x}.$$

Assuming $\lambda \geq 1$, an elementary computation yields

$$h(0) = 0, \qquad h'(x) = \lambda e^{-\lambda x} - \lambda e^{-x} \leq 0 \qquad \text{for all } x \geq 0.$$

Therefore $h(x) \leq 0$ for all $x \geq 0$. Using this inequality we obtain

$$\begin{aligned} S^{\mathbf{n}}(\lambda\mu) - \lambda S^{\mathbf{n}}(\mu) &= \int_{E_{\mathbf{n}}^{\perp}} \Big(1 - \exp\{-\lambda \Phi^{\mathbf{n}}(y)\}\Big) dy - \int_{E_{\mathbf{n}}^{\perp}} \lambda\Big(1 - \exp\{-\Phi^{\mathbf{n}}(y)\}\Big) dy \\ &= \int_{E_{\mathbf{n}}^{\perp}} h(\Phi^{\mathbf{n}}(y))\, dy \leq 0. \end{aligned} \tag{2.14}$$

5. To prove (v), we first compute the density function $\Phi^{\mathbf{n},\lambda}$ for the projected measure $(\lambda^{d-1}\mu^{\lambda})^{\mathbf{n}}$ on the $(d-1)$-dimensional subspace $E_{\mathbf{n}}^{\perp}$. From the identity

$$\int_A \Phi^{\mathbf{n},\lambda}(y)\, dy = \int_{\lambda^{-1}A} \lambda^{d-1} \Phi^{\mathbf{n}}(\tilde{y})\, d\tilde{y} = \int_A \Phi^{\mathbf{n}}(\lambda^{-1}y)\, dy$$

valid for every open set $A \subseteq E_{\mathbf{n}}^{\perp}$, we deduce

$$\Phi^{\mathbf{n},\lambda}(y) = \Phi^{\mathbf{n}}(\lambda^{-1}y)$$

for every $y \in E_{\mathbf{n}}^{\perp}$. Therefore, using the change of variable $\tilde{y} = \lambda^{-1}y$, one obtains

$$\begin{aligned} \mathcal{S}^{\mathbf{n}}(\lambda^{d-1}\mu^{\lambda}) &= \int_{E_{\mathbf{n}}^{\perp}} \Big(1 - \exp\{-\Phi^{\mathbf{n},\lambda}(y)\}\Big)\, dy \\ &= \int_{E_{\mathbf{n}}^{\perp}} \lambda^{d-1}\Big(1 - \exp\{-\Phi^{\mathbf{n}}(\tilde{y})\}\Big)\, d\tilde{y} \\ &= \lambda^{d-1}\mathcal{S}^{\mathbf{n}}(\mu). \end{aligned} \tag{2.15}$$

6. It remains to prove the two limits in (2.11). Assume that the positive measure μ has density f w.r.t. Lebesgue measure on $\mathbb{R}^d$. Then

$$\frac{\mathcal{S}^{\mathbf{n}}(\lambda\mu)}{\lambda} = \int_{E_{\mathbf{n}}^{\perp}} \frac{\Big(1 - \exp\Big\{-\lambda \int_{-\infty}^{\infty} f(y + t\mathbf{n}) dt\Big\}\Big)}{\lambda}\, dy$$

By Fubini's theorem, for almost every $y \in E_{\mathbf{n}}^{\perp}$ we have $\int_{-\infty}^{\infty} f(y + t\mathbf{n})\, dt < \infty$. At such a point y we have

$$\lim_{\lambda \to 0+} \frac{1 - \exp\Big\{-\lambda \int_{-\infty}^{\infty} f(y + t\mathbf{n})\Big\}}{\lambda} = \int_{-\infty}^{\infty} f(y + t\mathbf{n})\, dt.$$

On the other hand,

$$\frac{1 - \exp\Big\{-\lambda \int_{-\infty}^{\infty} f(y + t\mathbf{n})\Big\}}{\lambda} \leq \int_{-\infty}^{\infty} f(y + t\mathbf{n})\, dt \tag{2.16}$$

Therefore, by dominated convergence theorem, we conclude

$$\lim_{\lambda\to 0+} \frac{\mathcal{S}^{\mathbf{n}}(\lambda\mu)}{\lambda} \;=\; \int_{E_{\mathbf{n}}^{\perp}} \int_{-\infty}^{\infty} f(y+t\mathbf{n})\;dtdy \;=\; \mu(I\!\!R^d). \tag{2.17}$$

To prove the second equality in (2.11), call $\Phi^{\mathbf{n},\lambda}$ the density function for the projected measure $(\mu^{\lambda})^{\mathbf{n}}$. For almost every $y \in E_{\mathbf{n}}^{\perp}$ we have

$$\Phi^{\mathbf{n},\lambda}(y) \;=\; \frac{1}{\lambda^{d-1}} \int_{-\infty}^{\infty} f\left(\frac{y}{\lambda}+t\mathbf{n}\right) dt \;<\; +\infty. \tag{2.18}$$

Therefore,

$$\begin{aligned} \mathcal{S}^{\mathbf{n}}(\mu^{\lambda}) \;&=\; \int_{E^{\perp}} \Big(1-\exp\Big\{-\frac{1}{\lambda^{d-1}}\int_{-\infty}^{\infty} f\Big(\frac{y}{\lambda}+t\mathbf{n}\Big)dt\Big\}\Big)\;dy \\ &=\; \int_{E^{\perp}} \lambda^{d-1}\Big(1-\exp\Big\{-\frac{1}{\lambda^{d-1}}\int_{-\infty}^{\infty} f(y+t\mathbf{n})dt\Big\}\Big)\;dy. \end{aligned} \tag{2.19}$$

For a.e. y we have $\int_{-\infty}^{\infty} f(y+t\mathbf{n})dt < \infty$, and hence

$$\lambda^{d-1}\left(1-\exp\left\{-\frac{1}{\lambda^{d-1}}\int_{-\infty}^{\infty} f(y+t\mathbf{n})dt\right\}\right) \;\leq\; \int_{-\infty}^{\infty} f(y+t\mathbf{n})\,dt.$$

On the other hand, by L'Hospital Rule,

$$\lim_{\lambda\to+\infty} \lambda^{d-1}\Big(1-\exp\Big\{-\frac{1}{\lambda^{d-1}}\int_{-\infty}^{\infty} f(y+t\mathbf{n})dt\Big\}\Big) \;=\; \int_{-\infty}^{\infty} f(y+t\mathbf{n})\;dt$$

Letting $\lambda \to +\infty$ in (2.19), by the dominated convergence theorem one obtains the second equality in (2.11). □

The formula (2.5) covers the case where there are no other obstacles to light propagation except μ. Next, we want to model the presence of other plants that capture part of the light, and determine how much light is actually collected by μ.

As a preliminary, consider two positive measures μ and ν, absolutely continuous with densities f, g w.r.t. Lebesgue measure on $I\!\!R^d$. Assuming that light comes from the direction $\mathbf{n}$, the same computation as in (2.2) shows that the total amount of light that reaches a point $x = y + s\mathbf{n}$ is

$$\exp\left\{-\int_s^{+\infty} \big(f(y+t\mathbf{n})+g(y+t\mathbf{n})\big)\,dt\right\}.$$

Integrating by parts, the total amount of light collected by the distribution μ with density f is computed by

$$\begin{aligned} \mathcal{S}^{\mathbf{n}}(\mu) \;&=\; \int_{E_{\mathbf{n}}^{\perp}} \left(\int_{-\infty}^{+\infty} f(y+s\mathbf{n})\,\exp\left\{-\int_s^{+\infty} \big(f(y+t\mathbf{n})+g(y+t\mathbf{n})\big)\,dt\right\}\,ds\right)dy \\ &=\; \int_{E_{\mathbf{n}}^{\perp}} \left(\int_{-\infty}^{+\infty} \frac{d}{ds}\exp\left\{-\int_s^{+\infty} f(y+t\mathbf{n})\,dt\right\}\cdot\exp\left\{-\int_s^{+\infty} g(y+t\mathbf{n})\,dt\right\}\,ds\right)dy \\ &=\; \int_{E_{\mathbf{n}}^{\perp}} \Bigg(1-\exp\left\{-\int_{-\infty}^{+\infty} f(y+t\mathbf{n})\,dt\right\}\exp\left\{-\int_{-\infty}^{+\infty} g(y+t\mathbf{n})\,dt\right\} \\ &\qquad -\int_{-\infty}^{+\infty} g(y+s\mathbf{n})\exp\left\{-\int_s^{+\infty} \big(f(y+t\mathbf{n})+g(y+t\mathbf{n})\big)\,dt\right\}\,ds\Bigg)dy\,. \end{aligned} \tag{2.20}$$

In essence, this says that

$$[\text{light collected by } \mu] \;=\; [\text{light collected by } \mu+\nu] - [\text{light collected by } \nu].$$

Notice that here the right hand side makes sense also if μ is is an arbitrary measure, not necessarily absolutely continuous w.r.t. Lebesgue measure. This fact can be used to define the total sunlight absorbed by any positive measure μ, in the presence of a second measure ν which is absolutely continuous with density $g(\cdot)$ w.r.t. Lebesgue measure on $I\!R^d$.

For any given a unit vector $\mathbf{n}$, we represent $I\!R^d = E_{\mathbf{n}} \oplus E_{\mathbf{n}}^{\perp}$, as the sum of the orthogonal spaces containing all vectors parallel and orthogonal to $\mathbf{n}$, respectively. We denote by $(t, y) \in E_{\mathbf{n}} \oplus E_{\mathbf{n}}^{\perp}$ the variable corresponding to this decomposition. As before, let $\pi_{\mathbf{n}} : I\!R^d \mapsto E_{\mathbf{n}}^{\perp}$ be the perpendicular projection, and call $\mu^{\mathbf{n}}$ be the projection of μ on $E_{\mathbf{n}}^{\perp}$, defined as in (2.4). By Theorem 2.28 in [1] (on the disintegration of the measure μ), there exists a family of 1-dimensional measures μ^y, $y \in E_{\mathbf{n}}^{\perp}$, such that the following holds.

(i) $\mu^y(E_{\mathbf{n}}) = 1$ for every $y \in E_{\mathbf{n}}^{\perp}$.

(ii) The map $y \mapsto \mu^y$ is $\mu^{\mathbf{n}}$-measurable.

(iii) For every $\phi \in \mathbf{L}^1(I\!R^d)$ one has

$$\int_{I\!R^d} \phi\, d\mu \;=\; \int_{E_{\mathbf{n}}^{\perp}} \left(\int_{-\infty}^{+\infty} \phi(t,y)\, d\mu^y(t) \right) d\mu^{\mathbf{n}}(y)\,. \tag{2.21}$$

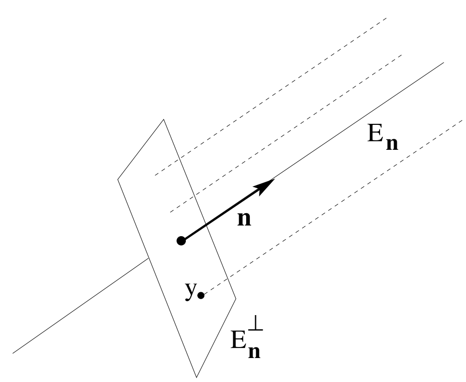

Figure 2: Disintegration of a measure μ on $I\!R^d$. According to (2.21), the integral $\int \phi\, d\mu$ can be computed first integrating ϕ along each line $\{y + t\mathbf{n}\,;\, t \in I\!R\}$ parallel to the unit vector $\mathbf{n}$, then integrating over the variable $y \in E_{\mathbf{n}}^{\perp}$.

To compute the total amount of light coming from the direction parallel to $\mathbf{n}$ which is captured by the measure μ, we proceed as follows.

Let $\Phi^{\mathbf{n}}$ be the density of the absolutely continuous part of $\mu^{\mathbf{n}}$ w.r.t. $(d-1)$-dimensional Lebesgue measure on $E_{\mathbf{n}}^{\perp}$, as in (2.5).

Now let ν be a second measure, absolutely continuous with density g w.r.t. Lebesgue measure on $I\!R^d$. Motivated by (2.20), for each $y \in E_{\mathbf{n}}^{\perp}$ we define

$$\mathcal{S}^{\mathbf{n}}_{\mu,\nu}(y) \doteq 1-\exp\{-\Phi^{\mathbf{n}}(y)\}\exp\left\{-\int g(y+s\mathbf{n})\,ds\right\}$$
$$-\int\left(g(y+s\mathbf{n})\exp\left\{-\int_s^{+\infty} g(y+t\mathbf{n})\,dt\right\}\cdot\exp\left\{-\Phi^{\mathbf{n}}(y)\cdot\mu^y([s,+\infty[)\right\}\right)ds\,, \tag{2.22}$$

$$\mathcal{S}^{\mathbf{n}}(\mu;\nu) \doteq \int_{E_{\mathbf{n}}^{\perp}} \mathcal{S}^{\mathbf{n}}_{\mu,\nu}(y)\,dy\,. \tag{2.23}$$

Definition 2.3 *Assume that light comes with variable intensity $\eta(\cdot)$ from different directions. The* **total sunlight $\mathcal{S}^{\eta}(\mu;\nu)$ absorbed by the measure μ in the presence of the absolutely continuous measure** *ν is then defined as*

$$\mathcal{S}^{\eta}(\mu;\nu) \doteq \int_{S^{d-1}} \mathcal{S}^{\mathbf{n}}(\mu;\nu)\,\eta(\mathbf{n})\,d\mathbf{n}\,. \tag{2.24}$$

We observe that the first three estimates in Lemma 2.1 remain valid in this more general situation.

