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FLOW STABILITY OF PATCHY VECTOR FIELDS AND ROBUST FEEDBACK STABILIZATION*

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Abstract. The paper is concerned with *patchy vector fields*, a class of discontinuous, piecewise smooth vector fields that were introduced by the authors to study feedback stabilization problems. We prove the stability of the corresponding solution set w.r.t. a wide class of impulsive perturbations. These results yield the robustness of *patchy feedback controls* in the presence of measurement errors and external disturbances.

Key words. patchy vector field, impulsive perturbation, feedback stabilization, discontinuous feedback, robustness

AMS subject classifications. 34A, 34D, 49E, 93D

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1. Introduction and basic notation. The aim of this paper is to establish the stability of the set of trajectories of a patchy vector field w.r.t. various types of perturbations and the robustness of patchy feedback controls.

Patchy vector fields were introduced in [A-B] in order to study feedback stabilization problems. The underlying motivation is the following: The analysis of stabilization problems by means of Lyapunov functions usually leads to stabilizing feedbacks with a wild set of discontinuities. On the other hand, as shown in [A-B], by patching together open-loop controls one can always construct a piecewise smooth stabilizing feedback whose discontinuities have a very simple structure. In particular, one can develop the whole theory by studying the corresponding discontinuous ODEs within the classical framework of Carathéodory solutions. We recall here the main definitions.

DEFINITION 1.1. *By a patch we mean a pair* (Ω, g)*, where* $\Omega \subset \mathbb{R}^n$ *is an open domain with smooth boundary* $\partial\Omega$ *and* g *is a smooth vector field defined on a neighborhood of the closure* $\overline{\Omega}$*, which points strictly inward at each boundary point* $x \in \partial\Omega$*.*

Calling $\mathbf{n}(x)$ the outer normal at the boundary point x, we thus require

$$\big\langle g(x),\ \mathbf{n}(x)\big\rangle < 0 \qquad \text{for all} \quad x \in \partial\Omega. \tag{1.1}$$

DEFINITION 1.2. *We say that* $g : \Omega \mapsto \mathbb{R}^n$ *is a patchy vector field on the open domain* Ω *if there exists a family of patches* $\big\{(\Omega_\alpha,\ g_\alpha);\quad \alpha \in \mathcal{A}\big\}$ *such that*

- $\mathcal{A}$ *is a totally ordered set of indices;*
- *the open sets* Ω_α *form a locally finite covering of* Ω*, i.e.,* $\Omega = \cup_{\alpha\in\mathcal{A}}\Omega_\alpha$ *and every compact set* $K \subset \mathbb{R}^n$ *intersect only a finite number of domains* Ω_α*,* $\alpha \in \mathcal{A}$*;*
- *the vector field* g *can be written in the form*

$$g(x) = g_\alpha(x) \qquad \text{if} \qquad x \in \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta. \tag{1.2}$$

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By setting

$$\alpha^*(x) \doteq \max\{\alpha \in \mathcal{A}\,;\quad x \in \Omega_\alpha\}, \tag{1.3}$$

we can write (1.2) in the equivalent form

$$g(x) = g_{\alpha^*(x)}(x) \qquad \text{for all} \quad x \in \Omega. \tag{1.4}$$

Remark 1.1. The patches $(\Omega_\alpha,\, g_\alpha)$ are not uniquely determined by the patchy vector field g. Indeed, whenever $\alpha < \beta$, by (1.2) the values of g_α on the set $\Omega_\alpha \cap \Omega_\beta$ are irrelevant. Therefore, if the open sets Ω_α form a locally finite covering of Ω and we assume that, for each $\alpha \in \mathcal{A}$, the vector field g_α satisfies (1.1) at every point $x \in \partial\Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta$, then the vector field g defined according with (1.2) is again a patchy vector field. To see this, it suffices to construct vector fields $\tilde{g}_\alpha$ which satisfy the inward-pointing property (1.1) at every point $x \in \partial\Omega_\alpha$ and such that $\tilde{g}_\alpha = g_\alpha$ on $\Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta$. To accomplish this, for each α we first consider a smooth vector field v_α such that $v_\alpha(x) = -\mathbf{n}(x)$ on $\partial\,\Omega_\alpha$. Then we construct a smooth scalar function $\psi_\alpha : \Omega \mapsto [0,1]$ such that

$$\psi_\alpha(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta, \\ 0 & \text{if} \quad x \in \partial\Omega_\alpha, \quad \big\langle g(x),\ \mathbf{n}(x)\big\rangle \geq 0. \end{cases}$$

Finally, for each $\alpha \in \mathcal{A}$ we define the interpolation

$$\tilde{g}_\alpha(x) \doteq \psi_\alpha(x) g_\alpha(x) + \big(1 - \psi_\alpha(x)\big) v_\alpha(x).$$

The vector fields $\tilde{g}_\alpha$ thus defined satisfy our requirements.

We shall occasionally adopt the longer notation $\big(\Omega,\ g,\ (\Omega_\alpha,\, g_\alpha)_{\alpha\in\mathcal{A}}\big)$ to indicate a patchy vector field, specifying both the domain and the single patches. If g is a patchy vector field, the differential equation

$$\dot{x} = g(x) \tag{1.5}$$

has many interesting properties. In particular, in [A-B] it was proved that the set of Carathéodory solutions of (1.5) is closed in the topology of uniform convergence but possibly not connected. Moreover, given an initial condition

$$x(t_0) = x_0, \tag{1.6}$$

the Cauchy problem (1.5)–(1.6) has at least one forward solution and at most one backward solution. For every Carathéodory solution $x = x(t)$ of (1.5), the map $t \mapsto \alpha^*(x(t))$ is left continuous and nondecreasing.

In this paper we study the stability of the solution set for (1.5) w.r.t. various perturbations. Most of our analysis will be concerned with impulsive perturbations, described by

$$\dot{y} = g(y) + \dot{w}. \tag{1.7}$$

Here $w = w(t)$ is any left continuous function with bounded variation. By a solution of the perturbed system (1.7) with an initial condition

$$y(t_0) = y_0, \tag{1.8}$$

we mean a measurable function $t \mapsto y(t)$ such that

$$y(t) = y_0 + \int_{t_0}^{t} g\big(y(s)\big)\, ds + \big[w(t) - w(t_0)\big] \tag{1.9}$$

(see [B1]). If $w(\cdot)$ is discontinuous, the system (1.7) has impulsive behavior and the solution $y(\cdot)$ will be discontinuous as well. We choose to work with (1.7) because it provides a simple and general framework to study robustness properties. Indeed, consider a system with both inner and outer perturbations of the form

$$\dot{x} = g\big(x + e_1(t)\big) + e_2(t). \tag{1.10}$$

The map $t \mapsto y(t) \doteq x(t) + e_1(t)$ then satisfies the impulsive equation

$$\dot{y} = g(y) + e_2(t) + \dot{e}_1(t) = g(y) + \dot{w},$$

where

$$w(t) = e_1(t) + \int_{t_0}^{t} e_2(s)\, ds.$$

Therefore, from the stability of solutions of (1.7) w.r.t. perturbations w that have small total variation, one can immediately deduce a result on the stability of solutions of (1.10), when Tot.Var.$\{e_1\}$ and $\|e_2\|_{\mathbf{L}^1}$ are suitably small. Here, Tot.Var.$\{e_1\}$ denotes the total variation of the function e_1 over the whole interval where it is defined, while Tot.Var.$\big\{e_1\,;\,J\big\}$ denotes the total variation of e_1 over a subset J. We shall also denote by BV the space of all functions of bounded variation. Any function of bounded variation $w = w(t)$ can be redefined up to $\mathbf{L}^1$-equivalence. For the sake of definiteness, throughout the paper we shall always consider left continuous representatives, so that $w(t) = w(t^-) \doteq \lim_{s \to t-} w(s)$ for every t. The Lebesgue measure of a Borel set $J \subset \mathbb{R}$ will be denoted by meas(J).

We observe that since the Cauchy problem for (1.5) does not have forward uniqueness and continuous dependence, one clearly cannot expect that a single solution of (1.5) be stable under small perturbations. What we establish here is a different stability property, involving not a single trajectory but the whole solution set: If the perturbation w is small in the BV norm, then every solution of (1.7) is close to some solution of (1.5). This is essentially an upper semicontinuity property of the solution set. Namely, we will prove in section 2 the following results.

THEOREM 1.3. *Let g be a patchy vector field on an open domain $\Omega \subset \mathbb{R}^n$. Consider a sequence of solutions $y_\nu(\cdot)$ of the perturbed system*

$$\dot{y}_\nu = g(y_\nu) + \dot{w}_\nu, \qquad t \in [0, T], \tag{1.11}$$

with Tot.Var.$\{w_\nu\} \to 0$ as $\nu \to \infty$. If the $y_\nu : [0,T] \mapsto \Omega$ converge to a function $y : [0,T] \mapsto \Omega$, uniformly on $[0,T]$, then $y(\cdot)$ is a Carathéodory solution of (1.5) *on* $[0, T]$.

COROLLARY 1.4. *Let g be a patchy vector field on an open domain $\Omega \subset \mathbb{R}^n$. Given any closed subset $A \subset \Omega$, any compact set $K \subset A$, and any $T, \varepsilon > 0$, there exists $\delta = \delta(A, K, T, \varepsilon) > 0$ such that the following holds. If $y : [0,T] \mapsto A$ is a solution of the perturbed system* (1.7), *with $y(0) \in K$ and Tot.Var.$\{w\} < \delta$, then there exists a solution $x : [0,T] \mapsto \Omega$ of the unperturbed equation* (1.5) *with*

$$\big\|x - y\big\|_{\mathbf{L}^\infty([0,T])} < \varepsilon\,. \tag{1.12}$$

We remark that the type of stability described above is precisely what is needed in many feedback control applications. As an example, consider the problem of stabilizing to the origin the control system

$$\dot{x} = f(x,u). \tag{1.13}$$

Given a compact set K and $\varepsilon > 0$, assume that there exists a piecewise constant feedback $u = U(x)$ such that $g(x) \doteq f\big(x, U(x)\big)$ is a patchy vector field and such that every solution of (1.5) starting from a point $x(0) \in K$ is steered inside the ball B_ε centered at the origin with radius ε, within a time $T > 0$. By Corollary 1.4, if the perturbation w is sufficiently small (in the BV norm), every solution of the perturbed system (1.7) will be steered inside the ball $B_{2\varepsilon}$ within time T. In other words, the feedback still performs well in the presence of small perturbations. Applications to feedback control will be discussed in more detail in section 3.