Lemma 2.2 *Let μ,ν be positive Radon measures on $I\!R^d$. Assume that ν is absolutely continuous w.r.t. Lebesgue measure. For any unit vector $\mathbf{n}\in S^{d-1}$, the following holds.*

(i) $\mathcal{S}^{\mathbf{n}}(\mu,\nu) \le \mathcal{S}^{\mathbf{n}}(\mu) \le \mu(I\!R^d)$.

(ii) If the measure μ is supported inside a ball of radius r, then $\mathcal{S}^{\mathbf{n}}(\mu;\nu) \le \omega_{d-1}\,r^{d-1}$.

(iii) For any positive measures μ_1,μ_2 one has

$$\mathcal{S}^{\mathbf{n}}(\mu_1,\nu) \le \mathcal{S}^{\mathbf{n}}(\mu_1+\mu_2,\nu) \le \mathcal{S}^{\mathbf{n}}(\mu_1,\nu)+\mu_2(I\!R^d). \tag{2.25}$$

Proof. 1. Let g be the density of ν w.r.t. Lebesgue measure on $I\!R^d$. By (2.22) we have

$$\begin{aligned}\mathcal{S}^{\mathbf{n}}_{\mu,\nu}(y) &\doteq 1-\exp\{-\Phi^{\mathbf{n}}(y)\}\exp\left\{-\int g(y+s\mathbf{n})\,ds\right\}\\ &\qquad -\int\left(g(y+s\mathbf{n})\exp\left\{-\int_s^{+\infty} g(y+t\mathbf{n})\,dt\right\}\cdot\exp\left\{-\Phi^{\mathbf{n}}(y)\cdot\mu^y([s,+\infty[)\right\}\right)ds\\ &\le 1-\exp\{-\Phi^{\mathbf{n}}(y)\}\exp\left\{-\int g(y+s\mathbf{n})\,ds\right\}\\ &\qquad -\int\left(\frac{d}{ds}\exp\left\{-\int_s^{+\infty} g(y+t\mathbf{n})\,dt\right\}\right)ds\cdot\exp\{-\Phi^{\mathbf{n}}(y)\}\\ &= 1-\exp\{-\Phi^{\mathbf{n}}(y)\} \le \Phi^{\mathbf{n}}(y)\,.\end{aligned} \tag{2.26}$$

Integrating over $E_{\mathbf{n}}^{\perp}$ we obtain the first inequality in (i). The second inequality is now a consequence of (2.12).

2. If μ is supported in a ball of radius r, then the estimate (ii) follows immediately from $\mathcal{S}^{\mathbf{n}}(\mu,\nu) \le \mathcal{S}^{\mathbf{n}}(\mu) \le \omega_{d-1} r^{d-1}$.

3. To prove (iii), let $\Phi_1^{\mathbf{n}}, \Phi_2^{\mathbf{n}}$, and $\Phi^{\mathbf{n}} = \Phi_1^{\mathbf{n}} + \Phi_2^{\mathbf{n}}$ be the densities of the absolutely continuous parts of $\mu_1^{\mathbf{n}}$, $\mu_2^{\mathbf{n}}$, and $\mu^{\mathbf{n}} = \mu_1^{\mathbf{n}} + \mu_2^{\mathbf{n}}$ w.r.t. the $(d-1)$-dimensional Lebesgue measure on $E_{\mathbf{n}}^{\perp}$, respectively. We claim that

$$\mathcal{S}^{\mathbf{n}}_{\mu_1+\mu_2,\nu}(y) - \mathcal{S}^{\mathbf{n}}_{\mu_1,\nu}(y) \ \le\ \Phi_2^{\mathbf{n}}(y) \tag{2.27}$$

for almost every $y \in E_{\mathbf{n}}^{\perp}$. Indeed, for a fixed y, assume $\Phi_2(y) \neq 0$ and define

$$\lambda \ \doteq\ \frac{\Phi_1^{\mathbf{n}}(y)}{\Phi_1^{\mathbf{n}}(y) + \Phi_2^{\mathbf{n}}(y)} \ <\ 1\,. \tag{2.28}$$

Call μ_1^y, μ_2^y, and μ^y the probability measures on the 1-dimensional space $E_{\mathbf{n}}$ corresponding to the disintegration of μ_1, μ_2, and $\mu = \mu_1 + \mu_2$, respectively. By (2.28) it follows

$$\mu^y \ =\ \lambda \mu_1^y + (1-\lambda)\mu_2^y\,. \tag{2.29}$$

We now compute

$$\begin{aligned}
&\mathcal{S}^{\mathbf{n}}_{\mu_1+\mu_2,\nu}(y) - \mathcal{S}^{\mathbf{n}}_{\mu_1,\nu}(y) \\
&\quad= \Big(\exp\{-\Phi_1^{\mathbf{n}}(y)\} - \exp\Big\{-\Phi_1^{\mathbf{n}}(y) - \Phi_2^{\mathbf{n}}(y)\Big\}\Big)\cdot \exp\Big\{-\int g(y+s\mathbf{n})\,ds\Big\} \\
&\qquad\quad + \int g(y+s\mathbf{n})\cdot \exp\Big\{-\int_s^\infty g(y+t\mathbf{n})dt\Big\} \\
&\qquad\qquad \cdot\Big(\exp\Big\{-\Phi_1^{\mathbf{n}}(y)\cdot \mu_1^y[s,\infty[\Big\} - \exp\Big\{-(\Phi_1^{\mathbf{n}}(y) + \Phi_2^{\mathbf{n}}(y))\,\mu^y[s,\infty[\Big\}\Big)ds \\
&\quad\doteq\ I + J.
\end{aligned} \tag{2.30}$$

The second term in the above expression can be estimated as

$$\begin{aligned}
J \ &=\ \int g(y+s\mathbf{n})\cdot \exp\Big\{-\int_s^\infty g(y+t\mathbf{n})dt\Big\} \\
&\cdot\Big(\exp\Big\{-\frac{\lambda}{1-\lambda}\Phi_2^{\mathbf{n}}(y)\cdot\mu_1^y[s,\infty[\Big\} - \exp\Big\{-\frac{1}{1-\lambda}\Phi_2^{\mathbf{n}}(y)\cdot(\lambda\,\mu_1^y + (1-\lambda)\,\mu_2^y)[s,\infty[\Big\}\Big)ds \\
&= \int g(y+s\mathbf{n})\cdot\exp\Big\{-\int_s^\infty g(y+t\mathbf{n})dt\Big\} \\
&\qquad\cdot\Big(\exp\Big\{-\frac{\lambda}{1-\lambda}\Phi_2^{\mathbf{n}}(y)\cdot\mu_1^y[s,\infty[\Big\}\cdot\Big(1-\exp\{\Phi_2^{\mathbf{n}}(y)\cdot\mu^y[s,\infty[\,\}\Big)\Big)ds \\
&\le \int g(y+s\mathbf{n})\cdot\exp\Big\{-\int_s^\infty g(y+t\mathbf{n})dt\Big\}\cdot\Phi_2^{\mathbf{n}}(y)\ ds \\
&= \Big(1-\exp\Big\{-\int g(y+s\mathbf{n})ds\Big\}\Big)\cdot\Phi_2^{\mathbf{n}}(y)
\end{aligned} \tag{2.31}$$

Combining (2.30) with (2.31) we obtain

$$
\begin{aligned}
&\mathcal{S}^{\mathbf{n}}_{\mu_1+\mu_2,\nu}(y) - \mathcal{S}^{\mathbf{n}}_{\mu_1,\nu}(y) \;=\; I+J \\
&\leq\; \Big(\exp\{-\Phi^{\mathbf{n}}_1(y)\} - \exp\{-\Phi^{\mathbf{n}}_1(y) - \Phi^{\mathbf{n}}_2(y)\} - \Phi^{\mathbf{n}}_2(y)\Big)\cdot \exp\left\{-\int g(y+s\mathbf{n})ds\right\} + \Phi^{\mathbf{n}}_2(y) \\
&\leq\; \Phi^{\mathbf{n}}_2(y).
\end{aligned}
$$

Integrating over the $(d-1)$-dimensional space $E^{\perp}_{\mathbf{n}}$ one obtains the desired estimate. This completes the proof of (iii). □

The next lemma, establishing the upper semicontinuity of the sunlight functional $\mathcal{S}$ w.r.t. weak convergence of measures, provides the main ingredient in the proof of existence of optimal measures. We recall that the weak convergence of measures $\mu_k \rightharpoonup \mu$ means

$$
\lim_{k\to\infty}\int \varphi\, d\mu_k \;=\; \int \varphi\, d\mu \qquad\qquad \text{for every} \quad \varphi\in\mathcal{C}^0_c(I\!R^d). \tag{2.32}
$$

In the following we consider a sequence of positive Radon measures $(\mu_k)_{k\geq 1}$, on $I\!R^d$, satisfying the usual assumptions

1 - Boundedness: there exists a constant C such that

$$
\mu_k(I\!R^d)\leq C \qquad \text{for all} \quad k\geq 1. \tag{2.33}
$$

2 - Tightness: for every $\varepsilon>0$ there exists a radius R_ε such that

$$
\mu_k\Big(\{x\in I\!R^d\,;\ |x|>R_\varepsilon\}\Big) \;<\; \varepsilon. \tag{2.34}
$$

By a well known compactness theorem [2, 8], this implies the existence of a weakly convergent subsequence: $\mu_{k_j}\rightharpoonup\mu$.

Lemma 2.3 *Consider a weakly convergent sequence of measures $\mu_k\rightharpoonup\mu$, satisfying the boundedness and tightness conditions (2.33)-(2.34). Then, for any unit vector $\mathbf{n}$ and every positive measure ν, absolutely continuous w.r.t. Lebesgue measure on $I\!R^d$, one has*

$$
\mathcal{S}^{\mathbf{n}}(\mu;\nu) \;\geq\; \limsup_{k\to\infty}\mathcal{S}^{\mathbf{n}}(\mu_k;\nu). \tag{2.35}
$$

Proof. 1. We start with the basic case where $\nu=0$ and all measures μ_k are supported inside a ball $B(0,R)\subset I\!R^d$.

From the assumption it follows the weak convergence $\mu^{\mathbf{n}}_k\rightharpoonup\mu^{\mathbf{n}}$ of the projected measures. Call $\Phi^{\mathbf{n}}_k$, $\Phi^{\mathbf{n}}$ respectively the density of the absolutely continuous part of $\mu^{\mathbf{n}}_k$ and $\mu^{\mathbf{n}}$ w.r.t. $(d-1)$-dimensional Lebesgue measure on $E^{\perp}_{\mathbf{n}}$.

Let $\varepsilon>0$ be given. According to the "biting lemma" [6, 14, 21], there exists a set $V_\varepsilon\subset B(0,R)\subset E^{\perp}_{\mathbf{n}}$, with

$$
meas(V_\varepsilon) \;<\; \varepsilon, \tag{2.36}
$$

and such that the following holds. Let $\hat\mu_k^{\mathbf n}$ be the absolutely continuous measure on $E_{\mathbf n}^\perp$ whose density (w.r.t. Lebesgue measure) is

$$\widehat\Phi_k^{\mathbf n}(y) \;=\; \begin{cases} \Phi_k^{\mathbf n}(y) & \text{if } \ y\in B(0,R)\setminus V_\varepsilon\,,\\ 0 & \text{otherwise.}\end{cases}$$

Then, by possibly extracting a subsequence, we have the weak convergence

$$\hat\mu_k^{\mathbf n} \;\rightharpoonup\; \hat\mu^{\mathbf n}, \qquad\qquad \widehat\Phi_k^{\mathbf n} \;\rightharpoonup\; \widehat\Phi^{\mathbf n}\,,$$

Here the second arrow denotes weak convergence in $\mathbf{L}^1$. Moreover, $\hat\mu^{\mathbf n}$ is the absolutely continuous measure having density $\widehat\Phi^{\mathbf n}$ w.r.t. $(d-1)$-dimensional Lebesgue measure. By (2.36) one has the obvious estimate

$$\int_{V_\varepsilon}\Big(1-\exp\{-\Phi^{\mathbf n}(y)\}\Big)\,dy \;\leq\; meas(V_\varepsilon) \;<\; \varepsilon. \tag{2.37}$$

Since $\mu^{\mathbf n}\geq\hat\mu^{\mathbf n}$, by (2.5) and (2.37) the total sunshine captured by the measure μ can now be estimated as

$$\begin{aligned}
\mathcal{S}^{\mathbf n}(\mu) \;&\geq\; \int_{E_{\mathbf n}^\perp}\Big(1-\exp\{-\widehat\Phi^{\mathbf n}(y)\}\Big)\,dy\\
&\geq\; \limsup_{k\to\infty}\int_{E_{\mathbf n}^\perp}\Big(1-\exp\{-\widehat\Phi_k^{\mathbf n}(y)\}\Big)\,dy\\
&=\; \limsup_{k\to\infty}\left(\int_{E_{\mathbf n}^\perp}\Big(1-\exp\{-\Phi_k^{\mathbf n}(y)\}\Big)\,dy-\int_{V_\varepsilon}\Big(1-\exp\{-\Phi_k^{\mathbf n}(y)\}\Big)\,dy\right)\\
&\geq\; \limsup_{k\to\infty}\,\mathcal{S}^{\mathbf n}(\mu_k)-\varepsilon.
\end{aligned}\tag{2.38}$$

Notice that the concavity of the function $x\mapsto(1-e^{-x})$ was here used in the estimate of the weak limit. Since $\varepsilon>0$ was arbitrary, this proves the lemma in the basic case.

2. Next, we still assume that the measures μ_k have uniformly bounded support, say

$$Supp(\mu_k)\;\subseteq\; B(0,R) \qquad\qquad \text{for all } k\geq 1\,, \tag{2.39}$$

but we allow the presence of an additional positive measure ν, having density $g\in\mathbf{L}^1_{loc}(I\!R^d)$ w.r.t. Lebesgue measure. In the following we consider the cylinder

$$\Gamma_R \;\doteq\; \{y+t\mathbf{n}\,;\;\; y\in E_{\mathbf n}^\perp\,,\;\; r\in I\!R\,,\;\; |y|\leq R,\; |t|\leq R\}. \tag{2.40}$$

Let $\varepsilon_0>0$ be given. Then there exists $\rho_0>0$ such that

$$\int_V g(x)\,dx \;\leq\; \varepsilon_0 \tag{2.41}$$

for every set $V\subseteq\Gamma_R$ such that $meas(V)\leq\rho_0$. Calling ω_{d-1} the volume of the unit ball in $I\!R^{d-1}$, we choose $\rho_1>0$ so that

$$\omega_{d-1}R^{d-1}\rho_1 \;<\; \rho_0\,. \tag{2.42}$$

Then we choose

$$-\infty \;=\; t_0 \;<\; t_1 \;<\; t_2 \;<\; \cdots \;<\; t_N \;<\; t_{N+1} \;=\; +\infty$$

such that

$$t_1 < -R, \qquad t_N > R, \qquad t_j - t_{j-1} < \rho_1 \qquad \text{for all } j = 2, \ldots, N, \tag{2.43}$$

$$\mu\Big(\{x \in I\!R^d\,;\ \ \langle x, \mathbf{n}\rangle = t_j\}\Big) \;=\; 0 \qquad \text{for all } j = 1, 2, \ldots, N. \tag{2.44}$$

3. Call $\mu_k^j \;=\; \chi_{\{\langle x,\mathbf{n}\rangle \geq t_j\}} \cdot \mu_k$ the restriction of the measure μ_k to the set where $\langle x, \mathbf{n}\rangle \geq t_j$, and let $\mu_k^{\mathbf{n},j}$, $\mu^{\mathbf{n},j}$ the projections of μ_k^j, μ^j on $E_{\mathbf{n}}^{\perp}$, as in (2.4). Moreover, call $\Phi_k^{\mathbf{n},j}$, $\Phi^{\mathbf{n},j}$ the densities of the absolutely continuous parts of $\mu_k^{\mathbf{n},j}$, $\mu^{\mathbf{n},j}$ w.r.t. the $(d-1)$-dimensional Lebesgue measure on $E_{\mathbf{n}}^{\perp}$.

The weak convergence $\mu_k \rightharpoonup \mu$, together with the assumption (2.44) implies the weak convergence

$$\mu_k^{\mathbf{n},j} \;\rightharpoonup\; \mu^{\mathbf{n},j} \qquad\qquad \text{for all } \ j = 0, 1, \ldots, N. \tag{2.45}$$

Using again the "biting lemma" [14], we can find a set $V \subseteq B(0, R) \subseteq E_{\mathbf{n}}^{\perp}$, with

$$\text{meas}(V) \;<\; \varepsilon_0, \qquad\qquad (t_N - t_1) \cdot \text{meas}(V) \;<\; \rho_0, \tag{2.46}$$

and such that the following holds. Let $\widehat{\mu}_k^{\mathbf{n},j}$ be the absolutely continuous measure on $E_{\mathbf{n}}^{\perp}$ whose density is

$$\widehat{\Phi}_k^{\mathbf{n},j}(y) \;=\; \begin{cases} \Phi_k^{\mathbf{n},j}(y) & \text{if } \ y \in B(0, R) \setminus V, \\[2mm] \quad 0 & \text{otherwise.} \end{cases} \tag{2.47}$$

Then, by possibly extracting a subsequence, for every $j = 1, \ldots, N$ we have the weak convergence

$$\widehat{\mu}_k^{\mathbf{n},j} \rightharpoonup \widehat{\mu}^{\mathbf{n},j}, \qquad\qquad \widehat{\Phi}_k^{\mathbf{n},j} \rightharpoonup \widehat{\Phi}^{\mathbf{n},j}\,.$$

Here the second arrow denotes weak convergence in $\mathbf{L}^1$. Moreover, $\widehat{\mu}^{\mathbf{n},j}$ is the absolutely continuous measure on $E_{\mathbf{n}}^{\perp}$ with density $\widehat{\Phi}^{\mathbf{n},j}$.

4. For each fixed $y \in E_{\mathbf{n}}^{\perp}$, the last integral in (2.22) can be estimated from above and from below in terms of Riemann sums. More precisely, for a given measure μ, call

$$\mu^j \;=\; \chi_{\{\langle x,\mathbf{n}\rangle \geq t_j\}} \cdot \mu$$

the restriction of μ to the set $\{x \in I\!R^d\,;\ \ \langle x, \mathbf{n}\rangle \geq t_j\}$. Let $\mu^{\mathbf{n},j}$ be the projection of μ^j on $E_{\mathbf{n}}^{\perp}$, and let $\Phi^{\mathbf{n},j}$ be the density of the absolutely continuous part of $\mu^{\mathbf{n},j}$. Since μ, ν are both positive measures, one has

$$\begin{aligned} L(y) \;&\doteq\; \sum_{j=2}^{N} \int_{t_{j-1}}^{t_j} \left(g(y + s\mathbf{n}) \exp\left\{ - \int_s^{+\infty} g(y + t\mathbf{n})\, dt \right\} \cdot \exp\Big\{ -\Phi^{\mathbf{n},j-1}(y) \Big\} \right) ds \\ &\leq\; \int_{t_1}^{t_N} \left(g(y + s\mathbf{n}) \exp\left\{ - \int_s^{+\infty} g(y + t\mathbf{n})\, dt \right\} \cdot \exp\Big\{ -\Phi^{\mathbf{n}}(y) \cdot \mu^y([s, +\infty[\,) \Big\} \right) ds \\ &\leq\; \sum_{j=2}^{N} \int_{t_{j-1}}^{t_j} \left(g(y + s\mathbf{n}) \exp\left\{ - \int_s^{+\infty} g(y + t\mathbf{n})\, dt \right\} \cdot \exp\Big\{ -\Phi^{\mathbf{n},j}(y) \Big\} \right) ds \;\doteq\; U(y)\,. \end{aligned} \tag{2.48}$$

The difference between the upper and lower Riemann sums, on the right and the left hand side of (2.48), can be estimated by

$$
\begin{aligned}
U(y)-L(y) &\leq \sum_{j=2}^{N}\int_{t_{j-1}}^{t_j} g(y+s\mathbf{n})\,ds\cdot\Big[\exp\{-\Phi^{\mathbf{n},j}(y)\}-\exp\{-\Phi^{\mathbf{n},j-1}(y)\}\Big]\\
&\leq \max_j\left(\int_{t_{j-1}}^{t_j} g(y+s\mathbf{n})\,ds\right)\cdot\sum_{j=2}^{N}\Big[\exp\{-\Phi^{\mathbf{n},j}(y)\}-\exp\{-\Phi^{\mathbf{n},j-1}(y)\}\Big]\\
&\leq \max_j\left(\int_{t_{j-1}}^{t_j} g(y+s\mathbf{n})\,ds\right).
\end{aligned}
\tag{2.49}
$$

By (2.49) and the choice of the points t_j it now follows

$$
\int_{|y|<R}\Big(U(y)-L(y)\Big)\,dy \;\leq\; \int_{|y|<R}\left(\sup_{2\leq j\leq N}\int_{t_{j-1}}^{t_j} g(y+s\mathbf{n})\,ds\right)dy \;\leq\; \varepsilon_0\,. \tag{2.50}
$$

Indeed, to prove the last inequality, consider a measurable subset $\Gamma^\sharp\subset\Gamma$ such that

$$
\Gamma^\sharp \;=\; \Big\{(y+s\mathbf{n})\,;\quad |y|<R,\quad s\in[t_{j(y)-1},t_{j(y)}]\Big\},
$$

where, for a.e. $y\in B(0,R)$,

$$
\int_{t_{j(y)-1}}^{t_{j(y)}} g(y+s\mathbf{n})\,ds \;=\; \max_{2\leq j\leq N}\int_{t_{j-1}}^{t_j} g(y+s\mathbf{n})\,ds.
$$

By (2.42),

$$
meas(\Gamma^\sharp)\leq\;\omega_{d-1}\cdot R^{d-1}\,\rho_1\;<\;\rho_0\,.
$$

Hence, by (2.41) and the definition of $\Gamma^\sharp$,

$$
\int_{|y|<R}\left(\sup_{2\leq j\leq N}\int_{t_{j-1}}^{t_j} g(y+s\mathbf{n})\,ds\right)dy\;\leq\;\int_{\Gamma^\sharp} g(x)\,dx\;\leq\;\varepsilon_0\,. \tag{2.51}
$$

5. Recalling (2.22)-(2.23) and using (2.49), we obtain

$$
\begin{aligned}
&\mathcal{S}^{\mathbf{n}}(\mu_k,\nu)-\mathcal{S}^{\mathbf{n}}(\mu,\nu) \;=\; \int_{E_{\mathbf{n}}^\perp}\Big(\mathcal{S}^{\mathbf{n}}_{\mu_k,\nu}(y)-\mathcal{S}^{\mathbf{n}}_{\mu,\nu}(y)\Big)\,dy\\
&\leq \int_{|y|<R}\exp\left\{-\int g(y+s\mathbf{n})\,dt\right\}\cdot\Big(\exp\{-\Phi^{\mathbf{n}}(y)\}-\exp\{-\Phi^{\mathbf{n}}_k(y)\}\Big)\,dy\\
&\quad+\int_{|y|<R}\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} g(y+s\mathbf{n})\exp\left\{-\int_s^{+\infty} g(y+t\mathbf{n})\,dt\right\}\,ds\\
&\qquad\qquad\cdot\Big(\exp\{-\Phi^{\mathbf{n},j-1}(y)\}-\exp\{-\Phi^{\mathbf{n},j}_k(y)\}\Big)\,dy\\
&\leq \int_{|y|<R}\exp\left\{-\int g(y+s\mathbf{n})\,dt\right\}\cdot\Big(\exp\{-\Phi^{\mathbf{n}}(y)\}-\exp\{-\Phi^{\mathbf{n}}_k(y)\}\Big)\,dy\\
&\quad+\int_{|y|<R}\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} g(y+s\mathbf{n})\exp\left\{-\int_s^{+\infty} g(y+t\mathbf{n})\,dt\right\}\,ds\\
&\qquad\qquad\cdot\Big(\exp\{-\Phi^{\mathbf{n},j}(y)\}-\exp\{-\Phi^{\mathbf{n},j}_k(y)\}\Big)\,dy\\
&\quad+\int_{|y|<R}(U(y)-L(y))\,dy\\
&=\; I_{1,k}+I_{2,k}+I_3\,.
\end{aligned}
\tag{2.52}
$$

As $k\to\infty$, the limits of the first two integrals can be estimated as in Step **1**. Indeed, recalling the properties (2.46)-(2.47) we obtain

$$\limsup_{k\to\infty}\int_{|y|<R,\ y\notin V}\exp\left\{-\int g(y+s\mathbf{n})\,dt\right\}\cdot\Big(\exp\{-\widehat\Phi^{\mathbf{n}}(y)\}-\exp\{-\widehat\Phi^{\mathbf{n}}_k(y)\}\Big)\,dy\ \le\ 0,\quad(2.53)$$

$$\begin{array}{l}\displaystyle\limsup_{k\to\infty}\int_{|y|<R,\ y\notin V}\int_{t_{j-1}}^{t_j}g(y+s\mathbf{n})\exp\left\{-\int_s^{+\infty}g(y+t\mathbf{n})\,dt\right\}\,ds\\ \displaystyle\qquad\qquad\qquad\cdot\Big(\exp\{-\widehat\Phi^{\mathbf{n},j}(y)\}-\exp\{-\widehat\Phi^{\mathbf{n},j}_k(y)\}\Big)\,dy\ \ \le\ 0.\end{array}\qquad(2.54)$$

Moreover, by (2.50) we already know that $I_3\le\varepsilon_0$. From (2.46) and the above inequalities we conclude

$$\limsup_{k\to\infty}\,(I_{1,k}+I_{2,k})+I_3\ \le\ \varepsilon_0+\varepsilon_0\,.\qquad(2.55)$$

Since $\varepsilon_0>0$ was arbitrary, this proves (2.35) in the case where the supports of the measures μ_k are uniformly bounded.

6. Finally, using the tightness assumption (2.34), we remove the assumption that the measures μ_k have uniformly bounded support.

For any given $\varepsilon>0$, by (2.34) there exists a radius R sufficient large such that

$$\mu_k\Big(\{x\in I\!R^d;\ |x|>R\}\Big)\ <\ \varepsilon$$

for every $k\ge1$. Without loss of generality, we can assume that

$$\mu\Big(\{x\in I\!R^d;\ |x|=R\}\Big)\ =\ 0.\qquad(2.56)$$

Calling B_R the open ball centered at the origin with radius R, we denote by $\mu_k^\flat,\mu_k^\sharp$ the restrictions of μ_k to B_R and $I\!R^d\setminus B_R$, respectively. The measures $\mu^\flat$, $\mu^\sharp$ are defined similarly. By the weak convergence $\mu_k\rightharpoonup\mu$ together with (2.56) it follows the weak convergence $\mu_k^\flat\rightharpoonup\mu^\flat$. By Lemma 2.2, for every k one has

$$\mathcal{S}^{\mathbf{n}}(\mu_k^\flat,\nu)\ \ge\ \mathcal{S}^{\mathbf{n}}(\mu_k,\nu)-\mu_k^\sharp\Big(I\!R^d\setminus B_R\Big)\ \ge\ \mathcal{S}^{\mathbf{n}}(\mu_k,\nu)-\varepsilon\,.$$

Since the measures $\mu_k^\flat$ have uniformly bounded support, by the previous analysis we conclude

$$\mathcal{S}^{\mathbf{n}}(\mu,\nu)\ \ge\ \mathcal{S}^{\mathbf{n}}(\mu^\flat,\nu)\ \ge\ \limsup_{k\to\infty}\mathcal{S}^{\mathbf{n}}(\mu_k^\flat,\nu)\ \ge\ \limsup_{k\to\infty}\mathcal{S}^{\mathbf{n}}(\mu_k,\nu)-\varepsilon.$$

Since ε is arbitrary, this completes the proof. □

From the above lemma one easily obtains the upper semicontinuity of the functional in (2.6).