Throughout the paper, by $B(x,r)$ we denote the closed ball centered at x with radius r and, for every given set A, we let $B(A,r) \doteq \cup_{x\in A} B(x,r)$. The closure, the interior, and the boundary of a set Ω are written as $\overline{\Omega}$, $\overset{\circ}{\to}\Omega$ and $\partial\Omega$, respectively.

2. Stability of patchy vector fields. We begin by proving a local existence result for solutions of the perturbed system (1.7).

PROPOSITION 2.1. *Let g be a patchy vector field on an open domain $\Omega \subset \mathbb{R}^n$. Given any compact set $K \subset \Omega$, there exists $\overline{\chi} = \overline{\chi}_K > 0$ such that, for each $y_0 \in K, t_0 \in \mathbb{R}$, and for every Lipschitz continuous function $w = w(t)$, with Lipschitz constant $\|\dot{w}\|_{\mathbf{L}^\infty} < \overline{\chi}$, the Cauchy problem* (1.7)–(1.8) *has at least one local forward solution.*

Proof. Fix some compact subset $K' \subset \Omega$ whose interior contains K. To prove the local existence of a forward solution to (1.7), first observe that because of the inward-pointing condition (1.1) and the smoothness assumptions on the vector fields g_α, one can find for any $\alpha \in \mathcal{A}$ some constant $\chi_\alpha > 0$ such that

$$\sup_{\substack{x\in\partial\Omega_\alpha\cap K' \\ |v|\le\chi_\alpha}} \big\langle g_\alpha(x) + v,\, \mathbf{n}_{\boldsymbol{\alpha}}(x)\big\rangle < 0, \tag{2.1}$$

where $\mathbf{n}_{\boldsymbol{\alpha}}(x)$ is the outer normal to $\partial\Omega_\alpha$ at the boundary point x. Since K' is a compact set and $\{\Omega_\alpha\}_\alpha$ is a locally finite covering of Ω, there will be only finitely many elements of $\{\Omega_\alpha\}_\alpha$ that intersect K'. Let

$$\big\{\alpha_1, \dots, \alpha_{N}\big\} = \big\{\alpha \in \mathcal{A} \ : \ \Omega_\alpha \cap K' \neq \emptyset\big\}, \tag{2.2}$$

and, by possibly renaming the indices α_i, assume that

$$\alpha_1 < \cdots < \alpha_{N}. \tag{2.3}$$

Choose a constant $\overline{\chi} > 0$ such that

$$\overline{\chi} \le \inf\big\{\chi_{\alpha_i} \ : \ i = 1, \dots, N\big\}. \tag{2.4}$$

For any fixed $y_0 \in K$, consider the index

$$\widehat{\alpha}(y_0) \doteq \max\big\{\alpha \ : \ y_0 \in \overline{\Omega}_\alpha\big\}.$$

By the definition of $\overline{\chi}$, any solution $y = y(\cdot)$ to the Cauchy problem

$$\dot{y} = g_{\widehat{\alpha}}(y) + \dot{w}, \qquad y(t_0) = y_0,$$

associated to a piecewise Lipschitz map $w = w(t)$ with $\|\dot{w}\|_{\mathbf{L}^\infty} < \overline{\chi}$, remains inside $\Omega_{\widehat{\alpha}}$ for all $t \in [t_0, t_0 + \delta]$ for some $\delta > 0$. Hence, it provides also a solution to (1.6) on some interval $[t_0, t_0 + \delta']$, $0 < \delta' \leq \delta$. □

Toward a proof of Theorem 1.3, we first derive an intermediate result. By the basic properties of a patchy vector field, for every solution $t \mapsto x(t)$ of (1.5) the corresponding map $t \mapsto \alpha^*\big(x(t)\big)$ in (1.3) is nondecreasing. Roughly speaking, a trajectory can move from a patch Ω_α to another patch Ω_β only if $\alpha < \beta$. This property no longer holds in the presence of an impulsive perturbation. However, the next proposition shows that for a solution y of (1.7) the corresponding map $t \mapsto \alpha^*\big(y(t)\big)$ is still nondecreasing, after a possible modification on a small set of times. Alternatively, one can slightly modify the impulsive perturbation w, say replacing it by another perturbation $w^\diamondsuit$, such that the map $t \mapsto \alpha^*\big(y^\diamondsuit(t)\big)$ is monotone along the corresponding trajectory $t \mapsto y^\diamondsuit(t)$.

Proposition 2.2. *Let g be a patchy vector field on an open domain $\Omega \subset \mathbb{R}^n$ determined by the family of patches $\big\{(\Omega_\alpha,\ g_\alpha);\ \alpha \in \mathcal{A}\big\}$. For any $T > 0$ and any compact set $K \subset \Omega$, there exist constants $C, \delta > 0$ and an integer N such that the following hold.*

(i) *For every $w \in \mathrm{BV}$ with Tot.Var.$\{w\} < \delta$, and for every solution $y : [0, T] \mapsto \Omega$ of the Cauchy problem (1.7)–(1.8) with $y_0 \in K$, there is a partition of $[0,\ T]$, $0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{N+1} = T$, and indices*

$$\alpha_1 < \alpha_2 < \cdots < \alpha_N \tag{2.5}$$

such that

$$\alpha^*(y(t)) \geq \alpha_i \qquad \text{for all } t \in\ ]\tau_i,\ \tau_{i+1}], \qquad i = 0, \ldots, N, \tag{2.6}$$

$$\mathrm{meas}\Big(\bigcup_{i \geq 0} \big\{t \in [\tau_i,\ \tau_{i+1}]\ :\ \alpha^*(y(t)) > \alpha_i\big\}\Big) < C \cdot \mathit{Tot.Var.}\{w\}. \tag{2.7}$$

(ii) *For every BV function $w = w(t)$ with Tot.Var.$\{w\} < \delta$, and for every solution $y : [0, T] \mapsto \Omega$ of the Cauchy problem (1.7)–(1.8) with $y_0 \in K$, there is a BV function $w^\diamondsuit = w^\diamondsuit(t)$ and a solution $y^\diamondsuit : [0, T] \mapsto \Omega$ of*

$$\dot{y}^\diamondsuit = g(y^\diamondsuit) + \dot{w}^\diamondsuit \tag{2.8}$$

so that the map $t \mapsto \alpha^*(y^\diamondsuit(t))$ is nondecreasing and left continuous, and there holds

$$\begin{aligned} \mathit{Tot.Var.}\{w^\diamondsuit\} &\leq C \cdot \mathit{Tot.Var.}\{w\}, \\ \big\|y^\diamondsuit - y\big\|_{\mathbf{L}^\infty([0,T])} &\leq C \cdot \mathit{Tot.Var.}\{w\}. \end{aligned} \tag{2.9}$$

Proof. (i) The proof of (i) will be given in three steps.

Step 1. Since each g_α is a smooth vector field and we are assuming a uniform bound on the total variation of every perturbation $w = w(t)$, there will be some compact subset $K' \subset \overline{\Omega}$ that contains every solution $y : [0, T] \mapsto \Omega$ of (1.7) starting at a point $y_0 \in K$. We will assume without loss of generality that every domain Ω_α is bounded since, otherwise, one can replace Ω_α with its intersection $\Omega_\alpha \cap \Omega'$ with a bounded domain $\Omega' \subset \Omega$ that contains K', preserving the inward-pointing condition (1.1) (cf. Remark 1.1). For each $\alpha \in \mathcal{A}$, define the map $\varphi_\alpha : \Omega \mapsto \mathbb{R}$ by setting

$$\varphi_\alpha(x) \doteq \begin{cases} d(x,\, \partial\Omega_\alpha) & \text{if} \quad x \in \Omega_\alpha, \\ -d(x,\, \partial\Omega_\alpha) & \text{otherwise,} \end{cases} \tag{2.10}$$

and let

$$\varphi_\alpha^+(x) \doteq \max\{\varphi_\alpha(x),\, 0\}$$

denote the positive part of $\varphi_\alpha(x)$. The regularity assumptions on the patch $(\Omega_\alpha, g_\alpha)$ guarantee that φ_α is smooth if restricted to a sufficiently small neighborhood of the boundary $\partial\Omega_\alpha$. Thus, if $\{\Omega_{\alpha_i} \ : \ i = 1, \dots, N\}$ denotes the finite collection of domains that intersect the compact set K' as in (2.2)–(2.3), there will be some constant $\overline{\rho} > 0$ so that, setting

$$\Omega_\alpha^{\overline{\rho}} \doteq \big\{x \in \Omega \ : \ d(x,\, \partial\Omega_\alpha) \geq \overline{\rho}\big\}, \tag{2.11}$$

the restriction of any map φ_{α_i} to the domain $\Omega \setminus \Omega_{\alpha_i}^{\overline{\rho}}$ will be smooth. In particular, for any $i = 1, \dots, N$, we will have

$$\nabla\varphi_{\alpha_i}(x) = -\mathbf{n}_{\boldsymbol{\alpha}_\mathbf{i}}\big(\pi_{\alpha_i}(x)\big) \qquad \text{for all} \quad x \in \Omega \setminus \Omega_{\alpha_i}^{\overline{\rho}}, \tag{2.12}$$

where $\mathbf{n}_{\boldsymbol{\alpha}_\mathbf{i}}$ represents as usual the outer normal to $\partial\Omega_{\alpha_i}$, while $\pi_{\alpha_i}(x)$ denotes the projection of the point x onto the set $\partial\Omega_{\alpha_i}$. On the other hand, thanks to the inward-pointing condition (1.1), we can choose the constant $\overline{\rho}$ so that

$$\sup_{\substack{i=1,\dots,N \\ x \in \Omega_{\alpha_i} \setminus \Omega_{\alpha_i}^{\overline{\rho}}}} \big\langle g_{\alpha_i}(x),\, \mathbf{n}_{\boldsymbol{\alpha}_\mathbf{i}}(\pi_{\alpha_i}(x))\big\rangle \leq -c' \tag{2.13}$$

for some $c' > 0$. Moreover, the smoothness of the fields g_α on $\overline{\Omega}$ implies the existence of some $c'' > 0$ such that

$$\sup_{\substack{i=1,\dots,N, j>i \\ x \in \Omega_{\alpha_i}}} \Big|\big\langle g_{\alpha_j}(x),\, \mathbf{n}_{\boldsymbol{\alpha}_\mathbf{i}}(\pi_{\alpha_i}(x))\big\rangle\Big| \leq c''. \tag{2.14}$$