Lemma 2.4 *Consider a weakly convergent sequence of measures $\mu_k\rightharpoonup\mu$, satisfying the boundedness and tightness conditions (2.33)-(2.34). Then, for any positive, integrable function $\eta\in\mathbf{L}^1(S^{d-1})$ and every positive measure ν, absolutely continuous w.r.t. Lebesgue measure on $I\!R^d$, one has*

$$\mathcal{S}^\eta(\mu;\nu)\ \ge\ \limsup_{k\to\infty}\mathcal{S}^\eta(\mu_k;\nu).\qquad(2.57)$$

Proof. By the boundedness assumption (2.33) and the estimate (i) in Lemma 2.2, for each $k \geq 1$ we have

$$\mathcal{S}^{\mathbf{n}}(\mu_k;\nu) \;\leq\; \mu_k(I\!\!R^d) \;\leq\; C. \tag{2.58}$$

This implies

$$\begin{array}{l}\displaystyle \limsup_{k\to\infty} \mathcal{S}^\eta(\mu_k,\nu) \;=\; \limsup_{k\to\infty} \int_{S^{d-1}} \eta(\mathbf{n})\mathcal{S}^{\mathbf{n}}(\mu_k,\nu)\,d\mathbf{n} \;\leq\; \int_{S^{d-1}} \limsup_{k\to\infty} \eta(\mathbf{n})\mathcal{S}^{\mathbf{n}}(\mu_k,\nu)\,d\mathbf{n} \\ \displaystyle \qquad \leq\; \int_{S^{d-1}} \eta(\mathbf{n})\mathcal{S}^{\mathbf{n}}(\mu,\nu)\,d\mathbf{n} \;=\; \mathcal{S}^\eta(\mu,\nu).\end{array} \tag{2.59}$$

Here the first inequality is valid because, by (2.58), all integrand functions are pointwise bounded by the function $C\,\eta(\cdot) \in \mathbf{L}^1(S^{d-1})$. The second inequality follows from Lemma 2.3. □

3 Harvest functionals

We now consider a utility functional associated with roots, whose the main goal is to collect moisture and nutrients from the ground. To model the efficiency of a root, consider a scalar function $u(\cdot)$ and a positive measure μ. We think of $u(x)$ as the density of water+nutrients at the point x, while μ is the density of root hair cells, which absorb fluids from the soil. Since these fluids diffuse through the ground and are harvested by the root, u will satisfy a parabolic equation of the form

$$u_t \;=\; \Delta u + f(x,u) - u\,\mu. \tag{3.1}$$

Remark 3.1 Here $f(x,u)$ is a term describing how nutrients are replenished within the soil. For example, this may be due to the presence of bacteria producing organic matter. A more accurate model should account for the concentration of water together with several different chemicals and bacteria. In this case, (3.1) would become a system of n reaction-diffusion equations for a vector-valued function $u = (u_1, \ldots, u_n)$. Throughout the following, for simplicity we shall assume that u is a scalar function. Similar results are expected to hold in the vector-valued case as well.

Since we are interested in average values over long periods of time, we look at the equilibrium states for (3.1). Throughout the following, we assume that

(A1) $\Omega \subset I\!\!R^d$ is a bounded, connected open set with $\mathcal{C}^2$ boundary.

(A2) $f : \overline{\Omega} \times I\!\!R \mapsto I\!\!R$ is a bounded, continuous function such that, for some constants M, K,

$$f(x,0) \;\geq\; 0, \qquad f(x,M) \;\leq\; 0, \qquad |f(x,u)| \;\leq\; K, \qquad\qquad \text{for all } x \in \overline{\Omega},\quad u \in [0,M]\,. \tag{3.2}$$

(A3) μ is a positive Radon measure supported on the compact set $\overline{\Omega}$.

We consider solutions $u : \overline{\Omega} \mapsto [0,M]$ of the elliptic problem with measure-valued coefficients

$$\Delta u + f(x,u) - u\,\mu \;=\; 0, \tag{3.3}$$

and Neumann boundary conditions

$$\partial_{\mathbf{n}(x)}u(x) \;=\; 0 \qquad\qquad x\in\partial\Omega. \tag{3.4}$$

Here $\mathbf{n}(x)$ denotes the unit outer normal vector at the boundary point $x\in\partial\Omega$, while $\partial_{\mathbf{n}}u = \mathbf{n}(x)\cdot\nabla u(x)$ is the derivative of u in the normal direction.

In alternative, we shall also consider Dirichlet boundary conditions

$$u(x) \;=\; 0 \qquad\qquad x\in\partial\Omega. \tag{3.5}$$

Observe that, if the measure μ has a smooth density $h(\cdot)$ w.r.t. Lebesgue measure, then the equation (3.3) takes the form

$$\Delta u + f(x,u) - h(x)\,u \;=\; 0. \tag{3.6}$$

By the assumption (A2), the constant function $u_*(x)=0$ is a subsolution, while $u^*(x)=M$ is a supersolution. A standard comparison argument now implies that the semilinear elliptic problem (3.6), (3.4) has at least one solution $u:\overline{\Omega}\mapsto[0,M]$.

Elliptic problems with measure data have been studied in several papers [9, 10, 15] and are now fairly well understood. A key fact is that, roughly speaking, the Laplace operator "does not see" sets with zero capacity. Following [9, 10] we thus call $\mathcal{M}_b$ the set of all bounded Radon measures on $\overline{\Omega}$. Moreover, we denote by $\mathcal{M}_0\subset\mathcal{M}_b$ the family of measures which vanish on Borel sets with zero capacity, so that

$$\mathrm{cap}_2(V) \;=\; 0 \qquad\Longrightarrow\qquad \mu(V) \;=\; 0. \tag{3.7}$$

For the definition and basic properties of capacity we refer to [18]. Every measure $\mu\in\mathcal{M}_b$ can be uniquely decomposed as a sum

$$\mu \;=\; \mu_0+\mu_s, \tag{3.8}$$

where $\mu_0\in\mathcal{M}_0$ while the measure μ_s is supported on a set with zero capacity. In the definition of solutions to (3.3), the presence of the singular measure μ_s is disregarded.

Remark 3.2 If μ is an arbitrary Radon measure and u is a measurable function defined up to a set of zero Lebesgue measure, the product $u\mu$ may not be well defined. In the present setting, however, we claim that the product measure $u\,\mu_0$ is uniquely defined. Indeed, calling

$$⨍_V u\,dx \;=\; \frac{1}{\mathrm{meas}(V)}\int_V u\,dx$$

the average value of u on a set V, for each $x\in\overline{\Omega}$ we can consider the limit

$$u(x) \;=\; \lim_{r\downarrow 0}\ ⨍_{\Omega\cap B(x,r)} u(y)\,dy. \tag{3.9}$$

As proved in [19], if $u\in H^1(\Omega)$ then the above limit exists at all points $x\in\overline{\Omega}$ with the possible exception of a set whose capacity is zero. Since $\mu\in\mathcal{M}_0$, we conclude that the measure $u\mu_0$ is well defined.

Definition 3.1 *Let μ be a measure in $\mathcal{M}_b$, decomposed as in (3.8).*

(i) A function $u \in \mathbf{L}^\infty(\Omega) \cap H^1(\Omega)$, with pointwise values given by (3.9), is a solution to the elliptic problem (3.3)-(3.4) if

$$-\int_\Omega \nabla u \cdot \nabla\varphi\, dx + \int_\Omega f(x,u)\varphi\, dx - \int_{\overline{\Omega}} u\varphi\, d\mu_0 \;=\; 0 \tag{3.10}$$

for every test function $\varphi \in \mathcal{C}_c^\infty(I\!R^d)$.

(ii) A function $u \in \mathbf{L}^\infty(\Omega) \cap H_0^1(\Omega)$, with pointwise values given by (3.9), is a solution to the elliptic problem (3.3), (3.5) if

$$-\int_\Omega \nabla u \cdot \nabla\varphi\, dx + \int_\Omega f(x,u)\varphi\, dx - \int_\Omega u\varphi\, d\mu_0 \;=\; 0 \tag{3.11}$$

for every test function $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

We can now state the main existence result for solutions to (3.3). The proof closely follows the arguments in [13].

Theorem 3.1 *Under the assumptions (A1)–(A3), the elliptic problem (3.3) with Neumann boundary conditions (3.4) has at least one solution $u : \Omega \mapsto [0, M]$. The same is true in the case of Dirichlet boundary conditions (3.5).*

Proof. Without loss of generality, we can assume that $\mu = \mu_0$, so that (3.7) holds.

1. We first consider the case of Neumann boundary conditions. Let $\Omega_\varepsilon = \{x \in I\!R^d\,;\ \ d(x, \overline{\Omega}) < \varepsilon\}$ be a neighborhood of radius $\varepsilon > 0$ around the compact set $\overline{\Omega}$. Following [17], we can construct a bounded, linear extension operator $E : H^1(\Omega) \mapsto H_0^1(\Omega_\varepsilon)$.

Since μ is a bounded Radon measure on Ω_ε which vanishes on sets of zero capacity, by the analysis in [15] it follows that $\mu \in \mathbf{L}^1(\Omega_\varepsilon) \oplus H^{-1}(\Omega_\varepsilon)$. More precisely, there exist functions $\phi_0 \in \mathbf{L}^1(\Omega_\varepsilon)$ and $\phi_1, \ldots \phi_d \in \mathbf{L}^2(\Omega_\varepsilon)$ such that

$$\int_{\Omega_\varepsilon} \varphi\, d\mu \;=\; \int_{\Omega_\varepsilon} \phi_0\, \varphi\, dx - \sum_{i=1}^d \int_{\Omega_\varepsilon} \phi_i\, \varphi_{x_i}\, dx \tag{3.12}$$

for every test function $\varphi \in \mathcal{C}_c^\infty(\Omega_\varepsilon)$. Hence the same holds for every $\varphi \in H_0^1(\Omega_\varepsilon)$.

2. By slightly shifting the measure μ in the interior of the domain Ω and performing a mollification, we construct sequences of smooth functions $\phi_{0,n}, \phi_{1,n}, \ldots, \phi_{d,n}$ such that

$$\lim_{n\to\infty} \|\phi_{0,n} - \phi_0\|_{\mathbf{L}^1(\Omega_\varepsilon)} \;=\; 0, \qquad \lim_{n\to\infty} \|\phi_{j,n} - \phi_j\|_{\mathbf{L}^2(\Omega_\varepsilon)} \;=\; 0, \qquad\qquad j = 1, \ldots, d. \tag{3.13}$$

Moreover, the measures μ_n with density $h_n = \phi_{0,n} + \sum_j (\phi_{j,n})_{x_j}$ w.r.t. Lebesgue measure are nonnegative and supported in the interior of Ω.

3. Since $u_* \equiv 0$ is always a subsolution to (3.14), by a standard comparison argument, for each $n \geq 1$ we obtain the existence of a classical solution $u_n : \overline{\Omega} \mapsto [0, M]$ to the elliptic equation

$$\Delta u + f(x,u) - h_n(x)u \;=\; 0 \qquad\qquad x \in \Omega\,, \tag{3.14}$$

with Neumann boundary conditions (3.4). Multiplying by u_n and integrating over Ω, one obtains

$$\int_\Omega \Big[\Delta u_n(x) + f(x, u_n(x)) - h_n(x)u_n(x)\Big]u_n(x)\,dx \;=\; 0 \tag{3.15}$$

Recalling that $h_n \geq 0$, $|f| \leq K$, and $u_n \in [0, M]$, we obtain

$$\int_\Omega |\nabla u_n(x)|^2\,dx \;\leq\; \int_\Omega |f(x, u_n(x))u_n(x)|\,dx \;\leq\; \text{meas}(\Omega)\cdot K\,M\,. \tag{3.16}$$

As a consequence, the norms $\|u_n\|_{H^1(\Omega)}$ remain uniformly bounded. Therefore, the norms of the extensions $\|Eu_n\|_{H^1(\Omega_\varepsilon)}$ are bounded as well.

4. Thanks to the previous estimates, by possibly taking a subsequence and relabeling, we can assume the strong convergence

$$\|u_n - u\|_{\mathbf{L}^2(\Omega)} \;\to\; 0 \tag{3.17}$$

and the weak convergence

$$u_n \rightharpoonup u \qquad\qquad \text{in } H^1(\Omega)\,, \tag{3.18}$$

$$Eu_n \rightharpoonup Eu \qquad\qquad \text{in } H^1_0(\Omega_\varepsilon)\,, \tag{3.19}$$

for some function $u \in H^1(\Omega)$. For every test function $\varphi \in \mathcal{C}^\infty_c(I\!R^d)$ we now have

$$\begin{aligned} 0 \;&=\; \int_\Omega \Delta u_n\,\varphi\,dx + \int_\Omega f(x,u_n)\varphi\,dx - \int_\Omega u_n\varphi\,d\mu_n \\ &=\; -\int_\Omega \nabla u_n\cdot\nabla\varphi\,dx + \int_\Omega f(x,u_n)\varphi\,dx - \int_{\Omega_\varepsilon}\left(\phi_{0,n} + \sum_{j=1}^d (\phi_{j,n})_{x_j}\right)(Eu_n)\,\varphi\,dx \\ &=\; -\int_\Omega \nabla u_n\cdot\nabla\varphi\,dx + \int_\Omega f(x,u_n)\varphi\,dx - \int_{\Omega_\varepsilon}\phi_{0,n}(Eu_n)\,\varphi\,dx + \sum_{j=1}^d\int_{\Omega_\varepsilon}\phi_{j,n}(Eu_n)_{x_j}\,\varphi\,dx\,. \end{aligned} \tag{3.20}$$

Letting $n \to \infty$, by the strong convergence in (3.13) and (3.17) and the weak convergence in (3.18)-(3.19), we obtain

$$\lim_{n\to\infty}\left(-\int_\Omega \nabla u_n\cdot\nabla\varphi\,dx + \int_\Omega f(x,u_n)\varphi\,dx\right) \;=\; -\int_\Omega \nabla u\cdot\nabla\varphi\,dx + \int_\Omega f(x,u)\varphi\,dx\,, \tag{3.21}$$

$$\begin{aligned} &\lim_{n\to\infty}\left(\int_{\Omega_\varepsilon}\phi_{0,n}(Eu_n)\,\varphi\,dx + \sum_{j=1}^d\int_{\Omega_\varepsilon}\phi_{j,n}(Eu_n)_{x_j}\,\varphi\,dx\right) \\ &\qquad =\; \int_{\Omega_\varepsilon}\phi_0(Eu)\,\varphi\,dx + \sum_{j=1}^d\int_{\Omega_\varepsilon}\phi_j(Eu)_{x_j}\,\varphi\,dx \;=\; \int_{\overline{\Omega}} u\varphi\,d\mu\,. \end{aligned} \tag{3.22}$$

Together, (3.21) and (3.22) yield (3.10), completing the proof in the case of Neumann boundary conditions.

5. In the case of Dirichlet boundary conditions, without loss of generality we can assume that the measure μ is entirely supported in the interior of Ω. Indeed, since $u = 0$ on the boundary, the part of μ supported on the boundary $\partial\Omega$ does not give any contribution to the right hand side of (3.11). We can thus use the representation theorem in [15] directly on the set Ω, and find functions $\phi_0 \in \mathbf{L}^1(\Omega)$, $\phi_1, \ldots, \phi_d \in \mathbf{L}^2(\Omega)$ such that

$$\int_\Omega \varphi \, d\mu \;=\; \int_\Omega \phi_0 \, \varphi \, dx - \sum_{i=1}^d \int_\Omega \phi_i \, \varphi_{x_i} \, dx \tag{3.23}$$

for every test function $\varphi \in \mathcal{C}_c^\infty(\Omega)$. The proof is then achieved by the same arguments as before. □

From the proof of the above theorem, one can also obtain a comparison result. As usual, we say that two Radon measures satisfy $\tilde\mu \leq \mu$ if $\tilde\mu(V) \leq \mu(V)$ for every Borel set V.

Lemma 3.1 *Let the assumptions (A1)-(A2) hold and consider two measures $\mu \geq \tilde\mu$, both satisfying (A3). Let $u : \overline{\Omega} \mapsto [0, M]$ be a solution of (3.3)-(3.4). Then, replacing μ with $\tilde\mu$, one can find a corresponding solution $\tilde u : \overline{\Omega} \mapsto [0, M]$ such that $\tilde u(x) \geq u(x)$ for every x.*

The same result holds in the case of Dirichlet boundary conditions (3.5).

Proof. Consider the case of Neumann boundary conditions. Define the positive measure $\mu^* = \mu - \tilde\mu$. Performing the construction described in step **2** of the proof of Theorem 3.1, with the same shifts and the same mollifications applied to all three measures, we obtain sequences of smooth functions $\phi_{i,n}$, $\tilde\phi_{i,n}$, $\phi^*_{i,n}$, for $i = 0, 1, \ldots, d$ and $n \geq 1$. Since $\mu = \tilde\mu + \mu^*$, this implies that the corresponding densities of the mollified measures satisfy

$$h_n(x) \;=\; \tilde h_n(x) + h_n^*(x) \;\geq\; \tilde h_n(x) \;\geq\; 0.$$

Since u_n is a solution to (3.14), it is a subsolution to

$$\Delta u + f(x, u) - \tilde h_n u \;=\; 0, \tag{3.24}$$

always with Neumann boundary conditions (3.4). By a standard comparison argument, there exists a solution $\tilde u_n : \overline{\Omega} \mapsto [0, M]$ to (3.24), (3.4). such that

$$\tilde u_n(x) \;\geq\; u_n(x) \qquad\qquad \text{for all} \quad x \in \overline{\Omega}.$$

By taking limits as $n \to \infty$, the result is proved.

The case of Dirichlet boundary conditions can be handled by the same technique. □

We can now introduce a harvest functional, defined for solutions of (3.3) with Neumann or Dirichlet boundary conditions.

Definition 3.2 *Given a positive Radon measure μ on $\overline{\Omega}$ and a solution u of (3.3)-(3.4), or (3.3), (3.5), the* **total harvest** *is defined as*

$$\mathcal{H}(u, \mu) \;\doteq\; \int_{\overline{\Omega}} u \, d\mu_0 \,, \tag{3.25}$$

where $\mu = \mu_0 + \mu_s$ is the decomposition introduced at (3.8).

In the case of Neumann boundary conditions, following [13] a more precise construction can be performed. Let $G = G(t, x; y)$ be the Green function for the heat equation

$$\begin{cases} w_t \;=\; \Delta w, & t > 0\,, \quad x \in \Omega\,, \\ \partial_{\mathbf{n}} w \;=\; 0, & t > 0\,, \quad x \in \partial\Omega\,. \end{cases} \tag{3.26}$$

As it is well known [20], for each fixed $y \in \Omega$ the function $G(\cdot, \cdot; y)$ provides a solution to (3.26) such that

$$\int_\Omega G(t, x; y)\, dx \;=\; 1\,, \qquad\qquad \lim_{t\downarrow 0} \int_\Omega G(t, x; y)\, \phi(x)\, dx \;=\; \phi(y) \tag{3.27}$$

for every $\phi \in \mathcal{C}(\overline{\Omega})$. The solution of (3.26) with a continuous initial data $w(0, x) = \phi(x)$ is thus given by

$$w(t, x) \;=\; \int_\Omega G(t, x; y)\, \phi(y)\, dy \qquad\qquad t > 0\,, \quad x \in \Omega\,.$$

Let now $u \geq 0$ be any function such that

$$\begin{array}{rcll} \Delta u & \geq & -K & \text{on } \Omega\,, \\ \partial_{\mathbf{n}} u & = & 0 & \text{on } \partial\Omega\,. \end{array} \tag{3.28}$$

In particular, if u is the solution to the elliptic problem (3.15) constructed in Theorem 3.1, then the condition (3.28) is satisfied. Indeed, by (3.2) all the classical solutions u_n of (3.14) satisfy

$$\Delta u_n \;=\; -f(x, u_n) + h_n(x) u_n \;\geq\; -f(x, u_n) \;\geq\; -K\,.$$

Taking the limit as $n \to \infty$ one obtains (3.15).

For any $t > 0$, consider the averaged function

$$u^{(t)}(x) \;\doteq\; \int_\Omega G(t, x; y)\, u(y)\, dy\,. \tag{3.29}$$

Using the boundary conditions in (3.26) and (3.28) to integrate by parts, by the first equations in (3.27) and (3.28) one obtains

$$\begin{aligned} \frac{d}{dt} u^{(t)}(x) \;&=\; \frac{d}{dt} \int_\Omega G(t, x; y)\, u(y)\, dy \;=\; \int_\Omega G_t(t, x; y)\, u(y)\, dy \\ &=\; \int_\Omega \Delta G(t, x; y)\, u(y)\, dx \;=\; \int_\Omega G(t, x; y)\, \Delta u(y)\, dy \;\geq\; -K\,. \end{aligned} \tag{3.30}$$

As a consequence, for every $x \in \Omega$ the map $t \mapsto u^{(t)}(x) + Kt$ is nondecreasing. Since every function $x \mapsto u^{(t)}(x)$ is uniformly continuous on Ω, it admits a continuous extension to the closure $\overline{\Omega}$. At each $x \in \overline{\Omega}$ we can thus uniquely define the value $u(x)$ by setting

$$u(x) \;=\; \lim_{t\downarrow 0} u^{(t)}(x) \;=\; \inf_{t>0} \Big(u^{(t)}(x) + Kt \Big). \tag{3.31}$$

The representation (3.31) shows that u is the infimum of a decreasing sequence of continuous functions. Hence u is upper semicontinuous.

We conclude this section by observing that, in the case of Neumann boundary conditions, the harvest functional can be equivalently written as

$$\mathcal{H}(u, \mu) \;\doteq\; \int_\Omega \; f(x, u(x))\, dx\,. \tag{3.32}$$

In the case of Dirichlet boundary conditions, assuming that the solution is $\mathcal{C}^1$ in a neighborhood of the boundary $\partial\Omega$, the harvest functional can be expressed as

$$\mathcal{H}(u,\mu) \doteq \int_\Omega f(x,u(x))\,dx + \int_{\partial\Omega} \partial_{\mathbf{n}(x)} u(x)\,d\sigma\,. \tag{3.33}$$

4 Optimal irrigation patterns

This section provides a brief review of ramified transport and optimal irrigation. To fix the ideas, throughout the following we assume

(A4) $\Omega \subset I\!R^d$ is a connected, open set with Lipschitz boundary, whose closure contains the origin: $\mathbf{0} \in \overline{\Omega}$.

Given $\alpha \in [0,1]$, to define the α-irrigation cost of a bounded, positive measure μ on $\overline{\Omega}$, we shall follow the Lagrangian approach of Maddalena, Morel, and Solimini [24].

Let $\kappa = \mu(\overline{\Omega})$ be the total mass to be transported and let $\Theta = [0,\kappa]$. We think of each $\theta \in \Theta$ as a "water particle". A measurable map

$$\chi : \Theta \times [0,1] \mapsto \overline{\Omega} \tag{4.1}$$

is called an **admissible irrigation plan** for the measure μ on $\overline{\Omega}$ if

(i) For a.e. $\theta \in \Theta$, the map $t \mapsto \chi(\theta,t)$ is Lipschitz continuous.

(ii) At time $t = 0$ all particles are at the origin: $\chi(\theta,0) = \mathbf{0} \in I\!R^d$ for all $\theta \in \Theta$.

(iii) At time $t = 1$ the push-forward of the Lebesgue measure on $[0,\kappa]$ through the map $\theta \mapsto \chi(\theta,1)$ coincides with the measure μ. In other words, for every open set $A \subset I\!R^d$ there holds

$$\mu(A) = \text{meas}\Big(\{\theta \in \Theta\,;\;\; \chi(\theta,1) \in A\}\Big). \tag{4.2}$$

To the irrigation plan χ we now attach a cost $\mathcal{E}^\alpha$. Toward this goal, given a point $x \in I\!R^d$ we first compute how many paths go through the point x. This is described by

$$|x|_\chi = \text{meas}\Big(\{\theta \in \Theta\,;\;\; \chi(\theta,t) = x \quad \text{for some} \;\; t \in [0,1]\}\Big). \tag{4.3}$$

We think of $|x|_\chi$ as the *total flux going through the point x*.

Definition 4.1 *For a given $0 < \alpha \le 1$, the total cost of the irrigation plan χ is*

$$\mathcal{E}^\alpha(\chi) \doteq \int_\Theta \left(\int_0^1 |\chi(\theta,t)|_\chi^{\alpha-1} \cdot |\chi_t(\theta,t)|\,dt \right) d\theta\,. \tag{4.4}$$

If μ is a positive, bounded Radon measure supported on $\overline{\Omega}$, the α-**irrigation cost** *of μ is defined as*

$$\mathcal{I}^\alpha(\mu) \doteq \inf_\chi \mathcal{E}^\alpha(\chi), \tag{4.5}$$

where the infimum is taken over all admissible irrigation plans.

Remark 4.1 In the optimal irrigation problem, water has to be transported from a central well located at the origin $\mathbf{0} \in \mathbb{R}^d$ to various locations inside $\overline{\Omega}$. We think of $\chi(\theta, t)$ as the position of the water particle θ at time t. The factor $|\chi(\theta,t)|_\chi^{\alpha-1}$ models the assumption that water is transported through a network of pipes, whose cost is proportional to the product

$$[\text{length}] \times [\text{flux}]^\alpha.$$

When $\alpha = 1$ the integral in (4.4) reduces to

$$\mathcal{E}^1(\chi) \doteq \int_\Theta \left(\int_0^1 |\chi_t(\theta,t)|\, dt \right) d\theta = \int_\Theta [\text{length of } \chi(\theta,\cdot)]\, d\theta\,.$$

If Ω is convex, the minimum irrigation cost is trivially achieved by transporting each particle along a straight line, hence

$$\mathcal{I}^1(\mu) = \int |x|\, d\mu(x). \tag{4.6}$$

On the other hand, when $\alpha < 1$, it becomes convenient to lump together several paths into a unique large pipe, and the optimal irrigation pattern can have a complicated structure.

Remark 4.2 In an irrigation plan, what matters are only the paths $\{\chi(\theta,t)\,;\ t \in [0,1]\} \subset \mathbb{R}^d$, not the time law with which these paths are traversed. Indeed, for each θ we could take a smooth bijection $\tau^\theta : [0,1] \mapsto [0,1]$ and consider the time-reparameterized path $\tilde{\chi}(\theta,t) = \chi(\theta, \tau^\theta(t))$. Then the irrigation plan $\tilde{\chi}$ has exactly the same cost as χ.