Step 2. Consider now a left continuous BV function $w = w(t)$ and let $y : [0, T] \mapsto \Omega$ be a solution of the corresponding Cauchy problem (1.7)–(1.8), with $y_0 \in K$. Observe that, for any $i = 1, \dots, N$, and for any interval $J \subset [0, T]$ such that

$$y(t) \in \Omega \setminus \Omega_{\alpha_i}^{\overline{\rho}} \qquad \text{for all} \quad t \in J,$$

the composed map $\varphi_{\alpha_i}^+ \circ y : J \mapsto \mathbb{R}$ is also a left continuous BV function whose distributional derivative $\mu_i \doteq D(\varphi_{\alpha_i}^+ \circ y)$ is a Radon measure, which can be decomposed into an absolutely continuous μ_i^{ac} and a singular part μ_i^s w.r.t. the Lebesgue measure dt. One can easily verify that for any Borel set $E \subset J$, the absolutely continuous part of μ_i is given by

$$\mu_i^{ac}(E) = \int_{E^+} \big\langle \nabla\varphi_{\alpha_i}(y(t)),\ g(y(t)) + \dot{w}(t)\big\rangle\, dt, \qquad E^+ \doteq \{t \in E \ ; \ y(t) \in \Omega_{\alpha_i}\}. \tag{2.15}$$

Moreover, calling μ_w^{ac} and μ_w^s, respectively, the absolutely continuous and the singular part of $\mu_w \doteq \dot{w}$, the following bounds hold:

$$\begin{aligned}\left|\int_{E^+}\big\langle\nabla\varphi_{\alpha_i}(y(t)),\ \dot{w}(t)\big\rangle\,dt+\mu_i^s(E)\right| &\le c'''\cdot\Big\{\big|\mu_w^{ac}(E)\big|+\big|\mu_w^s(E)\big|\Big\}\\ &\le c'''\cdot\text{Tot.Var.}\{w\}\end{aligned}\tag{2.16}$$

for some constant $c'''>0$ that depends only on the compact set K' and on the time interval $[0,T]$. Let $C_i,\ell_i,\ i=N,N-1,\dots,1$, be the constants recursively defined by

$$C_N\doteq 1+c''',\qquad \ell_N\doteq\frac{2C_N}{c'},\tag{2.17}$$

$$C_i\doteq c''\cdot\ell_{i+1}+\sum_{j=i+1}^{N}C_j,\qquad \ell_i\doteq\frac{1}{c'}\left(2C_i+c''\cdot\sum_{j=i+1}^{N}\ell_j\right)\qquad\text{if}\quad i<N.\tag{2.18}$$ □

LEMMA 2.3. *Assume that*

$$\textit{Tot.Var.}\{w\}<\delta\doteq\frac{\overline{\rho}}{2C_1},\tag{2.19}$$

and assume that there exists some interval $[t_1,\,t_2]\subset[0,\,T]$ *and some index* $i\in\{1,\dots,N\}$ *such that*

$$\text{meas}\big\{t\in[t_1,\,t_2]\ :\ \alpha^*(y(t))=\alpha_j\big\}\le\ell_j\cdot\textit{Tot.Var.}\{w\}\qquad\textit{for all}\ \ j>i\tag{2.20$_i$}$$

together with one of the following two conditions:

(a)$_i$

$$\varphi_{\alpha_i}(y(t))<2C_i\cdot\textit{Tot.Var.}\{w\}\qquad\textit{for all}\ \ t\in[t_1,\,t_2],\tag{2.21$_i$}$$

$$\text{meas}\big\{t\in[t_1,\,t_2]\ :\ \alpha^*(y(t))=\alpha_i\big\}>\ell_i\cdot\textit{Tot.Var.}\{w\}.\tag{2.22$_i$}$$

(b)$_i$ *There exists* $\tau\in[t_1,\,t_2]$ *such that*

$$\varphi_{\alpha_i}(y(\tau))\ge 2C_i\cdot\textit{Tot.Var.}\{w\}.\tag{2.23$_i$}$$

Then one has

$$\varphi_{\alpha_i}(y(t_2))\ge C_i\cdot\textit{Tot.Var.}\{w\}.\tag{2.24$_i$}$$

Proof of Lemma 2.3. Observe first that the recursive definition (2.17)–(2.18) of the constants C_i, ℓ_i and the bound (2.19) clearly imply

$$C_i\ge 1+c'''+c''\cdot\sum_{j=i+1}^{N}\ell_j,\tag{2.25}$$

$$2C_i\cdot\text{Tot.Var.}\{w\}<\overline{\rho}.\tag{2.26}$$

Assume now that (2.20_i)–(2.22_i) hold. Then, using (2.13)–(2.16) and recalling (2.25)–(2.26), we obtain

$$\begin{aligned}
\varphi^+_{\alpha_i}(y(t_2)) &\geq \varphi^+_{\alpha_i}(y(t_1)) + \int_{\{t\in[t_1,t_2]\ :\ \alpha^*(t)=\alpha_i\}} \big\langle \nabla\varphi_{\alpha_i}(y(t)),\ g_{\alpha_i}(y(t))\big\rangle\, dt \\
&\quad - \sum_{j=i+1}^{N} \int_{\{t\in[t_1,t_2]\ :\ \alpha^*(t)=\alpha_j\}} \Big|\big\langle \nabla\varphi_{\alpha_i}(y(t)),\ g_{\alpha_j}(y(t))\big\rangle\Big|\, dt - c''' \cdot \mathrm{Tot.Var.}\{w\} \\
&\geq \int_{\{t\in[t_1,t_2]\ :\ \alpha^*(t)=\alpha_i\}} -\big\langle \mathbf{n}_{\boldsymbol{\alpha}_\mathbf{i}}(\pi_{\alpha_i}(y(t))),\ g_{\alpha_i}(y(t))\big\rangle\, dt \\
&\quad - \Big(c'' \cdot \sum_{j=i+1}^{N} \ell_j + c'''\Big) \cdot \mathrm{Tot.Var.}\{w\} \\
&\geq \Big(\ell_i \cdot c' - c'' \cdot \sum_{j=i+1}^{N} \ell_j - c'''\Big) \cdot \mathrm{Tot.Var.}\{w\} \\
&\geq C_i \cdot \mathrm{Tot.Var.}\{w\},
\end{aligned} \tag{2.27}$$

proving (2.24_i). Next, assume that (2.20_i) and (2.23_i) hold, and let

$$\tau' \doteq \sup\big\{t \in [t_1,\, t_2]\ :\ \varphi_{\alpha_i}(y(t)) > 2C_i \cdot \mathrm{Tot.Var.}\{w\}\big\}. \tag{2.28}$$

Clearly, the bound (2.24_i) is satisfied if $\tau' = t_2$ since the map φ_{α_i} is left continuous. Next, consider the case $\tau' < t_2$. By computations similar to those in (2.27), using (2.13)–(2.16), and thanks to (2.20_i), (2.25)–(2.26), we get

$$\begin{aligned}
\varphi^+_{\alpha_i}(y(t_2)) &\geq \varphi^+_{\alpha_i}(y(\tau')) + \int_{\{t\in[\tau',t_2]\ :\ \alpha^*(t)=\alpha_i\}} \big\langle \nabla\varphi_{\alpha_i}(y(t)),\ g_{\alpha_i}(y(t))\big\rangle\, dt \\
&\quad - \sum_{j=i+1}^{N} \int_{\{t\in[\tau',t_2]\ :\ \alpha^*(t)=\alpha_j\}} \Big|\big\langle \nabla\varphi_{\alpha_i}(y(t)),\ g_{\alpha_j}(y(t))\big\rangle\Big|\, dt - c''' \cdot \mathrm{Tot.Var.}\{w\} \\
&\geq \left(2C_i - 1 - c''' - c'' \cdot \sum_{j=i+1}^{N} \ell_j\right) \cdot \mathrm{Tot.Var.}\{w\} \\
&\geq C_i \cdot \mathrm{Tot.Var.}\{w\},
\end{aligned} \tag{2.29}$$

thus concluding the proof of Lemma 2.3. □

Step 3. Assume that the bound (2.19) on the total variation of $w = w(t)$ holds. Set $\tau_1 = 0$, $\tau_{N+1} \doteq T$, and define recursively the points $\tau_N, \tau_{N-1}, \dots, \tau_2$ by setting, for every $1 < i \leq N$,

(2.30_i)

$$\mathcal{T}_i \doteq \Big\{t \in [0,\, \tau_{i+1}]\ :\ \varphi_{\alpha_i}(y(s)) \geq C_i \cdot \mathrm{Tot.Var.}\{w\} \qquad \text{for all}\ \ s \in [t,\, \tau_{i+1}]\Big\},$$

$$\tau_i \doteq \begin{cases} \inf \mathcal{T}_i & \text{if}\ \ \mathcal{T}_i \neq \emptyset, \\ \tau_{i+1} & \text{if}\ \ \mathcal{T}_i = \emptyset. \end{cases}$$

Applying Lemma 2.3 and proceeding by backward induction on $i = N, N-1, \dots, 2$, we show now that for any $t < \tau_i$, $i = 2, \dots, N$, one has

$$(2.31_i)\qquad \begin{gathered} \text{meas}\big\{s \in [0,\, t] \ :\ \alpha^*(y(s)) = \alpha_i\big\} \le \ell_i \cdot \text{Tot.Var.}\{w\}, \\ \varphi_{\alpha_i}(y(t)) < 2C_i \cdot \text{Tot.Var.}\{w\}. \end{gathered}$$

Indeed, if (2.31_i) is not satisfied, one of the two conditions $(\text{a})_\text{N}$ or $(\text{b})_\text{N}$ must be true on some interval $[0,\, \bar{t}\,]$, $\bar{t} < \tau_N$. But then, by $(2.24)_N$, we have

$$\varphi_{\alpha_N}(y(s)) \ge C_N \cdot \text{Tot.Var.}\{w\} \qquad \text{for all} \ \ s \in [\,\bar{t},\, T],$$

which contradicts the definition (2.30_i). On the other hand, if we assume that $(2.31)_j$ holds for $j = i+1, \dots, N$ but not for $j = i$, then one of the two conditions $(\text{a})_i$ or $(\text{b})_i$ must be true on some interval $[0,\, \bar{t}\,]$, $\bar{t} < \tau_i$. Moreover, the inductive assumptions $(2.31)_j$, $j > i$, imply (2.20_i) and hence, as above, thanks to (2.24_i), we get

$$\varphi_{\alpha_i}(y(s)) \ge C_i \cdot \text{Tot.Var.}\{w\} \qquad \text{for all} \ \ s \in [\bar{t},\, T],$$

reaching a contradiction with the definition (2.30_i).

To conclude the proof of property (i) stated in Proposition 2.2, observe that, thanks to (2.31_i), $i = 2, \dots, N$, we have

$$(2.32)\qquad \text{meas}\big\{s \in [\tau_i,\, \tau_{i+1}] \ :\ \alpha^*(y(s)) > \alpha_i\big\} \le \left(\sum_{j>i} \ell_j\right) \cdot \text{Tot.Var.}\{w\} \qquad \text{for all} \ \ i \ge 1.$$

Therefore, recalling the definitions of the map φ_{α_i} at (2.10), taking δ as in (2.19), and

$$(2.33)\qquad C > (N+1) \cdot \sum_{j=1}^{N} \ell_j,$$

from (2.31_i) and (2.32) we deduce that the partition $\tau_1 = 0 \le \tau_2 \le \cdots \le \tau_{N+1} = T$ of $[0,\, T]$, defined at (2.30_i), satisfies the properties (2.5)–(2.7).

(ii) Concerning (ii), let $C, \delta > 0$ be the constants defined according to (i) and, given a BV function $w = w(t)$ with $\text{Tot.Var.}\{w\} < \delta$, and a solution $y : [0, T] \mapsto \Omega$ of the Cauchy problem (1.7)–(1.8) with $y_0 \in K$, consider the partition $0 = \tau_1 \le \tau_2 \le \cdots \le \tau_{N+1} = T$, of $[0,\, T]$, with the properties in (i). Setting

$$\tau_i' \doteq \inf\Big\{t \in [\tau_i,\, \tau_{i+1}] \ :\ \alpha^*(y(t)) = \alpha_i\Big\}, \qquad i = 1, \dots, N,$$

define the map

$$(2.34)\qquad \tau(t) \doteq \begin{cases} \tau_i' & \text{if} \quad t \in \,]\tau_i,\, \tau_i'], \\ \sup\big\{s \in [\tau',\, t] \ :\ \alpha^*(y(s)) = \alpha_i\big\} & \text{if} \quad t \in \,]\tau_i',\, \tau_{i+1}] \end{cases}$$

over any interval $]\tau_i,\, \tau_{i+1}]$, $i = 1, \dots, N$. Notice that in the particular case where $\alpha^*(y(t)) > \alpha_i$ for all $t \in \,]\tau_i,\, \tau_{i+1}]$, by the above definitions one has $\tau(t) = \tau_i' = \tau_{i+1}$ for any $t \in \,]\tau_i,\, \tau_{i+1}]$. Then let $y^\diamondsuit : [0, T] \mapsto \Omega$ be the map recursively defined by setting

$$(2.35)\qquad y^\diamondsuit(t) \doteq y(t) \qquad \text{for all} \ \ t \in \,]\tau_N,\, T],$$

(2.36)
$$y^\diamondsuit(t) \doteq \begin{cases} y^\diamondsuit(\tau_{i+1}+) & \text{if} \quad \tau_i' = \tau_{i+1}, \\ y(\tau(t)+) & \text{if} \quad \tau_i' < \tau_{i+1}, \quad \alpha^*(y(\tau(t))) > \alpha_i, \\ y(\tau(t)) & \text{if} \quad \tau_i' < \tau_{i+1}, \quad \alpha^*(y(\tau(t))) = \alpha_i, \end{cases} \qquad \text{for all} \ \ t \in \,]\tau_i,\, \tau_{i+1}], \quad i < N,$$

$$y^\diamondsuit(0) \doteq y^\diamondsuit(0^+), \tag{2.37}$$

and let $w^\diamondsuit = w^\diamondsuit(t)$ be the function defined as

$$w^\diamondsuit(t) \doteq y^\diamondsuit(t) - \int_0^t g\big(y^\diamondsuit(s)\big)\, ds \qquad \text{for all} \ \ t \in [0,\, T]. \tag{2.38}$$

Clearly $y^\diamondsuit, w^\diamondsuit$ are both BV functions as well as y, w. Moreover, $y^\diamondsuit$ is a solution of the perturbed equation (2.8). By construction, for every $1 \leq i \leq N$ there holds

(2.39)
$$\alpha^*(y^\diamondsuit(t)) = \begin{cases} \alpha_i & \text{if} \quad \tau_i' < \tau_{i+1}, \\ \alpha^*(y^\diamondsuit(\tau_{i+1}{}^+)) & \text{if} \quad \tau_i' = \tau_{i+1}, \end{cases} \qquad \text{for all} \ \ t \in \,]\tau_i,\, \tau_{i+1}].$$

Hence the map $t \mapsto \alpha^*(y^\diamondsuit(t))$ is nonincreasing and left continuous. Next, recalling (2.6) and observing that

$$\alpha^*(y(t)) = \alpha_i \qquad \Longrightarrow \qquad \begin{matrix} \tau(t) = t, \\ y^\diamondsuit(t) = y(t), \end{matrix} \qquad \text{for all} \ \ t \in \,]\tau_i,\, \tau_{i+1}] \tag{2.40}$$

and defining

$$\mathcal{I} \doteq \bigcup_i \big\{ t \in \,]\tau_i,\, \tau_{i+1}] \ : \ \alpha^*(y(t)) > \alpha_i \big\}, \tag{2.41}$$

we have

$$y(t) = y^\diamondsuit(t) \qquad \text{for all} \ \ t \in (0,\, T) \setminus \mathcal{I}. \tag{2.42}$$

On the other hand, by the above definitions, calling $M \doteq \sup_{y \in \Omega} |g(y)|$, we derive

$$\big|\tau(t) - t\big| \leq \text{meas}(\mathcal{I}) \qquad \text{for all} \ \ t \in \mathcal{I}, \tag{2.43}$$

$$\begin{aligned} \big|y^\diamondsuit(t) - y(t)\big| &\leq \int_{\tau(t)}^t \big|g(y(s))\big|\, ds + \text{Tot.Var.}\big\{w\,;\, [0, t]\big\} \\ &\leq M \cdot \text{meas}(\mathcal{I}) + \text{Tot.Var.}\{w\} \qquad \text{for all} \ \ t \in \mathcal{I}, \end{aligned} \tag{2.44}$$

and

$$\begin{aligned} \Big|\text{Tot.Var.}\{y^\diamondsuit\} - \text{Tot.Var.}\{y\}\Big| &\leq \text{Tot.Var.}\big\{y\,;\, \mathcal{I}\big\} \\ &\leq M \cdot \text{meas}(\mathcal{I}) + \text{Tot.Var.}\{w\}. \end{aligned} \tag{2.45}$$

Then, using (2.44)–(2.45), we obtain

$$\begin{aligned}\Big|\text{Tot.Var.}\{w^{\diamondsuit}\} - \text{Tot.Var.}\{w\}\Big| &\le \int_{\mathcal{I}} \Big| \big|g\big(y^{\diamondsuit}(s)\big)\big| - \big|g\big(y(s)\big)\big| \Big| \, ds \\ &\quad + \Big|\text{Tot.Var.}\{y^{\diamondsuit}\} - \text{Tot.Var.}\{y\}\Big| \\ &\le M' \cdot \Big\{\text{meas}(\mathcal{I}) + \text{Tot.Var.}\{w\}\Big\} \end{aligned} \tag{2.46}$$

for some constant $M' > 0$, depending only on the field g. Hence, from (2.42), (2.44), (2.46), and applying (2.7), it follows that $y^{\diamondsuit}(\cdot)$ satisfies the estimates in (2.9) for some constant $C' > 0$, which concludes the proof of (ii).

We can now take δ as in (2.19) and choose $C > C'$ according to (2.33). Both properties (i) and (ii) are then satisfied, completing the proof of Proposition 2.2. □

Proof of Theorem 1.3. For a given sequence of solutions $y_\nu : [0,T] \mapsto \Omega$ of the perturbed system (1.11) with Tot.Var.$\{w_\nu\} \le \delta_\nu$, $\delta_\nu \to 0$ as $\nu \to \infty$, assume that the $y_\nu(\cdot)$ converge to a function $y : [0,T] \mapsto \Omega$ uniformly on $[0,T]$ and that $y_\nu(0)$ belongs to some compact set $K \subset \Omega$ for every ν. Thanks to property (ii) of Proposition 2.2, in connection with any pair $w_\nu(\cdot)$, $y_\nu(\cdot)$, there will be a BV function $w_\nu^{\diamondsuit}(\cdot)$ and a solution $y_\nu^{\diamondsuit}(\cdot)$ of (2.8) that satisfy

$$\text{Tot.Var.}\{w_\nu^{\diamondsuit}\} \le C' \cdot \delta_\nu, \qquad \big\|y_\nu^{\diamondsuit} - y_\nu\big\|_{\mathbf{L}^\infty([0,T])} \le C' \cdot \delta_\nu \tag{2.47}$$

for some constant $C' > 0$ independent of ν. Moreover there exists a partition $0 = \tau_{1,\nu} \le \tau_{2,\nu} \le \cdots \le \tau_{N+1,\nu} = T$ of $[0,\, T]$ such that

$$\alpha^*(y_\nu^{\diamondsuit}(t)) = \alpha_i \qquad \text{for all } \ t \in \,]\tau_{i,\nu},\, \tau_{i+1,\nu}], \qquad i = 1, \ldots, N. \tag{2.48}$$