Remark 4.3 As suggested by intuition, irrigation plans with minimum cost do not have loops. Namely:

$$\chi(\theta,t_1) = \chi(\theta,t_2) \qquad \Longrightarrow \qquad \chi(\theta,t) = \chi(\theta,t_1) \quad \text{for all } t \in [t_1,t_2]. \tag{4.7}$$

A further, useful property of optimal irrigation plans is

$$\chi(\theta_1,t_1) = \chi(\theta_2,t_2) \qquad \Longrightarrow \qquad \{\chi(\theta_1,t)\,;\ t \in [0,t_1]\} = \{\chi(\theta_2,t)\,;\ t \in [0,t_2]\}. \tag{4.8}$$

For the basic theory of ramified transport we refer to [12, 24, 25, 31, 32], or to the monograph [7]. The next lemmas review the existence and some basic properties of the irrigation functional.

Lemma 4.1 *Let Ω be a domain satisfying (A4), let $\alpha \in [0,1]$, and let μ be a bounded, positive measure on $\overline{\Omega}$. If there exists an admissible irrigation plan with finite cost $\mathcal{E}^\alpha(\chi) < +\infty$, then the measure μ admits an optimal irrigation plan.*

For a proof, see Proposition 3.41 in [7].

Lemma 4.2 *Let $\Omega \subset \mathbb{R}^d$ satisfy the assumptions in (A4) and let μ, μ_1, μ_2 be bounded, positive measures on $\overline{\Omega}$. Then*

$$\mathcal{I}^\alpha(\mu) \geq [\mu(\overline{\Omega})]^{\alpha-1} \int_{\overline{\Omega}} |x|\, d\mu\,, \tag{4.9}$$

$$\mathcal{I}^\alpha(\mu_1) \leq \mathcal{I}^\alpha(\mu_1+\mu_2) \leq \mathcal{I}^\alpha(\mu_1) + \mathcal{I}^\alpha(\mu_2). \tag{4.10}$$

Proof. 1. The first inequality follows immediately from

$$\begin{aligned}\mathcal{E}^\alpha(\chi) &\doteq \int_\Theta \left(\int_0^1 |\chi(\theta,t)|_\chi^{\alpha-1}\cdot|\chi_t(\theta,t)|\,dt\right)d\theta \;\geq\; [\mu(\overline{\Omega})]^{\alpha-1}\int_\Theta\left(\int_0^1|\chi_t(\theta,t)|\,dt\right)d\theta\\ &\geq [\mu(\overline{\Omega})]^{\alpha-1}\int_\Theta|\chi(\theta,1)|\,d\theta \;=\; [\mu(\overline{\Omega})]^{\alpha-1}\int_{\overline{\Omega}}|x|\,d\mu(x)\,.\end{aligned}$$

2. Next, for $i=1,2$ let $\kappa_i = \mu_i(\overline{\Omega})$ and let $\chi_i : [0,\kappa_i]\times[0,1]\mapsto\overline{\Omega}$ be an admissible irrigation plan for μ_i. Then the map $\chi : [0,\kappa_1+\kappa_2]\times[0,1]\mapsto\overline{\Omega}$ defined by

$$\chi(\theta,t) \;=\; \begin{cases}\chi_1(\theta,t) & \text{if}\quad \theta\in[0,\kappa_1],\\ \chi_2(\theta-\kappa_1,t) & \text{if}\quad \theta\in\,]\kappa_1,\,\kappa_1+\kappa_2],\end{cases}$$

is an admissible irrigation plan for $\mu_1+\mu_2$. Its cost is

$$\begin{aligned}\mathcal{E}^\alpha(\chi) &= \int_0^{\kappa_1+\kappa_2}\left(\int_0^1|\chi(\theta,t)|_\chi^{\alpha-1}\cdot|\chi_t(\theta,t)|\,dt\right)d\theta\\ &\leq \int_0^{\kappa_1}\left(\int_0^1|\chi(\theta,t)|_{\chi_1}^{\alpha-1}\cdot|\chi_t(\theta,t)|\,dt\right)d\theta+\int_{\kappa_1}^{\kappa_1+\kappa_2}\left(\int_0^1|\chi(\theta,t)|_{\chi_2}^{\alpha-1}\cdot|\chi_t(\theta,t)|\,dt\right)d\theta\\ &= \mathcal{E}^\alpha(\chi_1)+\mathcal{E}^\alpha(\chi_2).\end{aligned}$$

This proves the second inequality in (4.10).

To prove the first inequality we shall use the representation (see Proposition 4.8 in [7])

$$\mathcal{E}^\alpha(\chi) \;=\; \int_\Theta\left(\int_0^1|\chi(\theta,t)|_\chi^{\alpha-1}\cdot|\chi_t(\theta,t)|\,dt\right)d\theta \;=\; \int_{I\!R^d}|x|_\chi^\alpha\,d\mathcal{H}^1(x), \tag{4.11}$$

where $d\mathcal{H}^1$ denotes integration w.r.t. the 1-dimensional Hausdorff measure.

Let $\chi : [0,\kappa_1+\kappa_2]\mapsto\overline{\Omega}$ be an admissible irrigation plan for $\mu_1+\mu_2$. By possibly performing a measure-preserving transformation of the interval $\Theta=[0,\kappa_1+\kappa_2]$ into itself, we can assume that the map $\chi_1 : [0,\kappa_1]\times[0,1]\mapsto\overline{\Omega}$, obtained by restricting χ to the subdomain where $\theta\in[0,\kappa_1]$, is an admissible irrigation plan for μ_1. Using (4.11) we obtain the obvious estimate

$$\mathcal{E}^\alpha(\chi_1) \;=\; \int_{I\!R^d}|x|_{\chi_1}^\alpha\,d\mathcal{H}^1(x) \;\leq\; \int_{I\!R^d}|x|_\chi^\alpha\,d\mathcal{H}^1(x) \;=\; \mathcal{E}^\alpha(\chi).$$

In other words, given any admissible irrigation plan for $\mu_1+\mu_2$, one can find an admissible irrigation plan for μ_1 with smaller or equal cost. This proves the first inequality in (4.10).
□

5 Optimal shape of tree branches

Based on the functionals introduced in the previous sections, we now consider a constrained optimization problem for a measure μ on $I\!R^d$, which we think as the distribution of leaves on a tree. The payoff will be the total amount of sunlight captured by the leaves. This will be supplemented by the cost of transporting nutrients from the base of the trunk, located at the origin $\mathbf{0} \in I\!R^d$ to all leaves of the tree.

To formulate this optimization problem, we consider:

(i) An open domain $\Omega \subseteq I\!R^d$ with Lipschitz boundary, such that $\mathbf{0} \in \overline{\Omega}$.

(ii) Constants $c, \kappa_0 > 0$, and an exponent $0 < \alpha \leq 1$ such that

$$1 - \frac{1}{d-1} \;<\; \alpha \;\leq\; 1\,. \tag{5.1}$$

(iii) A non-negative, integrable function $\eta : \mathcal{S}^{d-1} \mapsto I\!R_+$, determining the intensity of light coming from various directions.

(iv) An absolutely continuous positive measure ν, with continuous density function $g : I\!R^d \mapsto I\!R_+$, describing the density of external vegetation.

We then consider the optimization problem

$$\text{maximize:} \quad \mathcal{S}^\eta(\mu;\nu) - c\mathcal{I}^\alpha(\mu). \tag{5.2}$$

subject to

$$Supp(\mu) \subseteq \overline{\Omega}, \qquad\qquad \mu(\overline{\Omega}) \;\leq\; \kappa_0\,. \tag{5.3}$$

Here $\mathcal{S}^\eta(\mu;\nu)$ is the *sunlight functional* introduced at (2.22)–(2.24), while $\mathcal{I}^\alpha(\mu)$ is the minimum cost to α-irrigate the measure μ, defined at (4.5).

Remark 5.1 One can think of (5.3) as a constraint on the size of the tree, i.e. on the total amount of leaves. Notice that the inequality in (5.3) is essentially equivalent to

$$\mu(\overline{\Omega}) \;=\; \kappa_0\,. \tag{5.4}$$

Indeed, given a measure μ with total mass $< \kappa_0$, we can always add to μ a Dirac mass at the origin, of size $\kappa_0 - \mu(I\!R^d)$. This would come at zero transportation cost, and zero additional payoff.

Remark 5.2 If μ is supported on a set of dimension $< d-1$, then $\mathcal{S}^{\mathbf{n}}(\mu,\nu) = 0$. On the other hand, if (5.1) fails, then $\mathcal{I}^\alpha(\mu) = +\infty$ for every measure μ whose support is NOT contained in a set of dimension $\leq d-1$. In this case, the above optimization problem would only have trivial solutions, where the measure μ is a point mass at the origin.

Using the semicontinuity of the functionals $\mathcal{S}^\eta$ and $\mathcal{I}^\alpha$, and deriving suitable a priori estimates, we now prove

Theorem 5.1 *In the above setting (i)–(iv), the constrained optimization problem (5.2)-(5.3) has at least one solution.*

Proof. 1. By Lemma 2.1 and the bound (5.3) it follows

$$\mathcal{S}^\eta(\mu,\nu) - c\mathcal{I}^\alpha(\mu) \ \leq\ \mathcal{S}^\eta(\mu) \ \leq\ \int_{S^{d-1}} \eta(\mathbf{n})\, d\mathbf{n} \cdot \sup\left\{\mathcal{S}^{\mathbf{n}}(\mu)\,;\ |\mathbf{n}|=1\right\} \ \leq\ \|\eta\|_{\mathbf{L}^1}\, \kappa_0\,, \tag{5.5}$$

showing that the functional in (5.2) has a finite upper bound. Hence there exists a sequence of positive measures $(\mu_n)_{n\geq 1}$, all satisfying the conditions in (5.3), and such that

$$\lim_{n\to\infty} \Big(\mathcal{S}^\eta(\mu_n;\nu) - c\mathcal{I}^\alpha(\mu_n)\Big) \ =\ \sup_{\mu} \Big\{\mathcal{S}^\eta(\mu;\nu) - c\mathcal{I}^\alpha(\mu)\Big\}. \tag{5.6}$$

The supremum on the right hand side is taken over all positive measures satisfying (5.3).

2. We claim that it is not restrictive to assume that the measures μ_n have uniformly bounded support. More precisely

$$Supp(\mu_n)\ \subseteq\ \overline{B}(0,r_0)\cap\overline{\Omega}, \tag{5.7}$$

where

$$r_0 \ =\ \frac{\kappa_0^{1-\alpha}}{c\,\alpha}\|\eta\|_{\mathbf{L}^1}\,. \tag{5.8}$$

Indeed, each measure μ_n can written as a sum: $\mu_n = \widehat{\mu}_n + \mu_n^*$, where $\widehat{\mu}_n$ is supported inside the closed ball $\overline{B}(0,r_0)$, while μ_n^* is supported outside this ball.

By Lemma 4.1, for each $n\geq 1$ there exists an optimal irrigation plan χ_n, i.e., a minimizer of the irrigation cost for the measure μ_n. By possibly performing a measure-preserving transformation of $\Theta_n = [0, \mu_n(\overline{\Omega})]$ into itself, it is not restrictive to assume that

$$\chi_n(\theta,\cdot) \ =\ \left\{\begin{array}{cl} \widehat{\chi}_n(\theta,\cdot) & \text{if}\quad \theta\in[0,\widehat{\kappa}_n], \\ \chi_n^*(\theta-\widehat{\kappa}_n,\cdot) & \text{if}\quad \theta\in\,]\widehat{\kappa}_n,\,\kappa_n], \end{array}\right.$$

where $\widehat{\chi}_n$ is an irrigation plan for $\widehat{\mu}_n$, while χ_n^* is an irrigation plan for μ_n^*. By (5.3) we have

$$\mu_n(\overline{\Omega}) \ =\ \widehat{\mu}_n(\overline{\Omega}) + \mu_n^*(\overline{\Omega}) \ \leq\ \kappa_0\,.$$

Since χ_n is optimal while $\widehat{\chi}_n$ is suboptimal, the difference between the minimal irrigation costs can be estimated as

$$\begin{aligned} \mathcal{I}(\mu_n) - \mathcal{I}(\widehat{\mu}_n) \ &\geq\ \int_{\overline{\Omega}} \Big(|x|_{\widehat{\chi}_n} + |x|_{\chi_n^*}\Big)^\alpha d\mathcal{H}^1 - \int_{\overline{\Omega}} |x|^\alpha_{\widehat{\chi}_n}\, d\mathcal{H}^1 \\ &\geq\ \int_{\overline{\Omega}} \alpha\,\kappa_0^{\alpha-1}\,|x|_{\chi_n^*}\, d\mathcal{H}^1 \ \geq\ \alpha\,\kappa_0^{\alpha-1}\, r_0\cdot\mu_n^*(\overline{\Omega}). \end{aligned} \tag{5.9}$$

The second inequality comes from the fact that $(x+y)^\alpha - x^\alpha \geq \frac{\alpha}{\kappa_0^{1-\alpha}}y$, when $x\geq 0$, $y\geq 0$, and $x+y\leq\kappa_0$.

On the other hand, by (2.25) the difference in the sunlight functional can be estimated by

$$\mathcal{S}^\eta(\mu_n,\nu) - \mathcal{S}^\eta(\widehat{\mu}_n,\nu) \ \leq\ \|\eta\|_{\mathbf{L}^1}\cdot\mu_n^*(\overline{\Omega}). \tag{5.10}$$

If the radius r_0 is chosen as in (5.8), then by (5.9)-(5.10) we have

$$\mathcal{S}^\eta(\widehat{\mu}_n,\nu) - c\mathcal{I}^\alpha(\widehat{\mu}_n) \ \geq\ \mathcal{S}^\eta(\mu_n,\nu) - c\mathcal{I}^\alpha(\mu_n) \tag{5.11}$$

for every $n \geq 1$. By replacing each μ_n with $\widehat{\mu}_n$ we thus obtain a maximizing sequence of measures whose supports are uniformly bounded.