Recalling (1.4) and (1.9), because of (2.48) we have

$$\begin{aligned} y_\nu^{\diamondsuit}(t) = y_\nu^{\diamondsuit}(0) \ + \ \sum_{\ell=1}^{i-1} \int_{\tau_{\ell,\nu}}^{\tau_{\ell+1,\nu}} g_{\alpha_\ell}(y_\nu^{\diamondsuit}(s)) \, ds \ + \int_{\tau_{i,\nu}}^{t} g_{\alpha_i}(y_\nu^{\diamondsuit}(s)) \, ds \ + \ [w_\nu^{\diamondsuit}(t) - w_\nu^{\diamondsuit}(0)] \\ \text{for all } \ t \in [\tau_{i,\nu},\, \tau_{i+1,\nu}], \qquad i = 1, \ldots, N. \end{aligned} \tag{2.49}$$

By possibly taking a subsequence, we can assume that every sequence $(\tau_{i,\nu})_{\nu \ge 1}$ converges to some limit point, say

$$\overline{\tau}_i \doteq \lim_{\nu \to \infty} \tau_{i,\nu}, \qquad i = 1, \ldots, N+1.$$

We now observe that

$$]\overline{\tau}_i,\, \overline{\tau}_{i+1}[\ \subseteq \bigcup_{\mu=1}^{\infty} \ \bigcap_{\nu=\mu}^{\infty}]\tau_{i,\nu},\, \tau_{i+1,\nu}\,] \qquad \text{for all } \ i.$$

Moreover, the second inequality in (2.47) and the uniform convergence $y_\nu(\cdot) \to y(\cdot)$ yield

$$\lim_{\nu \to \infty} \big\|y_\nu^{\diamondsuit} - y\big\|_{\mathbf{L}^\infty([0,T])} = 0. \tag{2.50}$$

From the first inequality in (2.47), and from (2.48)–(2.49), we now deduce

(2.51)
$$\begin{aligned} & y(t) \in \overline{\Omega}_{\alpha_i} \setminus \bigcup_{\beta > \alpha_i} \Omega_\beta, \\ & y(t) = y(0) + \sum_{\ell=1}^{i-1} \int_{\overline{\tau}_\ell}^{\overline{\tau}_{\ell+1}} g_{\alpha_\ell}(y(s))\, ds + \int_{\overline{\tau}_i}^{t} g_{\alpha_i}(y(s))\, ds \end{aligned} \qquad \text{for all} \ \ t \in \,]\overline{\tau}_i, \overline{\tau}_{i+1}], \ \text{for all} \ \ i.$$

In particular, on each interval $[\overline{\tau}_i, \overline{\tau}_{i+1}]$, the function $y(\cdot)$ is a classical solution of $\dot{y} = g_{\alpha_i}(y)$ and satisfies

$$\dot{y}(s^-) = g_{\alpha_i}(y(s)) \qquad \text{for all} \ \ s \in \,]\overline{\tau}_i, \overline{\tau}_{i+1}].$$

Moreover, observe that because of the inward-pointing condition (1.1), the set $\big\{t \in [\overline{\tau}_i, \overline{\tau}_{i+1}] \,:\, y(t) \in \partial\Omega_{\alpha_i}\big\}$ is nowhere dense in $[\overline{\tau}_i, \overline{\tau}_{i+1}]$. Thus, if s is any point in $]\overline{\tau}_i, \overline{\tau}_{i+1}]$ such that $y(s) \in \partial\Omega_{\alpha_i}$, there will be some increasing sequence $(s_n)_n \subset\,]\overline{\tau}_i, \overline{\tau}_{i+1}[$ converging to s and such that $y(s_n) \in \Omega_{\alpha_i}$ for any n. But this yields a contradiction with (1.1), because

$$0 \le \lim_{n\to\infty} \Big\langle \frac{y(s) - y(s_n)}{s - s_n},\ \mathbf{n}_{\boldsymbol{\alpha}_\mathbf{i}}\big(y(s)\big) \Big\rangle = \Big\langle \dot{y}(s-),\ \mathbf{n}\big(y(s)\big) \Big\rangle = \Big\langle g_{\alpha_i}\big(y(s)\big),\ \mathbf{n}_{\boldsymbol{\alpha}_\mathbf{i}}\big(y(s)\big) \Big\rangle.$$

Hence, recalling the definition (1.2), from (2.51) we conclude

$$y(t) \in \Omega_{\alpha_i} \setminus \bigcup_{\beta > \alpha_i} \Omega_\beta \qquad \text{for all} \ \ t \in \,]\overline{\tau}_i, \overline{\tau}_{i+1}], \qquad i = 1, \dots, N,$$

$$y(t) = y(0) + \int_0^t g\big(y(s)\big)\, ds \qquad \text{for all} \ \ t \in [0, T],$$

proving that $y : [0, T] \mapsto \Omega$ is a Carathéodory solution of (1.5) on $[0, T]$. □

Proof of Corollary 1.4. Assuming that statement is false, we shall reach a contradiction. Fix any closed subset $A \subset \Omega$, any compact set $K \subset A$, and assume that, for some $T, \varepsilon > 0$, there exists a sequence of solutions $y_\nu : [0,T] \mapsto A$ of the perturbed system (1.7), with $y_\nu(0) \in K$, Tot.Var.$\{w_\nu\} \le \delta_\nu$, $\delta_\nu \to 0$ as $\nu \to \infty$, such that the following property holds.

(P) Every solution $x : [0,T] \mapsto \Omega$ of the unperturbed equation (1.5) satisfies

$$\big\|x - y_\nu\big\|_{\mathbf{L}^\infty([0,T])} \ge \varepsilon \qquad \text{for all} \ \ \nu. \tag{2.52}$$

For each ν, call $y_\nu^\diamondsuit : [0,T] \mapsto \mathbb{R}^n$ the polygonal curve with vertices at the points $y_\nu(\ell\delta_\nu), \ell \ge 0$, defined by setting

$$\begin{aligned} & y_\nu^\diamondsuit(t) \doteq y_\nu\big(\ell\delta_\nu\big) + \frac{t - \ell\delta_\nu}{\delta_\nu} \cdot \Big(y_\nu\big((\ell+1)\delta_\nu\big) - y_\nu\big(\ell\delta_\nu\big)\Big) \\ & \text{for all} \ \ t \in [\ell\delta_\nu, (\ell+1)\delta_\nu] \cap [0,T], \qquad 0 \le \ell \le \lfloor T/\delta_\nu \rfloor, \end{aligned} \tag{2.53}$$

where $\lfloor T/\delta_\nu \rfloor$ denotes the integer part of T/δ_ν. Since every $y_\nu(\cdot)$ is a BV function that solves (1.7), it follows that there will be some constant $C > 0$, independent of ν, such that

$$\text{Tot.Var.}\{y_\nu \ ; \ J\} \le C \cdot \text{meas}(J) + \text{Tot.Var.}\{w_\nu \ ; \ J\} \tag{2.54}$$

for any interval $J \subset [0, T]$. Then, using (2.54), we derive for any fixed $0 \le \ell < \ell' \le \lfloor T/\delta_\nu \rfloor$ the bound

$$\begin{aligned}\Big|y_\nu^\diamondsuit(\ell'\delta_\nu) - y_\nu^\diamondsuit(\ell\delta_\nu)\Big| &= \Big|y_\nu(\ell'\delta_\nu) - y_\nu(\ell\delta_\nu)\Big| \\ &\le \text{Tot.Var.}\big\{y_\nu\ ;\ [\ell\delta_\nu,\ \ell'\delta_\nu]\big\} \\ &\le (1+C)\cdot(\ell'-\ell)\delta_\nu. \end{aligned} \tag{2.55}$$

Therefore $y_\nu^\diamondsuit(\cdot)$ is a uniformly bounded sequence of Lipschitz maps, having Lipschitz constant $\text{Lip}(y_\nu^\diamondsuit) \le (1+C)$. Hence, applying the Ascoli–Arzelà theorem, we can find a subsequence, which we still denote $y_\nu^\diamondsuit(\cdot)$, that converges to some function $y : [0, T] \mapsto \mathbb{R}^n$, uniformly on $[0, T]$. On the other hand, by construction, and thanks to (2.54), for any fixed $0 \le t \le T$, with $\ell\delta_\nu \le t < (\ell+1)\delta_\nu$, the following holds:

$$\begin{aligned}\Big|y_\nu(t) - y_\nu^\diamondsuit(t)\Big| &\le \Big|y_\nu(t) - y_\nu\big(\ell\delta_\nu\big)\Big| + \Big|y_\nu\big(\ell\delta_\nu\big) - y_\nu^\diamondsuit(t)\Big| \\ &\le \Big|y_\nu(t) - y_\nu(\ell\delta_\nu)\Big| + \Big|y_\nu\big((\ell+1)\delta_\nu\big) - y_\nu\big(\ell\delta_\nu\big)\Big| \\ &\le 2\cdot\text{Tot.Var.}\big\{y_\nu\ ;\ [\ell\delta_\nu,\ (\ell+1)\delta_\nu]\big\} \\ &\le 2(1+C)\cdot\delta_\nu. \end{aligned} \tag{2.56}$$

Thus, since $\delta_\nu \to 0$ as $\nu \to \infty$, the uniform convergence of $y_\nu^\diamondsuit(\cdot)$ to $y(\cdot)$ implies

$$\lim_{\nu\to\infty} \big\|y_\nu - y\big\|_{\mathbf{L}^\infty([0,T])} = 0. \tag{2.57}$$

By assumption, $\text{Range}(y_\nu) \subset A \subset \Omega$ for every ν, and hence from (2.57) we deduce also that the limit function $y(\cdot)$ takes values inside Ω. We can thus apply Theorem 1.3 to the sequence $y_\nu(\cdot)$ and conclude that the function $y : [0, T] \mapsto \Omega$ is a Carathéodory solution of the unperturbed equation (1.5) with

$$\big\|y - y_\nu\big\|_{\mathbf{L}^\infty([0,T])} < \varepsilon$$

for all ν sufficiently large. We thus obtain a contradiction with (2.52), concluding the proof. □

3. Robustness of patchy feedbacks. In this section we apply the previous results on patchy vector fields with impulsive perturbations and construct (discontinuous) stabilizing feedback controls that enjoy robustness properties in the presence of measurement errors and external disturbances. Consider the nonlinear control system on $\mathbb{R}^n$

$$\dot{x} = f(x, u), \qquad u(t) \in \mathcal{K}, \tag{3.1}$$

assuming that the control set $\mathcal{K} \subset \mathbb{R}^m$ is compact and that the map $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is smooth. We seek a feedback control $u = U(x) \in \mathcal{K}$ that stabilizes the trajectories of the closed-loop system

$$\dot{x} = f\big(x,\ U(x)\big) \tag{3.2}$$

at the origin. It is well known that even if every initial state $\overline{x} \in \mathbb{R}^n$ can be steered to the origin by an open-loop control $u = u^{\overline{x}}(t)$, a topological obstruction can prevent the existence of a continuous feedback control $u = U(x)$ which (locally) stabilizes the system (3.1). This fact was first pointed out by Sussmann [Su] for a two-dimensional system ($n = 2$, $\mathcal{K} = \mathbb{R}^2$) and by Sontag and Sussmann [SS] for one-dimensional systems ($n = 1$, $\mathcal{K} = \mathbb{R}$). For general nonlinear systems, it was further analyzed by Brockett [Bro] and Coron [Cor1], [Cor2]. It is thus natural to look for a stabilizing control within a class of discontinuous functions. However, this leads to a theoretical difficulty, because when the function U is discontinuous, the differential equation (3.2) may not admit any Carathéodory solution. To cope with this problem, two different approaches have been pursued.

1. An algorithm is defined which constructs approximate trajectories in connection with an arbitrary (discontinuous) feedback control function. For example, one can sample the feedback control at a discrete set of times. The resulting trajectory, called a sampling solution, was first studied by Krasovskii and Subbotin in the context of positional differential games (see [KS]). In this case, one is not concerned with the existence of exact solutions but only in the asymptotic stabilization properties of all approximate solutions.

2. Alternatively, by the asymptotic controllability to the origin of system (3.1) by means of open-loop controls, one proves the existence of a stabilizing feedback $u = U(x)$ having only a particular type of discontinuities. This feedback thus generates a *patchy vector field*, and the corresponding system (3.2) always admits Carathéodory solutions.

The first approach was initiated in [CLSS] and further developed in [Ri1], [Ri2], [Ri3], [CLRS], [So2]. The second was introduced in [A-B], defining the following class of piecewise constant feedback controls:

Definition 3.1. *Let $\big(\Omega,\ g,\ (\Omega_\alpha,\, g_\alpha)_{\alpha\in\mathcal{A}}\big)$ be a patchy vector field. Assume that there exist control values $k_\alpha \in \mathcal{K}$ such that for each $\alpha \in \mathcal{A}$, there holds*

$$g_\alpha(x) \doteq f(x,\, k_\alpha) \qquad \textit{for all } x \in D_\alpha \doteq \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta. \tag{3.3}$$

Then the piecewise constant map

$$U(x) \doteq k_\alpha \qquad \textit{if} \qquad x \in D_\alpha \tag{3.4}$$

is called a patchy feedback control on Ω and referred to as $\big(\Omega,\ U,\ (\Omega_\alpha,\, k_\alpha)_{\alpha\in\mathcal{A}}\big)$.

Remark 3.1. From Definitions 1.2 and 3.1, it is clear that the field

$$g(x) = f\big(x,\, U(x)\big)$$

defined in connection with a given patchy feedback $\big(\Omega,\ U,\ (\Omega_\alpha,\, k_\alpha)_{\alpha\in\mathcal{A}}\big)$ is precisely the patchy vector field $\big(\Omega,\ g,\ (\Omega_\alpha,\, g_\alpha)_{\alpha\in\mathcal{A}}\big)$ associated with a family of fields $\big\{g_\alpha : \alpha \in \mathcal{A}\big\}$ satisfying (1.1) Clearly, the patches $(\Omega_\alpha,\, g_\alpha)$ are not uniquely determined by the patchy feedback U. Indeed, whenever $\alpha < \beta$, by (3.3) the values of g_α on the set $\Omega_\alpha \setminus \Omega_\beta$ are irrelevant. Moreover, recalling the notation (1.3) we have

$$U(x) = k_{\alpha^*(x)} \qquad \text{for all } \ x \in \Omega. \tag{3.5}$$

Here, we address the issue of robustness of a stabilizing feedback law $u = U(x)$ w.r.t. small internal and external perturbations

$$\dot{x} = f\big(x,\ U(x + \zeta(t))\big) + d(t), \tag{3.6}$$

where $\zeta = \zeta(t)$ represents a state measurement error and $d = d(t)$ represents an external disturbance of the system dynamics (3.2). Since we are dealing with a discontinuous ODE, one cannot expect the full robustness of the feedback $U(x)$ w.r.t. measurement errors because of possible chattering behavior that may arise at discontinuity points (see [He1], [Ry], [So1], [L-S1], [L-S2], [CR]). Therefore, we shall consider state measurement errors which are small in BV norm, avoiding such phenomena.

Before stating our main result in this direction, we recall here some basic definitions and Proposition 4.2 in [A-B]. This provides the semiglobal practical stabilization (steering all states from a given compact set of initial data into a prescribed neighborhood of zero) of an asymptotically controllable system by means of a patchy feedback control which is robust w.r.t. external disturbances. We consider as (open-loop) *admissible controls* all the measurable functions $u : [0, \infty) \to \mathbb{R}^m$ such that $u(t) \in \mathcal{K}$ for a.e. $t \geq 0$.

DEFINITION 3.2. *The system* (3.1) *is globally asymptotically controllable to the origin if the following holds.*

1. Attractiveness. *For each $\overline{x} \in \mathbb{R}^n$ there exists some admissible (open-loop) control $u = u^{\overline{x}}(t)$ such that the corresponding trajectory of*

$$\dot{x}(t) = f\big(x(t),\, u^{\overline{x}}(t)\big), \qquad x(0) = \overline{x}, \tag{3.7}$$

either reaches the origin in finite time or tends to the origin as $t \to \infty$.

2. Lyapunov stability. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. For every $\overline{x} \in \mathbb{R}^n$ with $|\overline{x}| < \delta$, there is an admissible control $u^{\overline{x}}$ as in* 1 *steering the system from $\overline{x}$ to the origin, so that the corresponding trajectory of* (3.7) *satisfies $|x(t)| < \varepsilon$ for all $t \geq 0$.*

PROPOSITION 3.3 (see [A-B, Proposition 4.1]). *Let system* (3.1) *be globally asymptotically controllable to the origin. Then, for every $0 < r < s$, one can find $T > 0$, $\chi > 0$, and a patchy feedback control $U : D \mapsto \mathcal{K}$ defined on some domain*

$$D \supset \big\{x \in \mathbb{R}^n \ ; \ r \leq |x| \leq s\big\} \tag{3.8}$$

so that the following holds. For any measurable map $d : [0, T] \mapsto \mathbb{R}^n$ such that

$$\big\|d\big\|_{\mathbf{L}^\infty([0,\,T])} \leq \chi,$$

and for any initial state x_0 with $r \leq |x_0| \leq s$, the perturbed system

$$\dot{x} = f\big(x,\, U(x)\big) + d(t) \tag{3.9}$$

admits a (forward) Carathéodory trajectory starting from x_0. Moreover, for any such trajectory $t \mapsto \gamma(t)$, $t \geq 0$, one has

$$\gamma(t) \in D \qquad \textit{for all} \ \ t \geq 0, \tag{3.10}$$

and there exists $\overline{t}_\gamma < T$ such that

$$\big|\gamma(\overline{t}_\gamma)\big| < r. \tag{3.11}$$

Relying on Corollary 1.4 of Theorem 1.3 and on Proposition 3.3, we obtain here the following result concerning robustness of a stabilizing feedback w.r.t. both internal and external perturbations.

THEOREM 3.4. *Let system* (3.1) *be globally asymptotically controllable to the origin. Then, for every $0 < r < s$, one can find $T' > 0$, $\chi' > 0$, and a patchy* *feedback control* $U' : D' \mapsto \mathcal{K}$ *defined on some domain* D' *satisfying* (3.8) *so that the following holds. Given any pair of maps* $\zeta \in BV([0, T'])$, $d \in \mathbf{L}^\infty([0, T'])$ *such that*

$$\text{(3.12)} \qquad \textit{Tot.Var.}\{\zeta\} \le \chi', \qquad \big\|d\big\|_{\mathbf{L}^\infty([0,T'])} \le \chi',$$

and any initial state x_0 *with* $r \le |x_0| \le s$, *for every solution* $t \mapsto x(t)$, $t \ge 0$, *of the perturbed system* (3.6) *starting from* x_0, *one has*

$$\text{(3.13)} \qquad x(t) \in D' \qquad \textit{for all } t \in [0, T'],$$

and there exists $\bar{t}_x < T'$ *such that*

$$\text{(3.14)} \qquad \big|x(\bar{t}_x)\big| < r.$$

Proof. 1. Fix $0 < r < s$. Then, according to Proposition 3.3, we can find $T' > 0$ and a patchy feedback control $U' : D' \mapsto \mathcal{K}$ defined on some domain

$$\text{(3.15)} \qquad D' \supset \big\{x \in \mathbb{R}^n \ ;\ r/3 \le |x| \le s\big\}$$

so that the following holds. For every Carathéodory solution $t \mapsto x(t)$, $t \ge 0$, of the unperturbed system (3.2) (with $U = U'$) starting from a point x_0 in the compact set

$$\text{(3.16)} \qquad K \doteq \big\{x \in \mathbb{R}^n \ ;\ r \le |x| \le s\big\},$$

one has

$$\text{(3.17)} \qquad x(t) \in D_\rho \doteq \big\{x \in D' \ :\ d\big(x, \partial D'\big) > \rho\big\} \qquad \text{for all } t \ge 0$$