3. Thanks to the uniform boundedness of the supports, by possibly taking a subsequence we can assume the weak convergence of measures: $\mu_n \rightharpoonup \bar{\mu}$, for some positive measure $\bar{\mu}$ satisfying (5.3) as well.

By the lower semicontinuity of the irrigation cost $\mathcal{I}^\alpha$ (see Proposition 3.40 in [7]), it follows

$$\mathcal{I}^\alpha(\bar{\mu}) \ \leq \ \liminf_{n\to\infty} \mathcal{I}^\alpha(\mu_n).$$

On the other hand, the upper semicontinuity of the sunlight functional $\mu \mapsto \mathcal{S}^\eta(\mu,\nu)$ proved in Lemma 2.4 yields

$$\mathcal{S}(\bar{\mu},\nu) \ \geq \ \limsup_{n\to\infty} \mathcal{S}(\mu_n,\nu).$$

We conclude that $\bar{\mu}$ is an optimal solution to (5.2)-(5.4). □

6 Optimal shape of tree roots

In this section we consider constrained optimization problems for a measure μ on $I\!R^d$, which we now think as the distribution of root hair in the soil. The payoff will be the total amount of water+nutrients collected by the roots. This will be supplemented by the cost of transporting water from the tips of the roots to the base of the trunk. Under the same assumptions (A1)-(A2) in Section 3, let constants $\alpha, c, \kappa_0 > 0$ be given, with

$$1 - \frac{1}{d-2} \ < \ \alpha \ \leq \ 1\,. \tag{6.1}$$

We then consider the optimization problem

$$\text{maximize:} \quad \mathcal{H}(u,\mu) - c\mathcal{I}^\alpha(\mu), \tag{6.2}$$

among all positive measures μ on $\overline{\Omega}$ satisfying the constraint

$$\mu(\overline{\Omega}) \ \leq \ \kappa_0\,, \tag{6.3}$$

and all functions u such that the couple (u,μ) provides a solution to the elliptic boundary value problem (3.3). Here $\mathcal{H}(u,\mu)$ is the *harvest functional* introduced at (3.25), while $\mathcal{I}^\alpha(\mu)$ is the minimum cost to α-irrigate the measure μ, defined at (4.5).

Remark 6.1 If μ is supported on a set of zero capacity, then $\mathcal{H}(u,\mu) = 0$. As shown in chapter 5.9 of [4], if a set $A \subset I\!R^d$ has Hausdorff dimension $\leq d-2$, then its capacity is zero. On the other hand (see [7]), the minimum irrigation cost $\mathcal{I}^\alpha(\mu)$ is bounded only if μ is supported on a set of dimension $< \frac{1}{1-\alpha}$. To achieve a nontrivial solution of (6.2), one thus needs the inequality $\frac{1}{1-\alpha} > d-2$. This motivates the condition (6.1).

Using the semicontinuity of the functionals $\mathcal{H}$ and $\mathcal{I}^\alpha$, we will prove the existence of optimal solutions. We begin with the case of Neumann boundary conditions.

Theorem 6.1 *Let the assumptions (A1)-(A2) hold. Then the maximization problem (6.2), over all couples (u,μ) which satisfy (3.3), (3.4), and (6.3), has an optimal solution.*

Proof. 1. Call $\mathcal{A}$ the set of all admissible couples (u,μ), satisfying (3.3), (3.4), and (6.3). Since every solution u of (3.3) satisfies $u(x)\in[0,M]$, calling $\overline{M}$ the supremum over all admissible couples we have

$$\overline{M} \;\doteq\; \sup_{(u,\mu)\in\mathcal{A}} \{\mathcal{H}(u,\mu)-\mathcal{I}^\alpha(\mu)\} \;\leq\; M\,\kappa_0\,. \tag{6.4}$$

Let $\{(u_n,\mu_n)\}_{n\geq 1}$ be a maximizing sequence. It is clearly not restrictive to assume that $\mu_n\in\mathcal{M}_0$ for every $n\geq 1$.

By (3.16) we have the bounds

$$\|u_n\|_{H^1}\;\leq\; C, \qquad\qquad u_n(x)\in[0,M],$$

for some constant C and every $n\geq 1$. As remarked in Section 3, the functions u_n can be uniquely defined at every point $x\in\overline{\Omega}$ in terms of the limit (3.31). Consider the sequence of measures $\nu_n\doteq u_n\,\mu_n$. By possibly taking a subsequence and relabeling we can assume

$$\left\{\begin{array}{lll} \nu_n\rightharpoonup\nu, \qquad \mu_n\rightharpoonup\mu & & \text{in the sense of weak convergence of measures,}\\[2mm] u_n\to u & & \text{strongly in } \mathbf{L}^2(\Omega) \text{ and a.e. in } \Omega,\\[2mm] u_n\rightharpoonup u & & \text{weakly in } H^1(\Omega). \end{array}\right. \tag{6.5}$$

In addition, by Ascoli's theorem we can assume that, for every fixed $t>0$,

$$u_n^{(t)}(x) \;=\; \int_\Omega G(t,x,y)u_n(y)\,dy \;\to\; \int_\Omega G(t,x,y)u(y)\,dy \;=\; u^{(t)}(x) \tag{6.6}$$

as $n\to\infty$, uniformly for $x\in\overline{\Omega}$. Indeed, by choosing a subsequence we can achieve the convergence in (6.6) for every rational $t>0$. By continuity, this same subsequence satisfies (6.6) for every $t>0$.

2. We claim that, without loss of generality, one can assume that each measure μ_n satisfies

$$Supp(\mu_n)\;\subseteq\;\Big\{x\in\overline{\Omega}\,;\quad u_n(x)\;\geq\; c\alpha\,\kappa_0^{\alpha-1}\,|x|\Big\}\;\doteq\; A_n\,, \tag{6.7}$$

where c is the constant in (6.2). Indeed, consider the decomposition

$$\mu_n \;=\; \widehat{\mu}_n+\mu_n^*,$$

where $\widehat{\mu}_n\doteq\chi_{A_n}\cdot\mu_n$ is concentrated on A_n, while μ_n^* is concentrated on $\overline{\Omega}\setminus A_n$. We notice that A_n is a closed set, because u_n is upper semicontinuous.

Observing that u_n is a subsolution to the problem

$$\Delta u+f(x,u)-u\,\widehat{\mu}_n \;=\; 0 \tag{6.8}$$

with Neumann boundary conditions, we conclude that (6.8), (3.4) has a solution $\widehat{u}_n\geq u_n$. For this solution, one has

$$\mathcal{H}(\widehat{u}_n,\widehat{\mu}_n)\;\geq\;\mathcal{H}(u_n,\mu_n)-\int_{\overline{\Omega}} u_n\,d\mu^*\;\geq\;\mathcal{H}(u_n,\mu_n)-c\alpha\kappa_0^{\alpha-1}\int_{\overline{\Omega}}|x|\,d\mu^*. \tag{6.9}$$

Next, define the constants

$$\widehat{\kappa}_n \;=\; \widehat{\mu}_n(\overline{\Omega}), \qquad \kappa_n^* \;=\; \mu_n^*(\overline{\Omega}), \qquad \kappa_n \;=\; \mu_n(\overline{\Omega}) \;=\; \hat{\kappa}_n + \kappa_n^*\,,$$

and consider an optimal irrigation plan for μ_n, say $\chi_n : [0,\kappa_n]\times[0,1] \mapsto \overline{\Omega}$. By possibly performing a measure-preserving transformation of the domain $\Theta_n = [0,\kappa_n]$ into itself, we can assume that the maps

$$\begin{cases} \widehat{\chi}_n : [0,\widehat{\kappa}_n]\times[0,1] \mapsto \overline{\Omega}, \\ \chi_n^* : [0,\kappa_n^*]\times[0,1] \mapsto \overline{\Omega}, \end{cases} \qquad \begin{cases} \widehat{\chi}_n(\theta,t) \;=\; \chi_n(\theta,t), \\ \chi_n^*(\theta,t) \doteq\; \chi_n(\widehat{\kappa}_n+\theta,\, t), \end{cases}$$

are admissible irrigation plans for $\widehat{\mu}_n$ and μ_n^*, respectively (possibly not optimal). We now have

$$\begin{aligned} \mathcal{I}^\alpha(\mu_n) - \mathcal{I}^\alpha(\widehat{\mu}_n) \;&\geq\; \int |x|_{\chi_n}^\alpha \, d\mathcal{H}^1 - \int |x|_{\widehat{\chi}_n}^\alpha \, d\mathcal{H}^1 \;\geq\; \alpha\,\kappa_0^{\alpha-1} \int \Big(|x|_{\chi_n} - |x|_{\widehat{\chi}_n}\Big) d\mathcal{H}^1(x) \\ &=\; \alpha\,\kappa_0^{\alpha-1} \int |x|_{\chi_n^*} d\mathcal{H}^1(x) \;\geq\; \alpha\,\kappa_0^{\alpha-1} \int |x|\, d\mu_n^*. \end{aligned} \tag{6.10}$$

Together, (6.9) and (6.10) imply

$$\mathcal{H}(\widehat{u}_n,\widehat{\mu}_n) - c\mathcal{I}^\alpha(\widehat{\mu}_n) \;\geq\; \mathcal{H}(u_n,\mu_n) - c\mathcal{I}^\alpha(\mu_n).$$

By replacing each pair (u_n,μ_n) with $(\widehat{u}_n,\widehat{\mu}_n)$, we thus obtain a new maximizing sequence for which (6.7) holds.

3. Using (6.7), we now show that

$$Supp(\mu) \;\subseteq\; \Big\{x\in\overline{\Omega}\,;\;\; u(x) \;\geq\; c\alpha\,\kappa_0^{\alpha-1}\,|x|\Big\}. \tag{6.11}$$

Indeed, assume that, on the contrary, there is a point $x_0 \in Supp(\mu)$ such that

$$u(x_0) \;\leq\; c\alpha\kappa^{\alpha-1}|x_0| - 4\varepsilon, \qquad \text{for some } \varepsilon > 0. \tag{6.12}$$

By (3.31) there exists $t > 0$ such that

$$u^{(t)}(x_0) + Kt \;\leq\; c\alpha\kappa^{\alpha-1}|x_0| - 3\varepsilon. \tag{6.13}$$

with $u^{(t)}$ defined as in (3.29). The continuity of $u^{(t)}$ implies

$$u^{(t)}(x) + Kt \;\leq\; c\alpha\kappa^{\alpha-1}|x| - 2\varepsilon \tag{6.14}$$

for all $x \in B(x_0,r)\cap\overline{\Omega}$, with $r > 0$ sufficiently small. In turn, by the convergence $u_n^{(t)}(x) \to u^{(t)}(x)$, uniformly for all $x\in\overline{\Omega}$, we have

$$u_n^{(t)}(x) + Kt \;\leq\; c\alpha\kappa^{\alpha-1}|x| - \varepsilon \tag{6.15}$$

for all $n \geq N_0$ large enough and for all $x \in B(x_0,r)\cap\overline{\Omega}$.

By (6.7), this implies that

$$Supp(\mu_n)\cap B(x_0,r) \;=\; \emptyset, \qquad \text{for all } n\geq N_0\,.$$

From the weak convergence $\mu_n \rightharpoonup \mu$ it follows that $Supp(\mu) \cap B(x_0, r) \;=\; \emptyset$ as well, contradicting the assumption $x_0 \in Supp(\mu)$.

4. Thanks to (6.11) we can now define

$$\mu^* \;\doteq\; \frac{\nu}{u}.$$

By (6.5), u satisfies

$$\Delta u + f(x,u) - u\,\mu^* \;=\; 0, \tag{6.16}$$

with Neumann boundary conditions (3.4). Following [13], we now establish the key inequality

$$\mu^* \;\leq\; \mu. \tag{6.17}$$

To prove that (6.17) holds, thanks to the upper semicontinuity of u it suffices to show that

$$\int_\Omega \frac{\phi}{\psi}\,d\nu \;\leq\; \int_\Omega \phi\,d\mu, \qquad \text{for every} \quad \phi,\psi \in \mathcal{C}(\overline{\Omega}), \quad \phi \geq 0\,, \quad \psi > u. \tag{6.18}$$

Since ψ is continuous on the compact set $\overline{\Omega}$, we can choose $t, \delta > 0$ small enough so that

$$u(x) \;\leq\; u^{(t)}(x) + Kt \;<\; \psi(x) - \delta \qquad\qquad \text{for all } x \in \overline{\Omega}\,. \tag{6.19}$$

By (6.6), as $n \to \infty$ the corresponding functions $u_n^{(t)}$ converge to $u^{(t)}$ uniformly on $\overline{\Omega}$. Hence for all n large enough we have

$$u_n(x) \;\leq\; u_n^{(t)}(x) + Kt \;\leq\; u^{(t)}(x) + Kt + \delta \;<\; \psi(x) \qquad\qquad \text{for all } x \in \overline{\Omega}\,.$$

This yields

$$\int_\Omega \frac{\phi}{\psi}\,d\nu \;=\; \lim_{n\to\infty} \int_\Omega \phi\,\frac{1}{\psi}\,d\nu_n \;=\; \lim_{n\to\infty} \int_\Omega \phi\,\frac{u_n}{\psi}\,d\mu_n \;\leq\; \lim_{n\to\infty} \int_\Omega \phi\,d\mu_n \;=\; \int_\Omega \phi\,d\mu\,,$$

proving (6.17).