for some constant $\rho > 0$. Moreover, there exists $\bar{t}_x < T'$ such that

$$\text{(3.18)} \qquad \big|x(\bar{t}_x)\big| < \frac{r}{3}.$$

According with Definition 3.1, the field

$$\text{(3.19)} \qquad g(x) \doteq f\big(x, U'(x)\big)$$

is a patchy vector field associated to the family of fields $\big\{g_\alpha : \alpha \in \mathcal{A}\big\}$ defined as in (3.3). The smoothness of f guarantees that for BV perturbations $w = w(t)$ having some uniform bound Tot.Var.$\{w\} \le \widehat{\chi}$ on the total variation, every (left continuous) solution $y : [0, T'] \mapsto \mathbb{R}^2$ of the impulsive equation (1.7), starting at a point $x_0 \in K$, takes values in the closed set

$$\text{(3.20)} \qquad A \doteq B(D_\rho, \rho/2).$$

Therefore, thanks to Corollary 1.4 of Theorem 1.3, there exists some constant

$$\text{(3.21)} \qquad 0 < \widehat{\chi}' = \widehat{\chi}'(A, K, T', r/3) < \widehat{\chi}$$

such that the following holds. If $y : [0, T'] \mapsto \mathbb{R}^2$ is a (left continuous) solution of the impulsive equation (1.7), with $y(0) \in K$ and Tot.Var.$(w) < \widehat{\chi}'$, then one has

$$\text{(3.22)} \qquad y(t) \in A \qquad \text{for all } t \in [0, T'],$$

and there exists $\bar{t}_y < T'$ such that

$$\big|y(\bar{t}_y)\big| < \frac{2r}{3}. \tag{3.23}$$

2. In connection with the patchy feedback U' introduced above, define the map

$$h(y,z) \doteq f\big(y-z,\, U'(y)\big) - f\big(y,\, U'(y)\big) \tag{3.24}$$

and observe that, by the smoothness of f, there will be some constant $\bar{c} > 0$ such that

$$\big|h(y,z)\big| \le \bar{c}\cdot|z| \qquad \text{for all } \; y \in A, \quad |z| \le \widehat{\chi}'. \tag{3.25}$$

Consider now a pair of maps $\zeta \in BV([0,\,T'])$, $d \in \mathbf{L}^\infty([0,\,T'])$ satisfying (3.12) with

$$\chi' < \min\left\{\frac{\widehat{\chi}'}{2(1+T'\bar{c}')},\ \frac{r}{3},\ \frac{\rho}{2}\right\}, \tag{3.26}$$

and let $x = x(t)$ be any Carathéodory solution of the perturbed system (3.6), with an initial condition $x(0) = x_0 \in K$. Then, as observed in the introduction, the map

$$t \mapsto y(t) \doteq x(t) + \zeta(t) \tag{3.27}$$

satisfies the impulsive equation (1.7), where

$$w(t) \doteq \zeta(t) + \int_0^t \Big(h\big(y(s), \zeta(s)\big) + d(s)\Big)\, ds. \tag{3.28}$$

But then, since (3.12), (3.25), (3.26) together imply

$$\begin{aligned}\text{Tot.Var.}\big\{w\ ;\ [0,\,T']\big\} &\le \text{Tot.Var.}\big\{\zeta\ ;\ [0,\,T']\big\} + T'\bar{c}\cdot\big\|\zeta\big\|_{\mathbf{L}^\infty([0,\,T'])} + T'\cdot\big\|d\big\|_{\mathbf{L}^\infty([0,\,T'])}\\ &\le \big(1+T'\bar{c}\big)\cdot \text{Tot.Var.}\big\{\zeta\ ;\ [0,\,T']\big\} + \big\|d\big\|_{\mathbf{L}^\infty([0,\,T'])}\\ &< \widehat{\chi}',\end{aligned}$$

from (3.22)–(3.23) and (3.12), (3.20), (3.26), (3.27) it follows that

$$x(t) \in B(A,\, \rho/2) \subset D' \qquad \text{for all } \; t \in [0,\,T'], \tag{3.29}$$

$$\big|x(\bar{t}_y)\big| < r \qquad \text{for some} \qquad \bar{t}_y < T',$$

which completes the proof of the theorem, taking χ' as in (3.26). □

Remark 3.2. For discontinuous stabilizing feedbacks constructed in terms of sampling solutions, an alternative concept of robustness was introduced in [CLRS], [So1], [So2]. In this case, one considers a partition of the time interval and applies a constant control between two consecutive sampling times. To preserve stability, the measurement error should be sufficiently small compared to the maximum step size. Moreover, each step size should be big enough to prevent possible chattering phenomena. The next result shows that the feedback provided by [A-B, Proposition 4.2] also enjoys this type of robustness. Before stating this result we describe now the concept of a sampling trajectory associated to the perturbed system (3.6) that was introduced in [CLRS], [So2].

Let an initial condition x_0 and a partition $\pi = \{0 = \tau_0 < \tau_1 < \cdots < \tau_{m+1} = T\}$ of the interval $[0, T]$ be given. A sampling trajectory x_π of the perturbed system (3.6), corresponding to a set of measurement errors $\{e_i\}_{i=0}^m$ and an external disturbance $d \in \mathbf{L}^\infty([0, T])$, is defined in a step-by-step fashion as follows. Between τ_0 and τ_1, let $x_\pi(\cdot)$ be a Carathéodory solution of

$$\dot{x} = f\big(x,\ U(x_0 + e_0)\big) + d(t), \qquad t \in [\tau_0, \tau_1], \tag{3.30}$$

with initial condition $x_\pi(0) = x_0$. Then, $x_\pi(\cdot)$ is recursively obtained by solving the system

$$\dot{x} = f\big(x,\ U(x_\pi(\tau_i) + e_i)\big) + d(t), \qquad t \in [\tau_i, \tau_{i+1}], \quad i > 0. \tag{3.31}$$

The sequence $\{x_\pi(\tau_i) + e_i\}_{i=0}^m$ corresponds to the nonexact measurements used to select control values.

Theorem 3.5. *Let system* (3.1) *be globally asymptotically controllable to the origin. Then, for every* $0 < r < s$, *one can find* $T'' > 0$, $\chi'' > 0$, $\overline{\delta} > 0$, $\overline{k} > 0$, *and a patchy feedback control* $U'' : D'' \mapsto \mathcal{K}$ *defined on some domain* D'' *satisfying* (3.8) *so that the following holds. Given an initial state* x_0 *with* $r \le |x_0| \le s$, *a partition* $\pi = \{\tau_0 = 0, \tau_1, \dots, \tau_{m+1} = T''\}$ *of the interval* $[0, T'']$ *having the property*

$$\frac{\delta}{2} \le \tau_{i+1} - \tau_i \le \delta \qquad \textit{for all } i \qquad \textit{for some} \qquad \delta \in\ ]0, \overline{\delta}\,], \tag{3.32}$$

a set of measurement errors $\{e_i\}_{i=0}^m$, *and an external disturbance* $d \in \mathbf{L}^\infty([0, T''])$ *that satisfy*

$$\max_i |e_i| \le \overline{k} \cdot \delta, \tag{3.33}$$

$$\big\|d\big\|_{\mathbf{L}^\infty} \le \chi'', \tag{3.34}$$

the resulting sampling solution $x_\pi(\cdot)$ *starting from* x_0 *has the property*

$$x_\pi(t) \in D'' \qquad \textit{for all } t \in [0, T'']. \tag{3.35}$$

Moreover, there exists $\overline{t}_{x_\pi} < T''$ *such that*

$$\big|x_\pi(\,\overline{t}_{x_\pi})\big| < r. \tag{3.36}$$

Proof. 1. Fix $0 < r < s$. Then, according with Proposition 3.4, we can find $T' > 0, \chi' > 0$, and a patchy feedback control $U'' : D'' \longmapsto \mathcal{K}$ defined on a domain

$$D'' \supset \big\{x \in \mathbb{R}^n\ ;\ r/3 \le |x| \le 2s\big\}$$

so that the following holds. For every external disturbance $d \in \mathbf{L}^\infty$ satisfying (3.34) with $\chi'' \le \chi'$, and for any Carathéodory solution $t \mapsto x(t)$, $t \ge 0$, of the perturbed system (3.9) (with $U = U''$), starting from a point x_0 with $r \le |x_0| \le s$, one has

$$x(t) \in D_{\rho_1} \doteq \big\{x \in D''\ :\ d\big(x, \partial D''\big) > \rho_1\big\} \qquad \text{for all } t \ge 0 \tag{3.37}$$

for some constant $\rho_1 > 0$. Moreover, there exists $\overline{t}_x < T'$ such that

$$\big|x(\,\overline{t}_x)\big| < \frac{r}{3}. \tag{3.38}$$

Let

$$\big\{(\Omega_\alpha,\ g_\alpha)\ :\ \alpha = 1,\dots,N\big\},\qquad g_\alpha(x) = f(x,k_\alpha),\qquad k_\alpha \in \mathcal{K}, \tag{3.39}$$

be the collection of patches associated with the patchy vector field

$$g(x) = f\big(x,\, U''(x)\big). \tag{3.40}$$

We may assume that every vector field g_α is defined on a neighborhood $B(\Omega_\alpha,\, \rho_2)$, $0 < \rho_2 \leq \rho_1$, of the domain Ω_α so that, setting

$$\Omega_\alpha^\rho \doteq \big\{x \in \Omega_\alpha\ ;\ d\big(x,\ \partial\Omega_\alpha\big) > \rho\big\}, \tag{3.41}$$

one has

$$\Omega_\alpha^{\rho_2} \neq \emptyset,$$

and that every g_α is uniformly nonzero on the domain D_α defined in (3.3). Moreover, thanks to the inward-pointing condition (1.1), we may choose the constants $0 < \rho_2 < r/3$ and $\chi'' \leq \chi'$ so that the following hold:

$$\big|g_\alpha(x)\big| \geq 2\chi'' \qquad \text{for all}\ \ x \in B(D_\alpha,\, \rho_2) \tag{3.42}$$

and

$$\big\langle g_\alpha(x) + v,\ \mathbf{n}(x)\big\rangle < 0 \qquad \text{for all}\ \ x \in B(\partial\Omega_\alpha,\, \rho_2),\qquad |v| \leq \chi''. \tag{3.43}$$

For every $d \in \mathbf{L}^\infty$, we denote by $t \mapsto x^\alpha\big(t;\, t_0,\, x_0,\ d\big)$ the solution of the Cauchy problem

$$\dot{x} = g_\alpha(x) + d(t),\qquad x(t_0) = x_0, \tag{3.44}$$

and let $[t_0,\, t^{\max}]$ be the domain of definition of the maximal (forward) solution of (3.44) that is contained in $B(D_\alpha,\, \rho_2)$.

Observe that since every Carathéodory solution of the perturbed system (3.9) (with $U = U''$), starting from a point $x_0 \in B(0,s)\backslash \circ \to B(0,\, r)$, reaches the interior of the ball $B(0,\, r/3)$ in finite time, and because of (3.42), for any $\alpha = 1,\dots,N$ one can find $T_\alpha > 0$ with the following property.

(P)₁ For every $x_0 \in B(D_\alpha,\, \rho/2)$, $0 < \rho < \rho_2$, and for any $d \in \mathbf{L}^\infty$ satisfying (3.34), there exists some time $t_\rho \doteq t_\rho(x_0,\, d) < T_\alpha$ such that either one has

$$\big|x^\alpha\big(t_0 + t_\rho;\, t_0,\, x_0,\, d\big)\big| < \frac{2r}{3} \tag{3.45}$$

or else the following holds:

$$x^\alpha\big(t;\, t_0,\, x_0,\, d\big) \in B(D_\alpha,\, \rho_2) \setminus B(D_\alpha,\, \rho) \qquad \text{for all}\ \ t \in [t_0 + t_\rho,\ t^{\max}]. \tag{3.46}$$

On the other hand, relying on the inward-pointing condition (3.43), we deduce two further properties of the solutions of (3.44).

(P)₂ The sets Ω_α^ρ, $0 < \rho \leq \rho_2$, defined in (3.41) are positive invariant regions for trajectories of (3.44), i.e., for every $x_0 \in \Omega_\alpha^\rho$, and for any $d \in \mathbf{L}^\infty$ satisfying (3.34), one has

$$x^\alpha\big(t;\, t_0,\, x_0,\, d\big) \in \Omega_\alpha^\rho \qquad \text{for all}\ \ t \geq t_0. \tag{3.47}$$

(P)$_3$ There exists some constant $\overline{c} > 0$ so that, for every $x_0 \in B(\Omega_\alpha, \rho)$, $0 < \rho \leq \rho_2$, such that $d\big(x_0,\ \partial\Omega_\alpha\big) \leq \rho$, and for any $d \in \mathbf{L}^\infty$ satisfying (3.34), one has

$$x^\alpha\big(t;\, t_0,\, x_0,\, d\big) \in \Omega_\alpha^{2\rho} \qquad \text{for all} \quad t \geq t_0 + \overline{c} \cdot \rho\,. \tag{3.48}$$

2. Consider an initial state $x_0 \in B(0, s) \backslash \circ \to B\big(0,\, r\big)$ and a partition $\pi = \{\tau_i\}_{i \geq 0}$ of $[0,\, \infty[$ having the property (3.32), with

$$0 < \delta \leq \overline{\delta} \doteq \min\Big\{\overline{c} \cdot \rho_2,\ \frac{\rho_1}{M}\Big\}, \tag{3.49}$$
$$M \doteq \sup\big\{|g_\alpha(x)|\ :\ x \in B(\Omega_\alpha,\, \rho_2), \quad \alpha = 1, \ldots, N\big\}.$$

Let $x_\pi : [0,\, \infty[\ \mapsto \mathbb{R}^n$ be a sampling solution starting from x_0 and corresponding to a set of measurement errors $\{e_i\}_{i=0}^m$ and to an external disturbance $d(\cdot) \in \mathbf{L}^\infty$ that satisfy (3.33)–(3.34) with

$$\overline{k} \doteq \frac{1}{2\overline{c}}. \tag{3.50}$$

We will first show the following.

Lemma 3.6. *The map*

$$i \longmapsto \alpha^*(\tau_i) \doteq \alpha^*\big(x_\pi(\tau_i) + e_i\big), \qquad i \geq 0, \tag{3.51}$$

is nondecreasing.

Indeed, assume that $\alpha^*(\tau_i) = \widehat{\alpha}$, which by definitions (1.3), (3.3), (3.5) implies

$$x_\pi(\tau_i) + e_i \in D_{\widehat{\alpha}}, \tag{3.52}$$

$$x_\pi(\tau_{i+1}) = x^{\widehat{\alpha}}\big(\tau_{i+1};\ \tau_i,\, x_\pi(\tau_i),\, d\restriction_{[\tau_i,\, \tau_{i+1}]}\big). \tag{3.53}$$

Then, because of (3.33), (3.49)–(3.50), one has

$$x_i \doteq x_\pi(\tau_i) \in B(D_{\widehat{\alpha}},\ \overline{k}\delta) \subset B(\Omega_{\widehat{\alpha}},\ \rho_2). \tag{3.54}$$

We shall consider separately the case in which

$$x_i \in D_{\widehat{\alpha}}^{\overline{k}\delta} \subset \Omega_{\widehat{\alpha}}^{\overline{k}\delta}, \qquad \overline{k}\delta \leq \rho_2, \tag{3.55}$$

and the case where

$$x_i \in B(D_{\widehat{\alpha}},\ \overline{k}\delta), \qquad d\big(x_i\,,\, \partial\Omega_{\widehat{\alpha}}\big) \leq \overline{k}\delta \leq \rho_2. \tag{3.56}$$

In the first case, using (3.53) and applying (P)$_2$ we deduce that $x_\pi(\tau_{i+1}) \in \Omega_{\widehat{\alpha}}^{\overline{k}\delta}$, which, in turn, because of (3.33), (3.49)–(3.50), implies

$$x_\pi(\tau_{i+1}) + e_{i+1} \in \Omega_{\widehat{\alpha}}. \tag{3.57}$$

From (3.57), by definition (1.3) we derive

$$\alpha^*(\tau_{i+1}) \geq \widehat{\alpha}, \tag{3.58}$$

proving the lemma whenever (3.55) holds. On the other hand, when (3.56) is verified, since by (3.32), (3.50) one has

$$\tau_{i+1} - \tau_i \geq \frac{\delta}{2} = \overline{c}\overline{k} \cdot \delta,$$

applying $(\mathrm{P})_3$, we deduce $x_\pi(\tau_{i+1}) \in \Omega_{\widehat{\alpha}}^{2\overline{k}\delta}$. This again implies (3.57)–(3.58), completing the proof of Lemma 3.6.

Next, relying on $(\mathrm{P})_1$ and setting

$$\begin{aligned} \text{(3.59)} \quad & i'_\alpha \doteq \min\left\{\, i \geq 0 \;\; ; \;\; \alpha^*(\tau_i) = \alpha, \quad x_\pi(\tau_i) \notin B(0,\, 2r/3) \,\right\}, \\ & i''_\alpha \doteq \max\left\{\, i \geq 0 \;\; ; \;\; \alpha^*(\tau_i) = \alpha\,, \quad x_\pi(\tau_i) \notin B(0,\, 2r/3) \,\right\}, \qquad \alpha \in \mathrm{Range}(\alpha^*), \end{aligned}$$

we deduce

$$\tau_{i''_\alpha} - \tau_{i'_\alpha} \leq T_\alpha \qquad \text{for all} \;\; \alpha \in \mathrm{Range}(\alpha^*). \tag{3.60}$$

Indeed, if (3.60) does not hold, by definitions (3.3), (3.5) one has

$$x_{i'_\alpha} \doteq x_\pi(\tau_{i'_\alpha}) \in B(D_\alpha,\; \overline{k}\delta) \subset B(\Omega_\alpha,\; \rho_2/2), \tag{3.61}$$

$$x_\pi(t) = x^\alpha\big(t;\; \tau_{i'_\alpha},\, x_{i'_\alpha},\, d\restriction_{[\tau_{i'_\alpha},\tau_{i''_\alpha+1}]}\big) \qquad \text{for all} \;\; t \in [\tau_{i'_\alpha},\; \tau_{i''_\alpha+1}]. \tag{3.62}$$

But then, applying $(\mathrm{P})_1$, one could find some $\widehat{i} \leq i''_\alpha$ such that

$$x_\pi(t) \in B(D_\alpha,\, \rho_2) \setminus B(D_\alpha,\, 2\overline{k}\delta) \qquad \text{for all} \;\; t \in [\tau_{\widehat{i}},\, \tau_{i''_\alpha+1}].$$

By definitions (1.3), (3.51) and because of (3.33), this implies

$$\alpha^*(\tau_i) > \alpha \qquad \text{for all}\; \widehat{i} \leq i \leq i''_\alpha,$$

providing a contradiction with (3.59).

To conclude the proof of Theorem 3.5, we observe that the monotonicity of the map (3.51), together with the estimate (3.60), implies that there exists some time $\overline{t}_{x_\pi} < T'' \doteq \sum_{\alpha=1}^{N} T_\alpha$ such that (3.36) is verified. Moreover, (3.35) clearly follows from (3.37) and (3.49). ☐

Remark 3.3. Consider a partition $\pi = \{\tau_0 = 0,\, \tau_1,\, \ldots,\, \tau_{m+1} = T\}$ of the interval $[0,\, T\,]$ having the property (3.32). If we associate to a set of measurement errors $\{e_i\}_{i=1}^m$ satisfying (3.33) the piecewise constant function $\zeta\;\; : [0,\, T] \mapsto \mathbb{R}^n$ defined as

$$\zeta(t) = e_i \qquad \text{for all}\; t \in \;\,]\tau_i,\, \tau_{i+1}],$$

then

$$\mathrm{Tot.Var.}\{\zeta\} \leq 4\overline{k} \cdot T.$$

Thus, taking the constant $\overline{k}$ sufficiently small we may reinterpret the *discrete* internal disturbance allowed for a sampling solution in Theorem 3 as a particular case of the measurement errors with small total variation considered in Theorem 3.4.

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