5. We conclude by proving the pair (u,μ^*) is optimal. Since $\{(u_n,\mu_n)\}_{n\geq 1}$ is a maximizing sequence, using (6.5) and the lower semicontinuity of the irrigation cost $\mathcal{I}^\alpha$, one obtains

$$\begin{aligned}
\overline{M} \;&=\; \lim_{n\to\infty} \Big[\mathcal{H}(u_n,\mu_n) - c\mathcal{I}^\alpha(\mu_n)\Big] \\
&\leq\; \lim_{n\to\infty} \int_\Omega f(x,u_n)\;dx - c \liminf_{n\to\infty} \mathcal{I}^\alpha(\mu_n) \;\leq\; \int_\Omega f(x,u)\;dx - c\mathcal{I}^\alpha(\mu) \\
&\leq\; \int_\Omega f(x,u)\;dx - c\mathcal{I}^\alpha(\mu^*).
\end{aligned}$$

The last inequality follows from (6.17) and the monotonicity of $\mathcal{I}^\alpha$, proved at (4.10). By (6.17) the weak convergence $\mu_n \rightharpoonup \mu$ it follows

$$\mu^*(\overline{\Omega}) \;\leq\; \mu(\overline{\Omega}) \;=\; \kappa_0.$$

This completes the proof of the optimality of (u,μ^*). $\square$

We now prove an analogous existence result in the case of Dirichlet boundary conditions.

Theorem 6.2 *Let the assumptions (A1)-(A2) hold. Then the maximization problem (6.2), over all couples* (u,μ) *which satisfy (3.3), (3.5), and (6.3), has an optimal solution.*

Proof. 1. Call $\mathcal{A}$ the set of all admissible couples (u,μ) which satisfy (3.3), (3.5), and (6.3). As in the previous case, the supremum $\overline{M}$ of the functional (6.2) over all admissible couples $(u,\mu)\in\mathcal{A}$ satisfies (6.4). Let $\{(u_n,\mu_n)\}_{n\geq 1}$ be a maximizing sequence. It is clearly not restrictive to assume that $\mu_n\in\mathcal{M}_0$ for every n.

2. Let $w^*:\overline{\Omega}\mapsto[0,M]$ be the largest solution to the elliptic problem with smooth coefficients

$$\begin{cases} \Delta w+f(x,w) \;=\; 0 & x\in\Omega,\\ \qquad\qquad w \;=\; 0 & x\in\partial\Omega. \end{cases} \tag{6.20}$$

By classical theory, w^* can be constructed as the supremum of all functions $w:\overline{\Omega}\mapsto[0,M]$ which are subsolutions to (6.20). Hence w^* is well defined.

For each $n\geq 1$, since $u_n\mu_n\geq 0$, by Lemma 3.1, the solution u_n of (3.3), (3.5) satisfies

$$u_n(x)\;\leq\; w^*(x)\qquad\qquad \text{for all } x\in\overline{\Omega}. \tag{6.21}$$

3. Consider the set (see Fig. 3)

$$\Omega^*\;\doteq\;\Big\{x\in\Omega\,;\;\; w^*(x)\geq c\,\alpha\kappa_0^{\alpha-1}|x|\Big\}.$$

Note that Ω^* is closed and

$$\Omega^*\cap\partial\Omega\;=\;\{0\}.$$

Denote by χ_{Ω^*} the characteristic function of Ω^* and, for each $n\geq 1$, consider the measure $\mu_n^*=\chi_{\Omega^*}\cdot\mu_n$ supported on Ω^*. Since $\mu_n^*\leq\mu_n$, by the comparison argument in Lemma 3.1, we can find a solution u_n^* of

$$\Delta u+f(x,u)-u\,\mu_n^*\;=\;0$$

with Dirichlet boundary conditions (3.5), such that

$$u_n\;\leq\;u_n^*\;\leq\;w^*.$$

We claim that $(u_n^*,\mu_n^*)_{n\geq 1}$ is another maximizing sequence. Indeed,

$$\mathcal{H}(u_n,\mu_n)-\mathcal{H}(u_n^*,\mu_n^*)\;\leq\;\int_{\Omega\setminus\Omega^*}u_n(x)\,d\mu_n\;\leq\;\int_{\Omega\setminus\Omega^*}c\,\alpha\kappa_0^{\alpha-1}|x|d\mu_n\,. \tag{6.22}$$

On the other hand, the same argument used at (6.10) shows that the difference in the irrigation costs can be estimated by

$$\mathcal{I}^\alpha(\mu_n)-\mathcal{I}^\alpha(\mu_n^*)\;\geq\;\alpha\kappa_0^{\alpha-1}\int_{\Omega\setminus\Omega^*}|x|\,d\mu_n\,. \tag{6.23}$$

Together, (6.22) and (6.23) yield

$$\mathcal{H}(u_n,\mu_n) \;-\; \mathcal{H}(u_n^*,\mu_n^*) \;\leq\; c\mathcal{I}^\alpha(\mu_n) \;-\; c\mathcal{I}^\alpha(\mu_n^*), \tag{6.24}$$

proving that $\{(u_n^*,\mu_n^*)\}_{n\geq 1}$ is also a maximizing sequence. Without loss of generality, from now on we shall thus assume that

$$Supp(\mu_n) \;\subseteq\; \Omega^* \qquad\qquad \text{for all } n\geq 1\,. \tag{6.25}$$

4. Consider the sequence of measures $\nu_n \doteq u_n\,\mu_n$. By possibly taking a subsequence, we can again assume that (6.5) holds, for suitable positive measures μ,ν, supported on Ω^*. Moreover, for every fixed radius $r>0$, we can assume the convergence of the averaged values

$$u_n^{(r)}(x) \;\doteq\; \fint_{B(x,r)\cap\Omega} u_n(y)\,dy \;\to\; \fint_{B(x,r)\cap\Omega} u(y)\,dy \;\doteq\; u^{(r)}(x) \tag{6.26}$$

as $n\to\infty$, uniformly for $x\in\overline{\Omega}$.

5. We claim that

$$u(x) \;\geq\; c\alpha\kappa_0^{\alpha-1}|x| \qquad \text{for all } x\in Supp(\mu). \tag{6.27}$$

Indeed, assume that, on the contrary, there is a point $x_0\in Supp(\mu)\subseteq\Omega^*$ such that

$$u(x_0) \;\leq\; c\alpha\kappa^{\alpha-1}|x_0|-4\varepsilon, \qquad \text{for some } \varepsilon>0. \tag{6.28}$$

Clearly, this can hold only if $x_0\neq 0$. Hence we can choose $r_0>0$ so that $B(x_0,2r_0)\subset\Omega$.

Since $|f(x,u)|\leq K$, all functions $u_n+\frac{K}{2d}|x|^2$, and $u+\frac{K}{2d}|x|^2$ are subharmonic on the open set Ω. Hence, there exists a constant C such that, for every $x\in B(x_0,r_0)$ and $0<r\leq r_0$, all maps

$$r \;\mapsto\; u_n^{(r)}(x)+Cr, \qquad\qquad r \;\mapsto\; u_n^{(r)}(x)+Cr,$$

are nondecreasing. Taking a sequence $r_k\to 0$, the pointwise values of u_n,u can thus be defined as the infimum of a decreasing sequence of continuous functions:

$$u_n(x) \;\doteq\; \inf_{r>0}\fint_{B(x,r)} u_n(y)\,dy\,, \qquad\qquad u(x) \;\doteq\; \inf_{r>0}\fint_{B(x,r)} u(y)\,dy. \tag{6.29}$$

A contradiction is now achieved by the same argument used at (6.13)–(6.15), replacing the weigthed averages $u_n^{(t)}$ defined at (6.6) with the standard averages $u_n^{(r)}$ in (6.26).

6. By the previous step, we can define a measure μ^* supported on the open set Ω, by setting

$$\mu^* \;\doteq\; \frac{\nu}{u}\,.$$

Notice that, in principle, μ may contain a point mass at the origin. In this case, to remove any ambiguity we define $\mu^*(\{0\})=0$. By (6.5), the limit function u satisfies (6.16) with Dirichlet boundary conditions (3.5).

The same arguments used in step **4** of the proof of Theorem 6.1 now show that $\mu^*\leq\mu$. Hence the couple (u,μ^*) is admissible.

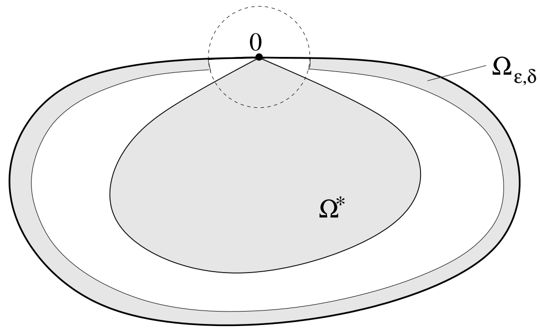

Figure 3: The sets Ω^* and $\Omega_{\varepsilon,\delta}$ considered in the proof of Theorem 6.2.

7. Finally, we prove that (u, μ^*) is optimal. Indeed, on the set $\Omega \setminus \Omega^*$ all functions u_n, u provide solutions to the semilinear elliptic equation with smooth coefficients

$$\Delta u + f(x, u) \;=\; 0.$$

For any $\varepsilon > 0$, using the Schauder regularity estimates [17, 20] up to the boundary, we can find $\rho > 0$ such that all solutions u_n are uniformly smooth on the set

$$\Omega_{\varepsilon,\delta} \;\doteq\; \Big\{x \in \Omega\,;\;\; dist(x, \partial\Omega) < \rho, \quad |x| > \varepsilon\Big\},$$

shown in Fig. 3. Hence, by Ascoli's theorem, by possibly taking a further subsequence we achieve the convergence of the normal derivatives along the boundary

$$\partial_{\mathbf{n}(x)} u_n(x) \;\to\; \partial_{\mathbf{n}(x)} u(x) \qquad\qquad \text{for all } x \in \partial\Omega \setminus \{0\}.$$

Notice that, for any $\varepsilon > 0$, the convergence is uniform on the set $\partial\Omega \setminus B(0, \varepsilon)$. Observing that $0 \le u_n \le w^*$ and similarly $0 \le u \le w^*$, we deduce

$$\partial_{\mathbf{n}(x)} u_n(x) \;\le\; 0, \qquad\qquad \partial_{\mathbf{n}(x)} w^*(x) \;\le\; \partial_{\mathbf{n}(x)} u(x) \;\le\; 0. \tag{6.30}$$

Using (6.30), for any fixed $\varepsilon > 0$ one obtains

$$\begin{aligned}
\lim_{n\to\infty} \int_{\partial\Omega} \partial_{\mathbf{n}} u_n \, d\sigma \;&\le\; \lim_{n\to\infty} \int_{\partial\Omega\setminus B(0,\varepsilon)} \partial_{\mathbf{n}} u_n \, d\sigma \;=\; \int_{\partial\Omega\setminus B(0,\varepsilon)} \partial_{\mathbf{n}} u \, d\sigma \\
&\le\; \int_{\partial\Omega} \partial_{\mathbf{n}} u \, d\sigma - \int_{\partial\Omega\cap B(0,\varepsilon)} \partial_{\mathbf{n}} w^* \, d\sigma.
\end{aligned}$$

Using the lower semicontinuity of the irrigation functional and the fact that $\mu^* \le \mu$, we thus conclude

$$\begin{aligned}
\overline{M} \;&=\; \lim_{n\to\infty} \mathcal{H}(u_n, \mu_n) - \lim_{n\to\infty} c\mathcal{I}^\alpha(\mu_n) \\
&\le\; \lim_{n\to\infty} \int_\Omega f(x, u_n)\, dx + \lim_{n\to\infty} \int_{\partial\Omega} \partial_{\mathbf{n}} u_n \, d\sigma - c\mathcal{I}^\alpha(\mu) \\
&\le\; \int_\Omega f(x, u)\, dx + \int_{\partial\Omega} \partial_{\mathbf{n}} u \, d\sigma + \int_{\partial\Omega\cap B(0,\varepsilon)} \partial_{\mathbf{n}} w^* \, d\sigma - c\mathcal{I}^\alpha(\mu^*) \\
&=\; \mathcal{H}(u, \mu^*) - c\mathcal{I}^\alpha(\mu^*) - \int_{\partial\Omega\cap B(0,\varepsilon)} \partial_{\mathbf{n}} w^* \, d\sigma.
\end{aligned} \tag{6.31}$$

By choosing $\varepsilon > 0$ small, the last integral on the right hand side of (6.31) can be made arbitrarily small. Hence $\mathcal{H}(u, \mu^*) - c\mathcal{I}^\alpha(\mu^*) \;\ge\; \overline{M}$, proving the optimality of (u, μ^*). □

7 Concluding remarks

In this paper we assumed that the primary goal of tree leaves (tree roots) is to gather sunlight (water and nutrients from the soil, respectively). We then tried to determine shapes that most efficiently achieve these goals. The search for these optimal shapes has been formulated as a maximization problem for certain functionals, in the spirit of the classical Calculus of Variations [5].

While our present analysis is purely theoretical, optimal shapes may be computed by the numerical algorithms recently developed in [26, 27, 28, 30]. It will then be of interest to compare numerical simulations with the shapes actually observed in nature. In this direction, we expect that root shapes which maximize our harvest functional will look very similar to the actual roots of biological trees.

On the other hand, we guess that in some cases the shapes which maximize the gathered sunlight will resemble an optimal disposition of solar panels, more than actual tree branches. If this is the case, it would indicate that the efficiency in capturing sunlight has not been the primary goal driving the evolution of plant shapes. In computer simulations of tree growth [3, 23, 29], the most realistic images are produced by algorithms based on the idea of conquering space. This suggests that tree shapes have evolved as the result of a competitive game among plants, rather than an optimization problem. A mathematical modeling of such a game remains to be worked out.

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